On certain partition bijections related to Euler's partition problem.

Aritro Pathak

Abstract

We give short elementary expositions of combinatorial proofs of some variants of Euler's partitition identity that were first addressed analytically by George Andrews [1], and later combinatorially by others [2, 3, 4]. Our methods, based on ideas from a previous paper by the author [5], enable us to state and prove new generalizations of two of these results.

1 Introduction and statement of results

There appeared a couple of conjectures [6],[7], on partition identities in the On-line Encyclopedia of Integer sequences, which were slight generalizations of Euler's famous partition result that the number of partitions of any integer n into all distinct parts equals the number of partitions where each part is odd. These results were first proven using generating functions by George Andrews [1].

We state the two conjectures here:

Theorem 1. *Let n be a positive integer, and let $a(n)$ be the number of partitions of n that contain exactly one even part, which may be repeated more than once. Then $a(n)$ is also the difference between the total number of parts in the odd partitions of n and the number of parts in the distinct partitions of n.*

Theorem 2. *Let n be a positive integer, and let $a_1(n)$ be the number of partitions of n such that there is exactly one part appearing three times and other parts appear exactly once. Then $a_1(n)$ is also the difference between the number of parts in the distinct partitions of n and the number of distinct parts in the odd partitions of n.*

After Andrews' analytical proof of these results, combinatorial proofs of the above conjectures were given in the papers of Yang [2], Fu and Tang [3], and Ballantine and Bielak [4]. Another direction in which we could study a variant of the statement of Theorem 1 is to see if we can allow more than one even part. This is what Andrews asked at the end of his paper [1]. Indeed, we have the following theorem, which is the $k = 2$ case of Theorem 1.4 of Fu and Tang's paper [3].

Theorem 3. *Let n and k be positive integers. Let $a_j(n)$ be the number of partitions of n where there are exactly k distinct even parts, each possibly repeated. Let $b_j(n)$ be the number of partitions that have exactly k repeated parts. Then $a_j(n) = b_j(n)$.*

There is a well known generalization of Euler's partition identity, which was first proven by Glaisher [8]. Glaisher originally used his technique to give the first known combinatorial proof of Euler's partition identity, and generalized it to the following:

Theorem 4. *Given an integer $d > 0$, the number of partitions $c(n)$ of an integer n such that no part is divisible by d, is the same as the number of partitions $e(n)$ of d where no part appears more than d times.*

Clearly the $d = 2$ case is Euler's partition theorem. Recently the author [5] gave a generalization of the original proof of Glaisher, by looking at the complementary problem: showing that there is a bijective correspondence between the set of partitions of n where at least one part appears d times and the set of partitions where at least one part is divisible by d. This seems to have not appeared in the literature before.

By looking at matrices such as those constructed in [5], in this paper we give quick elementary expositions of combinatorial proofs of Theorem 1, Theorem 2, and Theorem 3 but which essentially reduce to being variants of the proofs earlier given in [2, 3, 4]. We then state and prove two new results, one of which extends Theorem 2 and the other extends Theorem 3. We state them below:

Theorem 5. *Given a fixed positive integer $p \geq 2$, consider the set of partitions $A_k(n)$ of any given positive integer n, so that there are exactly k parts that are divisible by an exponent of 2 that is greater than or equal to p. Consider the set of partitions $B_k(n)$ of n that are such that exactly k parts appear at least 2^p times. Then there is a family of bijections between these two sets of partitions.*

Theorem 6. *For a positive integer n, let $f(n)$ be the number of partitions of n such that there is exactly one part appearing five times, and all other parts appear once. Also consider the set $G(n)$ of distinct partitions of n with the property that if $\alpha \cdot 2^i (i \geq 0)$ appears in the partition with α any odd integer, then $\alpha \cdot 2^{i+1}$ does not appear in the partition. Also consider the set $H(n)$ of partitions of n with only odd parts such that in the base 2 expansion of the number of times any odd number β appears, there are no two consecutive 1's. Then $f(n)$ is exactly the difference of the number of distinct parts appearing in $G(n)$ and the number of distinct parts in $H(n)$.*

2 Combinatorial proofs of Theorems 3 and a generalization.

For a given partition $\lambda \vdash n$, any odd integer x, for any $t \geq 0$, call the number of times that $x \cdot 2^t$ appears in $\lambda \vdash n$ as $n^{\lambda}_{(x,t)}$.

For every odd integer x, construct the following matrix $M^{(x)}_{ij}$ with $i, j \in \{0\} \cup \mathbb{N}$, where the j'th column from the left contains the description of $n^{\lambda}_{(x,j)}$: the (i, j) th cell contains 0 if in the base 2 expansion of $n^{\lambda}_{(x,j)}$, the coefficient of 2^i is 0, and is 1 otherwise.

As in [5], consider for any integer $k \geq 0$ the "diagonal" $D_k = \{(i, j) : i + j = k\}$. When we permute within any such fixed diagonal of any specific matrix corresponding to some odd number x, the contribution to the net sum remains the same, but we get new partitions of n. We will call the row with indexed by some $i \geq 0$ as row$_i$, and the column indexed by $j \geq 0$ as column$_j$.

We will begin with the proof of Theorem 3. By looking at the matrix construction from [5], the proof will be immediate.

Proof of Theorem 3. Consider the set $A_j(n)$ of partitions that have exactly k distinct even parts. The binary description of the number of appearances of each of these distinct even parts are distributed among k different columns in total, distributed among one or more of the matrices constructed above. Thus we are only concerned with columns with indices $j \geq 1$ whereas column$_0$ (corresponding to $j = 0$) may have any number of entries that are 1.

For each of the filled columns, push each entry of the specific column diagonally one place down and left (*i.e* $(i, j) \to (i + 1, j - 1)$), and for each $i \geq 1$, take the $(i, 0)$ entry (entries in the column$_0$) to the $(0, i)$ entry (entries in row$_0$), and keep the $(0, 0)$ entry constant. It is clear that this is a bijection, taking an element of $A_k(n)$ to an element of $B_k(n)$, which is the set of partitions where exactly k different parts are repeated as the set $B_k(n)$. □

Building on the above argument, we give a broader family of bijections where the partitions satisfy the more restrictive condition of Theorem 5.

Proof of Theorem 5. The argument here is a generalization of that of the previous proof. Consider any partition in $a_k(n)$, and any arbitrary matrix corresponding to an arbitrary odd number x.

Given any $1 \leq t \leq p$, for $j \geq (p - 1)$, we transfer the element in the cell (i, j) to the cell $(i + t, j - t)$. In the $t \times t$ block in the top left hand corner, we can permute the elements so that each element in any cell (a, b) only moves along the diagonal $D_{(a+b)}$ containing it, similar to the case in the proof in [5]. It remains to permute the elements in the set of cells $\{(i, j) : 0 \leq j < t - 1, i \geq t - 1\}$ to the set of cells $\{(i, j) : j \geq t - 1, 0 \leq i < t - 1\}$. This can be done in several different ways, as can be very easily verified. One option is to simply swap the elements $(i, j) \to (j, i)$ between the two aforementioned sets. The more general family of options is for each integer $m \geq 2$ to take the blocks $\{(i, j) : mp \leq i \leq (m + 1)p - 1, 0 \leq j \leq p - 1\}$, permute the elements within each "southwest-northeast" diagonal within this block, and transplant it to the set $\{(i, j) : mp \leq j \leq (m + 1)p - 1, 0 \leq i \leq p - 1\}$. Just as in the previous proof, the inverse maps are also obvious for each of this family of bijections, and we have defined the family of bijections. □

It should be clear in general that any of the general family of bijections for Theorem 5 stated above will not work for proving Theorem 3. Also, instead of considering powers of 2, we could easily deal with the powers of any arbitrary integer d and the statements of the above theorems are suitably modified. This analogous generalization for Theorem 3 was carried out in Fu and Tang's paper [3], and the similar generalization of Theorem 5 can easily be formulated, but we leave it for the interested reader.

3 Combinatorial proof of Theorem 2 and generalizations.

We first give a direct short proof of Theorem 2 using the same method of [5] as before. After that we state and prove two different generalizations of this result.

Proof of Theorem 2. It is clear that the number that is the difference between the total number of parts in the distinct partitions of n and the distinct parts in the odd partitions of n, can be split up as a sum of smaller parts.

Call the set of partitions of n with all distinct parts D_n, and the set of partitions of n with all odd parts as O_n. Given any fixed odd number x, in D_n the matrix $M_{ij}^{(x)}$ has a sequence of 1's in the first row, and the matrix $M_{(ij)}^{(x)}$ is otherwise empty. Call this number of 1's in the first row as d_x. In O_n, the entries of the previous matrix are essentially flipped to come to the firrst column through the Euler bijection. Each distinct part in any partition in O_n corresponds to this specific matrix. Thus, per fixed matrix corresponding to any given odd integer x, we get a contribution $(d_x - 1)$, and when we sum over all the possible matrices, we get the difference we need in the previous paragraph.

Hence, for a specific partition of n into distinct parts, and a specific odd part x and its corresponding matrix $M_{ij}^{(x)}$, consider only the cells $(0, j_1), (0, j_2), \dots, (0, j_m)$ to have 1's, for some positive integer m, and between $(0, j_a)$ and $(0, j_{a+1})$ let there be $t_a \geq 0$ empty cells, where $1 \leq a \leq m - 1$ are integers.

We show that corresponding to this matrix, we can get $m - 1$ many partitions in which there is exactly one part that appears three times and all others appear once, and corresponding to distinct matrices, we get all separate elements of this set of partitions with the above property.

Indeed, in the above matrix, consider $1 \leq n \leq m - 1$, and the partition where all the filled and unfilled cells prior to the $(0, j_n)$ cell remain unchanged, and the cells $(0, j_n)$ and the cell $(1, j_n)$ are filled with ones, and all the originally empty cells in between $(0, j_n)$ and $(0, j_{n+1})$ are filled with ones and the originally filled cell $(0, j_{n+1})$ is made empty. It is easy to check, of course, that the sum of this configuration is the same as the sum of the original configuration, and this is the only way to keep the sum constant and to have a case where there is one part appearing three times and all other parts appearing once.

Thus we have shown that corresponding to this matrix configuration, we can get $(m - 1)$ many partitions in which there's exactly one part that appears three times and all others appear once. Also it is clear that corresponding to distinct matrices, we get all separate elements of this set of partitions with the above property. Thus the proof is complete. $\square$

Now we generalize the statement of Theorem 2 to give the proof of Theorem 6.

Proof of Theorem 6. The proof of this theorem essentially uses the same argument as the previous proof. Again for a specific odd x, if there are m many appearances in the matrix $M_{ij}^{(x)}$ corresponding to the distinct partitions in $G(n)$, say in the cells $(0, j_1), (0, j_1), (0, j_3), .., (0, j_m)$ where now there is at least one empty cell between each of these filled cells, then in order to get distinct elements in the set of partitions with only one part repeated five times, all we do is choose a specific $1 \leq a \leq m - 1$, put 1's in the cells $(0, j_a)$ and $(2, j_a)$, keep the cell $(0, j_a + 1)$ empty and put 1's in all the cells from $(0, j_a + 2)$ to $(0, j_{a+1} - 1)$ and make $(0, j_{a+1})$ empty. In that process, again clearly the sum remains invariant, and we are done, as before. $\square$

In Yang's paper [2], there is a generalization of Theorem 2 in the Glaisher like setting with an arbitrary integer d, in Theorem 1.7 of that paper, where the set of partitions is considered where exactly one part appears more than d times and less than $2d$ times whereas all other parts appear less than d times. The proof of this would essentially follow our proof technique of Theorem 2 above. Combining this with our Theorem 6 above, we can formulate other interesting statements in the Glaisher setting.

4 Combinatorial proof of Theorem 1.

In this last section, we give quick combinatorial arguments for Theorem 1.

First proof of Theorem 1. In this case, consider the number of times n_α, that a particular odd number α appears in one of the odd partitions of n, and write it in the base 2 expansion: $n_\alpha = \gamma_0 + 2 \cdot \gamma_1 + 2^2 \cdot \gamma_2 + \cdots + 2^m \cdot \gamma_m$, with $\gamma_m = 1$, where each of the other γ_i are either 0 or 1. By an argument similar to that used in the previous proof, it should be clear that the total difference under consideration, can be broken up into summands, per partition and per matrix $M_{ij}^{(x)}$ corresponding to the odd number x, into the sums $(\gamma_0 + 2 \cdot \gamma_1 + 2^2 \cdot \gamma_2 + \cdots + 2^m \cdot \gamma_m) - (\gamma_0 + \gamma_1 + \gamma_2 + \cdots + \gamma_m = (2-1) \cdot \gamma_1 + (2^2 - 1) \cdot \gamma_2 + \cdots + (2^m - 1) \cdot \gamma_m$.

Now we show that per partition, and per matrix $M_{ij}^{(x)}$ corresponding to x in that partition, we will find $(2-1) \cdot \gamma_1 + (2^2 - 1) \cdot \gamma_2 + \cdots + (2^m - 1) \cdot \gamma_m$ partitions that contain exactly one even part, that may be repeated more than once. Consider the odd partitions of n where x appears $\gamma_0 + 2 \cdot \gamma_1 + 2^2 \cdot \gamma_2 + \cdots + 2^m \cdot \gamma_m$ number of times. The matrix thus has 1's in only the first column. We wish to introduce exactly one even part here; i.e. introduce some 1's into exactly one more column, while keeping the sum constant.

To do this, consider only the γ_i that equal one, and label them as $\gamma_{i_1}, \gamma_{i_2}, \gamma_{i_3}, \ldots$. Consider first γ_{i_1}. In the second column, you could potentially put any number of 1's in the cells $(0,1), (1,1), \ldots, (i_1 - 2, 1)$ (remembering that the indexing of the cells begins at 0, and the coefficient of 2^i in the first column lies in the $(i, 0)$ cell); then we will adjust the difference until the 2^ith coefficient in the first column, keeping the coefficient of 2^{i_1} equal to 0 (i.e. the cell $(i_1 - 1, 1)$ having the entry 0) and the coefficients of 2^j in the first column for $j \geq i_1 + 1$ constant. Otherwise we could simply put a 1 in the $(i_1 - 1, 1)$ cell, with all other cells below it empty. (i.e. the intersection of the i'th row with the 2nd column.). Either way, we have $(2^{i_1 - 1} - 1) + 1 = 2^{i_1 - 1}$ choices to fill up the second column, and keeping the coefficients of 2^j for $j \geq i + 1$ unchanged in the first column.

Now consider $k = 2$ and the second column. In this case, for $j \geq i_2 + 1$ we keep the coefficients of 2^j constant in the first column. We count the cases where in the second column we can put any number of 1's in the first $i_1 - 1$ cells from the top, while keeping the coefficient of 2^{i_1} equal to 1; (i.e. the cell $(i_1 - 1, 1)$ having the entry 1), the coefficient of 2^{i_2} equal to 0; (i.e. the cell $(i_2 - 1, 1)$ having the entry 0). Next we count the cases where we have at least one entry being 1 among the cells of the second column with row numbers greater than $i_1 - 1$ and less than $i_2 - 1$. Its obvious that we have thus $2^{i_2 - 1} - 1$ many new cases here. Finally consider the case where you have everything below the $i_2 - 1, 1$ cell in the second column having an entry 0 and just the $i_2 - 1, 1$ cell having the entry 1. Thus in total you have $2^{i_2 - 1}$ cases.

It should be clear that for the second column, corresponding to each of the i_k's we would have $2^{i_k - 1}$ cases to count.

Now if we wanted to fill the m'th column instead, we would have for any given k, $2^{i_k - (m-1)}$ choices corresponding to i_k. Thus for each fixed i_k, summing over all the columns, we get $2^{i_k - 1} + 2^{i_k - 2} + \cdots + 1 = 2^{i_k} - 1$ choices. Thus finally we find the number $(2-1) \cdot \gamma_1 + (2^2 - 1) \cdot \gamma_2 + \cdots + (2^m - 1) \cdot \gamma_m$. □

Below we give a quick outline of a slight variation of the argument of the above proof to arrive at the same count.

Second proof of Theorem 1. Consider again the number $n_\alpha = \gamma_0 + 2\cdot\gamma_1 + 2^2\cdot\gamma_2 + \cdots + 2^m\cdot\gamma_m$, from the previous proof, which is the number of times the odd number α appears in one of the odd partitions of n. Assume that we want to break this up into two parts, with α appearing some number of times, and $2^j\cdot\alpha$ (for any $j \geq 1$) appearing some other number of times, and these together summing up to give $\alpha\cdot n_\alpha$. Rewrite n_α as $n_\alpha = (\gamma_0 + 2\gamma_1 + \cdots + \gamma_{j-1}2^{j-1}) + 2^j(\gamma_j + 2\cdot\gamma_{j+1} + \dots)$. It is clear that we have $(\gamma_j + 2\gamma_{j+1} + \cdots + 2^{m-j}\gamma_m)$ many choices, which is the number in the second bracket above. Thus by an easy double counting, for any fixed coefficient γ_t, as we vary the j's above, we have $1 + 2 + 2^2 + \cdots + 2^{t-1} = 2^t - 1$ appearances of γ_t. Thus as before we have the total number $(2-1)\cdot\gamma_1 + (2^2-1)\cdot\gamma_2 + \cdots + (2^m-1)\cdot\gamma_m$. $\square$

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