THE WORPITZKY IDENTITY FOR THE GROUPS OF SIGNED AND EVEN-SIGNED PERMUTATIONS

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Abstract. The well-known Worpitzky identity provides a connection between two bases of $\mathbb{Q}[x]$: The standard basis $(x+1)^n$ and the binomial basis $\binom{x+n-i}{n}$, where the Eulerian numbers for the Coxeter group of type A (the symmetric group) serve as the entries of the transformation matrix. Brenti has generalized this identity to the Coxeter groups of types B and D (signed and even-signed permutations groups, respectively) using generating function techniques.

Motivated by Foata-Schützenberger and Rawlings' proof for the Worpitzky identity in the symmetric group, we provide combinatorial proofs of this identity and for their $q-$analogues in the Coxeter groups of types B and D.

1. Introduction

The well-known Worpitzky identity involves the Eulerian numbers, the original definition of which was given by Euler in an analytic context [5, §13]. Later, these numbers began to appear in combinatorial problems, and this is the context we choose to present them here.

Let S_n be the symmetric group on n elements. For any permutation $\pi \in S_n$, we say that π has a *descent* at position i if $\pi(i) > \pi(i+1)$, and we denote by $\mathrm{Des}(\pi)$ the set of descents:

$$\mathrm{Des}(\pi) := \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}. \tag{1}$$

We denote the number of descents in π by $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$.

The Eulerian number $A(n,k)$ counts the number of permutations in S_n having k descents:

$$A_{n,k} = |\{\pi \in S_n : \mathrm{des}(\pi) = k\}|.$$

One of the celebrated identities involving Eulerian numbers is the Worpitzky identity:

$$(k+1)^n = \sum_{i=0}^{n-1} A_{n,i} \binom{k+n-i}{n}. \tag{2}$$

An excellent overview of the Worpitzky identity and the Eulerian numbers can be found in Petersen's book [9, Chap. 1].

Worpitzky's identity was generalized to the Coxeter groups of types B and D in Borowiec and Młotkowski [2], though they used a different set of Eulerian numbers.

Generalizations of the Worpitzky identity, using the algebraic definition of the descents in these groups, were introduced by Brenti [3, Theorem 3.4(iii) for $q = 1$ and Corollary 4.11]:

$$(2x+1)^n = \sum_{k=0}^{n} \binom{x+n-k}{n} B_{n,k} \qquad \text{(type } B\text{)},$$

$$(2x+1)^n - 2^{n-1}(\mathscr{B}_n(x+1) - \mathscr{B}(n)) = \sum_{k=0}^{n} \binom{x+n-k}{n} D_{n,k} \qquad \text{(type } D\text{)},$$

where $B_{n,k}$ and $D_{n,k}$ are the Eulerian numbers of types B and D respectively (these notations will be defined in the next section), $\mathscr{B}(n)$ is the n-th Bernoulli number and $\mathscr{B}_n(x)$ is the n-th Bernoulli polynomial (see [7] for the definitions of these concepts).

Combinatorial identities usually have more than one possible proof. Some of them are analytic, some algebraic in nature, but the most beautiful ones are combinatorial, meaning that both sides of the identity count the same set of elements in different ways.

In our context, Foata and Schützenberger [6, p. 40] have proved the Worpitzky identity for the Coxeter group of type A in a combinatorial way (see also Rawlings [10] and Petersen [9, p. 366]). On the other hand, Brenti's proofs for the generalizations of Worpitzky's identities are non-combinatorial, and use generating function techniques.

Our contribution in this paper is combinatorial proofs for the q-analogues of the Worpitzky identity for types B and D:

$$(1+(1+q)m)^n = \sum_{k=0}^{n} \binom{n+m-k}{n} B_{n,k}(q) \qquad \text{(type } B\text{)}$$

$$\begin{aligned}(1+2m)((1+q)m)^{n-1} - (1+q)^{n-1}(\mathscr{B}_n(m+1) - \mathscr{B}(n)) &= \\ = \sum_{k=0}^{n} \tbinom{n+m-k}{n} D_{n,k}(q) & \qquad \text{(type } D\text{)}\end{aligned}$$

These q-analogues appear in Brenti [3] (the identity for type B is Theorem 3.4(iii) and the identity for type D is referred to implicitly before Theorem 4.10).

Our combinatorial proofs are in the same spirit of the proof of Foata-Schützenberger [6] for type A. In both types, we count vectors of length

n over a certain set, and in the case of type D we encounter a problem of missing vectors, which is corrected by adding a term of the form of a Bernoulli polynomial. This phenomenon appears regarding other statistics in Coxeter groups of type D, see e.g. [1].

The proof of the identity for type D integrates direct combinatorial arguments together with formal manipulations of algebraic expressions, sometimes referred to as "manipulatorics", see [9, p. 10]. This approach is, in general and in our case as well, less satisfying, but is often necessary to reformulate a complex bijective argument as an elegant formula.

The paper is organized as follows. In Section 2 we give some preliminaries, including the definitions of the Coxeter groups of types B and D and the Eulerian numbers associated with them. Sections 3 and 4 present the combinatorial proofs of the identities for types B and D respectively.

2. Preliminaries and definitions

In this section, we provide some background on the Coxeter groups of types B and D, and on the sets of vectors which we will count in our proofs of the Worpitzky identities for these groups. A general reference is Chapter 8 of Bjorner-Brenti's book [4].

2.1. **The Coxeter group of type B.** Define B_n as the group of *signed permutations* on $\{1,\dots,n\}$, i.e., the set of permutations π on $\{\pm 1, \pm 2, \dots, \pm n\}$ such that $\pi(-i) = -\pi(i)$ for $1 \leq i \leq n$. We consent that $\pi(0) = 0$ and occasionally write π_i instead of $\pi(i)$. This is the standard combinatorial realization of the Coxeter groups of type B.

We define some statistics on the group B_n. First, for a permutation $\pi \in B_n$, define:

$$\mathrm{Des}_A(\pi) = \{i : \pi(i) > \pi(i+1), 1 \leq i \leq n-1\},$$

and then denote: $\mathrm{des}_A(\pi) = |\mathrm{Des}_A(\pi)|$.

Now, for $\pi \in B_n$, we define:

$$\mathrm{Des}_B(\pi) = \begin{cases} \mathrm{Des}_A(\pi) \cup \{0\} & \pi(1) < 0 \\ \mathrm{Des}_A(\pi) & \pi(1) > 0 \end{cases}$$

As before, we denote: $\mathrm{des}_B(\pi) = |\mathrm{Des}_B(\pi)|$.

Example 2.1. Let $\pi = [\bar{1}2\bar{5}43]$. Then $\mathrm{Des}_B(\pi) = \{0, 2, 4\}$

Let $B_{n,k} = |\{\pi \in B_n : \text{des}_B(\pi) = k\}|$. The number $B_{n,k}$ is called the *Eulerian number of type* B. These numbers constitute the sequence A060187 in OEIS [8].

We define also another statistic:

$$\text{neg}(\pi) = |\{i : \pi(i) < 0, 1 \leq i \leq n\}|,$$

and the q-analogue of $B_{n,k}$ is:

$$B_{n,k}(q) = \sum_{\substack{\pi \in B_n \\ \text{des}_B(\pi) = k}} q^{\text{neg}(\pi)}.$$

Example 2.2. Let $\pi = [\bar{1}2\bar{5}43]$. Then $\text{neg}(\pi) = 2$.

2.2. **The Coxeter group of type D.** Denote by D_n the group of signed permutations on $\{1, \dots, n\}$ with an even number of negative elements. This is the standard combinatorial realization of the Coxeter group of type D.

Before presenting the Eulerian numbers for D_n, we need the following definitions: For $\pi \in D_n$, define:

$$\text{Des}_D(\pi) = \begin{cases} \text{Des}_A(\pi) \cup \{0\} & \pi(1) + \pi(2) < 0 \\ \text{Des}_A(\pi) & \pi(1) + \pi(2) > 0 \end{cases}$$

and denote: $\text{des}_D(\pi) = |\text{Des}_D(\pi)|$.

Example 2.3. Let $\pi = [\bar{3}26\bar{5}14]$. Then: $\text{Des}_D(\pi) = \{0, 3\}$ and $\text{des}_D(\pi) = 2$.

Let $D_{n,k} = |\{\pi \in D_n : \text{des}_D(\pi) = k\}|$ be the *Eulerian number of type* D (sequence A262226 in OEIS [8]). For the $q-$analogue, let

$$D_{n,k}(q) = \sum_{\substack{\pi \in D_n \\ \text{des}_D(\pi) = k}} q^{\text{neg}_2(\pi)},$$

where

$$\text{neg}_2(\pi) = |\{i \in \{2, \dots, n\} \mid \pi(i) < 0\}|.$$

In the last example, $\text{neg}_2(\pi) = 1$.

2.3. **Definitions for vectors.** Define on the alphabet

$$\Sigma = \{\mathbf{0}, \pm 1, \pm 2, \dots, \pm m\},$$

the following linear order (which will henceforth be referred to simply as the "defined order"):

$$\mathbf{0} \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots \prec -m \prec m.$$

In the context of the vectors, we define two different versions of the parameter neg, which counts the number of negative elements, one for type B and the other for type D.

Definition 2.4. The parameter $\text{neg}(\vec{v})$ is defined to be the number of negative entries in $\vec{v}$.

The parameter $\text{neg}_2(\vec{v})$ is defined to be the number of negative entries in $\vec{v}$, excluding the smallest element of $\vec{v}$ with respect to the order defined above (note the difference between this definition and the definition of neg_2 for a permutation).

3. The Worpitzky identity for type B

For the Coxeter group B_n, the following identity was proven by Brenti [3, Theorem 3.4(iii)]:

Theorem 3.1.

$$(1+(1+q)m)^n = \sum_{k=0}^{n} \binom{n+m-k}{n} B_{n,k}(q).$$

The proof of this identity in [3] uses manipulations of generating functions (see the proof of Theorem 3.4(ii)). Here, we present a combinatorial proof based on a direct counting argument.

We start by presenting an algorithm which associates with each vector a signed permutation.

Algorithm 3.2. Let $\vec{v} = (a_1, a_2, \ldots, a_n) \in (\{\mathbf{0}, \pm 1, \ldots, \pm m\})^n$. First write the entries of $\vec{v}$ in a row according to the defined order. This yields a permutation $\pi = [\pi_1, \ldots, \pi_n] \in S_n$ satisfying $a_{\pi_1} \le a_{\pi_2} \le \cdots \le a_{\pi_n}$. Moreover, if we have $a_{\pi_i} = a_{\pi_{i+1}}$, we construct the permutation π so the following condition holds:

(3) If $a_{\pi_i} \ge \mathbf{0}$, then $\pi_i < \pi_{i+1}$, and $\pi_i > \pi_{i+1}$ otherwise.

This means that π reads equal entries of $\vec{v}$ from left to right if they are nonnegative, and from right to left otherwise.

For example, let $m = 3$, $n = 6$, and $\vec{v} = (1, -2, \mathbf{0}, -1, 3, -2)$. In the defined order we have: $a_3 \prec a_4 \prec a_1 \prec a_2 = a_6 \prec a_5$, and hence (since $a_2 = a_6 < 0$) we have $\pi = [3, 4, 1, 6, 2, 5]$.

Finally, define $\sigma \in B_n$ in the following way: For each i satisfying $a_{\pi_i} \ge \mathbf{0}$, define $\sigma_i = \pi_i$ and for each i satisfying $a_{\pi_i} < \mathbf{0}$, define $\sigma_i = -\pi_i$. In the example above, we have $\sigma = [3, -4, 1, -6, -2, 5]$.
Note that $\pi_i = |\sigma_i|$.

Furthermore, if $|a_{\pi_j}| = |a_{\pi_{j+1}}|$, we have $\sigma_j < \sigma_{j+1}$. Indeed:

- If $a_{\pi_j} = a_{\pi_{j+1}} > 0$, by Condition (3) above, we have $\pi_j < \pi_{j+1}$.
- If $a_{\pi_j} = a_{\pi_{j+1}} < 0$, by Condition (3) we have $\pi_j > \pi_{j+1}$, and thus $\sigma_j = -\pi_j < -\pi_{j+1} = \sigma_{j+1}$.
- If $a_{\pi_j} = -a_{\pi_{j+1}}$, according to the defined order, we must have $a_{\pi_j} < 0$, $a_{\pi_{j+1}} > 0$, and hence $\sigma_j < 0 < \sigma_{j+1}$.

Therefore, by the definition of the descents of type B and by the construction of π, we conclude the following:

$$\text{If } j \in \mathrm{Des}_B(\sigma), \text{ then } |a_{\pi_j}| < |a_{\pi_{j+1}}| \tag{4}$$

(while assuming $\pi_0 = 0$, $a_{\pi_0} = a_0 = 0$ and recall that $\pi_i = |\sigma_i|$).

Proof of Theorem 3.1. The left hand side counts the number of vectors of the form

$$\vec{v} = (a_1, a_2, \ldots, a_n) \in (\{0, \pm 1, \ldots, \pm m\})^n,$$

where each vector $\vec{v}$ contributes $q^{\mathrm{neg}(\vec{v})}$.

As we show below, the right hand side counts the same set of vectors, where they are classified by signed permutations.

Denote by $\phi_{n,m}$ the mapping $\vec{v} \mapsto \sigma$ defined in Algorithm 3.2 and note that $\mathrm{neg}(\vec{v}) = \mathrm{neg}(\phi_{n,m}(\vec{v}))$.

We show that the number of vectors associated by the algorithm to a given permutation $\sigma \in B_n$ is exactly $\binom{n+m-\mathrm{des}_B(\sigma)}{n}$, i.e.,

$$\left|\phi_{n,m}^{-1}(\sigma)\right| = \binom{n+m-\mathrm{des}_B(\sigma)}{n},$$

from which the theorem immediately follows.

We start with an example: let $\sigma = [2, -1, 4, -5, 3] \in B_5$ and $m = 3$. Note that $\mathrm{Des}_B(\sigma) = \{1, 3\}$.

We have to find the vectors

$$\vec{v} = (a_1, a_2, a_3, a_4, a_5) \in (\{0, \pm 1, \pm 2, \pm 3\})^5$$

satisfying $a_1 < 0, a_5 < 0$ and $a_2 \geq 0, a_3 > 0, a_4 > 0$ (in the usual integer order) and

$$0 \leq |a_2| < |a_1| \leq |a_4| < |a_5| \leq |a_3| \leq 3, \tag{5}$$

and we have to show that there are $\binom{5+3-2}{5} = 6$ such vectors.

The sequence of inequalities (5) is equivalent in turn to

$$1 \leq \underbrace{|a_2| + 1}_{=b_1} < \underbrace{|a_1| + 1}_{=b_2} < \underbrace{|a_4| + 2}_{=b_3} < \underbrace{|a_5| + 2}_{=b_4} < \underbrace{|a_3| + 3}_{=b_5} \leq 6, \tag{6}$$

so we can conclude that the number of vectors satisfying the sequence of inequalities (5) is $\binom{6}{5} = 6$ as claimed.

Here is the argument in general: Let $\sigma = [\sigma_1, \dots, \sigma_n] \in B_n$. We have to find the number of vectors

$$\vec{v} = (a_1, a_2, \dots, a_n) \in (\{0, \pm 1, \dots, \pm m\})^n$$

such that for each $j \in \{1, \dots, n\}$ satisfying $\sigma_j < 0$, one has $a_{|\sigma_j|} < 0$ and:

$$0 \leq |a_{|\sigma_1|}| \leq |a_{|\sigma_2|}| \leq \cdots \leq |a_{|\sigma_n|}| \leq m,$$

with the property that the i^{th} order sign in this sequence of inequalities is strict if $i \in \mathrm{Des}_B(\sigma)$, for $0 \leq i \leq n-1$.

By adding 1 to each term in this sequence of inequalities, we obtain:

$$1 \leq |a_{|\sigma_1|}| + 1 \leq |a_{|\sigma_2|}| + 1 \leq \cdots \leq |a_{|\sigma_n|}| + 1 \leq m+1.$$

Now, in order to convert to strict order signs, we add 1 to the right hand side of each non-strict inequality (and to each inequality to the right of it). Since the number of strict order signs in the original sequence of inequalities is $\mathrm{des}_B(\sigma)$, at the end of this process, we have:

$$1 \leq b_1 < b_2 < \cdots < b_n \leq m + n - \mathrm{des}_B(\sigma),$$

where $b_i = \left|a_{|\sigma_i|}\right| + |\{j \in \mathrm{Des}_B(\sigma) \mid j < i\}| + 1$.

The number of integer solutions of this sequence of inequalities is: $\binom{m+n-\mathrm{des}_B(\sigma)}{n}$. Note that for each i, after fixing the value of b_i, the value of $a_{|\sigma_i|}$ is uniquely determined. □

4. Worpitzky identity for Coxeter groups of type D

The following generalization of Worpitzky identity for type D is due to Brenti [3, Coro. 4.11]:

Proposition 4.1. For $n \geq 2$, we have:

$$(1+2x)^n - 2^{n-1}(\mathcal{B}_n(x+1) - \mathcal{B}(n))) = \sum_{k=0}^{n} \binom{n+x-k}{n} D_{n,k}, \tag{7}$$

where $\mathcal{B}_n(\cdot)$ is the n^{th} Bernoulli polynomial and $\mathcal{B}_n$ is the n^{th} Bernoulli number.

By [7, Equation (5.12)], Equation (7) above can also be written as follows:

$$(1+2m)^n - 2^{n-1}(n(1^{n-1} + \cdots + m^{n-1})) = \sum_{k=0}^{n} \binom{n+m-k}{n} D_{n,k}.$$

Brenti [3] also alludes to the following q-analogue:

Theorem 4.2. *For $n \geq 2$, we have:*

$$(1+2m)((1+q)m)^{n-1} - (1+q)^{n-1}(n(1^{n-1}+\cdots+m^{n-1})) = \sum_{k=0}^{n} \binom{n+m-k}{n} D_{n,k}(q). \tag{8}$$

Before proving Theorems 4.1 and 4.2 combinatorially, we describe an algorithm which for a given vector $\vec{v} \in (\{0, \pm 1, \ldots, \pm m\})^n$ either associates with it a D_n-permutation or decides not to associate it with any D_n-permutation. When a D_n-permutation σ is associated to $\vec{v}$, it should satisfy the following condition, similar to the corresponding Condition (4) above for B_n:

$$\text{If } j \in \mathrm{Des}_D(\sigma), \text{ then } |a_{\sigma_j}| < |a_{\sigma_{j+1}}| \tag{9}$$

(while assuming again $\sigma_0 = 0$ and $a_{\sigma_0} = a_0 = 0$).

Algorithm 4.3. Let $\vec{v} = (a_1, \ldots, a_n) \in ([-m, m])^n$ and let $\sigma \in B_n$ be the permutation associated to $\vec{a}$ by Algorithm 3.2 above.

We distinguish between two cases, depending on whether or not the value $\mathbf{0}$ appears in $\vec{v}$.

First case: The number $\mathbf{0}$ appears in $\vec{v}$. Let i be the smallest index satisfying $a_i = \mathbf{0}$, and therefore $\sigma_1 = i$. We consider two sub-cases:

(a) If the number of negative signs in $\vec{v}$ is even, then $\sigma \in D_n$.
- If $0 \notin \mathrm{Des}_D(\sigma)$, i.e. $\sigma_1 + \sigma_2 > 0$, then we associate σ to $\vec{v}$.
- Otherwise, if $0 \in \mathrm{Des}_D(\sigma)$, then we do not associate any D_n-permutation to $\vec{v}$, since by Condition (9), we have $a_{\sigma_1} > a_{\sigma_0} = \mathbf{0}$, and so $\mathbf{0}$ cannot appear in $\vec{v}$.

(b) If the number of negative signs in $\vec{v}$ is odd (and so $\sigma \notin D_n$), then we modify σ by inverting the sign of σ_1 (and thus considering the first appearance of $\mathbf{0}$ in $\vec{v}$ as negative) and denote the resulting D_n-permutation by σ'.
- If $0 \notin \mathrm{Des}_D(\sigma')$, then associate σ' to $\vec{v}$.
- Otherwise, if $0 \in \mathrm{Des}_D(\sigma')$, then we do not associate any D_n-permutation to $\vec{v}$, again, in order to prevent a contradiction with Condition (9).

Example 4.4. Given $n = 3, m = 2$ and $\vec{v} = (-2, 0, 0)$. Then $\sigma = [2, 3, -1] \notin D_3$, and so we invert the sign of σ_1 to obtain $\sigma' = [-2, 3, -1] \in D_3$. The D_3-permutation σ' will be associated with $\vec{v}$, since $0 \notin \mathrm{Des}_D(\sigma')$.

On the other hand, if we take $\vec{v} = (2, 0, -1)$, then we get $\sigma = [2, -3, 1] \notin D_3$, so we must set $\sigma' = [-2, -3, 1] \in D_3$. Since $0 \in \mathrm{Des}_D(\sigma')$, we refrain from associating $\vec{v}$ with a D_3-permutation.

Second case: The value $\mathbf{0}$ does not appear in $\vec{v}$. In this case:

- If the number of negative signs in $\vec{v}$ is even, then we associate $\sigma \in D_n$ to $\vec{v}$;
- Otherwise, we do not associate any D_n-permutation to $\vec{v}$, since the obtained permutation σ is not in D_n.

We summarize Algorithm 4.3 in the following flowchart:

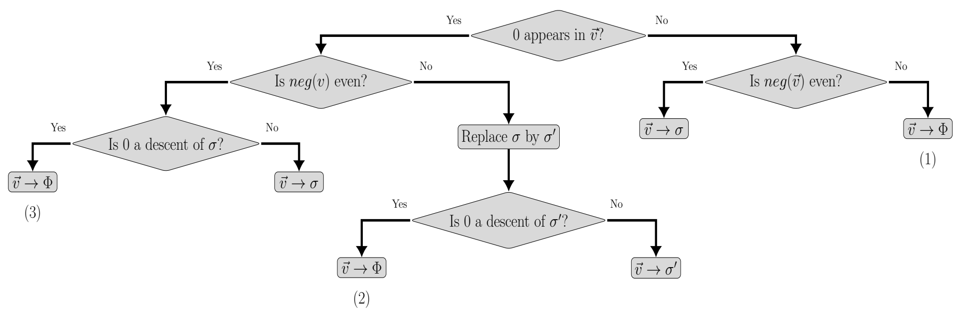

FIGURE 1. Flowchart of Algorithm 4.3

Denote by $\psi_{n,m}$ the (partial) mapping that associates a vector $\vec{v}$ with its permutation $\sigma \in D_n$ according to Algorithm 4.3.

We point out that any vector $\vec{v}$ not associated to a D_n-permutation by this algorithm must contain at most one zero, since if it contains more than one zero, then Algorithm 3.2 (for B_n) yields a permutation σ with $\sigma_2 > \sigma_1 > 0$ (since $\mathbf{0}$'s are read from left to right in that algorithm). Hence even if the sign of σ_1 is changed, we still have $0 \notin \mathrm{Des}_D(\sigma)$ and so $\vec{v}$ will be associated to some permutation (σ or σ') in D_n.

The proof of Theorem 4.1 consists of the following two lemmas:

Lemma 4.5. *For each $\sigma \in D_n$, we have $|\psi^{-1}_{n,m}(\sigma)| = \binom{n+m-\mathrm{des}_D(\sigma)}{n}$.*

Lemma 4.6. *The number of vectors not associated to any D_n-permutation by Algorithm 4.3 (called* 'missing' vectors*) is*

$$2^{n-1} n \sum_{j=0}^{m-1} (j+1)^{n-1}.$$

These two lemmas together clearly prove Theorem 4.1.

Proof of Lemma 4.5. As in the proof of Theorem 3.1 for type B, we need to show that the number of vectors associated by $\psi_{n,m}$ to each

D_n-permutation σ is equal to $\binom{n+m-\mathrm{des}_D(\sigma)}{n}$. The proof is identical to the parallel proof for type B, using the principle in Equation (9) above, so that if $0 \in \mathrm{Des}_D(\sigma)$ then $|a_{\sigma_1}| > 0$ (recall that $0 \in \mathrm{Des}_D(\sigma)$ if $\sigma_1 + \sigma_2 < 0$). Note that $\mathbf{0}$ can be considered as a negative value for the associated D_n-permutation, provided that $0 \notin \mathrm{Des}_D(\sigma)$, as in sub-case (b) of the first case in the algorithm. □

Example 4.7.
(a) Let $\sigma = [2, -3, 1, 4, -5] \in D_5$ and assume that $m = 4$. Note that $\mathrm{Des}_D(\sigma) = \{0, 1, 4\}$, hence, by Condition (9) we have $|a_{\sigma_1}| > 0$, so that the value $\mathbf{0}$ does not appear in any vector associated to σ.

We have to find the set of vectors

$$\vec{v} = (a_1, a_2, a_3, a_4, a_5) \in (\{\mathbf{0}, \pm 1, \pm 2, \pm 3, \pm 4\})^5,$$

satisfying $a_3 < 0, a_5 < 0$ and $a_1 \geq 0, a_2 \geq 0, a_4 \geq 0$ (in the usual integer order) and

$$0 < |a_2| < |a_3| \leq |a_1| \leq |a_4| < |a_5| \leq 4, \tag{10}$$

and one should find $\binom{5+4-3}{5} = 6$ such vectors.

The sequence of inequalities (10) is equivalent in turn to

$$1 \leq \underbrace{|a_2|}_{=b_1} < \underbrace{|a_3|}_{=b_2} < \underbrace{|a_1| + 1}_{=b_3} < \underbrace{|a_4| + 2}_{=b_4} < \underbrace{|a_5| + 2}_{=b_5} \leq 6, \tag{11}$$

so we can conclude that the number of vectors satisfying the sequence of inequalities (10) is $\binom{6}{5} = 6$ as needed. One can check that these vectors are: $(2, 1, -2, 2, -3)$, $(2, 1, -2, 2, -4)$, $(2, 1, -2, 3, -4)$, $(3, 1, -2, 3, -4)$, $(3, 1, -3, 3, -4)$ and $(3, 2, -3, 3, -4)$.

(b) We present also an example in which $0 \notin \mathrm{Des}_D(\sigma)$, so that σ may be associated with vectors which contain the value $\mathbf{0}$. Let $m = 2$ and let $\sigma = [-1, 2, -3] \in D_3$. The requirements here are that $a_1 \leq 0$ and $a_3 \leq 0$ and also that $0 \leq |a_1| \leq |a_2| < |a_3| \leq 2$.
The vectors associated with $[-1, 2, -3]$ are therefore $(\mathbf{0}, \mathbf{0}, -1)$, $(\mathbf{0}, \mathbf{0}, -2)$, $(\mathbf{0}, 1, -2)$ and $(-1, 1, -2)$. Note that the initial value $\mathbf{0}$ in the first three vectors are considered as negative (these vectors undergo the modification from $[1, 2, -3]$ to $[-1, 2, -3]$ in the first case of Algorithm 4.3).

Proof of Lemma 4.6. There are three types of 'missing' vectors:

(1) **Vectors which do not contain 0 and having an odd number of negative signs:** In this case, no correction of the number of signs is possible due to the lack of $\mathbf{0}$ (the presence of which could be used to add one to the number of negative signs), so

this type of vectors is missing (Leaf (1) in the flowchart appearing in Figure 1).

(2) Vectors which contain $\mathbf{0}$ (i.e. $a_{\sigma_1} = \mathbf{0}$) and have **an odd number of negative signs**, such that after the modification (of σ to σ') we get $0 \in \mathrm{Des}_D(\sigma')$ (Leaf (2) in the flowchart appearing in Figure 1): In this case, we must have (after the modification) $\sigma'_1 < 0$, and the condition $\sigma'_1 + \sigma'_2 < 0$ implies one of the following two possibilities:

   (a) $|\sigma_1| > |\sigma_2|$, i.e., **$\mathbf{0}$ is to the right of the element of $\vec{v}$ which follows $\mathbf{0}$ in the defined order.** In this sub-case, the sign of σ_2 is **arbitrary.**

   (b) $|\sigma_1| < |\sigma_2|$, but $\sigma_2 < 0$. In this sub-case, **the element of $\vec{v}$ following $\mathbf{0}$ in the defined order must be negative and must be located to the right of $\mathbf{0}$ in $\vec{v}$.**

(3) Vectors which contain $\mathbf{0}$ (again, recall that this means $a_{\sigma_1} = \mathbf{0}$), have **an even number of negative signs**, and their associated D_n-permutation σ satisfies $0 \in \mathrm{Des}_D(\sigma)$ (Leaf (3) in the flowchart appearing in Figure 1): In this case, we have $\sigma_1 > 0$, and since the vector has an even number of negative signs, the sign of σ_1 has not been changed by the algorithm. Combining with the fact that $0 \in \mathrm{Des}_D(\sigma)$, we conclude that $\sigma_2 < 0$ and $|\sigma_2| > \sigma_1$. These two requirements mean that **the element of $\vec{v}$ following $\mathbf{0}$ in the defined order must be negative and must be located to the right of $\mathbf{0}$.**

Before counting the 'missing' vectors, we show that none of them is already counted in the pre-image of $\psi_{n,m}$ in Lemma 4.5 for any D_n-permutation. Indeed, if $\sigma \in D_n$ satisfies that $\vec{v} \in \psi^{-1}_{n,m}(\sigma)$ and the value $\mathbf{0}$ does not appear in $\vec{v}$, then the number of negative elements in σ equals the number of negative elements in $\vec{v}$. On the other hand, if the value $\mathbf{0}$ does appear in $\vec{v}$, but $0 \in \mathrm{Des}_D(\pi)$, then by Condition (9) $|a_{\sigma_1}| > 0$ which contradicts the fact that $\mathbf{0}$ appears in $\vec{v}$.

We proceed by counting the above 'missing' vectors. We will have occasion throughout to refer to the absolute value of the element of $\vec{v}$ which is smallest in the defined order after $\mathbf{0}$, which we will call the "second-smallest element" of $\vec{v}$.

We start by counting the vectors appearing in the second case ($2a$), where **$\mathbf{0}$ is to the right of the second-smallest element** and the sign of σ_2 is **arbitrary**.

Note that an element having the same absolute value as the second-smallest element but negative cannot appear to the right of **0**. Indeed, if $\sigma_2 > 0$ this will contradict the fact that we read two elements with the same absolute value but different signs starting with the negative one. On the other hand, if $\sigma_2 < 0$, this contradicts the fact that two negative elements are read from the right to the left.

Our count has the form of a triple sum. We first choose the absolute value of the second-smallest element j, which is located to the left of **0** (the outer sum), then we choose the location i of **0**, counted from the right (the middle sum, see Figure 2), and then we choose the number of appearances k of the absolute value of the second-smallest element appearing to the right of **0** (the inner sum). The multiplied terms in the sum are as follows:

- The term $[(m-j+1)^{n-i} - (m-j)^{n-i}]$ is the number of ways to fill the places to the left of the **0** such that the value j will appear at least once among them.
- The term 2^{n-k-2} counts the number of ways to sign the $n-k-1$ elements, which are not **0** and are not among the k positive appearances of the second-smallest element which are located to the right of **0**, with the additional requirement of Case (2) that the total number of signs is odd.
- The term $\binom{i-1}{k}$ is the number of ways to choose the k positions, out of the first $i-1$, which are to the right of **0**, to be occupied by the positive appearances of the second-smallest element.
- The term $(m-j)^{i-1-k}$ counts the number of ways to fill the $i-1-k$ remaining places to the right of **0** with elements of larger absolute value than the second-smallest value.

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \overset{\text{X}}{\underline{\quad}}\ \overset{0}{\underline{\quad}}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{i-1}$$

FIGURE 2.

$$\sum_{j=1}^{m}\sum_{i=1}^{n-1}\sum_{k=0}^{i-1} 2^{n-k-2}\left[(m-j+1)^{n-i} - (m-j)^{n-i}\right]\binom{i-1}{k}(m-j)^{i-1-k} =$$

$$\overset{(j \leftarrow m-j)}{=} \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\sum_{k=0}^{i-1} 2^{n-k-2}\left[(j+1)^{n-i} - j^{n-i}\right]\binom{i-1}{k} j^{i-1-k} =$$

$$\stackrel{(binom)}{=} \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\left[(j+1)^{n-i}-j^{n-i}\right]\cdot 2^{n-2}\left(j+\frac{1}{2}\right)^{i-1} =$$

$$= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\left[(2j+2)^{n-i}-(2j)^{n-i}\right](2j+1)^{i-1} =$$

$$= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\left[(2j+2)^{n-i}(2j+1)^{i-1}-(2j)^{n-i}(2j+1)^{i-1}\right] =$$

$$= \frac{1}{2}\sum_{j=0}^{m-1}\left(\underbrace{-\left[(2j+1)^{n-1}-(2j+1)^{n-1}\right]}_{i=n}+\sum_{i=1}^{n}\left[(2j+2)^{n-i}(2j+1)^{i-1}-(2j)^{n-i}(2j+1)^{i-1}\right]\right) =$$

$$= \frac{1}{2}\sum_{j=0}^{m-1}\left(\sum_{i=1}^{n}(2j+2)^{n-i}(2j+1)^{i-1}-\sum_{i=1}^{n}(2j)^{n-i}(2j+1)^{i-1}\right) =$$

$$\stackrel{(*)}{=} \frac{1}{2}\sum_{j=0}^{m-1}\left(\frac{(2j+2)^n-(2j+1)^n}{(2j+2)-(2j+1)}-\frac{(2j+1)^n-(2j)^n}{(2j+1)-(2j)}\right) =$$

$$= \frac{1}{2}\sum_{j=0}^{m-1}\left((2j+2)^n-2(2j+1)^n+(2j)^n\right) = A,$$

where in $(*)$ we used the short multiplication formula: $\frac{a^n-b^n}{a-b}=\sum_{i=1}^{n}a^{n-i}b^{i-1}$.

Next, we concentrate on Cases $(2b)$ and (3) together: in both cases, **0 is to the left of the second-smallest element, which is negative**. Since in Case $(2b)$ the total number of negative signs is odd, while in Case (3) the total number of negative signs is even, in considering both cases together we may assume that the total number of negative signs is arbitrary.

As in the previous part, our count of these two cases has the form of a triple sum. We first choose the second-smallest element j, which is located to the right of **0** (the outer sum), then we choose the location i of **0**, counted from the left (the middle sum, see Figure 3), and then the number of appearances k of the second-smallest element to the right of **0** (the inner sum), where its appearances to the left of **0** are counted in the terms of the sum. The multiplied terms in the sum are as follows:

- The term $\binom{n-i}{k}$ is the number of ways to choose the k places out of the $n-i$ places to the right of **0**, occupied by the elements having the same absolute value as the second-smallest element, including itself.

- The term 2^k-1 counts the number of ways to sign the k elements having the same absolute value as the second-smallest element appearing to the right of $\mathbf{0}$, excluding the unique possibility to sign all the elements as positive, since at least one of the appearances of the second-smallest element to the right of $\mathbf{0}$ should be negative.
- The term $(2m-2j)^{n-i-k}$ is the number of ways to fill the places to the *right* of $\mathbf{0}$ with elements of larger absolute value than the second-smallest and with arbitrary signs.
- The term $(2m-2j+2)^{i-1}$ counts the number of ways to fill the $i-1$ remaining places to the *left* of $\mathbf{0}$ with elements of larger *or equal* absolute value to the second-smallest, and with arbitrary signs.

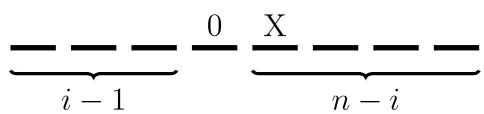

FIGURE 3.

$$\sum_{j=1}^{m}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}(2^k-1)\binom{n-i}{k}(2m-2j)^{n-i-k}(2m-2j+2)^{i-1} =$$

$$\overset{(j\leftarrow m-j)}{=} \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}(2^k-1)\binom{n-i}{k}(2j)^{n-i-k}(2j+2)^{i-1} =$$

$$= \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}(2j+2)^{i-1}\sum_{k=1}^{n-i}2^k\binom{n-i}{k}(2j)^{n-i-k} - \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}(2j+2)^{i-1}\sum_{k=1}^{n-i}\binom{n-i}{k}(2j)^{n-i-k} =$$

$$\overset{(binom)}{=} \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}(2j+2)^{i-1}[(2j+2)^{n-i}-\underbrace{(2j)^{n-i}}_{k=0}] - \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}(2j+2)^{i-1}[(2j+1)^{n-i}-\underbrace{(2j)^{n-i}}_{k=0}] =$$

$$= \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}(2j+2)^{i-1}\left[(2j+2)^{n-i}-(2j+1)^{n-i}\right] =$$

$$= \sum_{j=0}^{m-1}\sum_{i=1}^{n-1}\left[(2j+2)^{n-1}-(2j+2)^{i-1}(2j+1)^{n-i}\right] =$$

$$= \sum_{j=0}^{m-1}\left((n-1)(2j+2)^{n-1}-\sum_{i=1}^{n-1}(2j+2)^{i-1}(2j+1)^{n-i}\right) =$$

$$= \sum_{j=0}^{m-1} \left((n-1)(2j+2)^{n-1} - \left(\sum_{i=1}^{n} (2j+2)^{i-1}(2j+1)^{n-i} \right) + \underbrace{(2j+2)^{n-1}}_{i=n} \right) =$$

$$\stackrel{(*)}{=} \sum_{j=0}^{m-1} \left(n(2j+2)^{n-1} - \frac{(2j+2)^n - (2j+1)^n}{(2j+2)-(2j+1)} \right) =$$

$$= \sum_{j=0}^{m-1} \left[n(2j+2)^{n-1} - (2j+2)^n + (2j+1)^n \right] = B,$$

where in $(*)$ we used the short multiplication formula: $\frac{a^n - b^n}{a-b} = \sum\limits_{i=1}^{n} a^{n-i} b^{i-1}$.

We now sum up together Cases (2) and (3) (counted by expressions A and B):

$$\underbrace{\frac{1}{2} \sum_{j=0}^{m-1} \left((2j+2)^n - \underline{2(2j+1)^n} + (2j)^n \right)}_{=A} + \underbrace{\sum_{j=0}^{m-1} \left[n(2j+2)^{n-1} - (2j+2)^n + \underline{(2j+1)^n} \right]}_{=B} =$$

$$= \sum_{j=0}^{m-1} \left(\frac{1}{2}((2j+2)^n + (2j)^n) + n(2j+2)^{n-1} - (2j+2)^n \right) =$$

$$= \sum_{j=0}^{m-1} \left(n(2j+2)^{n-1} + \frac{1}{2}((2j)^n - (2j+2)^n) \right) =$$

$$= \sum_{j=0}^{m-1} 2^{n-1} n (j+1)^{n-1} + \sum_{j=0}^{m-1} \frac{1}{2}((2j)^n - (2j+2)^n) \stackrel{telescopic}{=}$$

$$= 2^{n-1} n \sum_{j=0}^{m-1} (j+1)^{n-1} - \frac{1}{2}(2m)^n = 2^{n-1} n \sum_{j=0}^{m-1} (j+1)^{n-1} - 2^{n-1} m^n$$

Adding the number of the vectors of Case (1), which is clearly $2^{n-1}m^n$ (half the total number of vectors not containing $\mathbf{0}$), yields the total number of vectors not associated to any D_n-permutation, namely:

$$2^{n-1} n \sum_{j=0}^{m-1} (j+1)^{n-1}.$$

This completes the proof of Theorem 4.1 as well. $\square$

The proof of Theorem 4.2 is based on Lemma 4.5 and on the following lemma:

Lemma 4.8. *The weight contributed by the vectors not associated to any D_n-permutation by Algorithm 4.3 is*

$$(1+q)^{n-1} n \sum_{j=0}^{m-1} (j+1)^{n-1}.$$

Proof of Lemma 4.8. As in Lemma 4.6, there are three types of 'missing' vectors. We now count their contribution with regard to the q-analogue.

Case (1): We put q for each negative term in $\vec{v}$ except for the smallest one and consider two cases:

(a) If the sign of the smallest element is **positive**, then we have m options to choose its value and the other elements contribute $\frac{((1+q)m)^{n-1}}{2} \cdot m$, since the number of negative elements must be odd.

(b) If the sign of the smallest element is **negative**, then after choosing the smallest element, the number of negatives among the remaining elements is even so we have again: $\frac{((1+q)m)^{n-1}}{2} \cdot m$.

In total we have $(1+q)^{n-1} m^n$.

Case (2a): We have the following triple sum:

$$\sum_{j=1}^{m} \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} (1+q)^{n-k-2} \left[(m-j+1)^{n-i} - (m-j)^{n-i}\right] \binom{i-1}{k} (m-j)^{i-1-k}.$$

Cases (2b)+(3): We have the following triple sum:

$$\sum_{j=1}^{m} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} ((1+q)^k - 1) \binom{n-i}{k} ((1+q)m - (1+q)j)^{n-i-k} ((1+q)m - (1+q)j + (1+q))^{i-1}.$$

Applying manipulations similar to the ones we used in Lemma 4.6, while replacing 2 by $(1+q)$, yields the required result. $\square$

Acknowledgments

In order to overcome the messy triple sums in Lemma 4.6, we used both the Mathematica and Maple symbolic computation abilities. We want to thank Aharon Naiman for writing a Mathematica program and Doron Zeilberger, Noah Dana-Picard and Ilias Kotsireas for helping us with the Maple.

References

[1] E. Bagno, R. Biagioli and D. Garber, *Some identities involving second kind Stirling numbers of types B and D*, Elect. J. Combin. **26**(3) (2019), P3.9, 20pp.

[2] A. Borowiec and W. Młotkowski, *New Eulerian numbers of type D*, Elect. J. Combin. **23**(1) (2016), P1.38.

[3] F. Brenti, *q-Eulerian polynomials arising from Coxeter groups*, Europ. J. Combin. **15** (1994), 417–441.

[4] A. Bjorner and F. Brenti, *Combinatorics of Coxeter groups*, Springer-Verlag, 2005.

[5] L. Eulero, *Methdus universalis series summandi ulterious promota*. Commentarii academi æscientiarum imperialis Petropolitan æ, **8** (1736), 147–158. Reprinted in his Opera Omnia, Series 1, Vol. 14, 124–137.

[6] D. Foata and M.-P. Schützenberger, *Theorie geomfrique des polynomes Euleriens*, Lecture Notes in Math. **138**, Springer-Verlag, Berlin, 1970.

[7] I. Mezo, Combinatorics and number theory of counting sequences, CRC press, FL.(2020)

[8] OEIS Foundation Inc. (2020), The On-Line Encyclopedia of Integer Sequences, http://oeis.org

[9] T.K. Petersen, *Eulerian numbers*, Birkhäuser Advanced Texts Basler Lehrbücher, Springer, New York (2015).

[10] D. Rawlings, *Generalized Worpitzky identities with applications to permutation enumeration*, Europ. J. Combin. **2** (1981), 67–78.

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