STAR FACTORIZATIONS AND NONCROSSING PARTITIONS

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Abstract. We develop the relationship between minimal transitive star factorizations and noncrossing partitions. This gives a new combinatorial proof of a result by Irving and Rattan, and a specialization of a result of Kreweras. It also arises in a poset on the symmetric group whose definition is motivated by the Subword Property of the Bruhat order.

1. Introduction

Let $\mathfrak{S}_n$ be the symmetric group on $[n] := \{1, \ldots, n\}$. For any $k \in [n]$, called the *pivot*, the group $\mathfrak{S}_n$ is generated by the transpositions

$$\circledast_{n;k} := \big\{(k\ i) : i \in [n]\backslash\{k\}\big\}.$$

For $\pi \in \mathfrak{S}_n$, a decomposition $\pi = g_1 \cdots g_r$ for $g_i \in \circledast_{n;k}$ is a *star factorization* of π. Following terminology of Pak in [7], we say that this star factorization is *transitive* if $\{g_1, \ldots, g_r\} = \circledast_{n;k}$. A *minimal* transitive star factorization refers to minimality of the length r. If $\pi \in \mathfrak{S}_n$ has m disjoint cycles, then a minimal transitive star factorization of π has length $n + m - 2$ [4, 7]. Let

$$\circledast_k(\pi)$$

denote the minimal transitive star factorizations of π with pivot k. Throughout this work, all "star factorizations" are assumed to be minimal transitive star factorizations.

Irving and Rattan showed in [4] that the number of star factorizations of a permutation $\pi \in \mathfrak{S}_n$ depended only on the cycle type of π: if π has cycles of lengths $\ell_1, \ldots, \ell_m$, then

$$|\circledast_k(\pi)| = (n + m - 2)_{m-2}\ell_1 \cdots \ell_m, \tag{1}$$

where $(a)_b$ is the falling factorial. Perhaps most striking about this result is its independence of the pivot k. In [11], we gave a combinatorial proof of that independence.

In this note, we give a new combinatorial proof of Equation (1), based on an interpretation of star-factorizations in terms of noncrossing partitions. This is indicative of a close relationship between these two classes of objects, which can be used to prove/recover several interesting results. First, using the enumeration given by [4] in Equation (1), we recover a result of Kreweras about enumerating noncrossing partitions with specified (labeled) part sizes [5]. This enumeration turns out to be independent of the part sizes themselves, and depends only on how many parts there are. Second, we show that in a Bruhat-style poset defined in terms of the minimal transitive star factorizations of elements of $\mathfrak{S}_n$, the poset's intervals are built out of noncrossing partition posets. Once again, the pivot value is immaterial, and the posets defined in this way for two different pivots are equal.

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This interplay between star factorizations and noncrossing partitions not only highlights their connection, but also a notion of "independent of ..." that certain features of each may share.

Section 2 reviews the main results of Irving and Rattan. These results are used to establish the relationship between star factorizations and labeled noncrossing partitions in Proposition 2.9 and Theorem 2.12. This yields that the number labeled noncrossing partitions of X into m parts of specified sizes is the falling factorial $(X)_{m-1}$ in Corollary 2.13. Section 3 revisits the enumeration of [4] and provides a new combinatorial proof of that result in Corollary 3.4. In Section 4, we define a poset on $\mathfrak{S}_n$ using star factorizations, motivated by the Subword Property of the Bruhat order. This is equivalent to a poset that can be defined in terms of permutation cycles (Theorem 4.3), and is thus yet another example of pivot independence arising in a property defined by star factorizations (Corollary 4.4). The relationship between this poset and noncrossing partitions is discussed in Lemma 4.5 and Corollary 4.6, and further elucidated in examples and discussions in that section. We conclude with two open questions in Section 5.

2. Star factorizations as labeled noncrossing partitions

Before envisioning star factorizations as labeled noncrossing partitions, we review two results of Irving and Rattan.

Fix positive integers $n \geqslant k$ and $\pi \in \mathfrak{S}_n$. Suppose that π consists of m disjoint cycles, $C_1, \ldots, C_m$, where C_i has length ℓ_i. Unless specified otherwise, cycles are index in order of their minimal elements. Let p be the index of the cycle containing the pivot k.

Lemma 2.1 ([4])**.**

(a) If $C_i = (a_1\ a_2\ \cdots\ a_t)$, where $i \neq p$, then some transposition $(k\ a_j)$ appears exactly twice in δ and all $(k\ a_h)$ with $h \neq j$ appear exactly once in δ. These appear as $(k\ a_j)(k\ a_{j-1})\cdots(k\ a_1)(k\ a_t)\cdots(k\ a_{j+1})(k\ a_j)$, from left to right in δ.

(b) If $C_p = (k\ b_2\ \cdots\ b_t)$, then each transposition $(k\ b_h)$ appears exactly once in δ. These appear as $(k\ b_t)\cdots(k\ b_2)$, from left to right in δ.

We can turn $\delta \in \circledast_k(\pi)$ into a word

$$\omega(\delta) \in [m]^{n+m-2}$$

recording the index of the cycle containing i for each factor $(k\ i)$.

Example 2.2. Let $\pi = (13)(285)(4)(67)$ and $k = 6$, with

$$\delta = (6\ 8)(6\ 1)(6\ 3)(6\ 1)(6\ 2)(6\ 5)(6\ 8)(6\ 7)(6\ 4)(6\ 4).$$

Then $p = 4$ because 6 appears in the fourth cycle, and

$$\omega(\delta) = 2111222433.$$

Irving and Rattan characterize the possible words $\omega(\delta)$ that are formed by $\delta \in \circledast_k(\pi)$.

Definition 2.3. A word $\omega \in [m]^{n+m-2}$ is *valid* if

- the symbol p appears $\ell_p - 1$ times,
- for all $j \in [m]\backslash\{p\}$, the symbol j appears $\ell_j + 1$ times,
- for $i \neq j$, there is no subword $ijij$ in $\omega(\delta)$, and
- for $i \neq p$, there is no subword ipi in $\omega(\delta)$.

Lemma 2.4 ([4])**.** A word $\omega \in [m]^{n+m-2}$ is valid if and only if $\omega = \omega(\delta)$ for some $\delta \in \circledast_k(\pi)$.

The third requirement in Definition 2.3 suggests a kind of noncrossing phenomenon, but the fourth requirement is not quite on target. However, if we add a copy of p to either end of a valid word ω, then we can indeed rephrase this as a noncrossing problem.

Definition 2.5. Given a valid word ω for π, let $\overline{\omega}$ be the necklace obtained by inserting p between the first and last letters of ω.

For consistency, we will read necklaces in counterclockwise order.

Example 2.6. The word $\omega = 2111222433$, valid for $(13)(285)(4)(67)$ with pivot $k = 6$, corresponds to the necklace $\overline{\omega}$ displayed in Figure 1, with the inserted copy of 4 circled.

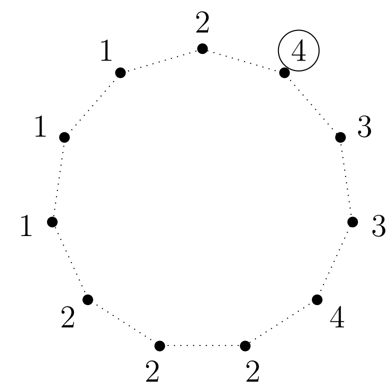

FIGURE 1. The necklace $\overline{2111222433}$, in which 4 (circled in the figure) has been inserted between the first and last letters.

These necklaces allow us to recharacterize valid words as noncrossing partitions.

Definition 2.7. Fix a sequence of positive integers $x = (x_1, \ldots, x_t)$. A *labeled noncrossing partition* of *type* x is a noncrossing partition of $[x_1 + \cdots + x_t]$ in which the parts have sizes $(x_1, \ldots, x_t)$, and the part of size x_i (equivalently, each element in that part) is labeled i. Let

$$\mathsf{L\text{-}NC}(x)$$

be the set of labeled noncrossing partitions of type x. The rotation classes of these objects are the *labeled noncrossing necklaces* of *type* x, denoted

$$\mathsf{L\text{-}NCN}(x) := \mathsf{L\text{-}NC}(x)/\text{rotation}.$$

Example 2.8. As illustrated in Figure 2, $|\mathsf{L\text{-}NC}(2,2)| = 4$ and $|\mathsf{L\text{-}NCN}(2,2)| = 1$.

Throughout this section, we will refer to the sequence $\ell' = (\ell'_1, \ldots, \ell'_m)$, where

$$\ell'_i := \begin{cases} \ell_i + 1 & \text{if } i \in [m]\backslash\{p\}, \text{ and} \\ \ell_p & \text{if } i = p. \end{cases} \tag{2}$$

Proposition 2.9. Consider a necklace α with ℓ'_i copies of i, for all $i \in [m]$. Then $\alpha \in \mathsf{L\text{-}NCN}(\ell')$ if and only if $\alpha = \overline{\omega}$ for some valid word $\omega \in [m]^{n+m-2}$.

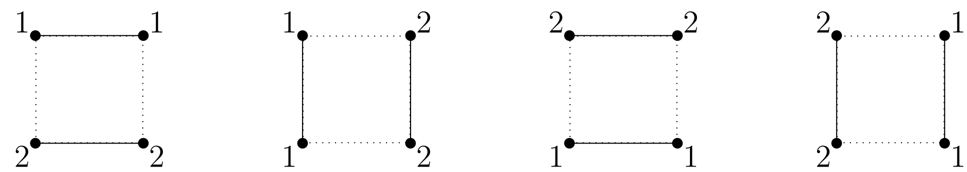

FIGURE 2. The four labeled noncrossing partitions of type $(2,2)$, which all are rotationally equivalent.

Proof. If $\alpha \in \mathsf{L\text{-}NCN}(\ell')$, then removing any copy of p and reading counterclockwise will produce a string ω that satisfies all requirements of Definition 2.3. Thus $\alpha = \overline{\omega}$, with that chosen copy of p being the inserted one. Now suppose that α has a crossing, and consider all ℓ'_p words obtained by removing one copy of p from α and reading the remaining letters in counterclockwise order. If the crossing in α involves values other than p, then that crossing will also appear in the resulting word, violating the third requirement for validity. On the other hand, if that crossing involves p and some letter i, then there is a substring ipi appearing in all ℓ'_p words obtained in this way, violating the fourth requirement for validity. $\square$

Proposition 2.9 is illustrated in Figure 3, continuing Example 2.6.

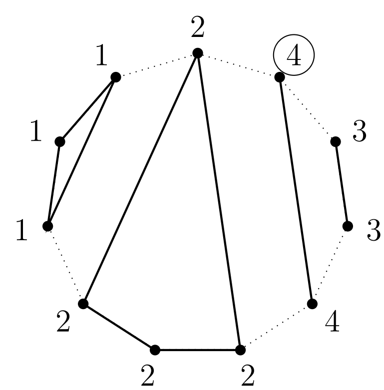

FIGURE 3. A labeled noncrossing necklace of type $(3,4,2,2)$, marked to match Figure 1.

Definition 2.10. Define the map

$$\mathsf{SF}_k : \mathsf{L\text{-}NCN}(\ell') \times [\ell_1] \times \cdots \times [\ell_m] \to \circledast_k(\pi)$$
$$(\alpha, d_1, \ldots, d_m) \mapsto \delta$$

as follows. First, fix a presentation of α and remove the d_pth appearance of p when reading the necklace in counterclockwise order from the topmost letter. Let ω be the (valid) word formed by reading the remainder of α in counterclockwise order from this removal (hence $\alpha = \overline{\omega}$). We know how to interpret each p in this word by Lemma 2.1(a). For each $j \in [m]\backslash\{p\}$ let $(k\ c_{d_j})$ be the factor that will appear twice in the δ, where c_{d_j} is the d_jth smallest value in cycle C_j. Thus, by Lemma 2.1(b), we have completely defined δ.

Example 2.11. Let $\pi = (13)(285)(4)(67)$ and $k = 6$, so $\ell' = (3,4,2,2)$. Let α be the necklace depicted in Figure 3. Let $(c_1, c_2, c_3, c_4) = (1,3,1,2)$. This c_4 removes the circled copy of 4 in Figure 3, and $\omega = 2111222433$. This c_1 means that the 1s in ω should be replaced by (6 1)(6 3)(6 1), while c_2 means that (6 8) should appear twice in δ, and c_3 indicates that (6 4) should appear twice in δ:

$$\delta = (6\ 8)(6\ 1)(6\ 3)(6\ 1)(6\ 2)(6\ 5)(6\ 8)(6\ 7)(6\ 4)(6\ 4) \in \circledast_6((13)(285)(4)(67)),$$

as in Example 2.2. If, instead, we had used $(c_1, c_2, c_3, c_4) = (2, 3, 1, 1)$, then the resulting star factorization would have been a cyclic rotation of this word δ:

$$(6\ 4)(6\ 4)(6\ 7)(6\ 8)(6\ 3)(6\ 1)(6\ 3)(6\ 2)(6\ 5)(6\ 8) \in \circledast_6((13)(285)(4)(67)).$$

Theorem 2.12. The map SF_k is a bijection.

Proof. Suppose that $\mathsf{SF}_k(\alpha, d_1, \ldots, d_m) = \mathsf{SF}_k(\beta, e_1, \ldots, e_m)$. The valid words defined by (α, d_p) and (β, e_p) must be the same in order to yield the same star factorization. Thus the necklaces α and β differ only by rotation and hence are the same. Suppose that d_p and e_p differ. Then, reading from the d_pth copy of p, the letters of the necklace are $p\omega_1\omega_2 \cdots \omega_t p\omega_1\omega_2 \cdots \omega_t$. If some $\omega_i \neq p$ then α would have a crossing, which is a contradiction. On the other hand, if all $\omega_i = p$, then we could rotate the necklace so that $d_p = e_p$. In order to have the correct factors appear twice in the factorization output by SF_k, the values d_i and e_i must be equal for all $i \in [m]\backslash\{p\}$. Thus SF_k is injective.

To show that SF_k is surjective, we can take any $\delta = \circledast_k(\pi)$ and define d_i for all $i \in [m]\backslash\{p\}$ as required by Definition 2.10. Form $\overline{\omega}$ from the word $\omega(\delta)$, and let d_p indicate the letter p that was added to form this necklace. Thus $\mathsf{SF}_k(\overline{\omega}, d_1, \ldots, d_m) = \delta$. □

We can use Theorem 2.12 to enumerate labeled noncrossing partitions with specified part sizes. This can also be shown using a result of Kreweras [5]. It is surprising that this depends only on the number of specified part sizes, not on the sizes themselves.

Corollary 2.13. The number of labeled noncrossing partitions of X into m parts of specified sizes is the falling factorial $(X)_{m-1}$.

Proof. If $m = 1$ then there is clearly $1 = (X)_{1-1}$ labeled noncrossing partition of X. Now assume that $m > 1$. Theorem 2.12 and Equation (1) yield

$$|\mathsf{L\text{-}NCN}(\ell')| = (n + m - 2)_{m-2}.$$

Recall the definition of ℓ' in Equation (2), and write $|\cdot|$ for the sum of the terms in a sequence. Thus, for any sequence $x = (x_1, \ldots, x_m)$ of positive integers with at most one $x_i = 1$, we have $|\mathsf{L\text{-}NCN}(x)| = (|x| - 1)_{m-2}$. We can expand this to allow more 1s by inducting on m. If $m = 2$, then certainly $|\mathsf{L\text{-}NCN}(x)| = 1 = (|x| - 1)_{2-2}$. Now suppose that we have a sequence $x = (x_1, \ldots, x_m)$ of positive integers, with $m \geqslant 2$. Consider $x^+ := (x_1, \ldots, x_m, 1)$. (Note, inserting the 1 anywhere else in x will work similarly.) Then each labeled noncrossing necklace of type x^+ can be obtained by inserting the label $m + 1$ anywhere among the $|x|$ letters in any element of $\mathsf{L\text{-}NCN}(x)$. Thus

$$|\mathsf{L\text{-}NCN}(x^+)| = |x| \cdot |\mathsf{L\text{-}NCN}(x)| = |x| \cdot (|x| - 1)_{m-2} = (|x|)_{m-1} = (|x^+| - 1)_{(m+1)-2},$$

as desired.

It remains now to convert noncrossing necklaces to noncrossing partitions. If $m = 1$, then $|\mathsf{L\text{-}NC}(x)| = 1 = |x|_{1-1}$. On the other hand, if $m > 1$, then the $|x|$ rotations of a necklace of type x are all distinct in $\mathsf{L\text{-}NC}(x)$. Therefore

$$\begin{aligned}|\mathsf{L\text{-}NC}(x)| &= |x| \cdot |\mathsf{L\text{-}NCN}(x)| \\ &= |x| \cdot (|x| - 1)_{m-2} \\ &= (|x|)_{m-1}.\end{aligned}$$

□

3. Pivot independence in star factorizations

Theorem 2.12 gives a correspondence between labeled noncrossing necklaces and star factorizations. We can use those partitions as an intermediary between elements of $\circledast_k(\pi)$ and elements of $\circledast_{k'}(\pi)$, giving a new combinatorial proof of Equation (1).

Fix positive integers $n \geqslant k > k'$ and a permutation $\pi \in \mathfrak{S}_n$. As before, suppose that π consists of m disjoint cycles, $C_1, \ldots, C_m$, where C_i has length ℓ_i. Let p be the index of the cycle containing the pivot k and let p' be the index of the cycle containing the pivot k'. Define sequences $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$ as

$$a_i := \begin{cases} \ell_i + 1 & \text{if } i \in [m]\backslash\{p\}, \text{ and} \\ \ell_p & \text{if } i = p, \end{cases} \qquad \text{and} \qquad b_i := \begin{cases} \ell_i + 1 & \text{if } i \in [m]\backslash\{p'\}, \text{ and} \\ \ell_{p'} & \text{if } i = p'. \end{cases}$$

Fix $\delta \in \circledast_k(\pi)$. Our goal will be to define a corresponding $\delta' \in \circledast_{k'}(\pi)$. We will do this using SF_k from the previous section, and the map Shift defined below.

Definition 3.1. Define the map

$$\begin{aligned}\mathsf{Shift} : \mathsf{L\text{-}NCN}(a) \times [\ell_1] \times \cdots \times [\ell_m] &\to \mathsf{L\text{-}NCN}(b) \times [\ell_1] \times \cdots \times [\ell_m] \\ (\alpha, d_1, \ldots, d_m) &\mapsto (\beta, d_1, \ldots, d_m)\end{aligned}$$

as follows. If $p = p'$, then set $\beta := \alpha$. Otherwise, first fix a presentation of α and locate the d_pth copy of p when reading the necklace in counterclockwise order from the topmost letter. Counterclockwise from there, locate the first copy of p', reading a string of the form $ps_1 \cdots s_t$, where $s_t = p'$. Let $h \in [1, t]$ be maximal such that the necklace obtained by replacing the string $ps_1 \cdots s_{t-1}p'$ by $ps_1 \cdots s_{h-1}ps_h \cdots s_{t-1}$ would not have any crossings. Let $\beta \in \mathsf{L\text{-}NCN}(b)$ be the necklace obtained by this replacement.

We know that the h described in Definition 3.1 exists because s_1 has the desired property.

Example 3.2. Let $n = 8$, $k = 6$, and $k' = 3$, and let $\pi = (13)(285)(4)(67)$. Then $p = 4$ and $p' = 1$. Let α be the necklace depicted in Figure 3. Then $\mathsf{Shift}(\alpha, 1, 3, 1, 2) = (\beta, 1, 3, 1, 2)$, where β is the necklace depicted in Figure 4, with the c_1st counterclockwise-from-top appearance of p' circled. Thus

Proposition 3.3. The map Shift is invertible.

Proof. Let $\mathsf{Shift}' : \mathsf{L\text{-}NCN}(b) \times [\ell_1] \times \cdots \times [\ell_m] \to \mathsf{L\text{-}NCN}(a) \times [\ell_1] \times \cdots \times [\ell_m]$ be defined exactly as Shift, except for replacing "counterclockwise from there" by "clockwise from there." Then $\mathsf{Shift}' \circ \mathsf{Shift}$ and $\mathsf{Shift} \circ \mathsf{Shift}'$ are the identity maps on their respective spaces. □

Corollary 3.4. The map

$$\mathsf{SF}_{k'} \circ \mathsf{Shift} \circ \mathsf{SF}_k^{-1} : \circledast_k(\pi) \to \circledast_{k'}(\pi)$$

is a bijection.

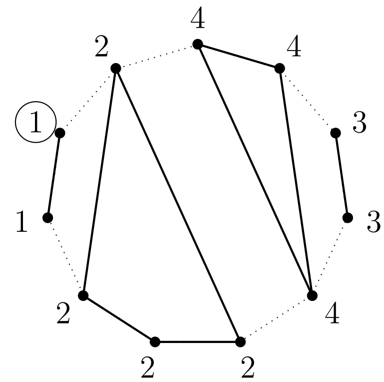

FIGURE 4. The image of $\mathsf{Shift}(\alpha, 1, 3, 1, 2)$, where α appeared in Figure 3.

Thus we have a bijection between star factorizations of π with pivot k, and star factorizations of π with pivot k', giving a new combinatorial proof of Equation (1).

Example 3.5. Continuing Example 3.2, the composition $\mathsf{SF}_3 \circ \mathsf{Shift} \circ \mathsf{SF}_6^{-1}$ would pair the star factorization

$$\delta = (6\ 8)(6\ 1)(6\ 3)(6\ 1)(6\ 2)(6\ 5)(6\ 8)(6\ 7)(6\ 4)(6\ 4) \in \ast_6((13)(285)(4)(67))$$

with the star factorization

$$\delta' = (3\ 1)(3\ 8)(3\ 2)(3\ 5)(3\ 7)(3\ 4)(3\ 4)(3\ 6)(3\ 7)(3\ 8) \in \ast_3((13)(285)(4)(67)).$$

They would be paired via $\delta \mapsto (\alpha, 1, 3, 1, 2) \mapsto (\beta, 1, 3, 1, 2) \mapsto \delta'$.

4. A Bruhat-style poset on star factorizations

The Subword Property of the Bruhat order says that we can fix a reduced decomposition of a permutation π, and find all $\sigma \leqslant \pi$ in the Bruhat order by deleting letters from that reduced decomposition [1]. This suggests a new poset structure to $\mathfrak{S}_n$ in terms of minimal transitive star factorizations.

Definition 4.1. Fix positive integers $n \geqslant k$. For $\sigma, \pi \in \mathfrak{S}_n$, say that $\sigma \preceq_k \pi$ if there exists $\gamma \in \ast_k(\sigma)$ that is a subword of $\delta \in \ast_k(\pi)$. Let $\mathsf{Star}_k(n)$ be the poset defined by $\preceq_k$ on $\mathfrak{S}_n$.

Note that Definition 4.1 is similar to the Subword Property of the Bruhat order, but not identical to it because it does not require the same $\delta \in \ast_k(\pi)$ to work for all $\sigma \preceq_k \pi$. The poset defined by $\preceq_1$ on $\mathfrak{S}_3$ appears in Figure 5.

Definition 4.2. Let $C = (c_1\ \cdots\ c_t)$ be a cycle. An *excerpt* D of C is a cycle of the form $D = (c_i\ c_{i+1}\ \cdots\ c_{i+j})$ for some positive integers i and j, where indices are taken modulo t. The cycle C has been *sliced*. A nonempty excerpt D is *proper* if $j < t$.

Characterization of the poset $\mathsf{Star}_k(n)$ follows, essentially, from Lemmas 2.1 and 2.4.

Theorem 4.3. $\sigma \preceq_k \pi$ if and only if every cycle of π is an excerpt of a cycle of σ.

Proof. Suppose that $\sigma \preceq_k \pi$, with γ and δ as in Definition 4.1. Because these are minimal *transitive* star factorizations, any letter that gets deleted from δ to form γ must have been duplicated in δ, by Lemma 2.1. Using the notation of that lemma, let $(k\ a_j)$ be a letter

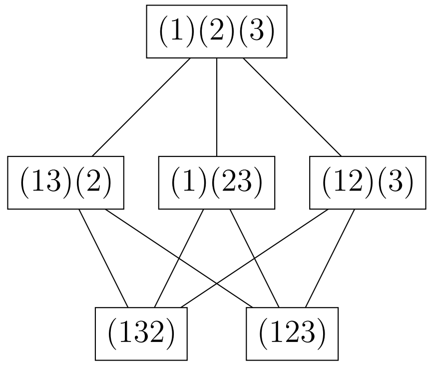

FIGURE 5. The poset $\mathsf{Star}_1(3)$.

deleted from δ to form γ. Suppose first that the factors from the C_i containing a_j had appeared between two factors from some other $C_q = (\cdots xy \cdots)$ in δ, as

$$(k\ y)\cdots(k\ a_j)(k\ a_{j-1})\cdots(k\ a_1)(k\ a_t)\cdots(k\ a_{j+1})(k\ a_j)\cdots(k\ x),$$

and let x and y be the nearest neighbors with this property. Then deleting either copy of $(k\ a_j)$ will merge the two cycles, forming

$$(\cdots x\ a_j\ a_{j+1}\cdots a_t\ a_1\cdots a_{j-1}\ y\cdots) \quad \text{or} \quad (\cdots x\ a_{j+1}\cdots a_t\ a_1\cdots a_{j-1}\ a_j\ y\cdots),$$

depending on which $(k\ a_j)$ factor was deleted. On the other hand, if the factors from this C_i do not appear between two factors from any other C_q, then C_i merges with the cycle C_p containing the pivot value, as indicated by Lemma 2.1.

Now suppose that every cycle of π is an excerpt of a cycle of σ. Orient the cycles in each permutation so that if the parentheses are deleted then the two resulting words are identical. (For a small example in $\mathfrak{S}_3$, we could write $\pi = (1)(32)$ and $\sigma = (132)$.) Construct $\gamma \in \circledast_k(\sigma)$ using Lemma 2.1, factoring each cycle from left to right. Do likewise to construct $\delta \in \circledast_k(\pi)$. This γ is necessarily a subword of δ, and so $\sigma \preceq_k \pi$. □

Reminiscent of the work of [4] and the previous section, Theorem 4.3 gives another situation in which the pivot value itself does not matter.

Corollary 4.4. For all $k, k' \in [n]$, $\mathsf{Star}_k(n) = \mathsf{Star}_{k'}(n)$.

In other words, we can call this poset

$$\mathsf{Star}(n)$$

and not specify the pivot k. Accordingly, we can write $\preceq$ for $\preceq_k$. The ordering characterization given in Theorem 4.3 allows us to describe the structure of this poset in detail. For example, the identity $(1)(2)\cdots(n)$ is the unique maximal element, and there are $(n-1)!$ minimal elements: the n-cycles.

Lemma 4.5. Fix $\sigma \preceq \pi$ in $\mathsf{Star}(n)$. Suppose that exactly one cycle of σ has been sliced to form π, and that it resulted in d proper excerpts in π. Then $[\sigma, \pi]$ is isomorphic to the noncrossing partition lattice $\mathrm{NC}(d)$.

Proof. Definition 4.2 and Theorem 4.3 allow us to reduce this to the problem of the interval $[(12\cdots d), (1)(2)\cdots(d)]$. Brady showed in [2] that the dual to this poset is isomorphic to $\mathrm{NC}(d)$. The lattice of noncrossing partitions is self-dual, completing the proof. □

Corollary 4.6. Suppose that $\sigma \preceq \pi$ in $\mathsf{Star}(n)$, that the cycles of σ that get sliced are $C_1, \ldots, C_s$ (meaning that all other cycles of σ appear identically in π), and that C_i results in d_i proper excerpts in π. Then $[\sigma, \pi] \cong \mathrm{NC}(d_1) \times \cdots \times \mathrm{NC}(d_s)$.

Proof. This follows from Lemma 4.5 and the fact that disjoint cycles act independently. □

Example 4.7. The interval $[(12345)(678), (15)(23)(4)(67)(8)]$ is isomorphic to $\mathrm{NC}(3) \times \mathrm{NC}(2)$, and is depicted in Figure 6.

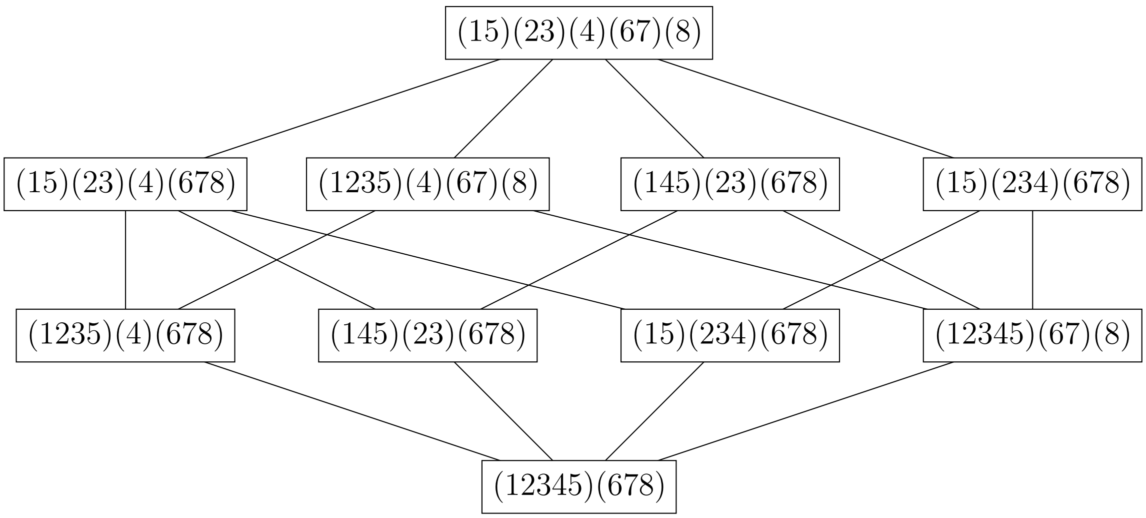

FIGURE 6. The interval $[(12345)(678), (15)(23)(4)(67)(8)] \subset \mathsf{Star}(8)$.

Suppose that $\pi \in \mathfrak{S}_n$ consists of m disjoint cycles, $C_1, \ldots, C_m$, where C_i has length ℓ_i. Additional conclusions we can draw from Theorem 4.3 include:

- $\mathsf{Star}(n)$ is not *bounded* because it has multiple ($(n-1)!$) minimal elements;
- $\mathsf{Star}(n)$ is *graded* because all maximal chains have n elements;
- as illustrated in Figure 5, two elements of $\mathsf{Star}(n)$ may not have a greatest lower bound (nor, even, a common lower bound) and thus $\mathsf{Star}(n)$ is not a lattice;
- π covers $\sum_{i<j}^{m} \ell_i \ell_j$ elements, formed by picking orientations of any two cycles and merging them in those orientations;
- π is covered by $\sum_i \binom{\ell_i}{2}$ elements, because a cycle of ℓ elements can be sliced into two proper excerpts by choosing two elements to serve as the first letters in each excerpt.

In previous work, we studied boolean elements in the Bruhat order [3, 8, 9, 10]. These had such interesting properties that we are motivated to look for boolean intervals in $\mathsf{Star}(n)$, as well. Note that this is a more general question than was studied for the Bruhat order, since we allow arbitrary intervals to be boolean, not just principal order ideals.

Definition 4.8. An interval $[\sigma, \pi]$ is *boolean* if it is isomorphic to a boolean algebra.

Due to Lemma 4.5, a study of boolean intervals in $\mathsf{Star}(n)$ amounts to understanding when the Catalan number C_n is equal to 2^{n-1}, and this happens only when $n \in \{1, 2\}$. This and Corollary 4.6 characterize boolean intervals.

Corollary 4.9. The interval $[\sigma, \pi]$ is boolean if and only if each cycle of σ is sliced into at most two excerpts to form π.

This characterization allows us to enumerate several interesting things:

- the number of boolean intervals with π as maximal element is
$$\sum_{\substack{\text{involutions } \alpha\in\mathfrak{S}_m\\ i<j \text{ with } \alpha(i)=j}} \ell_i\ell_j;$$
- the number of boolean intervals with π as minimal element is
$$\prod_i \left(1+\binom{\ell_i}{2}\right).$$

Thus, for example, the number of boolean intervals of the form $[\sigma,(1)(2)\cdots(n)]$ is the number of involutions in $\mathfrak{S}_n$, which is sequence A000085 of [6], and there are $1+3\cdot 1=4$ boolean intervals of the form $[\sigma,(123)(4)]$:
$$[(123)(4),(123)(4)],\ [(1423),(123)(4)],\ [(1243),(123)(4)],\ \text{and } [(1234),(123)(4)].$$

5. Future research

Two main avenues for future research emerge from this work. The first it to establish how else star factorizations and noncrossing partitions interact, and to determine if that interaction can be leveraged in some way. Secondly, several of the results above exhibited a property of star factorizations that was independent of the pivot value. Thus we ask, how else can this pivot independence arise, and what does it mean in those settings?

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