

GENERATING FUNCTIONS FOR A LATTICE PATH MODEL INTRODUCED BY DEUTSCH

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ABSTRACT. The lattice path model suggested by E. Deutsch is derived from ordinary Dyck paths, but with additional down-steps of size $-3, -5, -7, \dots$. For such paths, we find the generating functions of them, according to length, ending at level i , both, when considering them from left to right and from right to left. The generating functions are intrinsically cubic, and thus (for $i = 0$) in bijection to various objects, like even trees, ternary trees, etc.

1. INTRODUCTION

Nonnegative lattice paths consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$ and ending at the x -axis again are enumerated by Catalan numbers. Richard Stanley [10] devoted a whole book on combinatorial objects enumerated by Catalan numbers.

Nonnegative lattice paths consisting of up-steps $(1, 1)$ and down-steps $(1, -2)$ and ending at the x -axis again are enumerated by the numbers $\frac{1}{2n+1} \binom{3n}{n}$. These generalized Catalan numbers appear less often than Catalan numbers but they are sequence A001764 in [7] and also enumerate many different combinatorial objects. It is recommended to watch Donald Knuth's christmas lecture [5].

Deutsch [6] suggested different down-steps: $(1, -1), (1, -3), (1, -5), \dots$. This still leads to the numbers $\frac{1}{2n+1} \binom{3n}{n}$, and, because of the plethora of combinatorial objects, allows for bijective methods. Indeed, Cameron [2] established a bijection to so-called even trees, which are plane trees with the property that each node has an *even* number of subtrees.

In this paper, we will construct generating functions for the enumeration of "Deutsch-paths". We also generalize the concept by allowing that such a path ends at level $i \geq 0$, $i = 0$ being the original model.

Since up-steps and down-steps are no longer symmetric (as in ordinary Dyck paths), the enumeration from right to left, working with up-steps $(1, 1), (1, 3), (1, 5), \dots$ and one down-step $(1, -1)$ leads (for $i \neq 0$) for different generating functions and thus enumerations. This right-to-left model is more difficult to treat.

The method consists by first assuming a horizontal barrier at level h , so that the Deutsch-paths live in a strip. Then the generating functions are rational, and linear algebra methods

During the Corona crisis, I don't have access to a library. A more polished version of the current material will hopefully be posted soon.

(Cramer's rule) can be employed. In the resulting formulæ it is then possible to perform the limit $h \rightarrow \infty$.

To know the enumeration both, from left to right and from right to left allows to compute the total area, summed over all Deutsch-paths returning to the x -axis. The area of a path is the sum of all its ordinates.

Similar methods were already used in [8], solving an open problem in Cameron's thesis [3]. However, the results from our earlier paper cannot be used for the present enumeration of Deutsch-paths.

In whatever follows, the variable z is used in generating functions to mark the length of the paths. The substitution $z^2 = t(1-t)^2$ is used throughout, always with this meaning. In this way, the cubic equations that we encounter become manageable. Compare this with [9].

We find explicit expressions for the generating functions $f_i(z)$ and $g_i(z)$ of Deutsch-paths ending on level i from left to right resp. from right to left. They are rational in an auxiliary variable t which in itself satisfies a cubic equation. Thus one might speak about cubic Deutsch-paths. We are also able to write down explicit expressions for the coefficients of these generating functions. As a corollary we find an explicit expression for the cumulative area, summed over all Deutsch-paths of a given length. This result is derived from the generating functions. It should be stated that computer algebra (Maple) played a major role in the current project.

2. TWO GENERATING FUNCTIONS

The following expansions will be used later. Set

$$\frac{1}{1 - X + z^2 X^3} = \sum_{n \geq 0} a_n X^n$$

and

$$r_1 = 1 - t, \quad r_{2,3} = \frac{t \pm \sqrt{4t - 3t^2}}{2}.$$

We sometimes find it useful to abbreviate $W = \sqrt{4t - 3t^2}$. A direct computation confirms that

$$(1 - r_1 X)(1 - r_2 X)(1 - r_3 X) = 1 - X + z^2 X^3.$$

Some background information why this cubic equations factors nicely can be found in [5, 9].

Then, adjusting the initial values, we get the explicit expression

$$a_n = \frac{1}{3t - 1} \left[-r_1^{n+1} + \frac{3t + W}{2W} r_2^{n+1} - \frac{3t - W}{2W} r_3^{n+1} \right].$$

The other expansion we need is related to

$$\frac{1}{1 - Y^2 - zY^3} = \sum_{n \geq 0} b_n Y^n.$$

which is best written as a matrix equation

$$\begin{bmatrix} 1 & -z & 0 & -z & 0 & -z & 0 & \dots \\ -z & 1 & -z & 0 & -z & 0 & -z & \dots \\ 0 & -z & 1 & -z & 0 & -z & 0 & \dots \\ & & & \vdots & & & & \\ & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now let d_m be the determinant of this matrix with m rows and columns. We have $d_0 = 1$, $d_1 = 1$, $d_2 = 1 - z^2$, and the recursion

$$d_m = d_{m-1} - z^2 d_{m-3}.$$

The characteristic equation of this recursion is the cubic equation

$$\lambda^3 - \lambda^2 + z^2 = 0.$$

Note also the generating function

$$\begin{aligned} R(X) &= \sum_{m \geq 0} d_{m-1} X^m = 1 + X + X^2 + \sum_{m \geq 3} (d_{m-2} - z^2 d_{m-4}) X^m \\ &= 1 + X + X^2 + X \sum_{m \geq 2} d_{m-1} X^m - z^2 X^3 R(X) \\ &= 1 + X + X^2 + X R(X) - X - X^2 - z^2 X^3 R(X), \end{aligned}$$

or

$$R(X) = \frac{1}{1 - X + z^2 X^3} = \sum_{j \geq 0} d_{j-1} X^j.$$

So we see that d_{n-1} are the numbers studied in the previous section. Therefore

$$d_m = \frac{1}{3t-1} \left[-r_1^{m+2} + \frac{3t+W}{2W} r_2^{m+2} - \frac{3t-W}{2W} r_3^{m+2} \right].$$

Cramer's rule now leads to

$$f_k = z^k \frac{d_{h-k}}{d_{h+1}},$$

which, when performing the limit $h \rightarrow \infty$, leads to

$$f_k(z) = z^k r_1^{-k-1} = \frac{z^k}{(1-t)^{k+1}}.$$

This form will be useful later, but we would also like to compute $[z^n] f_k(z)$, i. e., the numbers $a_{n,k}$. We can only have contributions (which is also clear for combinatorial reasons) if $n \equiv k \pmod{2}$. So let us set $n = 2N + i$, $k = 2K + i$ for $i = 0, 1$, and compute

$$[z^{2N+i}] \frac{z^{2K+i}}{(1-t)^{2K+i+1}} = [z^{2N-2K}] \frac{1}{(1-t)^{2K+i+1}}.$$

It helps, as mentioned before, to set $z^2 = x = t(1-t)^2$. Then we can continue

$$\begin{aligned}
 a_{2N+i,2K+i} &= [x^{N-K}] \frac{1}{(1-t)^{2K+i+1}} \\
 &= \frac{1}{2\pi i} \oint \frac{dx}{x^{N-K+1}} \frac{1}{(1-t)^{2K+i+1}} \\
 &= \frac{1}{2\pi i} \oint \frac{dt(1-t)(1-3t)}{t^{N-K+1}(1-t)^{2N-2K+2}} \frac{1}{(1-t)^{2K+i+1}} \\
 &= [t^{N-K}] (1-3t) \frac{1}{(1-t)^{2N+i+2}} \\
 &= \binom{3N-K+i+1}{N-K} - 3 \binom{3N-K+i}{N-K-1}.
 \end{aligned}$$

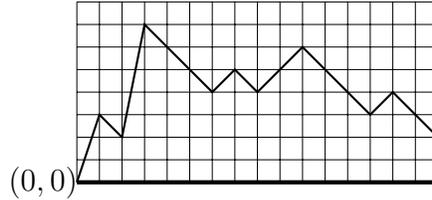
Notice in particular the enumeration of paths ending at the x -axis:

$$a_{2N,0} = \binom{3N+1}{N} - 3 \binom{3N}{N-1} = \frac{1}{2N+1} \binom{3N}{N},$$

a generalized Catalan number, enumerating many different combinatorial objects, as mentioned in the Introduction.

4. ENUMERATION OF TERNARY PATHS FROM RIGHT TO LEFT

We still prefer to work from left to right, so we change our setting as follows:



The picture shows a reversed Deutsch path ending in $(16, 2)$ and being bounded by 7 (or higher).

For the notation, we switch from $a_{n,k}$ to $b_{n,k}$ and from $f_k(z)$ to $g_k(z)$, and give a short list as an illustration.

$$\begin{aligned}
 g_0 &= 1 + z^2 + 3z^4 + 12z^6 + 55z^8 + 268z^{10} + 1338z^{12} + 6741z^{14} + 34075z^{16} + \dots \\
 g_1 &= z + 3z^3 + 12z^5 + 55z^7 + 273z^9 + 1428z^{11} + 7752z^{13} + 43263z^{15} + \dots \\
 g_2 &= 2z^2 + 9z^4 + 43z^6 + 218z^8 + 1155z^{10} + 6324z^{12} + 35511z^{14} + \dots \\
 g_3 &= z + 6z^3 + 31z^5 + 163z^7 + 882z^9 + 48967z^{11} + 27759z^{13} + \dots \\
 g_4 &= 3z^2 + 19z^4 + 108z^6 + 609z^8 + 3468z^{10} + 20007z^{12} + \dots \\
 g_5 &= z + 10z^3 + 65z^5 + 391z^7 + 2313z^9 + 13683z^{11} + \dots \\
 g_6 &= 4z^2 + 34z^4 + 228z^6 + 1431z^8 + 8787z^{10} + \dots
 \end{aligned}$$

We can now continue with the computation, according to Cramer's rule:

$$g_i = \frac{\Delta_{h+1, i+1}}{d_{h+1}}.$$

The instance $\Delta_{m, m}$ is less interesting, since we will eventually push the barrier h to infinity, and then it plays no role anymore. The instance $\Delta_{m, 1}$ will show as that $f_0 = g_0$, which is clear from combinatorial reasons, since it describes the same objects when left and right are switched. So we concentrate on the remaining cases and compute some generating functions:

$$\begin{aligned} \Delta_{m, q} &= z[X^{m-q}Y^{q-2}] \frac{(1+zY)(1+zXY)}{(1-X+z^2X^3)(1-Y^2-zY^3)} \\ &= za_{m-q}[Y^{q-2}] \frac{1+zY}{1-Y^2-zY^3} + z^2a_{m-q-1}[Y^{q-3}] \frac{1+zY}{1-Y^2-zY^3} \\ &= za_{m-q}(b_{q-2} + zb_{q-3}) + z^2a_{m-q-1}(b_{q-3} + zb_{q-4}) \end{aligned}$$

and we will rewrite this according to $g_i = \frac{\Delta_{h+1, i+1}}{d_{h+1}}$.

At that stage, we will let $h \rightarrow \infty$ and assume that $i \geq 1$.

$$\begin{aligned} g_i &= \frac{z}{(1-t)^{i+2}}(b_{i-1} + zb_{i-2}) + \frac{z^2}{(1-t)^{i+3}}(b_{i-2} + zb_{i-3}) \\ &= \frac{1}{3t-1}A\mu_1^{i-3} \left[\frac{z\mu_1^2}{(1-t)^{i+2}} + \frac{z^2\mu_1}{(1-t)^{i+2}} + \frac{z^2\mu_1}{(1-t)^{i+3}} + \frac{z^3}{(1-t)^{i+3}} \right] \\ &\quad + \frac{1}{3t-1}B\mu_2^{i-3} \left[\frac{z\mu_2^2}{(1-t)^{i+2}} + \frac{z^2\mu_2}{(1-t)^{i+2}} + \frac{z^2\mu_2}{(1-t)^{i+3}} + \frac{z^3}{(1-t)^{i+3}} \right] \\ &\quad + \frac{1}{3t-1}C\mu_3^{i-3} \left[\frac{z\mu_3^2}{(1-t)^{i+2}} + \frac{z^2\mu_3}{(1-t)^{i+2}} + \frac{z^2\mu_3}{(1-t)^{i+3}} + \frac{z^3}{(1-t)^{i+3}} \right] \\ &= \frac{t + (2t-1)W}{2(3t-1)(1-t)^{i+1}W} \mu_2^{i-3} \left[\frac{z\mu_2^2}{1-t} + \frac{z^2\mu_2}{1-t} + \frac{z^2\mu_2}{(1-t)^2} + \frac{z^3}{(1-t)^2} \right] \\ &\quad + \frac{-t + (2t-1)W}{2(3t-1)(1-t)^{i+1}W} \mu_3^{i-3} \left[\frac{z\mu_3^2}{1-t} + \frac{z^2\mu_3}{1-t} + \frac{z^2\mu_3}{(1-t)^2} + \frac{z^3}{(1-t)^2} \right] \\ &= z \frac{t + (2t-1)W}{4(3t-1)(1-t)^{i+2}W} \mu_2^{i-3} (-3t^2 + 3t + 2 - (t-3)W) \\ &\quad + z \frac{-t + (2t-1)W}{4(3t-1)(1-t)^{i+2}W} \mu_3^{i-3} (-3t^2 + 3t + 2 + (t-3)W) \\ &= \frac{z}{2(1-t)^{i+2}W} \mu_2^{i-3} (t^3 - 5t^2 + 5t - (t^2 - t - 1)W) \\ &\quad + \frac{z}{2(1-t)^{i+2}W} \mu_3^{i-3} (-t^3 + 5t^2 - 5t - (t^2 - t - 1)W) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4W(1-t)^{i+5}} \mu_2^i (t^3 - 5t^2 + 5t - (t^2 - t - 1)W)(2t^2 - 3t + W) \\
&+ \frac{1}{4W(1-t)^{i+5}} \mu_3^i (-t^3 + 5t^2 - 5t - (t^2 - t - 1)W)(2t^2 - 3t - W) \\
&= \frac{t}{2(1-t)^{i+1}W} \mu_2^i (-t + 2 + W) + \frac{t}{2(1-t)^{i+1}W} \mu_3^i (t - 2 + W) \\
&= \frac{t}{2(1-t)^{i+1}} (\mu_2^i + \mu_3^i) - \frac{z(t-2)}{2(1-t)^{i+2}} \frac{\mu_2^i - \mu_3^i}{\mu_2 - \mu_3}.
\end{aligned}$$

So we managed to compute the functions g_i for all $i \geq 0$.

5. EXTRACTION OF COEFFICIENTS

We start by noticing the pleasant formulae

$$\mu_2 \mu_3 = t - 1, \quad \mu_2 + \mu_3 = \frac{z}{1-t}.$$

We need the Girard-Waring formula, see e. g. [4]:

$$X^m + Y^m = \sum_{0 \leq k \leq m/2} (-1)^k \binom{m-k}{k} \frac{m}{m-k} (XY)^k (X+Y)^{m-2k}.$$

In our instance, we need

$$\mu_2^i + \mu_3^i = \sum_{0 \leq k \leq i/2} z^{i-2k} \binom{i-k}{k} \frac{i}{i-k} (1-t)^{3k-i}.$$

The other version is

$$\frac{X^m - Y^m}{X - Y} = \sum_{0 \leq k \leq (m-1)/2} (-1)^k \binom{m-1-k}{k} (XY)^k (X+Y)^{m-1-2k}$$

and in the special instance

$$\frac{\mu_2^i - \mu_3^i}{\mu_2 - \mu_3} = \sum_{0 \leq k \leq (i-1)/2} z^{i-1-2k} \binom{i-1-k}{k} (1-t)^{3k+1-i}.$$

So we get for $i \geq 1$

$$\begin{aligned}
g_i &= \frac{t}{2(1-t)^{i+1}} (\mu_2^i + \mu_3^i) - \frac{z(t-2)}{2(1-t)^{i+2}} \frac{\mu_2^i - \mu_3^i}{\mu_2 - \mu_3} \\
&= \frac{t}{2} \sum_{0 \leq k \leq i/2} z^{i-2k} \binom{i-k}{k} \frac{i}{i-k} (1-t)^{3k-2i+1} \\
&\quad - \frac{t-2}{2} \sum_{0 \leq k \leq (i-1)/2} z^{i-2k} \binom{i-1-k}{k} (1-t)^{3k-2i+1}
\end{aligned}$$

$$\begin{aligned}
 &= t \sum_{1 \leq k \leq (i-1)/2} z^{i-2k} \binom{i-1-k}{k-1} (1-t)^{3k-2i+1} \\
 &+ \sum_{0 \leq k \leq (i-1)/2} z^{i-2k} \binom{i-1-k}{k} (1-t)^{3k-2i+1}.
 \end{aligned}$$

Now we distinguish the two cases i even resp. odd. Set $i = 2I + \delta$, with $\delta \in \{0, 1\}$.

$$\begin{aligned}
 [z^{2N+\delta}]g_i &= [z^{2N}]t \sum_{1 \leq k \leq (i-1)/2} z^{2I-2k} \binom{i-1-k}{k-1} (1-t)^{3k-2i+1} \\
 &+ [z^{2N}] \sum_{0 \leq k \leq (i-1)/2} z^{2I-2k} \binom{i-1-k}{k} (1-t)^{3k-2i+1} \\
 &= [x^N] \sum_{1 \leq k \leq (i-1)/2} t^{I-k+1} \binom{i-1-k}{k-1} (1-t)^{k-i+1-\delta} \\
 &+ [x^N] \sum_{0 \leq k \leq (i-1)/2} t^{I-k} \binom{i-1-k}{k} (1-t)^{k-i+1-\delta} \\
 &= \sum_{1 \leq k \leq (i-1)/2} [t^N] (1-t)^{-2N-1} (1-3t) t^{I-k+1} \binom{i-1-k}{k-1} (1-t)^{k-i+1-\delta} \\
 &+ \sum_{0 \leq k \leq (i-1)/2} [t^N] (1-t)^{-2N-1} (1-3t) t^{I-k} \binom{i-1-k}{k} (1-t)^{k-i+1-\delta} \\
 &= \sum_{1 \leq k \leq (i-1)/2} \binom{i-1-k}{k-1} [t^{N-I+k-1}] (1-3t) (1-t)^{-n+k-i} \\
 &+ \sum_{0 \leq k \leq (i-1)/2} \binom{i-1-k}{k} [t^{N-I+k}] (1-3t) (1-t)^{-n+k-i} \\
 &= \sum_{1 \leq k \leq (i-1)/2} \binom{i-1-k}{k-1} \left[\binom{3N+2\delta+I-2}{N-I+k-1} - 3 \binom{3N+2\delta+I-3}{N-I+k-2} \right] \\
 &+ \sum_{0 \leq k \leq (i-1)/2} \binom{i-1-k}{k} \left[\binom{3N+2\delta+I-1}{N-I+k} - 3 \binom{3N+2\delta+I-2}{N-I+k-1} \right]
 \end{aligned}$$

6. THE AREA

Each contribution c_i to the area of a path $(0, c_0 = 0), \dots, (2n, c_{2n} = 0)$ can be seen as splitting the path into a path of length k (left to right) ending at level i and a path of length $2n - k$ (right to left) also ending at level i . Since we are working with generating

functions, all possible such splittings are taken into account when taking the product of two such generating functions.

The cumulated area is thus given as (write again $z^2 = x = t(1-t)^2$)

$$\text{AREA} = \sum_{i \geq 0} i f_i(z) g_i(z) = \frac{t(1+3t)}{(1-t)(1-3t)^2}.$$

Then

$$\begin{aligned} [z^{2n}]_{\text{AREA}} &= [x^n] \frac{t(1+3t)}{(1-t)(1-3t)^2} \\ &= \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \frac{t(1+3t)}{(1-t)(1-3t)^2} \\ &= \frac{1}{2\pi i} \oint \frac{dt(1-t)(1-3t)}{t^{n+1}(1-t)^{2n+2}} \frac{t(1+3t)}{(1-t)(1-3t)^2} \\ &= [t^{n-1}] \frac{1}{(1-t)^{2n+2}} \frac{1+3t}{1-3t} \\ &= \sum_{k \geq 0} 3^k [t^{n-1-k}] \frac{1+3t}{(1-t)^{2n+2}} \\ &= \sum_{k \geq 0} 3^k \left[\binom{3n-k}{n-1-k} + 3 \binom{3n-1-k}{n-2-k} \right]. \end{aligned}$$

The paper [1] contains general results about the area of lattice paths, of a less explicit nature than what we are doing here. Anyway, since Deutsch-paths have an infinite set of possible steps, they do not fall into the framework studied in [1].

We hope that the gentle reader will find our analysis of cubic Deutsch-paths exciting; we are sure that there are many questions left for future research, possibly also of a bijective nature.

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