FRACTAL PROJECTIONS WITH AN APPLICATION IN NUMBER THEORY

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ABSTRACT. In this paper, we discuss a connection between geometric measure theory and number theory. Among other results, we show that a problem posed by Graham on prime factors of binomial coefficients can be answered by considering a problem on projected images of fractal sets. From this point of view, we revisit a result obtained by Erdős, Graham, Ruzsa and Straus, also on prime factors of binomial coefficients.

1. PRIME FACTORS OF BINOMIAL COEFFICIENTS: GRAHAM'S QUESTION

In 1970s, Erdős, Graham, Ruzsa and Straus proved that there are infinitely many integers n such that $\binom{2n}{n}$ is coprime with $3 \times 5 = 15$, see [4]. Motivated by this result, Graham asked the following question.

Question 1.1 (Graham's binomial coefficients problem)**.** *Are there infinitely many integers $n \geq 1$ such that the binomial coefficient $\binom{2n}{n}$ is coprime with $105 = 3 \times 5 \times 7$?*

Remark 1.2. *According to* [17]*, Graham offers* 1000$ *to the first person with a solution.*

This problem turns out to be related with digit expansions of numbers in different bases. To be precise, let $b_1, \dots, b_k \geq 2$ be $k \geq 2$ different integers. For each $i \in \{1, \dots, k\}$, let $B_i \subset \{0, \dots, b_i - 1\}$ be a subset of digits in base b_i. Let $n, b \geq 2$ be integers, we write $D_b(n)$ for the set of digits used in representing n in base b. We define the following set of integers:

$$N^{B_1,\dots,B_k}_{b_1,\dots,b_k} = \{n \in \mathbb{N} : \forall i \in \{1, \dots, k\}, D_{b_i}(n) \subset B_i\}.$$

Thus $N^{B_1,\dots,B_k}_{b_1,\dots,b_k}$ contains integers with very special digit expansion simultaneously in many different bases. We will call such numbers to be with resitrcted digits. The original motivation of this type of problems is to study prime factors of $\binom{2n}{n}$. The connection between prime factors of $\binom{2n}{n}$ and digits expansions of n was established by Kummer in [12].

Theorem 1.3 (Kummer)**.** *Let p be a prime number. Then $p \not| \binom{2n}{n}$ if and only if the p-ary expansion of n contains only digits $\leq (p-1)/2$.*

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Due to Kummer's theorem, we see that for Graham's question one needs to study the set $N^{B_3,B_5,B_7}_{3,5,7}$ where

$$B_3 = \{0,1\}, B_5 = \{0,1,2\}, B_7 = \{0,1,2,3\}.$$

It is precisely the set of integers n with $\binom{2n}{n}$ being coprime with $3,5,7$. Graham's question is widely open but there are some progresses. Following the arguments in [3], it can be proved that

$$\#N^{B_3,B_5,B_7}_{3,5,7} \cap [1,N] \leq N^{0.026}$$

for all sufficiently large N. So we can say that there are not 'too many' integers n such that $\binom{2n}{n}$ is coprime with $3,5,7$. The results in [3] also lead us to the following conjecture whose statement answers Graham's question in a strong way.

Conjecture 1.4. *Let $p_1,\dots,p_k$ be $k\geq 2$ different prime numbers. For each $i\in\{1,\dots,k\}$, let*

$$B_i = \{0,\dots,(p_i-1)/2\}.$$

Denote the following number

$$s=\sum_{i=1}^k \frac{\log \#B_i}{\log p_i}=\sum_{i=1}^k \frac{\log(p_i+1)-\log 2}{\log p_i}.$$

If $s\in(k-1,k)$, then for each $\epsilon>0$ there is a constant $C_\epsilon>1$ such that

(*) $$C^{-1}_\epsilon N^{s-\epsilon}\leq \#N^{B_1,\dots,B_k}_{p_1,\dots,p_k}\cap[1,N]\leq C_\epsilon N^{s+\epsilon}$$

for all integers $N\geq 2$. If $s<k-1$ then $N^{B_1,\dots,B_k}_{p_1,\dots,p_k}$ is finite.

Remark 1.5. *The rightmost inequality of $(*)$ was proved in* [3]. *Thus the open problem is the leftmost inequality of $(*)$ and the finiteness statement. This is closely related to a Furstenberg's problem, see* [7], [20] *and* [18].

2. Results in this paper

In this paper, we provide a different approach to Graham's question and other problems related to digit expansions of number in different bases. We will relate it to projections of fractal sets, a well studied topic in geometric measure theory. We will provide a more detailed discussion on this topic in Sections 3 and 4 including the notion of self-similar sets, the strong separation condition, the Hausdorff dimension as well as radial projections. Here, we only need to know that Π_x for $x\in\mathbb{R}^d$ stands for the map

$$y\in\mathbb{R}^d\setminus\{x\}\to\Pi_x(y)=\frac{x-y}{|x-y|}\in S^{d-1}.$$

Intuitively speaking, let $A\subset\mathbb{R}^d$. Then $\Pi_x(A)$ is what an observer can see of A at a certain position $x\in\mathbb{R}^d$.

In what follows, we say that a list of numbers $a_1,\dots,a_d$ are multiplicatively independent if they are not 0 nor 1 and $1,\log a_2/\log a_1,\dots,\log a_d/\log a_1$ are linearly independent over the field of rational numbers.

Conjecture 2.1. *Let $A \subset \mathbb{R}^d, d \geq 2$ be a Cartesian product of self-similar sets in $\mathbb{R}$ with strong separation condition and uniform contraction ratios. Suppose further that the contraction ratios are multiplicatively independent. If $\dim_{\mathrm{H}} A > d-1$, then $\Pi_x(A)$ contains non-empty interiors for all $x \in \mathbb{R}^d$.*

In Section 4, we will provide several supporting heuristics. In Section 6 we will discuss some special situations when the conclusion of the above Conjecture can be checked to be true. Conjecture 2.1 turns out to be closely related to Graham's question.

Theorem 2.2. *Assuming Conjecture 2.1, there are infinitely many integers n such that $\binom{2n}{n}$ is coprime with $3 \times 5 \times 7$.*

Currently, we are not able to prove Conjecture 2.1. Nonetheless, the strategy for proving Theorem 2.2 can be adapted to prove many other (unconditional) results concerining numbers with restricted digits.

First, we shall prove the following quantitative version of a result in [4]. The number $15 = 3$ has no special significance. In pact, any two different odd primes will do.

Theorem 2.3. *Let $A = \{n \in \mathbb{N} : gcd(15, \binom{2n}{n}) = 1\}$. We have the following lower estimate for all large enough integers N,*

$$A \cap [1, N] \geq c \log N$$

where $c > 0$ is a constant.

Next, we prove the following result concerning linear forms of numbers with resitrcted digits.

Theorem 2.4. *There are infinitely many integers triples $(x, y, z) \in N_3^{\{0,1\}} \times N_4^{\{0,1\}} \times N_5^{\{0,1\}}$ with*

$$x + y = z.$$

Remark 2.5. *Let N be an integer. It is natural to consider the number $S(N)$ of solutions with x, y, z are at most N and study this number as $N \to \infty$. Our proof for this theorem actually shows that $S(N) \gtrsim \log N$. However, it might be true that $S(N) \gtrsim N^\delta$ for a constant $\delta > 0$.*

Remark 2.6. *This result says that there are infinitely many sums of powers of five that can be written as sums of powers of three and four. We list a few examples:*

$$5 = 4 + 1,$$
$$5^2 = 4^2 + 3^2,$$
$$5^3 + 5^2 = 3^4 + 4^3 + 4 + 1,$$
$$5^4 + 5^2 = 3^5 + 4^4 + 3^4 + 4^3 + 4 + 1 + 1.$$

Here, the choice of $3, 4, 5$ is by no means the only possible one. In fact, from the proof, the theorem holds for multiplicative independent integers b_1, b_2, b_3 with

$$\frac{1}{b_1 - 1} + \frac{1}{b_2 - 1} + \frac{1}{b_3 - 1} \geq 1. \tag{C}$$

In the statement of the theorem we chose $b_1 = 3, b_2 = 4, b_3 = 5$ just for concreteness. On the other hand, we believe that Condition (C) is essential. For example, we suspect that there are only finitely many integer triples $(x, y, z) \in N_9^{\{0,1\}} \times N_{10}^{\{0,1\}} \times N_{11}^{\{0,1\}}$ with $x + y = z$.

To prove the above results, we need to use Newhouse's gap lemma, see Section 3.3. This is a powerful tool for checking whether two Cantor sets intersect each other. We record here a simple but interesting observation which can be considered as a Waring type result for the middle third Cantor set

$$C_3 = \{x \in [0,1] : \text{the ternary expansion of } x \text{ contains only digits } 0, 2\}.$$

Its proof can be found in Section 3.3.

Theorem 2.7. [1] *For all $x \in [0,4]$, there exist $x_1, x_2, x_3, x_4 \in C_3$ such that*

$$x = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

More generally, for each integer $k \geq 2$ there is a number $n(k)$ such that all $x \in [0, n(k)]$ can be written as

$$x = \sum_{i=1}^{n(k)} x_i^k$$

where $x_1, \dots, x_{n(k)} \in C_3$. Moreover, $n(k) \leq 2^k$.

3. Some basics in geometric measure theory

3.1. **Hausdorff dimension of sets and measures.** For all $\delta > 0$ and $s > 0$, define the δ-approximate s-dimensional Hausdorff measure of a set $F \subset \mathbb{R}^n$ by

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \bigcup_i U_i \supset F, \operatorname{diam}(U_i) \leq \delta\right\},$$

and the s-dimensional Hausdorff measure of F by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

We then define the Hausdorff dimension of F to be

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, the space of Borel Probability measures on $\mathbb{R}^n$. The Hausdorff dimension of μ is defined to be

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : A \subset \mathbb{R}^n, \mu(A) > 0\}.$$

For more details see [5].

[1] We were told by S. Chow that this result (for $k = 2$) was conjectured in [2, Conjecture 13] and answered in [11]. We thank him for providing the references.

3.2. **Self-similar sets, and the strong separation condition.** Let $\mathcal{F} = \{f_i\}_{i\in\Lambda}$ be a finite collection of linear maps on $\mathbb{R}$. We can write down each linear maps explicitly as $f_i(x) = r_i x + a_i$. We assume that $r_i \in (0,1)$ for all $i \in \Lambda$. We call such a collection of linear maps to be a *linear IFS*. The parameters $r_i, i \in \Lambda$ are called *contraction ratios* and $a_i, i \in \Lambda$ are called *translations*. In case when all the contraction ratios are equal to $r \in (0,1)$, we call r to be the *uniform contraction ratio*.

By [10], there is a unique compact set F such that

$$F = \bigcup_{i\in\Lambda} f_i(F).$$

We call such a set F to be a *self-similar set*.

Given a positive probability vector $\{p_i\}_{i\in\Lambda}$, i.e. for $i \in \Lambda, p_i > 0$ and $\sum_{i\in\Lambda} p_i = 1$, then there is a uniquely defined measure μ such that

$$\mu = \sum_{i\in\Lambda} p_i f_i(\mu).$$

Here $f_i(\mu) = \mu \circ f_i^{-1}$ is the pushed forward measure of μ via the map f_i. It can be checked that μ is a probability measure supported on F, i.e. $\mu(F) = 1$. We call such a measure μ to be a *self-similar measure*.

We say that $\mathcal{F}$ satisfies the *strong separation condition* if the unique compact set F satisfies

$$f_i(F) \cap f_j(F) = \emptyset$$

for all $i, j \in \Lambda, i \neq j$.

3.3. **Thickness, intersection and sums of Cantor sets.** Let $A \subset \mathbb{R}$ be a compact, totally disconnect set. We shall call A a Cantor set. It is of no loss of generality to assume that $A \subset [0,1]$ and the convex hall of A is $[0,1]$. In this case, we see that $[0,1] \setminus A$ is a countable union of disjoint open intervals $\{I_i\}_{i\geq 1}$. We call those intervals to be the *bounded gaps* of A. Thus, A can be constructed by iteratively chopping out open intervals from $[0,1]$. We assume that the two end points of each interval in $\{I_i\}_{i\geq 1}$ are contained in A. We can also add $(-\infty, 0)$ and $(1,\infty)$ into the set of intervals. Thus, the set $\{I_i\}_{i\geq 1}$ is uniquely determined. Let $I = (a,b)$ be one of those bounded open intervals. We find the interval $I^- \in \{I_i\}_{i\geq 0}$ and $I^- \subset (-\infty, a)$ such that $|I^-| \geq |I|$ and there is no other such intervals between I^- and I. Similarly, we can find $I^+ \subset (b, \infty)$ to the right of I. Suppose that $I^- = (c,d)$ and $I^+ = (e,f)$. We see that

$$-\infty \leq c < d < a < b < e < f \leq \infty.$$

Let $g_L = a - d$, $g_R = e - b$ and

$$C(I) = \min\{g_L, g_R\}/|I|.$$

We define $C(A) = \inf_{I\in\{I_i\}_{i\geq 1}, I \text{ bounded}} C(I)$. This number $C(A)$ is called *the thickness* of A. We also define *the normalized thickness* of A is

$$S(A) = \frac{C(A)}{C(A)+1}.$$

Notice that $C(A), S(A)$ is unchanged if we replace A by an affine copy $aA+b$ with $a, b \in \mathbb{R}$. We have the following result due to Newhouse, see [14].

Theorem 3.1. *[Newhouse's gap lemma] Let A, B be two compact, totally disconnect sets. Suppose that A is not contained in any of the gaps of B and vice versa. If $S(A) + S(B) \geq 1$, then $A \cap B \neq \emptyset$.*

The following generalization can be found in [1, Theorem 2.4].

Theorem 3.2. *Let $A_1, \dots, A_k$ be $k \geq 2$ Cantor sets. Suppose that their convex halls are $I_1, I_2, \dots, I_k$ and the size of their largest gaps are $g_1, \dots, g_k$ respectively. Suppose further that $\sum_{i=1}^k S(A_i) \geq 1$ and $\min\{|I_1|, \dots, |I_k|\} > \max\{g_1, \dots, g_k\}$. Then $A_1 + \dots + A_k$ is an interval.*

For later use, we shall reformulate the above result in terms of intersections. Let A_1, A_2, A_3 be three Cantor sets satisfying the hypothesis of the above theorem. Consider the Cartesian product $A = A_1 \times A_2 \times A_3$. Let $v \in S^2$ be a direction vector of $\mathbb{R}^3$ and let H_v be the plane passing through the origin and normal to v. The family of planes parallel to H_v can be parametrized by $\mathbb{R}$, more precisely, $\{H_v(a) = H_v + av\}_{a\in\mathbb{R}}$. Denote $\Pi_v : \mathbb{R}^3 \to \mathbb{R}v$ to be the corresponding orthogonal projection on direction ν. The above theorem says that $\Pi_{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})}(A)$ is an interval. Moreover, since the normalized thickness is invariant under affine maps, there is a neighbourhood O of $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ in S^2 such that whenever $v \in O$, the size of the minimum convex hall is larger than the size of the maximum gap, thus $\Pi_v(A)$ is an interval.

Now if $\Pi_v(A)$ is an interval, we see that

$$\{a \in \mathbb{R} : H_v(a) \cap A \neq \emptyset\}$$

is an interval. More precisely, if $H_v(a) \cap I_1 \times I_2 \times I_3 \neq \emptyset$ then we have $H_v(a) \cap A \neq \emptyset$. We now conclude the above discussion into the following corollary.

Corollary 3.3. *Let A_1, A_2, A_3 be 3 Cantor sets. Suppose that their convex halls are I_1, I_2, I_3 and the size of their largest gaps are g_1, g_2, g_3 respectively. If*

$$\min\{|I_1|, |I_2|, |I_3|\} > \max\{g_1, g_2, g_3\}$$

then there is an open set $O \subset S^2$ such that whenever $v \in O$, we have

$$H_v(a) \cap I_1 \times I_2 \times I_3 \neq \emptyset \implies H_v(a) \cap A_1 \times A_2 \times A_3 \neq \emptyset.$$

We now compute the normalized thickness of some examples of Cantor sets. First, let $b > 2$ be an integer and let $B = \{0, 1, \dots, l\}$ where $l < b - 1$. We consider the set

$$A_b^B = \{x \in [0,1] : b\text{-ary expansion of } x \text{ contains only digits in } B\}.$$

Lemma 3.4. *Let b, B, A_b^B be as above. The normalized thickness of A_b^B is*

$$S(A_b^B) = \frac{l}{b-1}.$$

Proof. The convex hall of A_b^B is $[0,a]$ where

$$a = \sum_{i=1}^{\infty} \frac{l}{b^i} = \frac{l}{b-1}.$$

The largest gaps of A_b^B are of size

$$\frac{b-1-l}{b(b-1)}.$$

Those gaps are located in each of the following intervals

$$[0,1/b], \dots, [(l-2)/b, (l-1)/b].$$

Let I be one of those gaps, then we see that

$$C(I) = \frac{l}{b-1-l}.$$

Following this argument and the fact that A_b^B is self-similar, we see that $C(A_b^B) = l/(b-1-l)$ and $S(A_b^B) = l/(b-1)$. □

Next, we consider the middle third Cantor set $C_3 = A_3^{\{0,2\}}$. Following the above steps we see that $C(C_3) = 1, S(C_3) = 1/2$. Let $k \geq 2$ be an integer. We consider the image of C_3 under the map $x \to x^k$. We write this image as C_3^k.

Lemma 3.5. *Let k, C_3^k be as above. Then*

$$S(C_3^k) = \frac{1}{2^k}.$$

Proof. Suppose that $I = (a, a+\Delta)$ is a bounded gap of C_3. Then we see that the next gap on the right of I which is not smaller than I has left end point $a + 2\Delta$. Similarly, the next gap on the left of I which is not smaller than I has right end point $a - \Delta$. For the middle third Cantor set C_3, we always have $a \geq \Delta$.

After taking the k-th power map, we have points

$$(a-\Delta)^k, a^k, (a+\Delta)^k, (a+2\Delta)^k.$$

The gap I is now transformed into a gap of size

$$|a^k - (a+\Delta)^k|.$$

This length is increasing as a function of a as well as Δ. Thus we see that the number g_L in the definition of thickness is at least $|(a-\Delta)^k - a^k|$. We need to take care of g_R. By the above argument we see that g_R is at most $(a+2\Delta)^k - (a+\Delta)^k$. However, we need a lower bound for g_R. The problem is that a gap inside $[a+\Delta, a+2\Delta]$ might become larger than $|a^k-(a+\Delta)^k|$ after taking the k-th power. Let $u\in[a+\Delta, a+2\Delta]$ such that

$$u^k-(a+\Delta)^k=-a^k+(a+\Delta)^k.$$

Then we see that $[(a+\Delta)^k, u^k]$ will not contain any gaps of C_3^k which is larger than $-a^k+(a+\Delta)^k$. Thus g_R is at least $-a^k+(a+\Delta)^k$.

We see that

$$(**)\qquad \min\{g_L, g_R\}/|a^k-(a+\Delta)^k|\geq\frac{a^k-(a-\Delta)^k}{(a+\Delta)^k-a^k}\geq\frac{1}{2^k-1}.$$

As this holds for all bounded gaps of C_3^k we see that

$$C(C_3^k)\geq\frac{1}{2^k-1}$$

and

$$S(C_3^k)\geq\frac{1}{2^k}.$$

On the other hand, let $n\geq 1$ be an integer. Then $g=[3^{-n}, 2\times 3^{-n}]$ is a bounded gap of C_3. Now, in C_3, the next gap on the left of g with length at least $|g|$ is an infinite gap, i.e. $(-\infty, 0]$. Thus in C_3^k, the next gap on the left of $[3^{-kn}, 2^k3^{-kn}]$ with at least the same length is again $(-\infty, 0]$. This shows that the inequality (**) is sharp and the proof concludes. □

From here we see that Theorem 2.7 follows.

Proof of Theorem 2.7. By Lemma 3.5 and Theorem 3.2, it is enough to check the gap conditions stated in Theorem 3.2. As the convex call of C_3^k is $[0,1]$ and the largest gap is strictly shorter than 1, the result follows. □

4. Projections of fractal sets: general overview

Projections of fractal sets play an important role in geometric measure theory, see [6]. Let $d\geq k\geq 1$ be integers. Let $\{\Pi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous functions $\mathbb{R}^d\to\mathbb{R}^k$ parametrized by an index set Λ. We call $\{\Pi_\lambda\}_{\lambda\in\Lambda}$ a family of projections. Here we will mention two types of projections which were studied extensively.

4.1. **Linear projections.** The most intuitive notion of projection is the linear projection. Let $d\geq k\geq 1$ be integers. We want to parametrize the family of linear maps $\mathbb{R}^d\to\mathbb{R}^k$. The most convenient way is to use $Gr(k,\mathbb{R}^d)$, the Grassmanian manifold consisting all k-dimensional linear subspaces of $\mathbb{R}^d$. For each $\gamma\in Gr(k,\mathbb{R}^d)$, we let Π_γ be the linear projection from $\mathbb{R}^d$ to γ. The family $\{\Pi_\gamma\}_{\gamma\in Gr(k,\mathbb{R}^d)}$ is called $(d\to k)$ linear projections, or simply linear projections if the underlying spaces are clear from the context. We have the following classical results by Marstrand and Mattila, see [6, Theorem 3.1]

Theorem 4.1. *Let $d \geq k \geq 1$ be integers. Let $F \subset \mathbb{R}^d$ be a Borel set with Hausdorff dimension s. With respect to the Lebesgue measure on $Gr(k, \mathbb{R}^d)$, almost all $\gamma \in Gr(k, \mathbb{R}^d)$ we have $\dim_{\mathrm{H}} \gamma(F) = \min\{s, k\}$. If $s > k$, then $\gamma(F)$ has positive Lebesgue measure for almost all γ.*

Intuitively speaking, if a set is large, then its projected images are in general as large as possible. The above result is a prototype of all results of this type.

4.2. **Radial projections.** Another very natural type of projection is radial projection. Let $d \geq 2$ be an integer. We recall the radial projection function here. Let $x \in \mathbb{R}^d$ be a point and let Π_x be defined as follows

$$\Pi_x(y) = \frac{y - x}{|x - y|} \in S^{d-1}$$

for $y \neq x$. We have the following analogue of Theorem 4.1, see [15].

Theorem 4.2. *Let $d \geq 2$ be an integer. Let $F \subset \mathbb{R}^d$ be a Borel set with $\dim_{\mathrm{H}} F = s$. Then $\dim_{\mathrm{H}} \Pi_x(F) = \min\{d - 1, s\}$ for almost all $x \in \mathbb{R}^d$. If $s > d - 1$, then $\Pi_x(F)$ has positive Lebesgue measure for almost all $x \in \mathbb{R}^d$.*

4.3. **Projections for measures.** It is also possible to talk about projections of measures which can be defined as the pushing forward map of measures derived from the corresponding projection map. Analogous projection results for measures can also be proved. We mention the following result.

Theorem 4.3. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be a Cartesian product of self-similar measures supported on self-similar sets with strong separation condition and uniform contraction ratios. Suppose further that the contraction ratios are multiplicatively independent. Then for all $x \in \mathbb{R}^d$, $\dim_{\mathrm{H}} \Pi_x(\mu) = \min\{\dim_{\mathrm{H}} \mu, d - 1\}$.*

The above result is not directly proved as stated here. However, it can be derived from [8, Theorems 1.4 and 1.12].

4.4. **A remark on the radial projection conjecture.** Theorem 4.3 implies that the projected set in the statement of Conjecture 2.1 has full Hausdorff dimension. This is much weaker than having non-empty interiors. One particular reason for posing Conjecture 2.1 is that radial projection is non-linear. Non-linear images of self-similar sets/measures are often 'smoother', see [13]. Another supporting heuristics is closely related to Palis' conjecture. We remark that by following the argument in [19] one can show that in the case when $d = 2$, and $A = A_1 \times A_2$ be a Cartesian product of two self-similar sets as in the statement of Conjecture 2.1. If $\dim_{\mathrm{H}} A > 1$ then for each $x \in \mathbb{R}^2$, it is possible that one can slightly modify A_1, A_2 to A'_1, A'_2 as in [19] that $\Pi_x(A'_1 \times A'_2)$ contains non-empty interiors.

5. Proofs of Theorems 2.2, 2.3 and 2.4

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Consider the set $N_3^{B_3} \times N_5^{B_5} \times N_7^{B_7}$ where

$$B_3 = \{0,1\}, B_5 = \{0,1,2\}, B_7 = \{0,1,2,3\}.$$

Now we construct the following self-similar sets

$$A_3 = \{x \in [1,3] : \text{3-ary expansion of } x \text{ contains only digits in } B_3\},$$

$$A_5 = \{x \in [1,5] : \text{5-ary expansion of } x \text{ contains only digits in } B_5\},$$

$$A_7 = \{x \in [1,7] : \text{7-ary expansion of } x \text{ contains only digits in } B_7\}.$$

Then we see that $\dim_{\mathrm{H}} A_3 \times A_5 \times A_7 = \log 2/\log 3 + \log 3/\log 5 + \log 4/\log 7 > 2$. For each integer $k \geq 1$, consider the line l_k passing through the origin with direction vector

$$(1, 5^{\{k\log 3/\log 5\}}, 7^{\{k \log 3/\log 7\}}).$$

If $l_k \cap A_3 \times A_5 \times A_7 \neq \emptyset$ then we take a point $(x,y,z) \in l_k \cap A_3 \times A_5 \times A_7$. Consider the point

$$(x', y', z') = (3^k x, 5^{[k\log 3/\log 5]} y, 7^{[k\log 3/\log 7]} z).$$

Since we know that $y = 5^{\{k\log 3/\log 5\}} x, z = 7^{\{k\log 3/\log 7\}} x$ we see that

$$5^{[k\log 3/\log 5]} y = 5^{k\log 3/\log 5} x = 3^k x, 7^{[k\log 3/\log 7]} z = 7^{k\log 3/\log 7} x = 3^k x.$$

Thus we see that $x' = y' = z'$. It is straightforward to see that the 3-ary expansion of x' contains only digits in B_3, the 5-ary expansion of y' contains only digits in B_5 and the 7-ary expansion of z' contains only digits in B_7. Taking the integer part we see that

$$[x'] = [y'] = [z'] \in N_{3,5,7}^{B_3, B_5, B_7}.$$

If $\Pi_0(A_3 \times A_5 \times A_7)$ contains non-empty interior, then we see that there are infinitely many integers $k \geq 1$ such that

$$l_k \cap A_3 \times A_5 \times A_7 \neq \emptyset.$$

This proves the result if we assume Conjecture 2.1. □

Theorem 2.3 follows by using a similar argument.

Proof of Theorem 2.3. The proof is very similar to the previous one. Let p, q be two odd primes. Consider the set $N_p^{B_p} \times N_q^{B_q}$ where

$$B_p = \{0, 1, \ldots, (p-1)/2\}, B_q = \{0, 1, \ldots, (q-1)/2\}.$$

We also construct the sets

$$A_p = \{x \in [1,p] : p\text{-ary expansion of } x \text{ contains only digits in } B_p\},$$

$$A_q = \{x \in [1,q] : q\text{-ary expansion of } x \text{ contains only digits in } B_q\}.$$

Now by Lemma 3.4 we see that $S(A_p) = S(A_q) = 1/2$. By Theorem 3.1 and the argument above Corollary 3.3 (which can be easily modified to two dimensional situation), we see that $\Pi_0(A)$ contains non-empty interiors. Following the same argument as in the previous proof we see that there is an interval $I \subset [0,1]$ such that whenever $\{k\log p/\log q\} \in I$, there is a number $n \in N_{p,q}^{B_p,B_q} \cap [p^k, p^{k+1}]$. Since $\{k\log p/\log q\} \in I$ happens for k inside a subset of integers with positive density, we see that there is a $c>0$ and for all large enough integers N, there are least cN many intervals within

$$[1,p),[p,p^2),\dots,[p^{N-1},p^N)$$

intersecting $N_{p,q}^{B_p,B_q}$. Thus the result follows by taking $p=3, q=5$. □

At this stage, Theorem 2.4 seems to be clear. For convenience, we provide full details.

Proof of Theorem 2.4. Consider the set $N_3^{B_3}\times N_4^{B_4}\times N_5^{B_5}$ where

$$B_3=\{0,1\}, B_4=\{0,1\}, B_5=\{0,1\}.$$

Now we construct the following self-similar sets

$$A_3=\{x\in[1,3] : \text{3-ary expansion of } x \text{ contains only digits in } B_3\},$$

$$A_4=\{x\in[1,4] : \text{4-ary expansion of } x \text{ contains only digits in } B_4\},$$

$$A_5=\{x\in[1,5] : \text{5-ary expansion of } x \text{ contains only digits in } B_5\}.$$

Now we see that $S(A_3)=1/2, S(A_4)=1/3$ and $S(A_5)=1/4$. Thus we see that

$$S(A_3)+S(A_4)+S(A_5)=\frac{13}{12}>1. \tag{@}$$

Let $k\geq 1$ be an integer and let H_k be the plane

$$\{x+4^{-\{k\log 3/\log 4\}}y-5^{-\{k\log 3/\log 5\}}=0\}.$$

Suppose that $H_k\cap A_3\times A_4\times A_5\neq\emptyset$. We take a point (x,y,z) in this intersection. Consider the point

$$(x',y',z')=(3^k x, 4^{[k\log 3/\log 4]}y, 5^{[k\log 3/\log 5]}z).$$

Since we have

$$x+4^{-\{k\log 3/\log 4\}}y-5^{-\{k\log 3/\log 5\}}=0$$

we see that

$$3^{-k}x'+4^{-\{k\log 3/\log 4\}-[k\log 3/\log 4]}y'-5^{-\{k\log 3/\log 5\}-[k\log 3/\log 5]}z'=0$$

Thus we have

$$x'+y'-z'=0.$$

From (@) and Corollary 3.3 we see that H_k intersects $A_3\times A_4\times A_5$ for k inside a set of integers with positive density. In particular, there are infinitely many such integers k. Now x',y',z' contains only $\{0,1\}$ in their $3,4,5$-ary expansions respectively. However, they might not be integers. If we take the integer parts we see that $[x'],[y'],[z']$ contains only $\{0,1\}$ in their $3,4,5$-ary expansions respectively and

$$[x']+[y']=[z']+\{z'\}-\{x'\}-\{y'\}.$$

The above equation tells us that

$$\{z'\}-\{x'\}-\{y'\}$$

is an integer. Observe that $\{x'\},\{y'\},\{z'\}$ are positive numbers whose $3,4,5$-ary expansions (respectively) contains only digits 0 and 1. Thus we see that

$$\{x'\}\in(0,1/2],\{y'\}\in(0,1/3],\{z'\}\in(0,1/4].$$

This implies that

$$-\frac{5}{6}<\{z'\}-\{x'\}-\{y'\}\le\frac{1}{4}.$$

The only integer in this range is 0. Thus we see that

$$\{z'\}-\{x'\}-\{y'\}=0$$

and

$$[x']+[y']=[z'].$$

From here the result follows. □

6. ADDITIONAL REMARK

Our approach to Theorem 2.3 suggests that we can look at intersections of three Cantor sets to attack Question 1.1. Towards this direction, it is natural to extend Newhouse's gap lemma for three or more sets in the intersections. Due to a result in [9], one can conclude that for two Cantor sets A_1,A_2 with $S(A_1),S(A_2)$ larger than $9/10$, $S(A_1\cap A_2)$ is positive. Furthermore, $S(A_1\cap A_2)$ can be made arbitrarily close to one if $S(A_1),S(A_2)$ are both sufficiently close to one. Thus, suppose we have three Cantor sets A_1,A_2,A_3 with $S(A_1),S(A_2),S(A_3)$ sufficiently close to one, we can conclude that $\Pi_x(A_1\times A_2\times A_3)$ contains non-trivial open sets for all $x\in\mathbb{R}^3$. The above argument can be made iteratively to study intersections of $k\ge 3$ Cantor sets. Using Lemma 3.4, we can construct missing digits sets with normalised thickness arbitrarily close to one. Then arguments in the previous section help us to conclude the following result. We omit its proof.

Theorem 6.1. *Let $k\ge 2$ be an integer. Then there is an integer $M\ge 1$ such that for all k-tuples of multiplicative independent integers $b_1,\dots,b_k$ that are at least M, there are infinitely many integers whose base $b_1,\dots,b_k$ expansions all omit the digit zero.*

Unfortunately, results in [9] is not enough to attack Question 1.1 as we can see in the proof of Theorem 2.3 that all the Cantor sets in consideration have normalised thickness equal to $1/2$. Nonetheless, this approach sheds some lights on Question 1.1.

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