# Polyomino matchings in generalised games of memory and linear $k$-chord diagrams

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April 17, 2020

**Abstract**

In the game of memory, pairs of matched cards are placed in an array. The present author has previously enumerated configurations by dominoes – where matched pairs are adjacent. In this paper, we introduce a generalised game of memory where pairs are replaced with matched sets of size $k$, and obtain several results for one-dimensional arrays. We generalise the notion of linear chord diagrams to the case of matched sets of size $k$, which we call $k$-chord diagrams. We provide formal generating functions and recursion relations enumerating these $k$-chord diagrams by the number of polyominoes, where the latter is defined as all members of the matched set being adjacent, and is the generalisation of a short chord or loop in a linear chord diagram. We also enumerate $k$-chord diagrams by the number of connected components built from polyominoes and provide the associated generating functions in this case. We show that the distributions of polyominoes and connected components are asymptotically Poisson, and provide the associated means. Finally, we provide recursion relations enumerating non-crossing $k$-chord diagrams by the number of polyominoes, generalising the Narayana numbers, and establish asymptotic normality, providing the associated means and variances.

## 1 Introduction

In the game of memory, $n$ distinct pairs of cards are placed in an array. The present author [9, 10] has enumerated configurations for $2 \times n$ rectangular arrays in which exactly $\ell$ of the pairs are found side-by-side, or over top of one another, thus forming $1 \times 2$ or $2 \times 1$ dominoes. The enumeration of these configurations always carries a factor of $n!$, which counts the orderings of the $n$ distinguishable pairs. It is therefore easier to drop this factor, and thus treat the pairs as indistinguishable.

In [9] the mean for the distribution of dominoes for general arrays, described by graphs on $2n$ vertices, was also obtained. For the case of $1 \times 2n$ arrays, the graph is the path of length $2n$, and the configuration of the cards is in one-to-one correspondence with linear chord diagrams, where the dominoes are chords formed on adjacent vertices, variously called "loops" as in [6], "short chords" as in [2], or originally "paires courtes" as in [7]. Kreweras and Poupard [7] solved the domino enumeration problem in this context, providing recursion relations and closed form expressions for the number of $\ell$-domino configurations. They also showed that the mean number of dominoes is 1, and that all higher factorial moments approach 1 in the $n \to \infty$ limit, thus establishing the Poisson nature of the asymptotic distribution.

In this paper we introduce a more general version of the game in which the $n$ pairs are enlarged to $n$ sets of $k$ matched cards. For a general array, the concept of a domino is replaced by a *polyomino matching*, see Figure 1.

**Definition 1.** A *polyomino matching* is *a configuration where all of the $k$ matching cards from a set form a connected subgraph with $k$ vertices, i.e. a polyomino.*

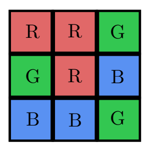


Figure 1: A 1-polyomino configuration in a generalised game of memory with $k = 3$ and $n = 3$, played on a square array.

For the case of $k = 2$ and a $1 \times 2n$ array, Kreweras and Poupard [7] also introduced the concept of a "free pair"[1]. A free pair is a chord which is not crossed by any other chord, and only contains other free chords, see Figure 2. A domino is therefore a free pair itself. In

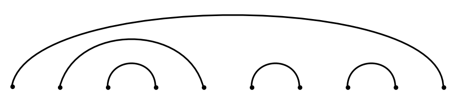

Figure 2: A configuration of free pairs.

Section 4 we will review Kreweras and Poupard's result that the number of configurations consisting entirely of free pairs is in bijection with Dyck paths and is hence counted by the Catalan numbers, and that the number of them with exactly $\ell$ dominoes is counted by the Narayana numbers. The concept of a free pair can be generalised to higher dimensions by considering the concept of a *non-crossing matching*, see Figure 3.

[1] They refer to this as "une paire libre".

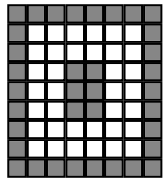

Figure 3: A configuration with one polyomino (the white cells) and two non-crossing matchings in the generalised game of memory with $k = 36$ and $n = 2$ played on the $9 \times 8$ rectangular array.

**Definition 2.** A *non-crossing matching* is *a matching which entirely surrounds a polyomino matching or another non-crossing matching. All polyomino matchings are themselves considered to be non-crossing matchings.*

By "entirely surrounds" we mean that every two vertices of a surrounded matching are connected by a path which does not pass through a vertex of the surrounding matching.

Finally, we introduce the notion of a *connected component* of polyomino matchings, see Figure 4.

**Definition 3.** A *connected component* in a generalised game of memory played on a graph with $kn$ vertices is a *connected set of polyomino matchings*.

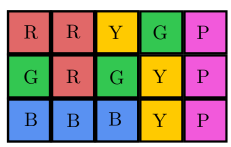


Figure 4: A configuration in a generalised game of memory with $k = 3$ and $n = 5$ with two connected components.

The problem of enumerating configurations in generalised games of memory by polyomino matchings, connected components, and non-crossing matchings is interesting, and likely very difficult to obtain exact results for in general. In this paper we take a first step towards studying these issues for the one dimensional case of $1 \times kn$ arrays. Before presenting the main results of the paper, it is useful to establish a few basic facts that apply to any array.

**Proposition 4.** *The number of ways of placing $n$ indistinguishable sets, each consisting of $k$ matched cards, on the $kn$ vertices of a graph is*

$$\mathcal{N}_{k,n} = \frac{(kn)!}{(k!)^n \, n!}.$$

**Theorem 5.** *The mean number of polyomino matchings in a generalised game of memory played on a graph with $kn$ vertices and $r$ connected subgraphs, each with $k$ vertices, is given by*

$$\binom{kn}{k}^{-1} nr.$$

*Proof.* The proof proceeds through the linearity of expectation. Let the random variable $X_j$ take the value 1 when the $j^{\text{th}}$ connected subgraph forms a polyomino matching and 0 otherwise. Once a polyomino is thusly placed, by Proposition 4, there are $\mathcal{N}_{k,n-1}$ ways of placing the remaining cards on the $kn-k$ remaining vertices of the graph. Thus $E(X_j) = \mathcal{N}_{k,n-1}/\mathcal{N}_{k,n}$. We therefore have that $E(\sum_{j=1}^{r} X_j) = \sum_{j=1}^{r} E(X_j) = r\,\mathcal{N}_{k,n-1}/\mathcal{N}_{k,n}$. □

It is interesting to consider how this mean scales with $n \to \infty$ for hypercubical grid (or other regular) graphs. For fixed $k$, we expect the number $r$ of subgraphs on $k$ vertices to scale as $n$. We thus expect the mean to scale as $n^2/n^k = n^{2-k}$. Indeed, for large hypercubical grid graphs of dimension $d$, it was shown in [9, Corollary 2] that the mean number of dominoes approaches $d$ (and is hence $\mathcal{O}(n^0)$). For $k > 2$, polyomino matchings are instead *suppressed* as $n$ grows large.

**Proposition 6.** *The number of configurations without any polyomino matchings in a generalised game of memory played on a graph $G$, with $kn$ vertices, is given by*

$$\sum_{j=0}^{n} (-1)^j \,\mathcal{N}_{k,n-j}\,\rho_j,$$

*where $\rho_j$ represents the number of ways of choosing $j$ disjoint connected subgraphs, each with $k$ vertices, from $G$. We define $\rho_0$ to be 1.*

*Proof.* The proof proceeds via inclusion-exclusion. We note that for each of the $\rho_j$ choices of $j$ sub-graphs on which to place $j$ polyominoes, there remains $\mathcal{N}_{k,n-j}$ configurations of the remaining $n-j$ matched sets. There will be some number of $q$-polyomino configurations among these $\mathcal{N}_{k,n-j}$. Then $\mathcal{N}_{k,n-j}\,\rho_j$ counts the $(q+j)$-polyomino configurations $\binom{q+j}{j}$ times. Let $N(q)$ be the number of $q$-polyomino configurations, we therefore have that

$$\begin{aligned}\sum_{j=0}^{n} (-1)^j \,\mathcal{N}_{k,n-j}\,\rho_j &= \sum_{j=0}^{n} (-1)^j \sum_{q=0}^{n-j} \binom{q+j}{j} N(q+j) \\ &= N(0) + \sum_{q+j=1}^{n} N(q+j) \sum_{j=0}^{q+j} (-1)^j \binom{q+j}{j},\end{aligned}$$

and so all but the 0-polyomino configurations cancel. □

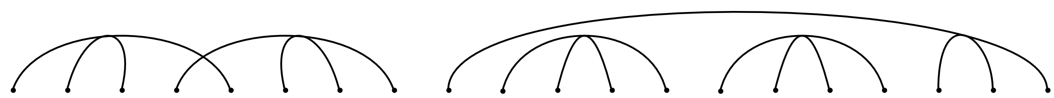

Figure 5: A linear $k$-chord diagram with $k = 4$ and $n = 5$. This is a configuration with 2 polyominoes, 3 non-crossing matchings, and 1 connected component (formed by the two adjacent polyominoes).

| $n \setminus \ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | | | | | |
| 2 | 7 | 2 | 1 | | | | |
| 3 | 219 | 53 | 7 | 1 | | | |
| 4 | 12861 | 2296 | 226 | 16 | 1 | | |
| 5 | 1215794 | 171785 | 13080 | 710 | 30 | 1 | |
| 6 | 169509845 | 19796274 | 1228655 | 53740 | 1835 | 50 | 1 |

Table 1: The number $d_{n,\ell}$ of configurations with exactly $\ell$ polyomino matchings in a generalised game of memory played on the path of length $kn$, for the case $k = 3$. OEIS reference A334056 (to appear).

# 2 Generalised memory on paths: linear $k$-chord diagrams

We introduce the notion of a $k$-chord which represents the matching of a set of $k$ vertices on the path of length $kn$, see Figure 5. We begin by enumerating configurations by number of polyominoes.

## 2.1 Enumeration by polyominoes

**Theorem 7.** *The number $d_{n,\ell}$ of configurations with exactly $\ell$ polyomino matchings in a generalised game of memory played on the path of length $kn$, is*

$$d_{n,\ell} = \frac{1}{\ell!} \sum_{j=\ell}^{n} \frac{(k(n-j)+j)!\,(-1)^{j-\ell}}{(k!)^{n-j}\,(n-j)!\,(j-\ell)!}.$$

*Proof.* This follows from direct calculation from Lemma 10. □

**Lemma 8.** *The number of ways of choosing $j$ pairwise non-adjacent subpaths, each of length $k$, from the path of length $kn$ is*

$$\binom{kn - j(k-1)}{j}.$$

*Proof.* For each of the $j$ subpaths, collapse the vertices of that subpath onto the left-most vertex, and mark it. We are left with $kn - j(k-1)$ vertices, j of which are marked. Thus there are $\binom{kn-j(k-1)}{j}$ choices for the positions of the marked vertices. $\square$

**Lemma 9.** *The number of configurations with at least $j$ polyominoes is given by*

$$\frac{(k(n-j))!}{(k!)^{n-j}\,(n-j)!}\binom{kn-j(k-1)}{j}.$$

*Proof.* We choose $j$ pairwise non-adjacent subpaths, enumerated according to Lemma 8, to place the polyominoes upon. By Proposition 4, for each such choice we have $\mathcal{N}_{k,n-j}$ ways of placing the remaining cards. $\square$

**Lemma 10.** *The number of configurations with exactly $\ell$ polyominoes is given by*

$$[z^\ell]\sum_{j=0}^{n}\frac{(k(n-j))!}{(k!)^{n-j}\,(n-j)!}\binom{kn-j(k-1)}{j}(z-1)^j.$$

*Proof.* This follows from inclusion-exclusion, c.f. [8, pg. 112]. $\square$

## 2.2 Two recursion relations

Kreweras and Poupard [7] gave a recursion relation for the $d_{n,\ell}$ for the case of $k=2$:

$$\ell\, d_{n,\ell} = (2n-\ell)\, d_{n-1,\ell-1} + \ell\, d_{n-1,\ell} \quad \Leftrightarrow \quad k=2.$$

They obtained this by considering the removal of a domino at a given position in the chord diagram. This domino is either nested inside another chord, and hence its removal does not change the number of dominoes, or it is not. A sum over all possible positions of the given domino then results in the recursion relation. We now give a generalisation of this recursion relation to the case of general $k$. Rather than attempt to repeat Kreweras and Poupard's arguments, which should be possible but rather complicated due to dealing with the ends of the path, we prove our recursion relation directly from Theorem 7.

**Theorem 11.** *The numbers $d_{n,\ell}$ satisfy the following recursion relation*

$$\ell\, d_{n,\ell} = (kn-\ell(k-1))\, d_{n-1,\ell-1} + \ell(k-1)\, d_{n-1,\ell}.$$

*Proof.* We use the result of Theorem 7 and consider $d_{n-1,\ell-1} - d_{n-1,\ell}$. In each of the two summands, we shift the summation variable $j \to j-1$. This procedure results in a relative sign difference between the summands. In $d_{n-1,\ell-1}$ the sum begins at $j=\ell$ as before, whereas in $d_{n-1,\ell}$, it now begins at $j = \ell+1$. We proceed by stripping off the $j = \ell$ term from $d_{n-1,\ell-1}$:

$$\begin{aligned}
d_{n-1,\ell-1} - d_{n-1,\ell} &= \frac{(k(n-\ell)+\ell-1)!}{(k!)^{n-\ell}\,(n-\ell)!}\frac{1}{(\ell-1)!} \\
&\quad + \sum_{j=\ell+1}^{n} \frac{(k(n-j)+j-1)!\,(-1)^{j-\ell}}{(k!)^{n-j}\,(n-j)!}\left(\frac{1}{(j-\ell)!(\ell-1)!} + \frac{1}{(j-\ell-1)!\ell!}\right) \\
&= \frac{(k(n-\ell)+\ell-1)!}{(k!)^{n-\ell}\,(n-\ell)!}\frac{1}{(\ell-1)!} + \sum_{j=\ell+1}^{n} \frac{(k(n-j)+j-1)!\,(-1)^{j-\ell}}{(k!)^{n-j}\,(n-j)!}\left(\frac{j/\ell}{(j-\ell)!(\ell-1)!}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&kn\, d_{n-1,\ell-1} - \ell(k-1)\,(d_{n-1,\ell-1} - d_{n-1,\ell}) \\
&\quad = \frac{(k(n-\ell)+\ell-1)!}{(k!)^{n-\ell}\,(n-\ell)!}\frac{(kn-\ell(k-1))}{(\ell-1)!} \\
&\qquad\qquad + \sum_{j=\ell+1}^{n} \frac{(k(n-j)+j-1)!\,(-1)^{j-\ell}}{(k!)^{n-j}\,(n-j)!}\left(\frac{kn-j(k-1)}{(j-\ell)!(\ell-1)!}\right) \\
&\quad = \frac{1}{(\ell-1)!}\sum_{j=\ell}^{n} \frac{(k(n-j)+j)!\,(-1)^{j-\ell}}{(k!)^{n-j}\,(n-j)!\,(j-\ell)!} = \ell\, d_{n,\ell}.
\end{aligned}$$

□

Krasko and Omelchenko [6] gave another recursion relation for the $d_{n,\ell}$ in the case of $k = 2$:

$$d_{n+1,\ell} = d_{n,\ell-1} + (2n-\ell)d_{n,\ell} + (\ell+1)d_{n,\ell+1} \quad \Leftrightarrow \quad k = 2.$$

This relation is found by considering the addition of new chord, with one end anchored outside the path. The other end of the chord could be placed in various positions: adjacent to the anchored end hence forming an external domino, in a location which disturbs no existing domino, or within an existing domino and hence breaking it up. These cases account, respectively, for the three terms in the recursion relation. To generalise this to a $k$-chord diagram we must account for a larger variety of cases.

**Theorem 12.** *The numbers $d_{n,\ell}$ satisfy the following recursion relation*

$$d_{n+1,\ell} = d_{n,\ell-1} + d_{n,\ell}\sum_{h=1}^{k-1}\binom{kn-(k-1)\ell+k-h-1}{k-h} + \sum_{p=1}^{k-1} C_{n,\ell,p,k}\, d_{n,\ell+p},$$

*where the $C_{n,\ell,p,k}$ are given by*

$$C_{n,\ell,p,k} = \sum_{h=1}^{k-p}\sum_{f=0}^{k-p-h}\binom{kn-(k-1)(\ell+p)+f-1}{f}[x^{k-h-f}y^p]\left(1+y-y(1-x)^{1-k}\right)^{-\ell-1}.$$

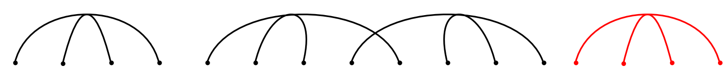

Figure 6: The first term in the recursion relation in Theorem 12. A new $k$-chord (shown in red) is added as a polyomino to the end of an existing configuration.

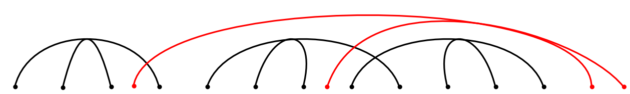

Figure 7: A general configuration from Theorem 12, with $k = 4$, $n = 3$, $\ell = 0$, $h = 2$, $f = 1$, and $p = 1$.

*Proof.* The recursion relation arises from the addition of a new $k$-chord, at least one strand of which is anchored to the right of all existing vertices, to a configuration of length $kn$. One of the ways we could add these vertices is as a polyomino placed on the end of the existing configuration, as in Figure 6. This accounts for the first term in the recursion relation.

We now consider the various positions which the strands of the new $k$-chord could occupy. If they remain to the right of all existing vertices, we consider them to be at *home*. If a strand is not at home, but is placed such that it does not disturb any existing polyomino, we say that it is in the *forest*. If the existing configuration has $\ell + p$ polyominoes, there are $kn - (k-1)(\ell + p)$ forest positions.

The second term in the recursion relation arises from leaving $h \geq 1$ strands at home, and placing the remaining $f = k - h$ strands into forest positions only, i.e. no strand which is not at home is placed within an existing polyomino. The number of ways of accomplishing this is the same as the number of ways of placing $f = k - h$ identical balls into $kn - (k-1)\ell$ distinguishable bins.

The third term in the recursion relation accounts for the general case: $h \geq 1$ strands are left at home, $f \geq 0$ occupy forest positions, and the remainder disturb existing polyominoes. An example is shown in Figure 7. The result follows from the preceding treatment of forest positions (there are now $kn-(k-1)(\ell+p)$ of these) in conjunction with Lemma 14, where the $j$ balls represent the $k - h - f$ strands which disturb polyominoes, and the $p$ bins represent the $p$ disturbed polyominoes, each of which can be disturbed in up to $k - 1$ positions. $\square$

**Example 13.** The recursion relation for the case $k = 3$ is given by

$$d_{n+1,\ell} = d_{n,\ell-1} + \frac{(3n-2\ell+3)(3n-2\ell)}{2}\, d_{n,\ell} + (\ell+1)(6n-4\ell+1)\, d_{n,\ell+1} + 2(\ell+1)(\ell+2)\, d_{n,\ell+2}.$$

**Lemma 14.** *The number of ways of placing $j$ identical balls into a selection of $p$, out of $\ell + p$ distinguishable bins, each containing $k-1$ distinguishable sub-bins, such that no bin is empty (though any sub-bin could be), is*

$$[x^j y^p] \left(1 + y - y(1-x)^{1-k}\right)^{-\ell-1}.$$

*Proof.* We first note that there are $\binom{\ell+p}{p}$ ways to choose the $p$ bins. To enumerate the possible ways of filling the sub-bins we consider the weak compositions of $1, 2, 3, \ldots$ into $k-1$ parts. For a given bin, let $f(x)$ be the generating function such that $[x^m]f(x)$ counts the number of ways of placing $m$ balls into the $k-1$ sub-bins. We have that

$$f(x) = \binom{k-1}{k-2} x + \binom{k}{k-2} x^2 + \binom{k+1}{k-2} x^3 + \cdots = (1-x)^{1-k} - 1.$$

To now account for $p$ bins, we take $f(x)^p$, and note that

$$\sum_p \binom{\ell+p}{p} f(x)^p y^p = \left(1 + y - y(1-x)^{1-k}\right)^{-\ell-1}.$$

□

# 3 Generating functions

In this section we establish formal generating functions enumerating configurations firstly by number of polyominoes (Theorem 17),

$$F_k(w,z) = \sum_{j \geq 0} \frac{(kj)!}{j!(k!)^j} \frac{w^j}{(1 + w(1-z))^{kj+1}},$$

where the power of $w$ corresponds to $n$ and the power of $z$ corresponds to the number of polyominoes, and secondly by the number of connected components (Theorem 21)

$$C_k(y,z) = \sum_{j \geq 0} \frac{(kj)!}{j!(k!)^j} y^j \left(\frac{1 - y(1-z)}{1 - y^2(1-z)}\right)^{kj+1},$$

where the power of $y$ corresponds to $n$ and the power of $z$ corresponds to the number of connected components. These generating functions are not convergent, but nevertheless provide compact expressions for the numbers they count.

## 3.1 Counting by number of polyominoes

**Proposition 15.** *The generating function which counts the number of ways of choosing $j$ pairwise non-adjacent subpaths, each of length $k$, from the path of length $kn$ is*

$$L_k(x,y) = \frac{1}{1 - y(1 + x^k y^{k-1})},$$

*where the power of $x$ corresponds to the total number of vertices in all chosen sub-paths, and the power of $y$ corresponds to the length of the path.*

*Proof.* By direct expansion $L_k(x,y) = \sum_{p\geq 0}\sum_{j=0}^{p}\binom{p}{j}x^{kj}\,y^{(k-1)j+p}$, so $[x^{kj}y^{kn}]L_k(x,y) = \binom{kn-j(k-1)}{j}$. By Lemma 8, the proposition is proven. □

**Lemma 16.** *The generating function which counts the number of configurations with at least $j$ polyominoes is given by*

$$N_k(w^k, z^k) = \int_0^\infty dt\, e^{-t}\,\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^k/k!}\,L_k\left(\frac{zx}{t},\frac{wt}{x}\right),$$

*where $[w^n z^j]\,N_k(w,z)$ is the number of configurations with at least $j$ polyominoes on the path of length $kn$.*

*Proof.* We begin by noting that

$$\begin{aligned}[w^{kn}z^{kj}]\int_0^\infty dt\, e^{-t}\,L_k\left(\frac{zx}{t},\frac{wt}{x}\right) &= \int_0^\infty dt\, e^{-t}\,t^{kn-kj}\binom{kn-j(k-1)}{j}x^{kj-kn}\\ &= (kn-kj)!\binom{kn-j(k-1)}{j}x^{kj-kn},\end{aligned}$$

where we have made use of Lemma 8. We now note that

$$[x^{kn-kj}]e^{x^k/k!} = \frac{1}{(k!)^{n-j}\,(n-j)!},$$

and so the contour integral selects precisely this term in the expansion of $e^{x^k/k!}$. We therefore have that

$$[w^{kn}z^{kj}]\int_0^\infty dt\, e^{-t}\,\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^k/k!}\,L_k\left(\frac{zx}{t},\frac{wt}{x}\right) = \frac{(kn-kj)!}{(k!)^{n-j}\,(n-j)!}\binom{kn-j(k-1)}{j},$$

and by Lemma 9 the proof is complete. □

**Theorem 17.** *The generating function which counts the number of configurations with exactly $\ell$ polyominoes on the path of length $kn$ is given by*

$$F_k(w,z) = \sum_{j\geq 0}\frac{(kj)!}{j!(k!)^j}\frac{w^j}{\left(1+w(1-z)\right)^{kj+1}},$$

*where the power of $w$ corresponds to $n$ and the power of $z$ corresponds to $\ell$.*

*Proof.* We use Lemma 16 to find the generating function for at least $j$ polyominoes, and then replacing $z \to z-1$ at the end, as per Lemma 10, yields the desired result. We note that

$$\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^k/k!}L_k\left(\frac{zx}{t},\frac{wt}{x}\right)=\frac{1}{2\pi i}\oint_{|x|=\epsilon}dx\,\frac{e^{x^k/k!}}{x\,(1-(wz)^k)-tw},$$

where we have made use of Proposition 15. We proceed by evaluating the residue at $x = tw/(1-(wz)^k)$

$$\begin{aligned}N_k(w^k,z^k)&=\int_0^\infty dt\,e^{-t}\,\frac{1}{1-(wz)^k}\exp\left(\frac{t^kw^k}{k!\,(1-(wz)^k)^k}\right)\\&=\int_0^\infty dt\,e^{-t}\,\frac{1}{1-(wz)^k}\sum_{j\geq 0}\frac{1}{j!}\left(\frac{t^kw^k}{k!\,(1-(wz)^k)^k}\right)^j,\end{aligned}$$

where the integration over $t$ is understood to be performed term-by-term in the expansion of the exponential, thus yielding a factor of $(kj)!$. Note that $w$ and $z$ appear uniformly with exponent $k$; we can therefore remove the exponents by considering $N_k(w,z)$ in place of $N_k(w^k,z^k)$. Finally, we have that

$$F_k(w,z)=N_k(w,z-1)=\sum_{j\geq 0}\frac{(kj)!}{j!(k!)^j}\frac{w^j}{(1+w(1-z))^{kj+1}}.$$

□

## 3.2 Counting by number of connected components

We now turn our attention to counting configurations on paths by the number of connected components, as defined in Definition 3. Let there be $q$ connected components; it is clear that there are therefore at least $q-1$ and at most $q+1$ regions devoid of polyominoes, depending on whether the first and last vertices are occupied by polyominoes or not. We will require the number $\rho_j$ of ways of choosing $j$ non-adjacent sub-paths from this disjoint collection of regions. Proposition 6 will then count the number of "zero-polyomino" configurations on these regions.

**Lemma 18.** *The numbers which count $\rho_j$ (c.f. Proposition 6) for the case of a single component of size $km$, where $m \geq 1$, are denoted $\rho_j^{(1)}(m)$. We have that*

$$\rho_j^{(1)}(m)=[x^{kj}y^{kn-km}]L_k(x,y)^2.$$

*Proof.* The result follows from Proposition 15. The sum over the position of the connected component is accounted for through the symbolic method. □

Each further component which is added produces a new region, which, rather than being bounded on one side by the ends of the path, is bounded by two connected components. It is clear that these new regions must not be allowed to have zero length; this is accomplished by subtracting 1 from $L_k(x,y)$.

**Lemma 19.** *The numbers $\rho_j^{(q)}(m_1,\dots,m_q)$ which count $\rho_j$ for the case of $q$ connected components, of sizes $km_1,\dots,km_q$, summed over positions, is*

$$\rho_j^{(q)}(m_1,\dots,m_q) = [x^{kj} y^{kn-k\sum m_p}]\, L_k(x,y)^2\, (L_k(x,y)-1)^{q-1}\,.$$

*Proof.* This follows from the symbolic method. □

**Lemma 20.**

$$\sum_{j=0}^{\tilde n} (-1)^j \mathcal{N}_{k,\tilde n - j}\, \rho_j^{(q)}(m_1,\dots,m_q)$$
$$= [y^{k\tilde n}] \int_0^\infty dt\, e^{-t}\, \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x}\, e^{x^k/k!}\, L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right)^2 \left(L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right) - 1\right)^{q-1},$$

where $\tilde n = n - \sum m_p$.

*Proof.* The scaling of the $x$ and $y$ variables, together with the result of Lemma 19, imply that

$$[y^{k\tilde n}] L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right)^2 \left(L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right) - 1\right)^{q-1}$$
$$= \sum_{j=0}^{\tilde n} (-1)^j x^{kj-k\tilde n}\, t^{k\tilde n - kj} \rho_j^{(q)}(m_1,\dots,m_q).$$

Note that the integration over $t$ then enacts the replacement $t^{k\tilde n - kj} \to (k\tilde n - kj)!$. We now consider

$$[x^0]\, e^{x^k/k!} \sum_{j=0}^{\tilde n} (-1)^j x^{kj-k\tilde n}\, (k\tilde n - kj)! \rho_j^{(q)}(m_1,\dots,m_q)$$
$$= [x^0] \sum_\ell \frac{x^{k\ell}}{(k!)^\ell \ell!} \sum_{j=0}^{\tilde n} (-1)^j x^{kj-k\tilde n}\, \rho_j^{(q)}(m_1,\dots,m_q)$$
$$= \sum_{j=0}^{\tilde n} (-1)^j \frac{(k\tilde n - kj)!}{(k!)^{\tilde n - j} (\tilde n - j)!} \rho_j^{(q)}(m_1,\dots,m_q),$$

and so the contour integral in $x$ picks out precisely this expression. □

| $n$ \ $q$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | | | |
| 2 | 7 | 3 | | | |
| 3 | 219 | 56 | 5 | | |
| 4 | 12861 | 2352 | 183 | 4 | |
| 5 | 1215794 | 174137 | 11145 | 323 | 1 |
| 6 | 169509845 | 19970411 | 1078977 | 30833 | 334 |

Table 2: The numbers $c_{n,q}$ of configurations with exactly $q$ connected components in a generalised game of memory played on the path of length $kn$, for the case $k = 3$. OEIS reference A334060 (to appear).

**Theorem 21.** *The numbers $c_{n,q} = [y^n z^q]C_k(y,z)$ count the number of configurations with exactly $q$ connected components, in the game of memory played on the path of length $kn$, where*

$$C_k(y,z) = \sum_{j\geq 0} \frac{(kj)!}{j!(k!)^j}\, y^j \left(\frac{1-y(1-z)}{1-y^2(1-z)}\right)^{kj+1}.$$

*Proof.* We now make use of Lemma 20 and sum over the sizes $m_1, \ldots, m_q$ of the connected components:

$$\begin{aligned}
C_k(y^k,z) - C_k(y^k,0) &= \sum_{q\geq 1} z^q \sum_{\{m_p\geq 1\}} y^{k\sum m_p} \sum_{j=0}^{\tilde{n}} (-1)^j \mathcal{N}_{k,\tilde{n}-j}\, \rho_j^{(q)}(m_1,\ldots,m_q) \\
&= \sum_{q\geq 1} z^q \left(\frac{y^k}{1-y^k}\right)^q \int_0^\infty dt\, e^{-t} \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x}\, e^{x^k/k!}\, L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right)^2 \left(L_k\left(\frac{xe^{i\pi/k}}{t}, \frac{yt}{x}\right) - 1\right)^{q-1} \\
&= \int_0^\infty dt\, e^{-t} \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x}\, e^{x^k/k!} \sum_{q\geq 1} \frac{-x^2(zy^k)^q(xy^k-ty)^{q-1}}{(y^k-1)^q(x-ty+xy^k)^{1+q}} \\
&= \int_0^\infty dt\, e^{-t} \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x}\, e^{x^k/k!} \frac{x^2 z\, y^k}{\big(x(1+y^k)-ty\big)\Big(x\big(1-y^{2k}(1-z)\big) - ty\big(1-y^k(1-z)\big)\Big)} \\
&= \int_0^\infty dt\, e^{-t} \left(\frac{1-y^k(1-z)}{1-y^{2k}(1-z)} \exp\frac{t^k}{k!}\left(\frac{y(1-y^k(1-z))}{1-y^{2k}(1-z)}\right)^k - \frac{1}{1+y^k}\exp\frac{t^k}{k!}\left(\frac{y}{1+y^k}\right)^k\right).
\end{aligned}$$

The integration over $t$ proceeds term-by-term in an expansion of the exponentials as was seen in Theorem 17. In going from the penultimate line to the last, the contour integral picks-up two residues, one of which produces precisely minus the generating function for the zero-polyomino configurations, i.e. $-F_k(y,0)$ from Theorem 17, corresponding to the case of zero connected components. Adding this back as a $z^0$ term, and replacing $y^k \to y$ we obtain the advertised result for $C_k(y,z)$. □

### 3.3 Asymptotic distributions

We expect the asymptotic distribution of polyominoes, i.e.

$$\lim_{n\to\infty} \frac{d_{n,\ell}}{\mathcal{N}_{k,n}},$$

to be Poisson with mean given by (the large-$n$ limit of) Theorem 5, with $r = kn - (k-1)$:

$$\lambda = \lim_{n\to\infty} \binom{kn}{k}^{-1} n\,(kn - (k-1)) = k!\, k^{1-k}\, n^{2-k}.$$

The distribution of connected components

$$\lim_{n\to\infty} \frac{c_{n,q}}{\mathcal{N}_{k,n}},$$

should have the same distribution. This is because for long paths, most $\ell$-polyomino configurations will also have $\ell$ trivially connected components. We begin by considering polyominoes.

**Theorem 22.** *The asymptotic distribution of polyomino matchings, in a generalised game of memory played on the path of length $kn$, is Poisson with mean $\lambda = k!\, k^{1-k}\, n^{2-k}$.*

*Proof.* Using Lemma 10, we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{d_{n,\ell}}{\mathcal{N}_{k,n}} &= [z^\ell] \lim_{n\to\infty} \sum_{j=0}^{n} (k!)^j \frac{n!}{(n-j)!} \frac{(k(n-j))!}{(kn)!} \binom{kn - j(k-1)}{j} (z-1)^j \\
&= [z^\ell] \sum_{j=0}^{\infty} (k!)^j n^j \frac{1}{(kn)^{kj}} \frac{(kn)^j}{j!} (z-1)^j = [z^\ell] \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{k!}{k^{k-1} n^{k-2}} \right)^j (z-1)^j \\
&= [z^\ell] \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (z-1)^j.
\end{aligned}$$

It follows that the $j^{\text{th}}$ factorial moment is $\lambda^j$, and hence the distribution is Poisson. $\square$

**Theorem 23.** *The asymptotic distribution of connected components, in a generalised game of memory played on the path of length $kn$, is Poisson with mean $\lambda = k!\, k^{1-k}\, n^{2-k}$.*

*Proof.* We begin by expanding the generating function given in Theorem 21:

$$\begin{aligned}
\frac{c_{n,q}}{\mathcal{N}_{k,n}} &= \frac{1}{\mathcal{N}_{k,n}}[y^n z^q]\sum_{j\geq 0}\frac{(kj)!}{j!(k!)^j}\,y^j\left(\frac{1-y(1-z)}{1-y^2(1-z)}\right)^{kj+1} \\
&= \sum_{j\geq 0}\frac{\mathcal{N}_{k,j}}{\mathcal{N}_{k,n}}[y^n z^q]\sum_{p,r} y^{j+p+2r}(z-1)^{p+r}\binom{kj+1}{p}\binom{kj+r}{r}(-1)^r \\
&= \sum_{j\geq 0}\frac{\mathcal{N}_{k,j}}{\mathcal{N}_{k,n}}[y^n z^q]\sum_{\ell,r} y^{j+\ell+r}(z-1)^{\ell}\binom{kj+1}{\ell-r}\binom{kj+r}{r}(-1)^r \\
&= [z^q]\sum_{\ell,r}\frac{\mathcal{N}_{k,n-\ell-r}}{\mathcal{N}_{k,n}}(z-1)^{\ell}\binom{k(n-\ell-r)+1}{\ell-r}\binom{k(n-\ell-r)+r}{r}(-1)^r.
\end{aligned}$$

We now take the $n\to\infty$ limit

$$\begin{aligned}
\lim_{n\to\infty}\frac{c_{n,q}}{\mathcal{N}_{k,n}} &= [z^q]\sum_{\ell,r}\frac{(k!)^{\ell+r}}{k^{k(\ell+r)}n^{(k-1)(\ell+r)}}\,(z-1)^{\ell}\,\frac{(kn)^{\ell-r}}{(\ell-r)!}\,\frac{(kn)^r}{r!}\,(-1)^r \\
&= [z^q]\sum_{\ell,r}\frac{(-1)^r}{\ell!}\binom{\ell}{r}\left(\frac{k!}{k^k\,n^{k-1}}\right)^r(z-1)^{\ell}\,\lambda^{\ell} = [z^q]\sum_{\ell}\left(1-\frac{k!}{k^k\,n^{k-1}}\right)^{\ell}\frac{(z-1)^{\ell}\,\lambda^{\ell}}{\ell!} \\
&\simeq [z^q]\sum_{\ell}\frac{(z-1)^{\ell}\,\lambda^{\ell}}{\ell!}.
\end{aligned}$$

It follows that the $j^{\text{th}}$ factorial moment is $\lambda^j$, and hence the distribution is Poisson. □

# 4 Non-crossing configurations

For the case of linear chord diagrams the enumeration of so-called *non-crossing configurations*, where no two chords cross each other, is by now standard combinatorical lore. For the sake of completeness, and in order to motivate the case for general $k$, we repeat the main elements of the arguments given by Kreweras and Poupard [7], who established a bijection with Dyck paths. The mapping is as follows: traversing the path of length $2n$ from left to right, we map the start of a chord to an up step $(0,+1)$, and the end of a chord with a down step $(+1,0)$. To establish the mapping in the other direction, we associate consecutive up steps to the starting vertices of successively nested chords. It is clear that there are $n$ up steps and $n$ down steps, and that the first step is always an up step. A domino is mapped to a peak, i.e. to an up step immediately followed by a down step. Therefore the Narayana numbers $\binom{n}{\ell}\binom{n}{\ell-1}/n$ give the number of non-crossing configurations with exactly $\ell$ dominoes, and the Catalan numbers $\binom{2n}{n}/(n+1)$ count the total number of non-crossing configurations. The bijection to lattice paths can be extended for general $k$, to paths which begin and end on the line $y=(k-1)x$. A polyomino, traversed left to right, is represented by $k-1$ up steps followed by a single down step, see Figure 8.

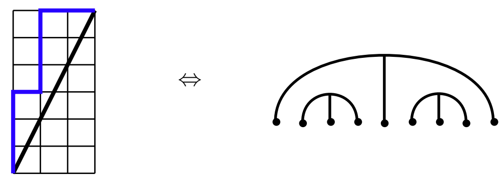

Figure 8: The bijection between non-crossing configurations and lattice paths for the case of $k = 3$.

In order to generalise the counting to $k$-chord diagrams we establish the following recursion.

**Theorem 24.** *The number $T_{m,\ell}$ of non-crossing linear $k$-chord diagrams on the path of length $km$, with exactly $\ell$ polyominoes, obeys the following recursion relation*

$$T_{m+1,\ell} = \sum_{\substack{\sum m_i = m \\ \sum \ell_i = \ell}} \prod_{i=1}^{k} T_{m_i,\ell_i} - T_{m,\ell} + T_{m,\ell-1}, \qquad T_{0,0} = 1.$$

*Proof.* The proof proceeds diagrammatically, see Figure 9. The first term accounts for all possible nestings of smaller non-crossing diagrams in the $k-1$ arches, and also to the left, of an additional $k$-chord. This term counts one set of configurations incorrectly, which are those pictured on the right in Figure 9. When the arches are empty, corresponding to $m_{i>1} = 0, \ell_{i>1} = 0$, the additional $k$-chord is also a polyomino. Thus this set of configurations must be subtracted, and hence the second term in the recursion relation. The third term adds back the correction. $\square$

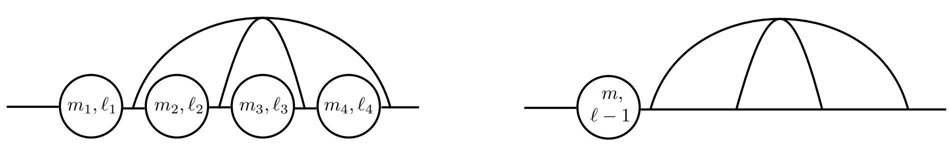


Figure 9: Configurations of non-crossing linear $k$-chord diagrams for the case of $k = 4$. The circles represent all non-crossing linear 4-chord diagrams with $m_i$ 4-chords, $\ell_i$ of which are polyominoes.

**Corollary 25.** *The generating function for the $T_{m,\ell}$ is $T(x, y) = \sum_{m,\ell \geq 0} T_{m,\ell}\, x^m y^\ell$ and obeys*

$$T(x, y) - 1 = xT(x, y)^k - x(1 - y)T(x, y).$$

*Proof.* This is shown by multiplying the recursion relation of Theorem 24 by $x^{m+1}y^\ell$ and summing over $m$ and $\ell$. $\square$

**Corollary 26.** *The total number of of non-crossing linear $k$-chord diagrams on the path of length $km$ is given by the Fuss-Catalan number*

$$T_m = \sum_{\ell=1}^{m} T_{m,\ell} = \frac{1}{(k-1)m+1}\binom{km}{m}.$$

*Proof.* This can be established via Lagrange inversion on Corollary 25, with $y = 1$. We have that

$$\begin{aligned}
x &= \frac{T(x,1)-1}{T(x,1)^k} = f(T(x,1)),\\
g_m &= \lim_{w\to 1}\frac{d^{m-1}}{dw^{m-1}}\left(\frac{w-1}{f(w)-f(1)}\right)^m = \lim_{w\to 1}\frac{d^{m-1}w^{km}}{dw^{m-1}} = \frac{(km)!}{((k-1)m+1)!},\\
T(x,1) &= 1+\sum_m g_m\frac{x^m}{m!} = 1+\sum_m \frac{x^m}{(k-1)m+1}\binom{km}{m}.
\end{aligned}$$

These numbers appear as A062993 in the OEIS. $\square$

Kreweras and Poupard [7] also give a closed expression for the number $d_{n,\ell,m}$ of linear chord diagrams on the path of length $2n$ with $m$ non-crossing chords and $\ell$ dominoes

$$d_{n,\ell,m} = \frac{2n-2m+1}{m}\binom{m}{\ell}\binom{2n-m}{\ell-1} d_{n-m,0} \;\Leftrightarrow\; k=2.$$

They obtain this by considering disconnected regions consisting solely of non-crossing chords. They note that upon removing these regions, and joining the remaining components, one is necessarily left with a zero-domino configuration on the path of length $2n-2m$; hence the appearance of the number of such configurations, i.e. $d_{n-m,0}$. For a general value of $k$, we have the following theorem.

**Theorem 27.** *The number $d_{n,\ell,m}$ of configurations with exactly $\ell$ polyominoes, and exactly $m$ non-crossing matchings, in a generalised game of memory played on the path of length $kn$, are given by*

$$d_{n,\ell,m} = [x^m y^\ell]\, T(x,y)^{kn-km+1}\, d_{n-m,0},$$

*where $d_{n-m,0}$ is the number of 0-polyomino configurations on the path of length $k(n-m)$.*

*Proof.* We consider a general linear $k$-chord diagram on the path of length $kn$. Let there be $p$ disjoint regions, each consisting solely of non-crossing $k$-chords. When we remove these regions, we are left with a path of length $kn-km$ where $m$ is the number of non-crossing $k$-chords. There are thus $\binom{kn-km+1}{p}$ distinct ways of placing the $p$ regions. We need to sum over all possible sizes of these regions, and also over all possible distributions of the $\ell$

| $n \backslash \ell$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | 2 | 1 | | | | | |
| 3 | 4 | 7 | 1 | | | | |
| 4 | 8 | 30 | 16 | 1 | | | |
| 5 | 16 | 104 | 122 | 30 | 1 | | |
| 6 | 32 | 320 | 660 | 365 | 50 | 1 | |
| 7 | 64 | 912 | 2920 | 2875 | 903 | 77 | 1 |

Table 3: The number $T_{n,\ell}$ of non-crossing configurations with exactly $\ell$ polyomino matchings in a generalised game of memory played on the path of length $kn$, for the case $k = 3$. OEIS reference A334062 (to appear).

polyominoes amongst the $p$ regions. These sums take place automatically using the symbolic method by adding a factor of $(T(x, y) - 1)^p$, where we subtract 1 because the regions cannot be empty. Finally, we multiply by $d_{n-m,0}$, because the configuration on the path of length $kn - km$ is necessarily one with no polyominoes. We therefore have that

$$d_{n,\ell,m} = d_{n-m,0} \sum_{p \geq 1} \binom{kn - km + 1}{p} [x^m y^\ell] \, (T(x, y) - 1)^p = [x^m y^\ell] \, T(x, y)^{kn-km+1} \, d_{n-m,0}.$$

□

## 4.1 Asymptotic distribution of polyominoes

In this section we consider the asymptotic distribution of polyominoes amongst non-crossing configurations, i.e.

$$\lim_{n \to \infty} \frac{T_{n,\ell}}{T_n}.$$

It has been established that the Narayana numbers (i.e. the $k = 2$ case) are asymptotically normally distributed, c.f. [5], with mean $\mu = n/2$ and variance $\sigma^2 = n/8$. In this section we appeal to the methods of Flajolet and Noy [3, Theorem 5] to establish the following generalisation.

**Theorem 28.** *The numbers $T_{n,\ell}$ of non-crossing linear $k$-chord diagrams on the path of length $kn$, with exactly $\ell$ polyominoes, are asymptotically normally distributed with mean $\mu$ and variance $\sigma^2$ given by*

$$\mu = \left(\frac{k-1}{k}\right)^{k-1} n, \qquad \sigma^2 = \left(\frac{k-1}{k}\right)^{2k} \frac{k}{(k-1)^2} \left(1 - 2k + (k-1)\left(\frac{k}{k-1}\right)^k\right) n.$$

*Proof.* The methods used by Flajolet and Noy [3, Theorem 5] extend the analytic combinatorics of implicitly defined generating functions, as treated in Flajolet and Sedgewick [4, Section VI.7], to the case of bivariate generating functions. In the univariate case (i.e. setting $y = 1$ in $T(x, y)$), it is straightforward to establish an asymptotic expansion

$$T(x,1) = d_0 + d_1\sqrt{1 - x/\rho} + \mathcal{O}(1 - x/\rho),$$

which then implies the asymptotic growth

$$[x^n]T(x,1) \sim \gamma\frac{\rho^{-n}}{\sqrt{\pi n^3}}\left(1 + \mathcal{O}(n^{-1})\right), \quad \rho = \frac{(k-1)^{k-1}}{k^k}, \quad \gamma = \sqrt{\frac{k}{2(k-1)^3}}.$$

The method of Flajolet and Noy is to extend this to the bivariate case by considering $y$ as a parameter. We begin by expressing the recursion relation of Theorem 24 as follows

$$T(x,y) = x\,\phi\left(T(x,y)\right) \Rightarrow \phi(u) = (1+u)^k - (1-y)(1+u).$$

We are then tasked with solving the so-called *characteristic equation*

$$\phi(\tau(y)) - \tau(y)\phi'(\tau(y)) = 0,$$

which in our case is

$$(1+\tau)^k - (1-y)(1+\tau) - \tau\left(k(1+\tau)^{k-1} - (1-y)\right) = 0.$$

Solving this equation in an expansion of $\tau(y)$ about $y = 1$, we find

$$\tau(y) = \frac{1}{k-1} + \frac{k}{(k-1)^2}\left(\frac{k-1}{k}\right)^k (y-1) - \frac{k}{(k-1)^2}\left(\frac{k-1}{k}\right)^{2k}(y-1)^2 + \mathcal{O}\left((y-1)^3\right),$$

which further implies an expansion for

$$\rho(y) = \frac{\tau(y)}{\phi\left(\tau(y)\right)}.$$

Flajolet and Noy establish that this implies the following asymptotic growth

$$[x^n]T(x,y) = \gamma(y)\,\rho(y)^{-n}\left(1 + \mathcal{O}(n^{-1/2})\right),$$

where $\gamma(y)$ is an analytic function of $y$. The asymptotic normality is then established through results due to Bender and Richmond [1]. The probability generating function for the distribution is given by

$$\frac{[x^n]T(x,y)}{[x^n]T(x,1)} = \frac{\gamma(y)}{\gamma(1)}\left(\frac{\rho(y)}{\rho(1)}\right)^{-n}\left(1 + \mathcal{O}(n^{-1/2})\right).$$

The mean and variance are computed in the usual way

$$\mu = -\frac{\rho'(1)}{\rho(1)}, \qquad \sigma^2 = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2,$$

and this results in

$$\mu = \left(\frac{k-1}{k}\right)^{k-1} n, \qquad \sigma^2 = \left(\frac{k-1}{k}\right)^{2k} \frac{k}{(k-1)^2}\left(1 - 2k + (k-1)\left(\frac{k}{k-1}\right)^k\right) n.$$

□

**Corollary 29.** *The mean $\mu$ and variance $\sigma^2$ are given by the following expressions in the $k \to \infty$ limit:*

$$\mu = \frac{n}{e}, \qquad \sigma^2 = \frac{e-2}{e^2}\, n.$$

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