Non-maximal sensitivity to synchronism in periodic elementary cellular automata: exact asymptotic measures

Pedro P. B. de Oliveira[1], Enrico Formenti[2], Kévin Perrot[3], Sara Riva[2], and Eurico L. P. Ruivo[1]

[1]Universidade Presbiteriana Mackenzie, FCI, São Paulo, Brazil
[2]Université Côte d'Azur, CNRS, I3S, France
[3]Université publique

Abstract

In [11] and [13] the authors showed that elementary cellular automata rules 0, 3, 8, 12, 15, 28, 32, 34, 44, 51, 60, 128, 136, 140, 160, 162, 170, 200 and 204 (and their conjugation, reflection, reflected-conjugation) are not maximum sensitive to synchronism, *i.e.* they do not have a different dynamics for each (non-equivalent) block-sequential update schedule (defined as ordered partitions of cell positions). In this work we present exact measurements of the sensitivity to synchronism for these rules, as functions of the size. These exhibit a surprising variety of values and associated proof methods, such as the special pairs of rule 128, and the connection to the bissection of Lucas numbers of rule 8.

1 Introduction

Cellular automata (CAs) are discrete dynamical systems with respect to time, space and state variables, which have been widely studied both as mathematical and computational objects as well as suitable models for real-world complex systems.

The dynamics of a CA is locally-defined: every agent (*cell*) computes its future state based upon its present state and those of their neighbors, that is, the cells that are connected to them. In spite of their apparent simplicity, they may display non-trivial global emergent behavior, some of them even reaching computational universality [5, 8].

Originally, CAs are updated in a synchronous fashion, that is, every cell of the lattice is updated simultaneously. However, over the last decade, *asynchronous* cellular automata have attracted increasing attention in its associated scientific community.

A comprehensive and detailed overview of asynchronous CAs is given in [7]. There are different ways to define asynchronism in CAs, be it deterministically or stochastically.

Here, we deal with a deterministic version of asynchronism, known as *block-sequential*, coming from the model of Boolean networks and first characterized for this more general model in [3, 2]. Under such an update scheme, the lattice of the CA is partitioned into blocks of cells, each one is assigned a priority of being updated, and this priority ordering is kept fixed throughout the time evolution. For the sake of simplicity, from now on, whenever we refer to *asynchronism*, we will mean *block-sequential*, deterministic asynchronism.

In previous works ([11, 13]), the notion of *maximum sensitivity to asynchronism* was established. Basically, a CA rule was said to present maximum sensitivity to asynchronism when, for any two different block-sequential update schedules, the rule would yield different dynamics. Out of the 88 dynamically independent elementary cellular automata (ECAs) rules, 59 possess maximum sensitivity to asynchronism, while the remaining 19 rules do not. Therefore, it is natural to try and define a *degree* of sensitivity to asynchronism to the latter.

Here, such a notion of a measure to the sensitivity to asynchronism is presented and general analytical formulas for sensitivities of the non-maximal sensitive rules are provided. The results (to be presented on Table 2 at the end of Section 2) exhibit an interesting range of values requiring the introduction of various technics, from measures tending to 0 (highly insensitive) to measures tending to 1 (almost maximal sensitive), with one rule tending to some surprising constant between 0 and 1.

This paper is organized as follows. In Section 2, fundamental definitions and results on Boolean networks, update digraphs and elementary cellular automata are given. Then, in Section 3, experimental measures of sensitivity to asynchronism are given for rules which do not possess maximal sensitivity to asynchronism. Such experimental measures pave the way to the theoretical results in Section 4, in which formal expressions to the sensitivity to asynchronism of such rules are provided for Boolean networks of arbitrary size. Finally, concluding remarks are made in Section 5.

2 Definitions

Elementary cellular automata will be presented in the more general framework of Boolean automata networks, for which the variation of update schedule benefits from useful considerations already studied in the literature. Figure 1 illustrates the definitions.

2.1 Boolean networks

A Boolean Network (BN) of size n is an arrangement of n finite Boolean automata (or components) interacting each other according to a *global rule* $f\colon \{\mathtt{0},\mathtt{1}\}^n \to \{\mathtt{0},\mathtt{1}\}^n$ which describes how the global state changes after one time step. Let $[\![n]\!] = \{0,\dots,n-1\}$. Each automaton is identified with a unique integer $i \in [\![n]\!]$ and x_i denotes the current state of the automaton i. A *configuration* $x \in \{\mathtt{0},\mathtt{1}\}^n$ is a snapshot of the current state of all automata and represents the global state of the BN.

For convenience sake, we identify configurations with words on $\{\mathtt{0},\mathtt{1}\}^n$. Hence, for example, $\mathtt{01111}$ or $\mathtt{01}^4$ both denote the configuration $(\mathtt{0},\mathtt{1},\mathtt{1},\mathtt{1},\mathtt{1})$. Remark that the global function $f : \{\mathtt{0},\mathtt{1}\}^n \to \{\mathtt{0},\mathtt{1}\}^n$ of a BN of size n induces a set of n *local functions* $f_i : \{\mathtt{0},\mathtt{1}\}^n \to \{\mathtt{0},\mathtt{1}\}$, one per each component, such that $f(x) = (f_0(x), f_1(x), \dots, f_{n-1}(x))$ for all $x \in \{\mathtt{0},\mathtt{1}\}^n$. This

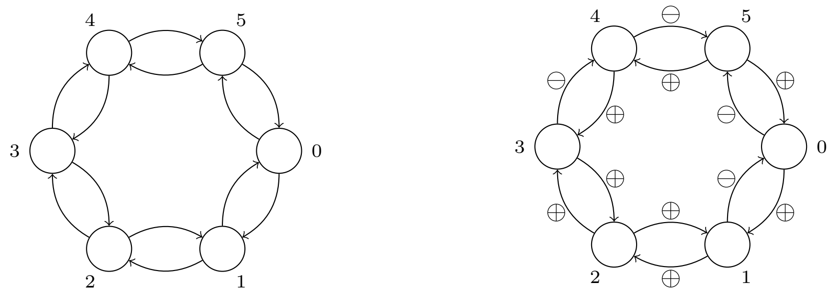


Figure 1: interaction digraph G_6^{ECA} of the ECA rule 128 for $n = 6$, with local functions $f_i(x) = x_{i-1} \wedge x_i \wedge x_{i+1}$ for all $i \in \{0, \ldots, 5\}$ (left). Update digraph corresponding to the update schedules $\Delta = (\{1,2,3\},\{0,4\},\{5\})$ and $\Delta' = (\{1,2,3\},\{0\},\{4\},\{5\})$, which are therefore equivalent ($\Delta \equiv \Delta'$). For example, $f^{(\Delta)}(\mathtt{111011}) = \mathtt{110000}$ whereas for the synchronous update schedule we have $f^{(\Delta^{\mathsf{sync}})}(\mathtt{111011}) = \mathtt{110001}$ (right).

gives a static description of a discrete dynamical system, and it remains to set the order in which components are updated in order to get a dynamics. Before going to update schedules, let us first introduce interaction digraphs.

The component i *influences* the component j if $\exists x \in \{\mathtt{0},\mathtt{1}\}^n : f_j(x) \neq f_j(\overline{x}^i)$, where $\overline{x}^i$ is the configuration obtained from x by flipping the state of component i. Note that in literature one may also consider *positive* and *negative* influences, but they will not be useful for the present study. The *interaction digraph* $G_f = (V, A)$ of a BN f represents the effective dependencies among its set of components

$$V = [\![n]\!] \quad \text{and} \quad A = \{(i,j) \mid i \text{ influences } j\}.$$

It will turn out to be pertinent to consider $\hat{G}_f = (V, A)$, obtained from G_f by removing the loops (arcs of the form (i,i)).

For $n \in \mathbb{N}$, denote $\mathcal{P}_n$ the set of ordered partitions of $[\![n]\!]$ and $|f|$ the size of a BN f. A *block-sequential update schedule* $\Delta = (\Delta_1, \ldots, \Delta_k)$ is an element of $\mathcal{P}_{|f|}$. It defines the following dynamics $f^{(\Delta)} : \{\mathtt{0},\mathtt{1}\}^n \to \{\mathtt{0},\mathtt{1}\}^n$,

$$f^{(\Delta)} = f^{(\Delta_k)} \circ \cdots \circ f^{(\Delta_2)} \circ f^{(\Delta_1)} \quad \text{with} \quad f^{(\Delta_j)}(x)_i = \begin{cases} f_i(x) & \text{if } i \in \Delta_j, \\ x_i & \text{if } i \notin \Delta_j. \end{cases}$$

In words, the components are updated in the order given by Δ: sequentially part after part, and in parallel within each part. The *parallel* or *synchronous* update schedule is $\Delta^{\mathsf{sync}} = ([\![n]\!])$ and we have $f^{(\Delta^{\mathsf{sync}})} = f$. In this article, since only block-sequential update schedules are considered, they are simply called *update schedule* for short. They are

- "*fair*" in the sense that all components are updated the exact same number of times,
- "*periodic*" in the sense that the same ordered partition is repeated.

Given a BN f of size n and an update schedule Δ, the *transition digraph* $D_{f^{(\Delta)}} = (V, A)$ is such that

$$V = \{\mathtt{0},\mathtt{1}\}^n \quad \text{and} \quad A = \{(x, f^{(\Delta)}(x)) \mid x \in \{\mathtt{0},\mathtt{1}\}^n\}.$$

It describes the *dynamics* of f under the update schedule Δ. The set of all possible dynamics of the BN f, at the basis of the measure of sensitivity to synchronism, is then defined as

$$\mathcal{D}(f) = \left\{ D_{f(\Delta)} \mid \Delta \in \mathcal{P}_{|f|} \right\}.$$

2.2 Update digraphs and equivalent update schedules

For a given BN, some update schedules always give the same dynamics. Indeed, if, for example, two components do not influence each other, their order of updating has no effect on the dynamics (see 1 for a detailed example). In [3], the notion of *update digraph* has been introduced in order to study update schedules.

Given a BN f with loopless interaction digraph $\hat{G}_f = (V, A)$ and an update schedule $\Delta \in \mathcal{P}_n$, define $lab_\Delta : A \to \{\oplus, \ominus\}$ as

$$\forall (i,j) \in A : lab_\Delta((i,j)) = \begin{cases} \oplus & \text{if } i \in \Delta_{k_i}, j \in \Delta_{k_j} \text{ with } i \geq j, \\ \ominus & \text{if } i \in \Delta_{k_i}, j \in \Delta_{k_j} \text{ with } i < j. \end{cases}$$

The *update digraph* $U_{f(\Delta)}$ of the BN f for the update schedule $\Delta \in \mathcal{P}_n$ is the loopless interaction digraph decorated with lab_Δ, *i.e.* $U_{f(\Delta)} = (V, A, lab_\Delta)$. Note that loops are removed because they bring no meaningful information: indeed, an edge (i,i) would always be labeled $\oplus$. Now we have that, if two update schedules define the same update digraph then they also define the same dynamics.

Theorem 1 ([3])**.** *Given a BN f and two update schedules Δ, Δ', if $lab_\Delta = lab_{\Delta'}$ then $D_{f(\Delta)} = D_{f(\Delta')}$.*

A very important remark is that not all labelings correspond to *valid* update digraphs (*i.e.* such that there are update schedules giving these labelings). For example, if two arcs (i,j) and (j,i) belong to the interaction digraph and are both labeled $\ominus$, it would mean that i is updated prior to j and j is updated prior to i, which is contradictory. Hopefully there is a nice characterisation of *valid* update digraphs.

Theorem 2 ([2])**.** *Given f with $\hat{G}_f = (V, A)$, the label function $lab : A \to \{\oplus, \ominus\}$ is valid if and only if there is no cycle $(i_0, i_1, \ldots, i_k)$, with $i_0 = i_k$ and $k > 0$, such that*

- $\forall 0 \leq j < k : ((i_j, i_{j+1}) \in A \wedge lab((i_j, i_{j+1})) = \oplus) \vee ((i_{j+1}, i_j) \in A \wedge lab((i_{j+1}, i_j)) = \ominus)$,
- $\exists 0 \leq i < k : lab((i_{j+1}, i_j)) = \ominus$.

In words, Theorem 2 states that a labeling is valid if and only if the multi-digraph where the labeling is unchanged but the orientation of arcs labeled $\ominus$ is reversed, does not contain a cycle with at least one arc label $\ominus$ (*forbidden cycle*).

According to Theorem 1, update digraphs define equivalence classes of update schedules: $\Delta \equiv \Delta'$ if and only if $lab_\Delta = lab_{\Delta'}$. Given a BN f, the set of equivalence classes of update schedules is therefore defined as

$$\mathcal{U}(f) = \left\{ U_{f(\Delta)} \mid \Delta \in \mathcal{P}_{|f|} \right\}.$$

2.3 Sensitivity to synchronism

The sensitivity to synchronism $\mu_s(f)$ of a BN f quantifies the proportion of distinct dynamics *w.r.t* non-equivalent update schedules. The idea is that when two or more update schedules are equivalent then $\mu_s(f)$ decreases, while it increase when distinct update schedules bring to different dynamics. More formally, given a BN f we define

$$\mu_s(f) = \frac{|\mathcal{D}(f)|}{|\mathcal{U}(f)|}.$$

Obviously, it holds that $\frac{1}{|\mathcal{U}(f)|} \leq \mu_s(f) \leq 1$, and a BN f is as much sensible to synchronism as it has different dynamics when the update schedule varies. The extreme cases are a BN f with $\mu_s(f) = \frac{1}{|\mathcal{U}(f)|}$ that has always the same dynamics $D_{f(\Delta)}$ for any update schedule Δ, and a BN f with $\mu_s(f) = 1$ which has a different dynamics for different update schedules (for each $\Delta \not\equiv \Delta'$ it holds $D_{f(\Delta)} \neq D_{f(\Delta')}$). A BN f is *max-sensitive* to synchronism iff $\mu_s(f) = 1$. Note that a BN f is max-sensitive if and only if

$$\forall \Delta \in \mathcal{P}_{|f|} \forall \Delta' \in \mathcal{P}_{|f|}\ (\Delta \not\equiv \Delta') \Rightarrow \exists x \in \{0,1\}^n \exists i \in [\![n]\!]\ f^{(\Delta)}(x)_i \neq f^{(\Delta')}(x)_i\ . \tag{1}$$

2.4 Elementary cellular automata

In this study we investigate the sensitivity to synchronism of *elementary cellular automata* (ECA) over periodic configurations. Indeed, they are a subclass of BN in which all components (also called *cells* in this context) have the same local rule. Given a size n, the ECA of local function $h : \{0,1\}^3 \to \{0,1\}$ is the BN f such that

$$\forall i \in [\![n]\!] : f_i(x) = h(x_{i-1}, x_i, x_{i+1})$$

where components are taken modulo n (this will be the case throughout all the paper without explicit mention). We use *Wolfram numbers* [14] to designate each of the 256 ECA local rule $h : \{0,1\}^3 \to \{0,1\}$ as the number

$$w(r) = \sum_{(x_1,x_2,x_3)\in\{0,1\}^3} h(x_1,x_2,x_3) 2^{2^2 x_1 + 2^1 x_2 + 2^0 x_3}.$$

Given a Boolean function $h : \{0,1\}^3 \to \{0,1\}$, consider the following transformations over local rules: $\tau_i(h)(x,y,z) = h(x,y,z)$, $\tau_r(h)(x,y,z) = h(z,y,x)$, $\tau_n(h)(x,y,z) = 1 - h(1 - z, 1-y, 1-x)$ and $\tau_{rn}(h)(x,y,z) = 1 - h(1-z, 1-y, 1-x)$ for all $x,y,z \in \{0,1\}$. In [4], it is proved the previous transformations preserve topological dynamics and hence, in our context, they preserve the sensitivity to synchronism. For this reason we consider ECA up to topological conjugacy. Table 1 reports the equivalence classes of ECA up to topological conjugacy, the smallest Wolfram number per class is indicated.

The definitions of Subsection 2.3 are applied to ECA rules as follows. Given a size n, the *ECA interaction digraph of size n* $G_n^{\text{ECA}} = (V, A)$ is such that $V = [\![n]\!]$ and $A = \{(i+1, i), (i, i+1) \mid i \in [\![n]\!]\}$.

In [11, 13], it is proved that

$$|\mathcal{U}^{\text{ECA}}(n)| = 3^n - 2^{n+1} + 2.$$

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 50, 51, 54, 56, 57, 58, 60, 62, 72, 73, 74, 76, 77, 78, 90, 94, 104, 105, 106, 108, 110, 122, 126, 128, 130, 132, 134, 136, 138, 140, 142, 146, 150, 152, 154, 156, 160, 162, 164, 168, 170, 172, 178, 184, 200, 204, 232

Table 1: ECA local rules up to topological conjugacy.

where $\mathcal{U}^{\text{ECA}}(n)$ is the set of valid labelings of G_n^{ECA}. The sensitivity to synchronism of ECAs is measured relatively to the family of ECAs, and therefore relatively to this count of valid labelings of G_n^{ECA}, even for rules where some arcs do not correspond to effective influences (one may think of rule 0). Except from this subtlety, the measure is correctly defined by considering, for an ECA rule number α and a size n, that $h_\alpha \colon \{0,1\}^3 \to \{0,1\}$ is its local rule, and that $f_{\alpha,n} \colon \{0,1\}^n \to \{0,1\}^n$ is its global function on periodic configurations of size n,

$$\forall x \in \{0,1\}^n \; f_{\alpha,n}(x)_i = h_\alpha(x_{i-1}, x_i, x_{i+1}).$$

Then, the sensitivity to synchronism of ECA rule number α is given by

$$\mu_s(f_{\alpha,n}) = \frac{|\mathcal{D}(f_{\alpha,n})|}{3^n - 2^{n+1} + 2}.$$

An ECA rule number α is (ultimately) *max-sensitive to synchronism* when

$$\lim_{n \to +\infty} \mu_s(f_{\alpha,n}) = 1.$$

The following result provides a first overview of sensitivity to synchronism in ECA.

Theorem 3 ([11, 13])**.** *For any size $n \geq 7$, the nineteen ECA rules 0, 3, 8, 12, 15, 28, 32, 34, 44, 51, 60, 128, 136, 140, 160, 162, 170, 200 and 204 are not max-sensitive to synchronism. The remaining sixty nine other rules are max-sensitive to synchronism.*

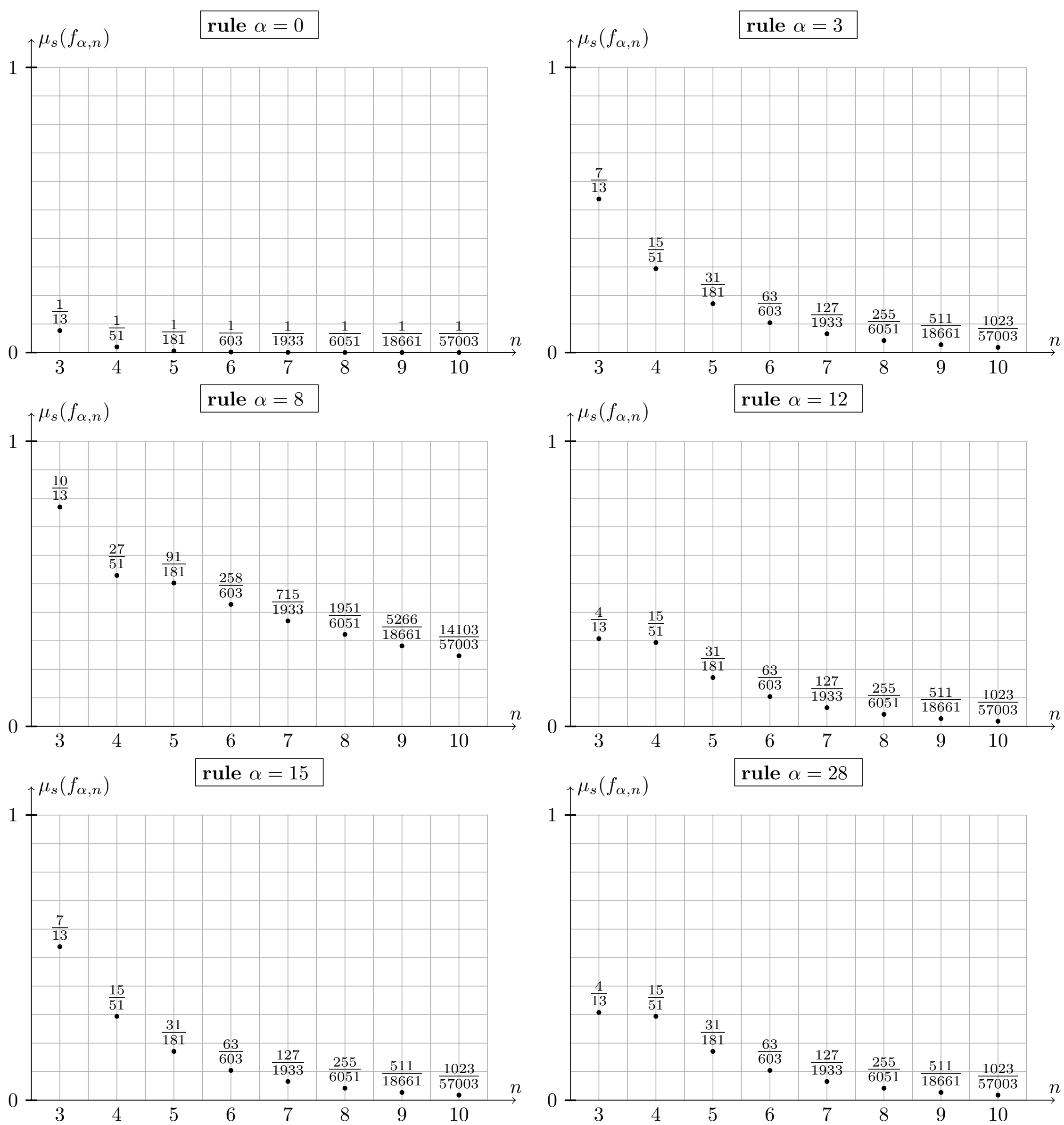
Theorem 3 gives a precise measure of sensitivity for the sixty nine maximum sensitive rules, for which $\mu_s(f_{\alpha,n}) = 1$ for all $n \geq 7$, but for the nineteen that are not maximum sensitive it only informs that $\mu_s(f_{\alpha,n}) < 1$ for all $n \geq 7$. In the rest of this paper we study the precise dependency on n of $\mu_s(f_{\alpha,n})$ for these rules, filling the huge gap between $\frac{1}{3^n - 2^{n+1} + 2}$ and $\frac{3^n - 2^{n+1} + 1}{3^n - 2^{n+1} + 2}$. This will offer a finer view on the sensitivity to synchronism of ECA. The results are summarized in Table 2.

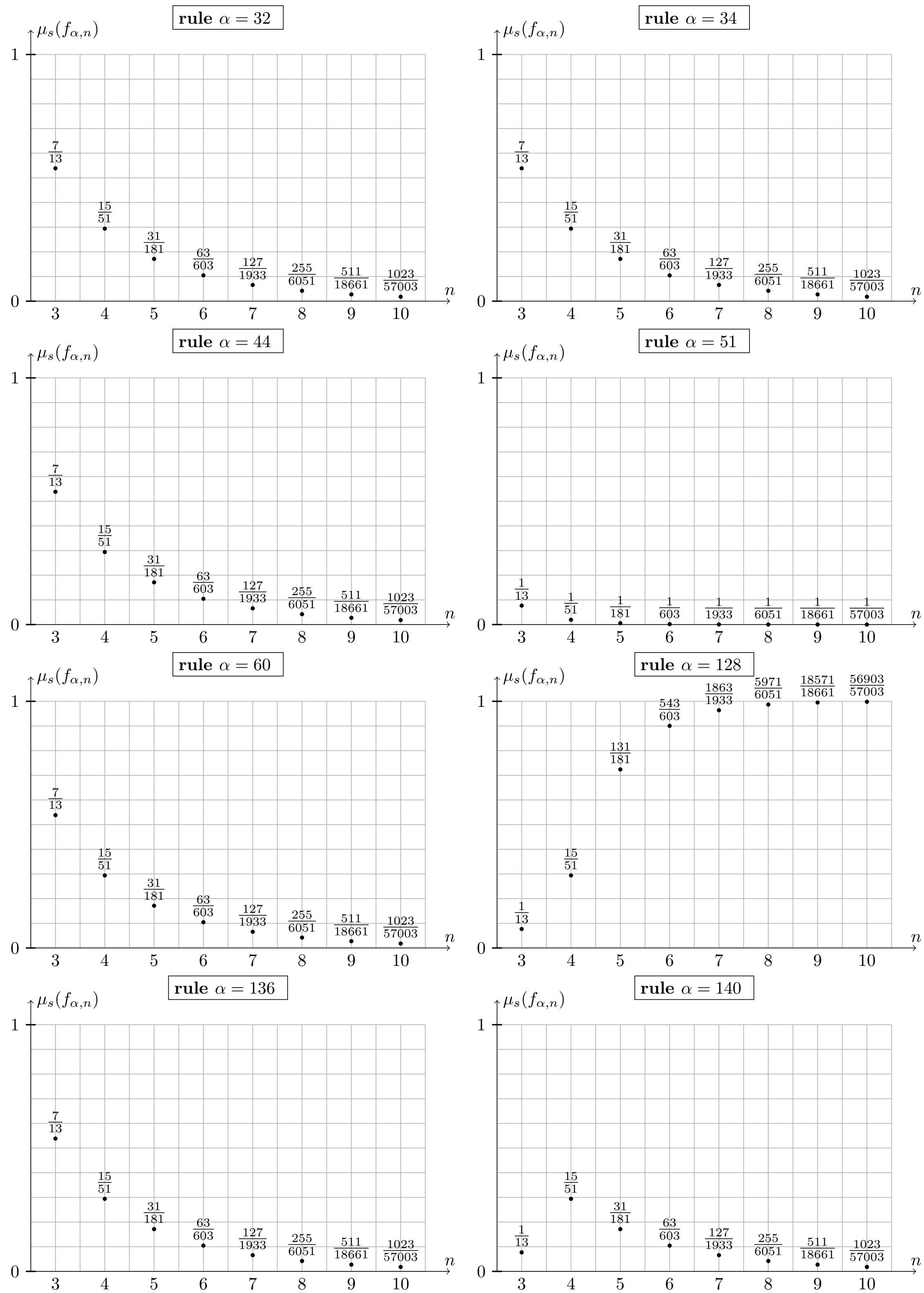
3 Experimental measures of sensitivity to synchronism

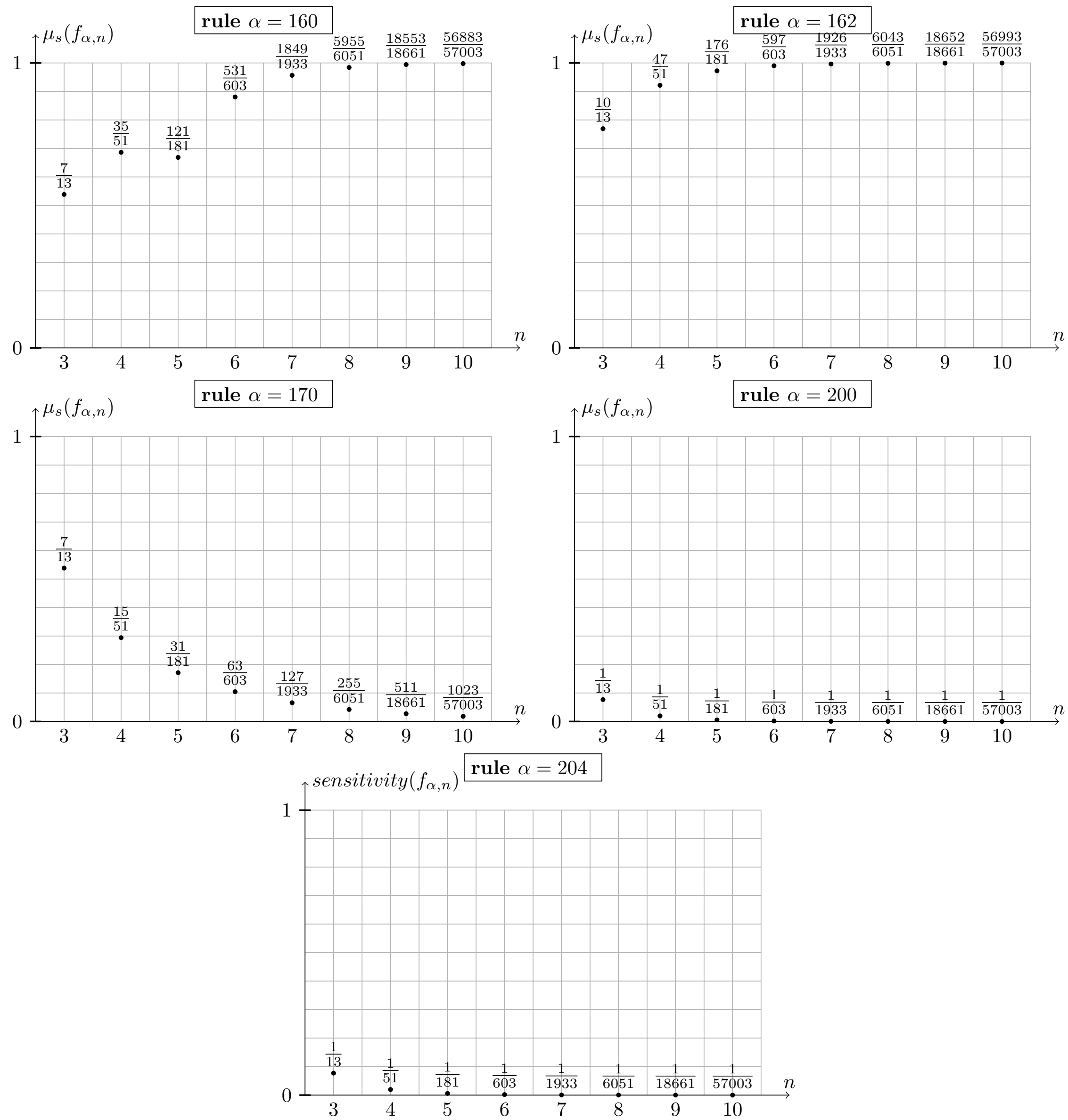
This section presents some numerical calculations of $\mu_s(f_{\alpha,n})$ for rules that are not max-sensitive to synchronism according to Theorem 3. An interesting variety of behaviors (for $n = 3$ to 10) is observed. It will be characterized in Section 4.

Rules (α)	Sections	Sensitivity ($\mu_s(f_{\alpha,n})$)
$0, 51, 200, 204$	4.1	$\frac{1}{3^n-2^{n+1}+2}$ for any $n \geq 3$
$3, 12, 15, 34, 60, 136, 170$	4.2.1	$\frac{2^n-1}{3^n-2^{n+1}+2}$ for any $n \geq 4$
$28, 32, 44, 140$	4.2.2	$\frac{2^n-1}{3^n-2^{n+1}+2}$ for any $n \geq 4$
8	4.2.3	$\frac{\phi^{2n}+\phi^{-2n}-2^n}{3^n-2^{n+1}+2}$ for any $n \geq 5$
$128, 160, 162$	4.3	$\frac{3^n-2^{n+1}-cn+2}{3^n-2^{n+1}+2}$ for any $n \geq 5$

Table 2: presentation of the results.




rule $\alpha = 32$
$\mu_s(f_{\alpha,n})$
n
$\frac{7}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$
rule $\alpha = 34$
$\mu_s(f_{\alpha,n})$
n
$\frac{7}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$
rule $\alpha = 44$
$\mu_s(f_{\alpha,n})$
n
$\frac{7}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$
rule $\alpha = 51$
$\mu_s(f_{\alpha,n})$
n
$\frac{1}{13}$ $\frac{1}{51}$ $\frac{1}{181}$ $\frac{1}{603}$ $\frac{1}{1933}$ $\frac{1}{6051}$ $\frac{1}{18661}$ $\frac{1}{57003}$
rule $\alpha = 60$
$\mu_s(f_{\alpha,n})$
n
$\frac{7}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$
rule $\alpha = 128$
$\mu_s(f_{\alpha,n})$
n
$\frac{1}{13}$ $\frac{15}{51}$ $\frac{131}{181}$ $\frac{543}{603}$ $\frac{1863}{1933}$ $\frac{5971}{6051}$ $\frac{18571}{18661}$ $\frac{56903}{57003}$
rule $\alpha = 136$
$\mu_s(f_{\alpha,n})$
n
$\frac{7}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$
rule $\alpha = 140$
$\mu_s(f_{\alpha,n})$
n
$\frac{1}{13}$ $\frac{15}{51}$ $\frac{31}{181}$ $\frac{63}{603}$ $\frac{127}{1933}$ $\frac{255}{6051}$ $\frac{511}{18661}$ $\frac{1023}{57003}$



4 Theoretical measures of sensitivity to synchronism

In this section contains the main results of the paper, regarding the dependency on n of $\mu_s(f_{\alpha,n})$ for ECA rules that are not max-sensitive to synchronism.

As illustrated in Table 2, the sensitivity functions of ECA can be divided into three main classes according to their asymptotic behavior. Each class will require specific proof techniques but all of them have interaction digraphs as a common denominator.

As a starting point, one can consider the case of ECA rules have an interaction digraph which is a proper subgraph of G_n^{ECA}. Indeed, when considering them as BN many distinct update schedules give the same labelings and hence, by Theorem 1 and the definition of $\mu_s(f_{\alpha,n})$, they cannot be max-sensitive. This is the case of the following set of ECA rules $\mathcal{S} = \{0, 3, 12, 15, 34, 51, 60, 136, 170, 204\}$. Indeed, denoting $G_{f_{\alpha,n}} = (\llbracket n \rrbracket, A_{f_{\alpha,n}})$ the interaction

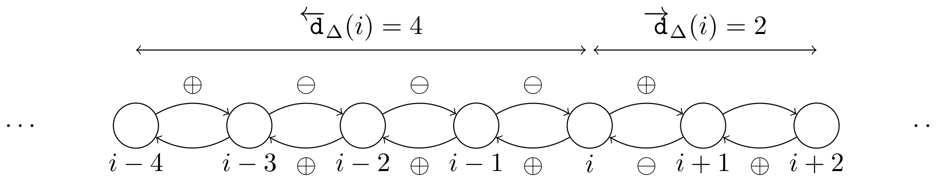


Figure 2: illustration of the chain of influences for some update schedule Δ.

digraph of ECA rule α of size n for $\alpha \in \mathcal{S}$, one finds $\forall n \geq 3$ and $\forall i \in [\![n]\!]$:

- $(i+1, i) \notin A_{f_{0,n}}$ and $(i-1, i) \notin A_{f_{0,n}}$,
- $(i+1, i) \notin A_{f_{3,n}}$,
- $(i+1, i) \notin A_{f_{12,n}}$,
- $(i+1, i) \notin A_{f_{15,n}}$,
- $(i, i+1) \notin A_{f_{34,n}}$,
- $(i+1, i) \notin A_{f_{51,n}}$ and $(i-1, i) \notin A_{f_{51,n}}$,
- $(i+1, i) \notin A_{f_{60,n}}$,
- $(i, i+1) \notin A_{f_{136,n}}$,
- $(i, i+1) \notin A_{f_{170,n}}$,
- $(i+1, i) \notin A_{f_{204,n}}$ and $(i-1, i) \notin A_{f_{204,n}}$.

Let us now introduce some useful results and notations that will be widely used in the sequel. Given an update schedule Δ, in order to study the chain of influences involved in the computation of the image at cell $i \in [\![n]\!]$, define

$$\begin{aligned}\overleftarrow{\mathtt{d}}_\Delta(i) &= \max\left\{k \in \mathbb{N} \mid \forall j \in \mathbb{N} : 0 < j < k \implies lab_\Delta((i-j, i-j+1)) = \ominus\right\}\\ \overrightarrow{\mathtt{d}}_\Delta(i) &= \max\left\{k \in \mathbb{N} \mid \forall j \in \mathbb{N} : 0 < j < k \implies lab_\Delta((i+j, i+j-1)) = \ominus\right\}.\end{aligned}$$

These quantities are well defined because $k = 1$ is always a possible value, and moreover, if $\overleftarrow{\mathtt{d}}_\Delta(i)$ or $\overrightarrow{\mathtt{d}}_\Delta(i)$ is greater than n, then there is a forbidden cycle in the update digraph of schedule Δ (Theorem 2). Note that for any $\Delta \in \mathcal{P}_n$

$$lab_\Delta((i - \overleftarrow{\mathtt{d}}_\Delta(i), i - \overleftarrow{\mathtt{d}}_\Delta(i) + 1)) = \oplus \quad \text{and} \quad lab_\Delta((i + \overrightarrow{\mathtt{d}}_\Delta(i), i + \overrightarrow{\mathtt{d}}_\Delta(i) - 1)) = \oplus.$$

See Figure 2 for an illustration.

The purpose of these quantities is of that for any $x \in \{\mathtt{0}, \mathtt{1}\}^n$ it holds

$$f_\alpha^{(\Delta)}(x)_i = r_\alpha(\underbrace{r_\alpha(\underbrace{\cdots}_{r_\alpha(x_{i-\overleftarrow{\mathtt{d}}_\Delta(i)}, x_{i-\overleftarrow{\mathtt{d}}_\Delta(i)+1}, x_{i-\overleftarrow{\mathtt{d}}_\Delta(i)+2})}, x_{i-1}, x_i)}, x_i, \underbrace{r_\alpha(x_i, x_{i+1}, \underbrace{\cdots}_{r_\alpha(x_{i+\overrightarrow{\mathtt{d}}_\Delta(i)-2}, x_{i+\overrightarrow{\mathtt{d}}_\Delta(i)-1}, x_{i+\overrightarrow{\mathtt{d}}_\Delta(i)})})}) \tag{2}$$

i.e. the quantities $\overleftarrow{\mathtt{d}}_\Delta(i)$ and $\overrightarrow{\mathtt{d}}_\Delta(i)$ are the lengths of the chain of influences at cell i for the update schedule Δ, on both sides of the interaction digraph. If the chains of influences at some cell i are identical for two update schedules, then the images at i we be identical for any configuration, as stated in the following lemma.

Lemma 4. *For any ECA rule α, any $n \in \mathbb{N}$, any $\Delta, \Delta' \in \mathcal{P}_n$ and any $i \in [\![n]\!]$, it holds that*

$$\overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_{\Delta'}(i) \wedge \overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i) \text{ implies } \forall x \in \{\mathtt{0}, \mathtt{1}\}^n\ f_{\alpha,n}^{(\Delta)}(x)_i = f_{\alpha,n}^{(\Delta')}(x)_i.$$

Proof. This is a direct consequence of Equation 2, because the nesting of local rules for Δ and Δ' are identical at cell i. $\square$

For any rule α, size n, and update schedules $\Delta, \Delta' \in \mathcal{P}_n$, it holds

$$\forall i \in [\![n]\!] : \overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_{\Delta'}(i) \wedge \overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i) \iff \Delta \equiv \Delta' \tag{3}$$

and this implies $D_{f_{\alpha,n}^{(\Delta)}} = D_{f_{\alpha,n}^{(\Delta')}}$. Remark that it is possible that $\overleftarrow{\mathtt{d}}_\Delta(i) + \overrightarrow{\mathtt{d}}_\Delta(i) \geq n$, in which case the image at cell i depends on the whole configuration. Moreover the previous inequality may be strict (meaning that the dependencies on both sides may overlap for some cell). This will be a key in computing the dependency on n of the sensitivity to synchronism for rule 128 for example. Let

$$\mathtt{d}_\Delta(i) = \{j \mid i - j \leq \overleftarrow{\mathtt{d}}_\Delta(i)\} \cup \{j \mid j - i \leq \overrightarrow{\mathtt{d}}_\Delta(i)\}$$

be the set of cells that i depends on under update schedule $\Delta \in \mathcal{P}_n$. When $\mathtt{d}_\Delta(i) \neq [\![n]\!]$ then cell i does not depend on the whole configuration, and $\mathtt{d}_\Delta(i)$ describes precisely Δ, as stated in the following lemma.

Lemma 5. *For any $\Delta, \Delta' \in \mathcal{P}_n$, it holds that*

$$\forall i \in [\![n]\!]\ d_\Delta(i) = d_{\Delta'}(i) \neq [\![n]\!] \textit{ implies } \Delta \equiv \Delta'.$$

Proof. If $\mathtt{d}_\Delta(i) \neq [\![n]\!]$ then $\overleftarrow{\mathtt{d}}_\Delta(i)$ and $\overrightarrow{\mathtt{d}}_\Delta(i)$ do not overlap. Moreover, remark that $\overleftarrow{\mathtt{d}}_\Delta(i)$ and $\overrightarrow{\mathtt{d}}_\Delta(i)$ can be deduced from $\mathtt{d}_\Delta(i)$. Indeed,

$$\begin{aligned}\overleftarrow{\mathtt{d}}_\Delta(i) &= \max\{j \mid \forall k \in [\![j]\!], i - j + k \in \mathtt{d}_\Delta(i)\}\\ \overrightarrow{\mathtt{d}}_\Delta(i) &= \max\{j \mid \forall k \in [\![j]\!], i + j - k \in \mathtt{d}_\Delta(i)\}\end{aligned}$$

The result follows since knowing $\overrightarrow{\mathtt{d}}_\Delta(i)$ and $\overleftarrow{\mathtt{d}}_\Delta(i)$ for all $i \in [\![n]\!]$ allows to completely reconstruct lab_Δ, which would be the same as $lab_{\Delta'}$ if $d_\Delta(i) = d_{\Delta'}(i)$ for all $i \in [\![n]\!]$ (Formula 3). $\square$

4.1 Class I: the sensitivity function tends to 0

This class contains the simplest dynamics and it is a good starting point for our analysis.

Theorem 6. $\mu_s(f_{0,n}) = \frac{1}{3^n - 2^{n+1} + 2}$ *for any $n \geq 1$ and for $\alpha \in \{0, 51, 204\}$.*

Proof. The result for ECA rule 0 is obvious since $\forall n \geq 1 : \forall x \in \{\mathtt{0}, \mathtt{1}\}^n : f_{0,n}(x) = \mathtt{0}^n$. The ECA Rule 51 is based on the boolean function $r_{51}(x_{i-1}, x_i, x_{i-1}) = \neg x_i$ and ECA rule 204 is the identity. Therefore, similarly to ECA rule 0, for any n their interaction digraph has no arcs. Hence, there is only one equivalence class of update digraph, and one dynamics. $\square$

The ECA rule 200 also belongs to Class I. It has the following local function $r_{200}(x_1, x_2, x_3) = x_2 \wedge (x_1 \vee x_3)$. Indeed, it is almost equal to the identity (ECA rule 204), except for $r_{200}(\mathtt{0}, \mathtt{1}, \mathtt{0}) = \mathtt{0}$. It turns out that, even if its interaction digraph has all of the $2n$ arcs, this rule produces always the same dynamics, regardless of the update schedule.

Theorem 7. *$\mu_s(f_{200,n}) = \frac{1}{3^n - 2^{n+1}+2}$ for any $n \geq 1$.*

Proof. We prove that $f^{(\Delta)}_{200,n}(x) = f^{(\Delta^{\texttt{sync}})}_{200,n}(x)$ for any configuration $x \in \{\texttt{0},\texttt{1}\}^n$ and for any update schedule $\Delta \in \mathcal{P}_n$. For any $i \in [\![n]\!]$ such that $x_i = \texttt{0}$, the ECA rule 200 is the identity, therefore it does not depend on the states of its neighbors which may have been updated before itself, *i.e.* $f^{(\Delta)}_{200,n}(x)_i = \texttt{0} = f^{(\Delta^{\texttt{sync}})}_{200,n}(x)_i$. Moreover, for any $i \in [\![n]\!]$ such that $x_i = \texttt{1}$, if its two neighbors x_{i-1} and x_{i+1} are both in state $\texttt{0}$ then from what preceeds these neighbors will remain in state $\texttt{0}$ and $f^{(\Delta)}_{200,n}(x)_i = \texttt{0} = f^{(\Delta^{\texttt{sync}})}_{200,n}(x)_i$, otherwise the ECA 200 is the identity map and the two neighbors of cell i also apply the identity, thus again $f^{(\Delta)}_{200,n}(x)_i = \texttt{1} = f^{(\Delta^{\texttt{sync}})}_{200,n}(x)_i$. □

4.2 Class II: the sensitivity function tends to a constant $0 < c < 1$

This is a very interesting class which demands us to develop specific arguments and tools. However, the starting point is always the interaction digraph.

4.2.1 Sensitivity tends to $c = 2/3$

The following result counts the number of equivalence classes of update schedules for ECA rules α having only arcs of the form $(i, i+1)$, or only arcs of the form $(i+1, i)$ in their interaction digraph $G_{f_{\alpha,n}}$.

Lemma 8. *For the ECA rules $\alpha \in \{3, 12, 15, 34, 60, 136, 170\}$, it holds $|\mathcal{U}(f_{\alpha,n})| \leq 2^n - 1$.*

Proof. The interaction digraph of these rules is the directed cycle on n vertices (with n arcs). There can be only a forbidden cycle of length n in the case that all arcs are labeled $\ominus$ (see Theorem 2). Except for the all $\oplus$ labeling (which is valid), any other labeling prevents the formation of an invalid cycle, since the orientation of at least one arc is unchanged (labeled $\oplus$), and the orientation of at least one arc is reversed (labeled $\ominus$). □

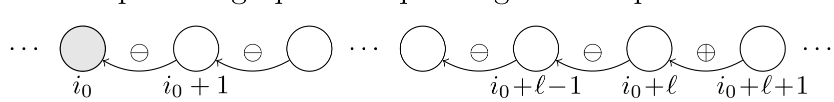
In the sequel we are going to exploit Lemma 8 to obtain one of the main results of this section. The ECA rule 170, which is based on the following Boolean function: $r_{170}(x_{i-1}, x_i, x_{i+1}) = x_{i+1}$, shows the pathway.

Theorem 9. *$\mu_s(f_{170,n}) = \frac{2^n-1}{3^n - 2^{n+1}+2}$ for any $n \geq 2$.*

Proof. Let $f = f_{170,n}$ and $n \geq 2$. By definition, one finds that for any two non-equivalent update schedules $\Delta \not\equiv \Delta'$ it holds

$$\exists i_0 \in [\![n]\!] \; lab_\Delta((i_0+1, i_0)) = \oplus \wedge lab_{\Delta'}((i_0+1, i_0)) = \ominus.$$

Furthermore, since having $lab_{\Delta'}((i+1, i)) = \ominus$ for all $i \in [\![n]\!]$ creates an invalid cycle of length n, there exists a minimal $\ell \geq 1$ such that $lab_{\Delta'}((i_0+\ell+1, i_0+\ell)) = \oplus$ (this requires $n > 1$). A part of the update digraph corresponding to Δ' is pictured below.



By definition of the labels and the minimality of ℓ we have that

$$\forall 0 \leq k < \ell : f^{(\Delta')}(x)_{i_0+k} = x_{i_0+\ell+1}.$$

Since for the update schedule Δ we have $f^{(\Delta)}(x)_{i_0} = x_{i_0+1}$, it is always possible to construct a configuration x with $x_{i_0+1} \neq x_{i_0+\ell+1}$ such that the two dynamics differ *i.e.* $f^{(\Delta)}(x)_{i_0} \neq f^{(\Delta')}(x)_{i_0}$. The result holds by Formula 1. □

Generalizing the idea behind the construction used for ECA rule 170 allows to find the precise sensitivity measure for the ECA rules $3, 12, 15, 34, 60, 136$.

Theorem 10. *$\mu_s(f_{\alpha,n}) = \frac{2^n-1}{3^n-2^{n+1}+2}$ for any $n \geq 2$ and for all $\alpha \in \{3, 12, 15, 34, 60, 136\}$.*

Proof. We present the case when the interaction digraph has only arcs of type $(i+1, i)$ (such as rule 170), the case $(i, i+1)$ is symmetric. Fix $n \geq 2$ and choose two update schedules $\Delta, \Delta' \in \mathcal{P}_n$ such that $\Delta \not\equiv \Delta'$, then it holds that

$$\exists i_0 \in [\![n]\!]\ lab_\Delta((i_0+1, i_0)) = \oplus \wedge lab_{\Delta'}((i_0+1, i_0)) = \ominus$$
$$\exists \ell \in [\![n]\!]\ (\forall 0 \leq k < \ell\ lab_{\Delta'}((i_0+k+1, i_0+k)) = \ominus) \wedge (lab_{\Delta'}((i_0+\ell+1, i_0+\ell)) = \oplus).$$

Fix $\alpha \in \{3, 12, 15, 34, 60, 136\}$ and let r be the corresponding Boolean function. Moreover let $f = f_{\alpha,n}$. We know that for any $x \in \{0,1\}^n$ we will have $f^{(\Delta)}(x)_{i_0} = r(b, x_{i_0}, x_{i_0+1})$ for any $b \in \{0,1\}$. Our goal is to construct a configuration $x \in \{0,1\}^n$ such that $f^{(\Delta)}(x)_{i_0} \neq f^{(\Delta')}(x)_{i_0}$. In order to start, we need

$$\exists x_{i_0}, x_{i_0+1}, o_1, o_2 \in \{0,1\}\ \forall b, b' \in \{0,1\} \qquad \begin{array}{c} r(b, x_{i_0}, x_{i_0+1}) \neq r(b', x_{i_0}, o_1) \\ \text{and } r(x_{i_0}, x_{i_0+1}, o_2) = o_1. \end{array} \tag{4}$$

In other words, we can choose x_{i_0}, x_{i_0+1} so that there is a target output o_1 for $f^{(\Delta')}(x)_{i_0+1}$, such that if $f^{(\Delta')}(x)_{i_0+1} = o_1$ then $f^{(\Delta)}(x)_{i_0} \neq f^{(\Delta')}(x)_{i_0}$ (the values of b and b' do not matter). Similarly, given x_{i_0}, x_{i_0+1}, o_1, there is a target output o_2 for $f^{(\Delta')}(x)_{i_0+2}$. We can now construct x by finite induction by expressing such target outputs. In order to continue we want (the idea is to use this by induction for all $0 < k \leq \ell$)

$$\forall x_{i_0+k-1}, o_k \in \{0,1\}\ \exists x_{i_0+k}, o_{k+1} \in \{0,1\}\ r(x_{i_0+k-1}, x_{i_0+k}, o_{k+1}) = o_k. \tag{5}$$

If Formulas 4 and 5 hold then we can construct the desired configuration x. Indeed, Formula 4 gives $x_{i_0}, o_1, x_{i_0+1}, o_2$, and by induction, knowing x_{i_0+k-1}, o_k Formula 4 gives x_{i_0+k}, o_{k+1} for $0 < k \leq \ell$. The construction ends with $x_{i_0+\ell+1} = o_{\ell+1}$. A sequence $(x_i, o_i) \in \{0,1\}^2$ for $i \in [\![l]\!]$ which satisfies both Formulas 4 and 5 is called a *witness* sequence. Given a witness sequence $(x_i, o_i)_{i \in [\![l]\!]}$ it holds $f^{(\Delta)}(x)_{i_0} \neq f^{(\Delta')}(x)_{i_0}$, and hence, by Formula 1 we have the result. We end by providing the witness sequences for all the local rules in the hypothesis. We start by those rules which have an interaction digraph made by arcs of type $(i+1, i)$.

- Rule 34 :
  - Formula (4): $x_{i_0} = 0$, $x_{i_0-1} = 1$, $o_1 = 1$, $o_2 = 1$.

- Formula (5): $(x_{i_0+k}, o_{k+1}) = \begin{cases} (\mathtt{0},\mathtt{0}), & \text{if } k \text{ is even} \\ (\mathtt{0},\mathtt{1}), & \text{otherwise} \end{cases}$

- Rule 136 :
  - Formula (4): $x_{i_0} = \mathtt{1}$, $x_{i_0-1} = \mathtt{1}$, $o_1 = \mathtt{0}$, $o_2 = \mathtt{0}$.
  - Formula (5): $(x_{i_0+k}, o_{k+1}) = \begin{cases} (\mathtt{0},\mathtt{0}), & \text{if } k \text{ is even} \\ (\mathtt{1},\mathtt{1}), & \text{otherwise} \end{cases}$

We conclude with the witness sequences for the rules which have interaction digraph made by arcs of type $(i, i+1)$.

- Rule 3 :
  - Formula (4): $x_{i_0} = \mathtt{0}$, $x_{i_0+1} = \mathtt{0}$, $o_{-1} = \mathtt{1}$, $o_{-2} = \mathtt{0}$.
  - Formula (5): $(x_{i_0-k}, o_{-k-1}) = \begin{cases} (\mathtt{0},\mathtt{1}), & \text{if } k \text{ is even} \\ (\mathtt{0},\mathtt{0}), & \text{otherwise} \end{cases}$
- Rule 12 :
  - Formula (4): $x_{i_0} = \mathtt{1}$, $x_{i_0+1} = \mathtt{1}$, $o_{-1} = \mathtt{0}$, $o_{-2} = \mathtt{1}$.
  - Formula (5): $(x_{i_0-k}, o_{-k-1}) = \begin{cases} (\mathtt{0},\mathtt{0}), & \text{if } k \text{ is even} \\ (\mathtt{1},\mathtt{0}), & \text{otherwise} \end{cases}$
- Rule 15 :
  - Formula (4): $x_{i_0} = \mathtt{0}$, $x_{i_0+1} = \mathtt{0}$, $o_{-1} = \mathtt{1}$, $o_{-2} = \mathtt{0}$.
  - Formula (5): $(x_{i_0-k}, o_{-k-1}) = \begin{cases} (\mathtt{0},\mathtt{1}), & \text{if } k \text{ is even} \\ (\mathtt{0},\mathtt{0}), & \text{otherwise} \end{cases}$
- Rule 60 :
  - Formula (4): $x_{i_0} = \mathtt{0}$, $x_{i_0+1} = \mathtt{0}$, $o_{-1} = \mathtt{1}$, $o_{-2} = \mathtt{1}$.
  - Formula (5): $(x_{i_0-k}, o_{-k-1}) = \begin{cases} (\mathtt{0},\mathtt{0}), & \text{if } k \text{ is even} \\ (\mathtt{0},\mathtt{1}), & \text{otherwise} \end{cases}$

□

Example 11. *Consider the ECA rule $\alpha = 34$ and a size $n = 6$. Given the two distinct update schedules $\Delta = (\{3,4\},\{5\},\{2\},\{0,1\})$ and $\Delta' = (\{3,4\},\{5\},\{0,1,2\})$. Let $i_0 = 1$ and $\ell = 2$. The following is a witness sequence (see the proof of Theorem 10): $x_{i_0-1} = \mathtt{1}, x_{i_0} = \mathtt{0}, o_1 = \mathtt{1}, x_{i_0+1} = \mathtt{0}, o_2 = \mathtt{1}, x_{i_0+2} = \mathtt{0}, o_3 = \mathtt{1}, x_{i_0+3} = \mathtt{0}, x_{i_0+4} = o_4 = \mathtt{1}$. By construction it ensures*

$$\begin{aligned} & f^{(\Delta')}(\mathtt{100010})_{i_0} = r(\mathtt{1},\mathtt{0},r(\mathtt{0},\mathtt{0},r(\mathtt{0},\mathtt{0},\mathtt{1}))) = r(\mathtt{1},\mathtt{0},r(\mathtt{0},\mathtt{0},\mathtt{1})) = r(\mathtt{1},\mathtt{0},\mathtt{1}) = \mathtt{1} \\ \neq & f^{(\Delta)}(\mathtt{100010})_{i_0} = r(\mathtt{100}) = \mathtt{0}. \end{aligned}$$

□

4.2.2 Exploiting patterns in the interaction digraph

In this subsection we are going to develop a proof technique which characterizes the number of non-equivalent update schedules according to the presence of specific patterns in their interaction digraph. This will concern ECA rules $28, 32, 44$ and 140. When n is clear from the context, we will simply denote f_α instead of $f_{\alpha,n}$ with $n \in \{28, 32, 44, 140\}$.

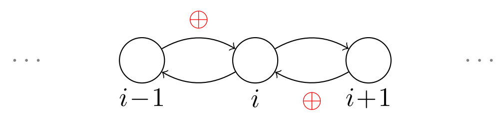


Figure 3: labeling presented in the Lemma 12. This is the only situation in which we can obtain a cell updated equal to 1.

We begin with the ECA Rule 32 which is based on the Boolean function $r_{32}(x_1, x_2, x_3) = x_1 \wedge \neg x_2 \wedge x_3$.

Lemma 12. *Fix $n \in \mathbb{N}$. For any update schedule $\Delta \in \mathcal{P}_n$, for any configuration $x \in \{0,1\}^n$ and for any $i \in [\![n]\!]$, the following holds:*

$$f_{32}^{(\Delta)}(x)_i = 1 \iff lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = \oplus \wedge (x_{i-1}, x_i, x_{i+1}) = (1,0,1).$$

Proof.
($\Leftarrow$) Since $lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = \oplus$ (see Figure 3) that means that cell i is not updated after cells $i-1$ and $i+1$, therefore $f_{32}^{(\Delta)}(x)_i = r_{32}(x_{i-1}, x_i, x_{i+1}) = r_{32}(1,0,1) = 1$.
($\Rightarrow$) Choose $x \in \{0,1\}^n$ such that $f_{32}^{(\Delta)}(x)_i = 1$. Assume that $lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = \oplus$ but $(x_{i-1}, x_i, x_{i+1}) \neq (1,0,1)$. By the same reasoning as above, we have $f_{32}^{(\Delta)}(x)_i = r_{32}(x_{i-1}, x_i, x_{i+1}) = 0$, since $(x_{i-1}, x_i, x_{i+1}) \neq (1,0,1)$. Now, assume $lab_\Delta((i+1,i)) = \ominus$ or $lab_\Delta((i-1,i)) = \ominus$. Then,

$$f_{32}^{(\Delta)}(x)_i = \begin{cases} r_{32}(0,0,1) = 0, \text{if } lab_\Delta((i-1,i)) = \ominus \wedge lab_\Delta((i+1,i)) = \oplus \\ r_{32}(1,0,0) = 0, \text{if } lab_\Delta((i-1,i)) = \oplus \wedge lab_\Delta((i+1,i)) = \ominus \\ r_{32}(0,0,0) = 0, \text{otherwise} \end{cases}$$

which contradicts the hypothesis. □

Corollary 13. *Fix $n \in \mathbb{N}$. For any update schedule $\Delta \in \mathcal{P}_n$, for any configuration $x \in \{0,1\}^n$ and $i \in [\![n]\!]$, if $lab_\Delta((i-1,i)) = \ominus$ or $lab_\Delta((i+1,i)) = \ominus$, then $f_{32}^{(\Delta)}(x)_i = 0$.*

Lemma 14. *For any $n \in \mathbb{N}$. Consider a pair of update schedules $\Delta, \Delta' \in \mathcal{P}_n$. Then, $D_{f_{32,n}^{(\Delta)}} \neq D_{f_{32,n}^{(\Delta')}}$ if and only if there exists $i \in [\![n]\!]$ such that one of the following holds:*

1. *$lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = \oplus$ and either $lab_{\Delta'}((i+1,i)) = \ominus$ or $lab_{\Delta'}((i-1,i)) = \ominus$;*
2. *$lab_{\Delta'}((i+1,i)) = lab_{\Delta'}((i-1,i)) = \oplus$ and either $lab_\Delta((i+1,i)) = \ominus$ or $lab_\Delta((i-1,i)) = \ominus$.*

Proof.
($\Leftarrow$) WLOG, suppose that $lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = \oplus$ and $lab_{\Delta'}((i+1,i)) = \ominus$ or $lab_{\Delta'}((i-1,i)) = \ominus$ (the other case is the same where Δ and Δ' are exchanged). Then, by Lemma 12 one can finds $f_{32}^{(\Delta)}(x)_i = 1$ and by Corollary 13, $f_{32}^{(\Delta')}(x)_i = 0$. Therefore, $D_{f_{32,n}^{(\Delta)}} \neq D_{f_{32,n}^{(\Delta')}}$.
($\Rightarrow$) Suppose that for every $i \in [\![n]\!]$ one of the following holds:

Figure 4: on the left, starting from a configuration $(y, \mathtt{1}, \mathtt{1}, \star)$ the schedule Δ updates before the cell i which becomes $\mathtt{0}$, after the rule is applied on the cell $i-1$, therefore $z_- = r_{44}(y, \mathtt{1}, \mathtt{0})$; on the right, the two cells are updated at the same time or after and $z_+ = r_{44}(y, \mathtt{1}, \mathtt{1})$.

(Case 1) $lab_\Delta((i+1,i)) = lab_\Delta((i-1,i)) = lab_{\Delta'}((i+1,i)) = lab_{\Delta'}((i-1,i)) = \oplus$
(Case 2) $(lab_\Delta((i+1,i)), lab_\Delta((i-1,i))) \neq (\oplus,\oplus) \neq (lab_{\Delta'}((i+1,i)), lab_{\Delta'}((i-1,i)))$.

We will show that in both cases $D_{f^{(\Delta)}_{32,n}} = D_{f^{(\Delta')}_{32,n}}$. Let $j \in [\![n]\!]$ and consider a configuration $x \in \{\mathtt{0},\mathtt{1}\}^n$ such that: $(x_{j-1}, x_j, x_{j+1}) \neq (\mathtt{1},\mathtt{0},\mathtt{1})$, then by Lemma 12, $f^{(\Delta)}_{32}(x)_j = f^{(\Delta')}(x)_j = \mathtt{0}$. Now suppose $(x_{j-1}, x_j, x_{j+1}) = (\mathtt{1},\mathtt{0},\mathtt{1})$. If we are in Case 1, then $f^{(\Delta)}_{32}(x)_j = f^{(\Delta')}_{32}(x)_j = \mathtt{1}$. If we are in Case 2, then $f^{(\Delta)}_{32}(x)_j = f^{(\Delta')}_{32}(x)_j = \mathtt{0}$. By the generality of j, $f^{(\Delta)}_{32}(x) = f^{(\Delta')}_{32}(x)$ and by the generality of x, $D_{f^{(\Delta)}_{32,n}} = D_{f^{(\Delta')}_{32,n}}$. □

For the ECA rules 28, 44 and 140 we are going to develop a similar construction as the one for ECA rule 32 but before let us recall the Boolean functions which they are based on. We start with ECA rule 44 which is based on the Boolean function $r_{44}(x_1,x_2,x_3) = (\neg x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2 \wedge x_3)$ which implies that $r_{44}(\mathtt{1},\mathtt{0},\mathtt{1}) = r_{44}(\mathtt{0},\mathtt{1},\mathtt{1}) = r_{44}(\mathtt{0},\mathtt{1},\mathtt{0}) = \mathtt{1}$.

Notation 15. *Let us call $\star$ a possible value of a cell in the configuration that has no effect on the result of the update procedure over the cells under consideration. At the same time, we will use a letter to represent the value of a cell in the configuration that is unknown but which has an impact on the result of the update procedure over the cells under consideration.*

Lemma 16. *Given two update schedules Δ and Δ', if there exists $i \in [\![n]\!]$ such that:*

- $lab_\Delta((i,i-1)) = \ominus \wedge lab_{\Delta'}((i,i-1)) = \oplus$
- $lab_\Delta((i-1,i)) = lab_{\Delta'}((i-1,i)) = \oplus$
- $lab_\Delta((j,j-1)) = lab_{\Delta'}((j,j-1)) \wedge lab_\Delta((j-1,j)) = lab_{\Delta'}((j-1,j))$ *for each* $j \neq i, j \in [\![n]\!]$

then $D_{f^{(\Delta)}_{44,n}} = D_{f^{(\Delta')}_{44,n}}$.

Proof. Given two update schedules Δ and Δ', we prove that $(f^{(\Delta)}(x)_{i-1} = f^{(\Delta')}(x)_{i-1}) \wedge (f^{(\Delta)}(x)_i = f^{(\Delta')}(x)_i)$ for every possible starting configuration x.
Starting from the case with $x_{i-1} = x_i = \mathtt{1}$ (see Figure 4), one obtains cells $i-1$ and i updated to states $r_{44}(y,\mathtt{1},\mathtt{0})$ and $\mathtt{0}$ (respectively) according to the Δ update schedule and to states $r_{44}(y,\mathtt{1},\mathtt{1})$ and $\mathtt{0}$ (respectively) according to the Δ' update schedule. According to the rule, we know that $r_{44}(\mathtt{0},\mathtt{1},\mathtt{0}) = r_{44}(\mathtt{0},\mathtt{1},\mathtt{1}) = \mathtt{1}$ and $r_{44}(\mathtt{1},\mathtt{1},\mathtt{0}) = r_{44}(\mathtt{1},\mathtt{1},\mathtt{1}) = \mathtt{0}$ consequently the equivalence holds in the case of $x_{i-1} = x_i = \mathtt{1}$.
If we consider $x_{i-1} = \mathtt{1}$ and $x_i = \mathtt{0}$ (see Figure 5), one obtains cells $i-1$ and i updated to states $r_{44}(y,\mathtt{1},r_{44}(\mathtt{1},\mathtt{0},w))$ and $r_{44}(\mathtt{1},\mathtt{0},w)$ (respectively) according to the Δ update schedule

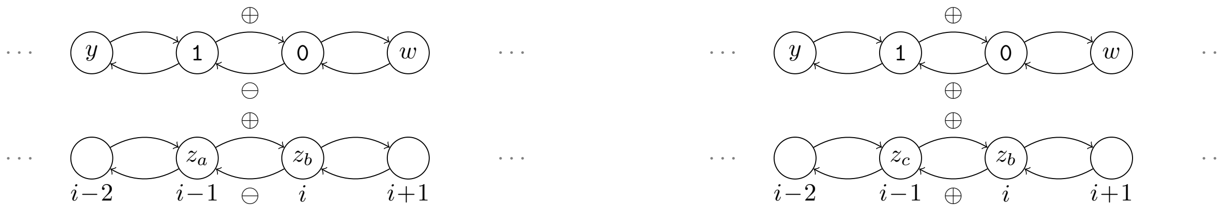


Figure 5: on the left, starting from a configuration $(y, \mathtt{1}, \mathtt{0}, w)$ the schedule Δ updates before the cell i to $z_b = r_{44}(\mathtt{1}, \mathtt{0}, w)$, after the rule is applied on the cell $i-1$ to obtain $z_a = r_{44}(y, \mathtt{1}, z_b)$; on the right, the two cells are updated at the same time or after, therefore $z_c = r_{44}(y, \mathtt{1}, \mathtt{0})$ and $z_b = r_{44}(\mathtt{1}, \mathtt{0}, w)$.

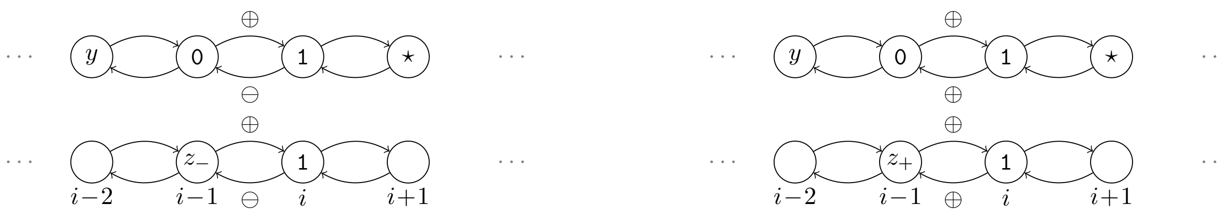


Figure 6: on the left, starting from a configuration $(y, \mathtt{0}, \mathtt{1}, \star)$ the schedule Δ updates before the cell i which becomes $r_{44} = (\mathtt{0}, \mathtt{1}, \star) = \mathtt{1}$, after the rule is applied on the cell $i-1$, where $z_- = r_{44} = (y, \mathtt{0}, \mathtt{1})$; on the right, the two cells are updated at the same time or after, we can obtain $x_i = \mathtt{1}$ and $x_{i-1} = z_+ = z_- = r_{44} = (y, \mathtt{0}, \mathtt{1})$.

and to states $r_{44}(y, \mathtt{1}, \mathtt{0})$ and $r_{44}(\mathtt{1}, \mathtt{0}, w)$ (respectively) according to the Δ' update schedule. Like in the previous case, the result of the update procedure depends only on the y value which will be the same in Δ and in Δ' consequently the equivalence holds in this case. If we consider the opposite case $x_{i-1} = \mathtt{0}$ and $x_i = \mathtt{1}$ (see Figure 6), one obtains cells $i-1$ and i updated to states $r_{44}(y, \mathtt{0}, \mathtt{1})$ and $\mathtt{1}$ (respectively) according to the Δ update schedule and to states $r_{44}(y, \mathtt{0}, \mathtt{1})$ and $\mathtt{1}$ (respectively) according to the Δ' update schedule, consequently the equivalence holds also in this case. The last case corresponds to $x_{i-1} = x_i = \mathtt{0}$ (see Figure 7), one obtains cells $i-1$ and i updated to states $\mathtt{0}$ and $\mathtt{0}$ according to Δ and Δ', consequently the equivalence holds. The two different update schedules give the same configurations independently from the initial configuration, in other words $D_{f_{44,n}^{(\Delta)}} = D_{f_{44,n}^{(\Delta')}}$. $\square$

Figure 7: on the left, starting from a configuration $(\star, \mathtt{0}, \mathtt{0}, \star)$ the schedule Δ updates before the cell i which becomes $\mathtt{0}$, after the rule is applied on the cell $i-1$; on the right, the two cells are updated at the same time or after. Consider that $r_{44} = (\mathtt{0}, \mathtt{0}, \star) = r_{44} = (\star, \mathtt{0}, \mathtt{0}) = \mathtt{0}$.

Lemma 17. *Given two update schedules* Δ *and* Δ', $D_{f_{44,n}^{(\Delta)}} \neq D_{f_{44,n}^{(\Delta')}} \iff \exists i, i \in [\![n]\!]$ *such that* $lab_\Delta((i-1,i)) = \ominus \wedge lab_{\Delta'}((i-1,i)) = \oplus$.

Proof. We can consider $lab_\Delta((i,i-1)) = lab_{\Delta'}((i,i-1)) = \oplus$ because according to Lemma 16 the value of $lab_{\Delta'}((i,i-1))$ cannot change the dynamics that we are considering and the value of $lab_\Delta((i,i-1))$ must be $\oplus$ given the $\ominus$ in the opposite sense.
We can consider equal labelings over the other transitions.
Let j be a cell such that $lab_{\Delta'}((j,j+1)) = \oplus$ and $lab_{\Delta'}((j+k,j+k+1)) = \ominus$ for all $1 \leq k \leq i-j-1$. Such a j must exist since otherwise we would have a $\ominus$ cycle of length n. Now, let $x \in \{\mathtt{0},\mathtt{1}\}^n$ be any configuration of length n such that

$$x_{[j,i+1]} = \begin{cases} \mathtt{1}(\mathtt{1})^{(i-1)-j-1}\mathtt{011}, \text{ if } (i-1)-j-1 \mod 2 = 0 \text{ or } j = i-2 \\ \mathtt{0}(\mathtt{1})^{(i-1)-j-1}\mathtt{011}, \text{ otherwise} \end{cases} \tag{6}$$

Then we have

$$\left(f^\Delta(x)_{[j+1,i]}\right) = \begin{cases} \mathtt{0}(\mathtt{10})^{\lfloor \frac{(i-1)-j-2}{2} \rfloor}\mathtt{110}, \text{ if } (i-1)-j-1 \mod 2 = 0 \\ \mathtt{10}, \text{ if } j = i-2 \\ \mathtt{1}(\mathtt{01})^{\lfloor \frac{(i-1)-j-2}{2} \rfloor}\mathtt{10}, \text{ otherwise} \end{cases}.$$

In general we can always obtain $f^\Delta(x)_i = \mathtt{0}$ (see Figure 8). The update schedule Δ' gives $f^{\Delta'}(x)_i = \mathtt{1}$. Therefore, $f^\Delta(x)_i \neq f^{\Delta'}(x)_i$ and $D_{f_{44,n}^{(\Delta)}} \neq D_{f_{44,n}^{(\Delta')}}$. ∎

Consider that the previous lemma is sufficient to determine that two update schedules that differ in at least one cell i such that $lab_\Delta((i-1,i)) = \ominus \wedge lab_{\Delta'}((i-1,i)) = \oplus$ generate two different dynamics. In fact, we can focus on one of these cells to build a configuration in which the cell updated in different values.

Consider now the ECA rule 28, it is based on $r_{28}(x_1,x_2,x_3) = (\neg x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3)$ and hence $r_{28}(\mathtt{1},\mathtt{0},\mathtt{0}) = r_{28}(\mathtt{0},\mathtt{1},\mathtt{1}) = r_{28}(\mathtt{0},\mathtt{1},\mathtt{0}) = \mathtt{1}$. Remark that the Lemma 16 holds also for this rule. The only difference is in the proof, for completeness we show the equivalence that holds for every possible starting configuration (see Figure 9).

For this rule also the Lemma 17 can be applied. The main idea is the same. In fact, let $x \in \{\mathtt{0},\mathtt{1}\}^n$ be any configuration of length n such that

$$x_{[j,i+1]} = \begin{cases} \mathtt{1}(\mathtt{1})^{(i-1)-j-1}\mathtt{100}, \text{ if } (i-1)-j-1 \mod 2 = 0 \text{ or } j = i-2 \\ \mathtt{0}(\mathtt{1})^{(i-1)-j-1}\mathtt{100}, \text{ otherwise} \end{cases} \tag{7}$$

Then we have

$$\left(f^\Delta(x_{[j+1,i]})\right) = \begin{cases} (\mathtt{01})^{\lfloor \frac{(i-1)-j}{2} \rfloor}\mathtt{00}, \text{ if } (i-1)-j-1 \mod 2 = 0 \\ \mathtt{00}, \text{ if } j = i-2 \\ \mathtt{1}(\mathtt{01})^{\lfloor \frac{(i-1)-j-2}{2} \rfloor}\mathtt{00}, \text{ otherwise} \end{cases}.$$

In general we obtain $f^\Delta(x)_i = \mathtt{0}$. The update schedule Δ' gives $f^{\Delta'}(x)_i = \mathtt{1}$. Therefore, $f^\Delta(x)_i \neq f^{\Delta'}(x)_i$ and $D_{f_{28,n}^{(\Delta)}} \neq D_{f_{28,n}^{(\Delta')}}$.

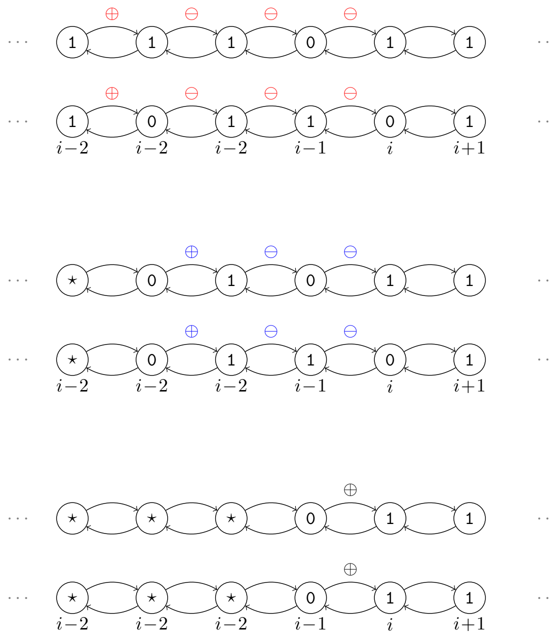


Figure 8: since we cannot have a $\ominus$ cycle in the labeled interaction digraph of update schedule Δ' there must be a cell j such that $lab_{\Delta'}((j, j+1)) = \oplus$ and $lab_{\Delta'}((j+k, j+k+1)) = \ominus$ for all $1 \leq k \leq i-j-1$. The blue and the red updates shows the different cases of equation 6 and the black one represent Δ'.

Consider that the previous lemma is sufficient to determine that two update schedules that differ in at least one cell i such that $lab_{\Delta}((i-1, i)) = \ominus \wedge lab_{\Delta'}((i-1, i)) = \oplus$ generate two different dynamics. In fact, we can focus on one of these cells to build a configuration in which the cell updated in different values.

Let us now focus our attention on ECA rule 140 which is based on the Boolean function $r_{140}(x_1, x_2, x_3) = (\neg x_1 \vee x_3) \wedge x_2$, that is to say $r_{140}(\mathtt{1}, \mathtt{1}, \mathtt{1}) = r_{140}(\mathtt{0}, \mathtt{1}, \mathtt{1}) = r_{140}(\mathtt{0}, \mathtt{1}, \mathtt{0}) = \mathtt{1}$.

Lemma 18. *For any $n > 3$, given two update schedules $\Delta, \Delta' \in \mathcal{P}_n$, if there exists $i \in [\![n]\!]$ such that*

- $lab_{\Delta}((i, i+1)) = \ominus$ *and* $lab_{\Delta'}((i, i+1)) = \oplus$
- $lab_{\Delta}((i+1, i)) = lab_{\Delta'}((i+1, i)) = \oplus$
- $lab_{\Delta}((j, j-1)) = lab_{\Delta'}((j, j-1))$ *and* $lab_{\Delta}((j-1, j)) = lab_{\Delta'}((j-1, j))$ *for each* $j \neq i+1, j \in [\![n]\!]$

then $D_{f^{(\Delta)}_{140,n}} = D_{f^{(\Delta')}_{140,n}}$.

Proof. Given the two update schedules $\Delta, \Delta' \in \mathcal{P}_n$, using the same reasoning as for Lemma 16, one can prove that $f^{\Delta}(x)_i = f^{\Delta'}(x)_i$ and $f^{\Delta}(x)_{i+1} = f^{\Delta'}(x)_{i+1}$ for every possible starting configuration $x \in \{\mathtt{0}, \mathtt{1}\}^n$. It is easy to see from the Figures 10, 11, 12 and 13 that the equivalence holds for every possible initial configuration. The two different update schedules give the same configurations independently from the initial configuration, in other words $D_{f^{(\Delta)}_{140,n}} = D_{f^{(\Delta')}_{140,n}}$. □

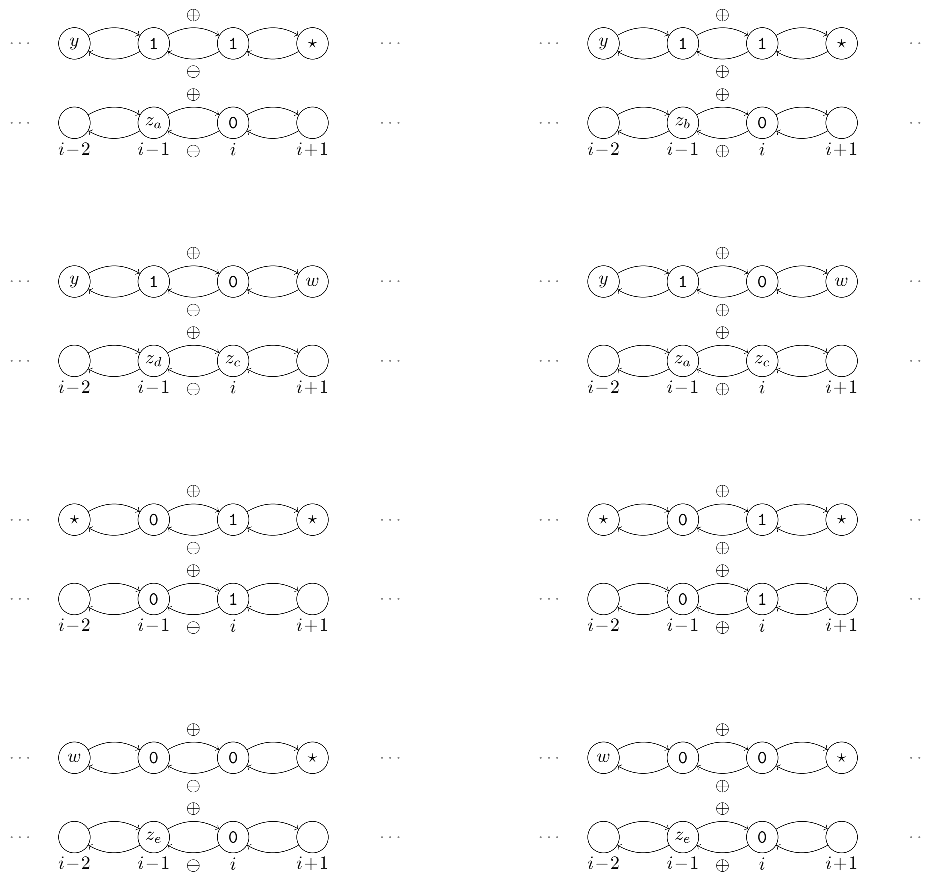


Figure 9: starting from every possible configuration, only a difference in a label over the edge between $i+1$ and i is not sufficient in order to obtain different final configurations for rule 28. In the figure: $z_a = r_{28}(y, \mathtt{1}, \mathtt{0})$, $z_b = r_{28}(y, \mathtt{1}, \mathtt{1})$, $z_c = r_{28}(\mathtt{1}, \mathtt{0}, w)$, $z_d = r_{28}(y, \mathtt{1}, z_c)$ and $z_e = r_{28}(w, \mathtt{0}, \mathtt{0})$. We need also to consider that $z_a = z_b = y$ and $z_c = z_d = \mathtt{0}$ if $y = \mathtt{1}$, $z_c = z_d = \mathtt{1}$ otherwise.

Figure 10: on the left, starting from a configuration $(\star, \mathtt{1}, \mathtt{1}, y)$ the Δ update updates before the cell i which becomes $r_{140} = (\star, \mathtt{1}, \mathtt{1}) = \mathtt{1}$, after the rule is applied on the cell $i+1$, where $z = r_{140}(\mathtt{1}, \mathtt{1}, y)$; on the right, the two cells are updated at the same time or after.

Figure 11: on the left, starting from a configuration $(y, 1, 0, \star)$ the Δ update updates before the cell i, $z_a = r_{140}(y, 1, 0)$, after the rule is applied on the cell $i+1$, therefore $r_{140}(r_{140}(y, 1, 0), 0, \star) = 0$; on the right, the two cells are updated at the same time or after. Remember that $r_{140}(1, 0, \star) = 0$.

Figure 12: on the left, starting from a configuration $(\star, 0, 1, \star)$ the Δ update updates before the cell i which becomes 0 (in fact $r_{140}(\star, 0, 1) = 0$), after the rule is applied on the cell $i+1$ updated to $r_{140}(0, 1, \star) = 1$; on the right, the two cells are updated at the same time or after.

Remark 19. *The ECA rule* 140 *is such that* $r_{140}(x_1, 0, x_2) = 0$ *for any* $x_1, x_2 \in \{0, 1\}$, *hence for any given update schedule* Δ *a cell that is in state* 0 *will remain in such a state throughout the whole evolution.*

Lemma 20. *For any* $n > 3$, *given two update schedules* $\Delta, \Delta' \in \mathcal{P}_n$, *it holds*

$$D_{f^{(\Delta)}_{140,n}} \neq D_{f^{(\Delta')}_{140,n}} \iff \exists i \in [\![n]\!] \text{ s. t. } lab_{\Delta}((i+1, i)) = \ominus \text{ and } lab_{\Delta'}((i+1, i)) = \oplus \ .$$

Proof. Choose n, Δ and Δ' as in the hypothesis. We are going to prove that there exists a configuration such that $f^{\Delta}(x)_i \neq f^{\Delta'}(x)_i$. Consider the following initial configuration $(x_{i-1}, x_i, x_{i+1}, x_{i+2}) = (1, 1, 1, 0)$ and assume that $lab_{\Delta}((i-1, i)) = \oplus$ (according to

Figure 13: on the left, starting from a configuration $(\star, 0, 0, \star)$ the Δ update updates before the cell i which becomes 0, after the rule is applied on the cell $i+1$; on the right, the two cells are updated at the same time or after. In fact, we know that $r_{140}(\star, 0, 0) = 0$ and $r_{140}(0, 0, \star) = 0$.

Lemma 18, this is not changing the dynamics). Moreover, assume that $lab_\Delta((i+1,i)) \neq lab_{\Delta'}((i+1,i))$ is the only difference between the two update schedules. According to Δ', i and $i+1$ are updated together, therefore the final configuration is $(\mathtt{1},\mathtt{1},\mathtt{0},\mathtt{0})$. In the case of Δ, the cell $i+1$ is updated before than i holding $r_{140}(\mathtt{1},\mathtt{1},\mathtt{0}) = \mathtt{0}$. In a second moment, the cell i is updated and $r_{140}(\mathtt{1},\mathtt{1},\mathtt{0}) = \mathtt{0}$. It follows that $f^{\Delta}(x)_i \neq f^{\Delta'}(x)_i$ and $D_{f^{(\Delta)}_{140,n}} \neq D_{f^{(\Delta')}_{140,n}}$. Remark that the cell $i+1$ can be influenced from a $\ominus$ chain, but a cell with value $\mathtt{0}$ is frozen at this state. $\square$

The previous lemma is sufficient to determine that two update schedules that differ in at least one cell i such that $lab_\Delta((i+1,i)) = \ominus$ and $lab_{\Delta'}((i+1,i)) = \oplus$ generate two different dynamics. Indeed, one can focus on one of these cells to build a configuration in which the cell updated produces different values.

Theorem 21. *$\mu_s(f_{\alpha,n}) = \frac{2^n-1}{3^n-2^{n+1}+2}$ for any $n > 3$ and for all ECA rules $\alpha \in \{28, 32, 44, 140\}$.*

Proof. Given a configuration of length $n > 3$, the patterns in Lemma 12 (ECA rule 32) (resp., Lemma 17 for ECA rule 44 and Lemma 20 for ECA rule 140) may be present in k cells out of n with $1 \leq k \leq n$ (it must be present in at least one cell because otherwise we would have a $\ominus$ cycle). Therefore, there are $\sum_{k=1}^{n} \binom{n}{i} = 2^n - 1$ different dynamics. $\square$

4.2.3 The sensitivity tends to $c = \frac{1+\phi}{3}$

This subsection is concerned uniquely with ECA Rule 8 which is based on the following Boolean function $r_8(x_1,x_2,x_3) = \neg x_1 \wedge x_2 \wedge x_3$. As we will see finding the expression of sensitivity function for this rule is a bit difficult and require to develop specific techniques.

Remark 22. *For any $x_1, x_3 \in \{\mathtt{0},\mathtt{1}\}$, it holds $r_8(x_1,\mathtt{0},x_3) = \mathtt{0}$. Hence, for any update schedule a cell that is in state $\mathtt{0}$ will remain in state $\mathtt{0}$ forever.*

We will first see in Lemma 23 that as soon as two update schedules differ on the labeling of an arc $(i,i-1)$, then the two dynamics are different. Then, given two update schedules Δ, Δ' such that $lab_\Delta((i,i-1)) = lab_{\Delta'}((i,i-1))$ for all $i \in [\![n]\!]$, Lemmas 24 and 25 will respectively give sufficient and necessary conditions for the equality of the two dynamics.

Lemma 23. *Consider two update schedules $\Delta, \Delta' \in \mathcal{P}_n$ for $n \geq 3$. If there exists $i \in [\![n]\!]$ such that $lab_\Delta((i,i-1)) \neq lab_{\Delta'}((i,i-1))$, then $D_{f^{(\Delta)}_{8,n}} \neq D_{f^{(\Delta')}_{8,n}}$.*

Proof. Choose $n \geq 3$ and fix some $i \in [\![n]\!]$. WLOG, assume that $lab_\Delta((i,i-1)) = \oplus$ and $lab_{\Delta'}((i,i-1)) = \ominus$ and take $x \in \{\mathtt{0},\mathtt{1}\}^n$ such that $(x_{i-2},x_{i-1},x_i) = (\mathtt{0},\mathtt{1},\mathtt{1})$. Cell $i-2$ will not change its state, hence when it is time for cell $i-1$ to be updated it will have a $\mathtt{0}$ at its left (cell $i-2$) in both cases. For Δ, when cell $i-1$ is to be updated, its neighborhood will be $(\mathtt{0},\mathtt{1},\mathtt{1})$, and its state will become $\mathtt{1}$ after the iteration. As for Δ', when cell i is to be updated, cell $i-1$ is still in state $\mathtt{1}$, therefore its state will become $\mathtt{0}$ and when its time for cell $i-1$ to be updated, it will have a $\mathtt{0}$ at its right (cell i) and its state will become $\mathtt{0}$ after the iteration. We conclude that $f^{(\Delta)}_{8,n}(x)_{i-1} \neq f^{(\Delta')}_{8,n}(x)_{i-1}$ and the result follows. $\square$

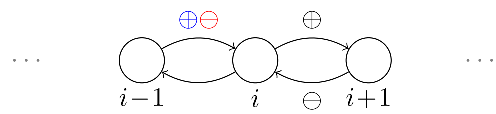


Figure 14: illustration of lab_Δ in blue/black and $lab_{\Delta'}$ in red/black, in Lemma 24. All other labels are equal (the label of arc $(i+1,i)$ is $\ominus$ in both update schedules by hypothesis).

Now consider two update schedules Δ, Δ' whose labelings are equal on all *counter-clockwise* arcs (*i.e.* of the form $(i, i-1)$). Lemma 24 states that, if Δ and Δ' differ only on one arc $(i-1, i)$ such that $lab_\Delta((i+1,i)) = lab_{\Delta'}((i+1,i)) = \ominus$, then the two dynamics are identical. By transitivity, if there are more differences but only on arcs of this form, then the dynamics are also identical.

Lemma 24. *Suppose Δ and Δ' are two update schedules over a configuration of length $n \geq 3$ and there is $i \in [\![n]\!]$ such that*

- $lab_\Delta((i+1,i)) = lab_{\Delta'}((i+1,i)) = \ominus$;
- $lab_\Delta((i-1,i)) \neq lab_{\Delta'}((i-1,i))$;
- $lab_\Delta((j_1,j_2)) = lab_{\Delta'}((j_1,j_2))$, *for all* $(j_1,j_2) \neq (i-1,i)$.

Then $D_{f_{8,n}^{(\Delta)}} = D_{f_{8,n}^{(\Delta')}}$.

Proof. Fix $n \geq 3$ and choose $i \in [\![n]\!]$ WLOG suppose that $lab_\Delta((i-1,i)) = \oplus$ and $lab_{\Delta'}((i-1,i)) = \ominus$. By Theorem 2 and the fact that $lab_\Delta((i+1,i)) = lab_{\Delta'}((i+1,i)) = \ominus$, it follows that $lab_\Delta((i,i+1)) = lab_{\Delta'}((i,i+1)) = \oplus$, otherwise a forbidden cycle of length two is created. See Figure 14 for an illustration of the setting.

The two update schedules Δ and Δ' are very similar. Indeed, for any cell $j \in [\![n]\!] \setminus \{i\}$ the chain of influences are identical, *i.e.* $\overleftarrow{\mathtt{d}}_\Delta(j) = \overleftarrow{\mathtt{d}}_{\Delta'}(j)$ and $\overrightarrow{\mathtt{d}}_\Delta(j) = \overrightarrow{\mathtt{d}}_{\Delta'}(j)$. We deduce from Lemma 4 that for any configuration $x \in \{\mathtt{0},\mathtt{1}\}^n$ and any $j \neq i$ the images under update schedules Δ and Δ', *i.e.* $f_{8,n}^{(\Delta)}(x)_j = f_{8,n}^{(\Delta')}(x)_j$. As a consequence, it only remains to consider cell i. Let $x \in \{\mathtt{0},\mathtt{1}\}^n$ be any configuration (if $n \leq 2$ then $i-1 = i+1$, but $lab_\Delta((i-1,i)) = \oplus$ whereas $lab_\Delta((i+1,i)) = \ominus$).

By Remark 22, if $x_i = \mathtt{0}$, then $f_{8,n}^{(\Delta)}(x)_i = f_{8,n}^{(\Delta')}(x)_i = \mathtt{0}$. Now suppose $x_i = \mathtt{1}$. Since $lab_\Delta((i,i+1)) = lab_{\Delta'}((i,i+1)) = \oplus$, by the time cell $i+1$ is updated, there is a $\mathtt{1}$ at its left (cell i) in both cases, hence $f_{8,n}^{(\Delta)}(x)_{i+1} = f_{8,n}^{(\Delta')}(x)_{i+1} = \mathtt{0}$. Then, when cell i is updated in both cases, there will be a $\mathtt{0}$ at its right (cell $i+1$), therefore $f_{8,n}^{(\Delta)}(x)_i = f_{8,n}^{(\Delta')}(x)_i = \mathtt{0}$.

We conclude that for all $x \in \{\mathtt{0},\mathtt{1}\}^n$ and all $j \in [\![n]\!]$ we have $f_{8,n}^{(\Delta)}(x)_j = f_{8,n}^{(\Delta')}(x)_j$, *i.e.* $D_{f_{8,n}^{(\Delta)}} = D_{f_{8,n}^{(\Delta')}}$. □

Lemma 25 states that, as soon as Δ and Δ' differ on arcs of the form $(i-1,i)$ such that $lab_\Delta((i+1,i)) = lab_{\Delta'}((i+1,i)) = \oplus$, then the two dynamics are different (remark that in this case we must have $lab_\Delta((i,i-1)) = lab_{\Delta'}((i,i-1)) = \oplus$ otherwise one of Δ or Δ' has an invalid cycle of length two between the nodes $i-1$ and i). This lemma can be applied if at least one cell of the configuration contains the pattern.

Lemma 25. *For $n \geq 5$, consider two update schedules $\Delta, \Delta' \in \mathcal{P}_n$. If there exists (at least one cell) $i \in [\![n]\!]$ such that*

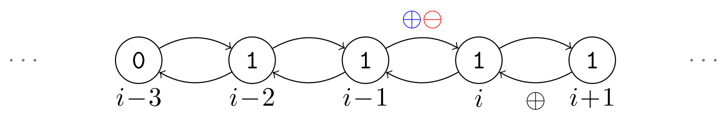


Figure 15: illustration of lab_Δ in blue/black, $lab_{\Delta'}$ in red/black, in Lemma 25. Other labels on arcs of the form $(j-1, j)$ are *a priori* unknown (they may be equal or different in Δ and Δ'), however labels on arcs of the form $(j, j-1)$ are equal by hypothesis. States inside the nodes correspond to configuration x such that the image of cell i under update schedule Δ is $\mathtt{0}$, whereas under update schedule Δ' it is $\mathtt{1}$.

- $lab_\Delta((i+1, i)) = lab_{\Delta'}((i+1, i)) = \oplus$;
- $lab_\Delta((i-1, i)) \neq lab_{\Delta'}((i-1, i))$;
- $lab_\Delta((j, j-1)) = lab_{\Delta'}((j, j-1))$, *for all* $j \in [\![n]\!]$;

then $D_{f_{8,n}^{(\Delta)}} \neq D_{f_{8,n}^{(\Delta')}}$.

Proof. Choose n, Δ and Δ' as in the hypothesis. WLOG, assume that $lab_\Delta((i-1, i)) = \oplus$ and $lab_{\Delta'}((i-1, i)) = \ominus$ for $i \in [\![n]\!]$. See Figure 15 for an illustration of the setting. We are going to construct a configuration $x \in \{\mathtt{0}, \mathtt{1}\}^n$ such that $f_{8,n}^{(\Delta)}(x)_i = \mathtt{0}$ whereas $f_{8,n}^{(\Delta')}(x)_i = \mathtt{1}$, *i.e.* such that the two dynamics differ in the image of cell i.

The construction of $x \in \{\mathtt{0}, \mathtt{1}\}^n$ only requires to set the pattern $(x_{i-3}, x_{i-2}, x_{i-1}, x_i, x_{i+1}) = (\mathtt{0}, \mathtt{1}, \mathtt{1}, \mathtt{1}, \mathtt{1})$. Regarding Δ, from the $\oplus$ labels of arcs $(i-1, i)$ and $(i+1, i)$ we have $f_{8,n}^{(\Delta)}(x)_i = r_8(x_{i-1}, x_i, x_{i+1}) = r_8(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$. Regarding Δ', let us deduce by denoting y the image of x (*i.e.* $y_i = f_{8,n}^{(\Delta')}(x)_i$) that whatever the value of $\overleftarrow{\mathtt{d}}_{\Delta'}(i)$ we have $f_{8,n}^{(\Delta')}(x)_i = \mathtt{1}$ (*i.e.* $y_i = \mathtt{1}$).

- If $\overleftarrow{\mathtt{d}}_{\Delta'}(i) = 2$ then cell $i-1$ is updated and then cell i,
  - $y_{i-1} = r_8(x_{i-2}, x_{i-1}, x_i) = r_8(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$,
  - $y_i = r_8(y_{i-1}, x_i, x_{i+1}) = r_8(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{1}$.
- if $\overleftarrow{\mathtt{d}}_{\Delta'}(i) = 3$ then cell $i-2$ is updated then cell $i-1$ and then cell i,
  - $y_{i-2} = r_8(x_{i-3}, x_{i-2}, x_{i-1}) = r_8(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{1}$,
  - $y_{i-1} = r_8(y_{i-2}, x_{i-1}, x_i) = r_8(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$,
  - $y_i = r_8(y_{i-1}, x_i, x_{i+1}) = r_8(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{1}$.
- if $\overleftarrow{\mathtt{d}}_{\Delta'}(i) \geq 4$ then cell $i-3$ is updated then cell $i-2$ then cell $i-1$ and then cell i,
  - $y_{i-3} = \mathtt{0}$ by Remark 22 since $x_{i-3} = \mathtt{0}$,
  - $y_{i-2} = r_8(y_{i-3}, x_{i-2}, x_{i-1}) = r_8(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{1}$,
  - $y_{i-1} = r_8(y_{i-2}, x_{i-1}, x_i) = r_8(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$,
  - $y_i = r_8(y_{i-1}, x_i, x_{i+1}) = r_8(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{1}$.

Remark that $n \geq 5$ is required by the consideration of cells $i-3$ to $i+1$ in the third case. □

Lemmas 23, 24 and 25 characterize completely for rule 8 the cases when two update schedules Δ, Δ' lead to

- the same dynamics, *i.e.* $\mathcal{D}(f_{8,n}^{(\Delta)}) = \mathcal{D}(f_{8,n}^{(\Delta')})$, or
- different dynamics, *i.e.* $\mathcal{D}(f_{8,n}^{(\Delta)}) \neq \mathcal{D}(f_{8,n}^{(\Delta')})$.

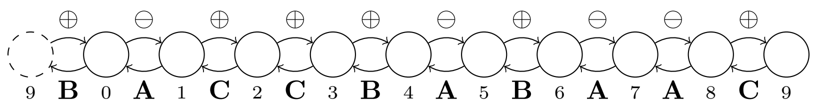


Figure 16: counting the number of different dynamics for ECA rule 8 when the labeling of arcs $(i-1,i)$ for $i \in [\![10]\!]$ is $(\oplus, \ominus, \oplus, \oplus, \oplus, \ominus, \oplus, \ominus, \ominus, \oplus)$. Labels **A** is enforced to be $\oplus$ by Theorem 2, labels **B** have no influence according to Lemma 24, and any combination of labels **C** gives a different dynamics according to Lemma 25.

Indeed, Lemma 23 shows that counting $|\mathcal{D}(f_{8,n})|$ can be partitioned according to the word given by $lab_\Delta((i,i-1))$ for $i \in [\![n]\!]$, and then for each labeling of the n arcs of the form $(i,i-1)$, Lemmas 24 and 25 provide a way of counting the number of dynamics. We first give an example of application, and then the general counting result establishing a relation to the bisection of Lucas numbers.

Example 26. *Consider the set of non-equivalent update schedules $\Delta \in \mathcal{P}_{10}$ such that*

$$\big(lab_\Delta((i,i-1))\big)_{i\in[\![10]\!]} = (\oplus, \ominus, \oplus, \oplus, \oplus, \ominus, \oplus, \ominus, \ominus, \oplus).$$

We have the following disjunction for $i \in [\![n]\!]$ (see Figure 16):

- ***A-*** *if $lab((i,i-1)) = \ominus$ then $lab((i-1,i)) = \oplus$ according to Theorem 2, else*
  - ***B-*** *if $lab((i+1,i)) = \ominus$ then $lab((i-1,i))$ does not change the dynamics according to Lemma 24,*
  - ***C-*** *if $lab((i+1,i)) = \oplus$ then the two possibilities for $lab((i-1,i))$ each lead to different dynamics according to Lemma 25.*

Therefore, on overall, there are $2^3 = 8$ different dynamics for such update schedules.

Theorem 27. $\mu_s(f_{8,n}) = \frac{\phi^{2n}+\phi^{-2n}-2^n}{3^n-2^{n+1}+2}$ *for any $n \geq 5$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio.*

Proof. According to Lemma 23, the set $\mathcal{D}(f_{8,n})$ can be partitioned as

$$\mathcal{D}_u(f_{8,n}) = \big\{D_{f^{(\Delta)}} \mid \Delta \in \mathcal{P}_n \text{ and } \big(lab_\Delta((i,i-1))\big)_{i\in[\![n]\!]} = u\big\} \text{ for } u \in \{\oplus,\ominus\}^n.$$

That is, in $\mathcal{D}_u(f_{8,n})$ the labels of arcs of the form $(i,i-1)$ for $i \in [\![n]\!]$ are fixed according to some word $u \in \{\oplus,\ominus\}^n$. Therefore we have

$$|\mathcal{D}(f_{8,n})| = \sum_{u\in\{\oplus,\ominus\}^n} |\mathcal{D}_u(f_{8,n})|.$$

Then, given some word $u \in \{\oplus,\ominus\}^n \setminus \{\oplus^n, \ominus^n\}$, according to Theorem 2 and Lemmas 24 and 25 we have (see Example 26 for details)

$$|\mathcal{D}_u(f_{8,n})| = 2^{|u|_\oplus - |u|_{\oplus\ominus}}$$

where $|u|_\oplus$ is the number of $\oplus$ in word u, and $|u|_{\oplus\ominus}$ is the number of $\oplus\ominus$ factors in word u considered periodically, *i.e.* $|u|_{\oplus\ominus} = |\{i \in [\![n]\!] \mid u_i = \oplus \text{ and } u_{i+1} = \ominus\}|$.

According to Theorem 2, the cases $u \in \{\oplus^n, \ominus^n\}$ are particular. Indeed, for any n:

- all labels $\ominus$ (*i.e.*, $u = \ominus^n$) is an invalid cycle hence $|\mathcal{D}_{\ominus^n}(f_{8,n})| = 0$,
- all labels $\oplus$ (*i.e.*, $u = \oplus^n$) forces the labels of all arcs of the form $(i-1, i)$ for $i \in [\![n]\!]$ to be also labeled $\oplus$ otherwise a forbidden cycle is created, hence $|\mathcal{D}_{\ominus^n}(f_{8,n})| = 1$.

Given that $2^{|\ominus^n|_\oplus - |\ominus^n|_{\oplus\ominus}} = 2^0 = 1$ (instead of 0) and $2^{|\oplus^n|_\oplus - |\oplus^n|_{\oplus\ominus}} = 2^n$ (instead of 1), we deduce that

$$|\mathcal{D}(f_{8,n})| = \left(\sum_{u \in \{\oplus,\ominus\}^n} 2^{|u|_\oplus - |u|_{\oplus\ominus}} \right) - 2^n. \tag{8}$$

In order to study the summation term in Equation 8, let us denote it $S(n)$. We will consider recurrence relations according to the following partition of the set $\{\oplus,\ominus\}^n$: for $\sigma, \sigma' \in \{\oplus,\ominus\}$ let $L_{\sigma\sigma'}(n)$ be the set of words beginning with label σ and ending with label σ', *i.e.* $L_{\sigma\sigma'}(n) = \{u = u_0 \dots u_{n-1} \in \{\oplus,\ominus\}^n \mid u_0 = \sigma \text{ and } u_{n-1} = \sigma'\}$. Denoting

$$S_{\sigma\sigma'}(n) = \sum_{u \in L_{\sigma\sigma'}} 2^{|u|_\oplus - |u|_{\oplus\ominus}}$$

the recurrence relations are, for all $n \geq 1$ (although Equation 8 holds only for $n \geq 5$, the value starting from which Lemmas 23, 24 and 25 hold),

- $S_{\oplus\oplus}(n+1) = 2S_{\oplus\oplus}(n) + 2S_{\oplus\ominus}(n)$,
- $S_{\oplus\ominus}(n+1) = \frac{1}{2}S_{\oplus\oplus}(n) + S_{\oplus\ominus}(n)$,
- $S_{\ominus\oplus}(n+1) = 2S_{\ominus\oplus}(n) + S_{\ominus\ominus}(n)$,
- $S_{\ominus\ominus}(n+1) = S_{\ominus\oplus}(n) + S_{\ominus\ominus}(n)$,

and we have $S(n) = S_{\oplus\oplus}(n) + S_{\oplus\ominus}(n) + S_{\ominus\oplus}(n) + S_{\ominus\ominus}(n)$. Indeed, for example regarding $S_{\oplus\oplus}(n+1)$, consider a word $u = u_0 \dots u_{n-1} \in \{\oplus,\ominus\}^n$ and the concatenation of a label $\sigma \in \{\oplus,\ominus\}$ at the end of u, then $u' = u_0 \dots u_{n-1}\sigma) \in L_{\oplus\oplus}(n+1)$ if and only if $\sigma = \oplus$ and $u_0 = \oplus$, *i.e.* $\sigma = \oplus$ and ($u \in L_{\oplus\oplus}(n)$ or $u \in L_{\oplus\ominus}(n)$). It follows that,

- if $u \in L_{\oplus\oplus}(n)$ then $|u'|_\oplus = |u|_\oplus + 1$ and $|u'|_{\oplus\ominus} = |u|_{\oplus\ominus}$,
- if $u \in L_{\oplus\ominus}(n)$ then $|u'|_\oplus = |u|_\oplus + 1$ and $|u'|_{\oplus\ominus} = |u|_{\oplus\ominus}$,

which gives the first recurrence. A similar reasoning lead to the three other recurrence relations. Also remark that by symmetry we always have $S_{\oplus\ominus}(n) = S_{\ominus\oplus}(n)$, though this fact will not be used in the coming proof.

In order to solve the recurrence, we establish a relation to known formulas by remarking that $S(n) = 3S(n-1) - S(n-2)$, which corresponds to the bisection of Fibonacci-like integer sequences (*aka* Lucas sequences):

$$\begin{aligned}
S(n) &= S_{\oplus\oplus}(n) + S_{\oplus\ominus}(n) + S_{\ominus\oplus}(n) + S_{\ominus\ominus}(n) \\
&= 2S_{\oplus\oplus}(n-1) + 2S_{\oplus\ominus}(n-1) + \frac{1}{2}S_{\oplus\oplus}(n-1) + S_{\oplus\ominus}(n-1) \\
&\quad + 2S_{\ominus\oplus}(n-1) + S_{\ominus\ominus}(n-1) + S_{\ominus\oplus}(n-1) + S_{\ominus\ominus}(n-1) \\
&= 3S_{\oplus\oplus}(n-1) + 3S_{\oplus\ominus}(n-1) + 3S_{\ominus\oplus}(n-1) + 3S_{\ominus\ominus}(n-1) \\
&\quad - \frac{1}{2}S_{\oplus\oplus}(n-1) - S_{\ominus\ominus}(n-1) \\
&= 3S(n-1) - S_{\oplus\oplus}(n-2) - S_{\oplus\ominus}(n-2) - S_{\ominus\oplus}(n-2) - S_{\ominus\ominus}(n-2) \\
&= 3S(n-1) - S(n-2)
\end{aligned}$$

Finally, since we have $S(1) = 2$ and $S(2) = 3$ we deduce that $S(n)$ is the bisection of Lucas numbers, sequence A005248 of OEIS [1]. The nice closed form involving the golden ratio is a folklore adaptation of Binet's formula to Lucas numbers, and Equation 8 gives the result. □

4.3 Class III: the sensitivity tends to 1.

This last class contains three ECA rules, namely 128, 160 and 162. The study of sensitivity to synchronism for these rules is based on the characterization of pairs of update schedule leading to the same dynamics. A pair of update schedules $\Delta, \Delta' \in \mathcal{P}_n$ is *special for rule* α if $\Delta \not\equiv \Delta'$ but $D_{f^{(\Delta)}_{\alpha,n}} = D_{f^{(\Delta)}_{\alpha,n}}$. We will count the special pairs for rules 128, 160 and 162.

Given an update schedule $\Delta \in \mathcal{P}_n$, define the *left rotation* $\sigma(\Delta)$ and the *left/right exchange* $\rho(\Delta)$ such that $\forall i \in [\![n]\!]$ it holds $lab_{\sigma(\Delta)}((i, j)) = lab_\Delta((i+1, j+1))$ and $lab_{\rho(\Delta)}((i, j)) = lab_\Delta((j, i))$. It is clear that if a pair of update schedules $\Delta, \Delta' \in \mathcal{P}_n$ is special then $\sigma(\Delta), \sigma(\Delta')$ is also special. Furthermore, when rule α is left/right symmetric (meaning that $\forall x_1, x_2, x_3 \in \{\mathtt{0}, \mathtt{1}\}$ we have $r_\alpha(x_1, x_2, x_3) = r_\alpha(x_3, x_2, x_1)$, which is the case of rules 128 and 162, but not 160) then $\rho(\Delta), \rho(\Delta')$ is also special. We say that special pairs in a set S are *disjoint* when no update schedule belongs to more than one pair *i.e.*, if three update schedules $\Delta, \Delta', \Delta'' \in S$ are such that both (Δ, Δ') and (Δ, Δ'') are special pairs then $\Delta' = \Delta''$. When it is clear from the context, we will omit to mention the rule relative to which some pairs are special.

4.3.1 ECA rule 128

The Boolean function associated with the ECA rule 128 is $r_{128}(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3$. Its simple definition will allow us to better illustrate the role played by special pairs.

Remark 28. *When* $\mathtt{d}_\Delta(i) = [\![n]\!]$ *for some cell* i*, the only possibility to get* $f^{(\Delta)}_{128}(x)_i = \mathtt{1}$ *is* $x = \mathtt{1}^n$*. However for* $x = \mathtt{1}^n$ *we have* $f^{(\Delta)}_{128}(x)_i = \mathtt{1}$ *for any* Δ*.*

The previous remark combined with an observation in the spirit of Lemma 5, gives the next characterization. Let us introduce the notation $\mathtt{d}_\Delta = \mathtt{d}_{\Delta'}$ for cases in which $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i)$ holds in every cell $i \in [\![n]\!]$.

Lemma 29. *For any* $n \in \mathbb{N}$*, choose* $\Delta, \Delta' \in \mathcal{P}_n$ *such that* $\Delta \not\equiv \Delta'$*. Then,* $\mathtt{d}_\Delta = \mathtt{d}_{\Delta'}$ *if and only if* $D_{f^{(\Delta)}_{128,n}} = D_{f^{(\Delta')}_{128,n}}$*.*

Proof. For $n = 1, 2, 3$ we have $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i) = [\![n]\!]$ for all $i \in [\![n]\!]$, and only one dynamics (see Section 3), therefore the result holds. Now, consider $n \geq 4$.

($\Rightarrow$) Assume $\mathtt{d}_\Delta = \mathtt{d}_{\Delta'}$. Given $i \in [\![n]\!]$, two cases are possible:

- $\mathtt{d}_\Delta(i) \neq [\![n]\!]$. In this case, one can deduce from $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i)$ that $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i)$ and $\overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_{\Delta'}(i)$ (see the proof of Lemma 5). From Formula 2 it follows $f^{(\Delta)}_{128}(x)_i = f^{(\Delta')}_{128}(x)_i$ for any $x \in \{\mathtt{0}, \mathtt{1}\}^n$.
- $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i) = [\![n]\!]$. In this case, in order to have $f^{(\Delta)}_{128}(x)_i \neq f^{(\Delta')}_{128}(x)_i$ one must have one of them equal to $\mathtt{1}$ and the other equal to $\mathtt{0}$. WLOG assume $f^{(\Delta)}_{128}(x)_i = \mathtt{1}$. From the definition of the ECA rule 128 and Formula 2, since $\mathtt{d}_\Delta(i) = [\![n]\!]$ the only possibility is $x = \mathtt{1}^n$, but this also implies $f^{(\Delta')}_{128}(x)_i = \mathtt{1}$.

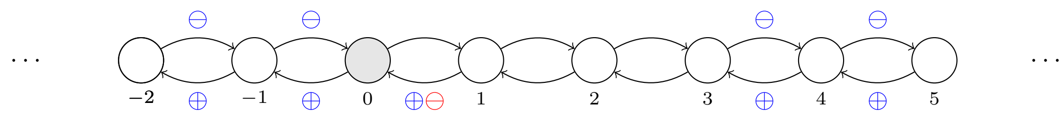


Figure 17: illustration of Lemma 31, with Δ in blue and Δ' in red: hypothesis on the labelings of arc $(1,0)$ imply many $\ominus$ labels on arcs of the form $(j,j+1)$, and $\oplus$ labels on arcs of the form $(j+1,j)$, for Δ.

We conclude that $f_{128}^{(\Delta)}(x)_i = f_{128}^{(\Delta')}(x)_i$ for any $x \in \{\mathtt{0},\mathtt{1}\}^n$ and $i \in [\![n]\!]$, which is equivalently formulated as $D_{f_{128,n}^{(\Delta)}} = D_{f_{128,n}^{(\Delta')}}$.

($\Leftarrow$) Assume $\mathtt{d}_\Delta(i) \neq \mathtt{d}_{\Delta'}(i)$ for some $i \in [\![n]\!]$. Then, WLOG there exists $j \in [\![n]\!]$ such that $j \in \mathtt{d}_\Delta(i) \setminus \mathtt{d}_{\Delta'}(i)$. The configuration x with $x_i = \mathtt{0} \iff i = j$ gives, again by Formula 2, that $f_{128}^{(\Delta)}(x)_i \neq f_{128}^{(\Delta')}(x)_i$. Indeed,

- the image of i under the update schedule Δ depends on $x_j = \mathtt{0}$ which, by the definition of the ECA rule 128, ensures that $f_{128}^{(\Delta)}(x)_i = \mathtt{0}$, and
- the image of i under the update schedule Δ' depends only on cells in state $\mathtt{1}$ which, by the definition of the ECA rule 128, ensures that $f_{128}^{(\Delta')}(x)_i = \mathtt{1}$.

Consequently, $D_{f_{128,n}^{(\Delta)}} \neq D_{f_{128,n}^{(\Delta')}}$. □

Lemma 29 characterizes exactly the pairs of non-equivalent update schedules for which the dynamics of rule 128 differ, *i.e.* the set of special pairs for rule 128, which are the set pairs $\Delta, \Delta' \in \mathcal{P}_n$ such that $\Delta \not\equiv \Delta'$ but $\mathtt{d}_\Delta = \mathtt{d}_{\Delta'}$. Computing $\mu_s(f_{128,n})$ is now a combinatorial problem, of computing the number of possible $\mathtt{d}_\Delta$ for $\Delta \in \mathcal{P}_n$.

Remark 30. *Lemma 29 does not hold for all rules, since some of them are max-sensitive even though there exist $\Delta \not\equiv \Delta'$ with $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i)$ for all $i \in [\![n]\!]$.*

We are going to prove that for any $n > 6$, there exist $10n$ disjoint special pairs of schedules of size n (Lemma 33). We will first argue that special pairs differ in the labeling of exactly one arc (Lemma 32), then exhibit $10n$ special pairs of schedules of size n (which come down to five cases up to rotation and left/right exchange) and finally argue that these pairs are disjoint. This will lead to Theorem 34. The coming proofs will make heavy use of the following lemma (see Figure 17).

Lemma 31. *For any $n \geq 4$, consider a special pair $\Delta, \Delta' \in \mathcal{P}_n$ for rule 128 such that $lab_\Delta((i+1,i)) = \oplus$ and $lab_{\Delta'}((i+1,i)) = \ominus$ for some $i \in [\![n]\!]$. For all $j \in [\![n]\!] \setminus \{i, i+1, i+2\}$, it holds $lab_\Delta((j,j+1)) = \ominus$ and $lab_\Delta((j+1,j)) = \oplus$.*

Proof. From Lemma 29, we must have $\mathtt{d}_\Delta = \mathtt{d}_{\Delta'}$. Hence, in particular, $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i)$. However, from the hypothesis on the labelings of arc $(i+1,i)$, the only possibility is that $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i) = [\![n]\!]$. Indeed, we have $i+2 \in \mathtt{d}_{\Delta'}(i)$, but on Δ to the right we have $\overrightarrow{\mathtt{d}}_\Delta(i) = 0$ thus for the chain of influences of cell i to contain cell $i+2$ we must have $\overleftarrow{\mathtt{d}}_\Delta(i) \geq n-2$, which corresponds to $lab_\Delta((j+1,j)) = \ominus$ for all $j \in [\![n]\!] \setminus \{i, i+1, i+2\}$. It follows that for these j we have $lab_\Delta((j,j+1)) = \oplus$ otherwise an invalid cycle of length two is created (Theorem 2). □

Lemma 32. *For any $n > 6$, if $\Delta, \Delta' \in \mathcal{P}_n$ is a special pair for rule* 128 *then Δ and Δ' differ on the labeling of exactly one arc.*

Proof. First, by definition of special pair, we have $\Delta \not\equiv \Delta'$. Hence, Δ and Δ' must differ on the labeling of at least one arc. Up to rotation and right/left exchange, let us suppose WLOG that $lab_\Delta((1,0)) = \oplus$ and $lab_{\Delta'}((1,0)) = \ominus$. Now, for the sake of contradiction, assume that they also differ on another arc, and consider the following cases disjunction (remark that the order of the case study is chosen so that cases make reference to previous cases).

(a) If $lab_\Delta((i,i+1)) = \oplus$ and $lab_{\Delta'}((i,i+1)) = \ominus$ for some $i \in [\![n]\!]$, then by applying Lemma 31 to the two arcs where Δ and Δ' differ leads to a contradiction on the labeling of some arc according to Δ. Indeed, Lemma 31 is applied to two arcs in different directions, one application leaves three arcs of the form $(j, j+1)$ not labeled $\ominus$ in Δ and three arcs of the form $(j+1, j)$ not labeled $\oplus$ in Δ, the converse for the other application, hence starting from $n = 7$ these labelings overlap in a contradictory fashion.

(b) If $lab_\Delta((i+1,i)) = \ominus$ and $lab_{\Delta'}((i+1,i)) = \oplus$ for some $i \in [\![n]\!] \setminus \{0\}$, then $i \in \{2,3\}$ otherwise there is a forbidden cycle of length two in Δ with some $\ominus$ label given by the application of Lemma 31 to the arc $(1,0)$. However, for $i \in \{2,3\}$ the application of Lemma 31 to the arc $(i+1,i)$ gives $lab_{\Delta'}((0,1)) = \ominus$, creating a forbidden cycle of length two in Δ'.

(c) If $lab_\Delta((i+1,i)) = \oplus$ and $lab_{\Delta'}((i+1,i)) = \ominus$ for some $i \in [\![n]\!] \setminus \{1\}$, then applying Lemma 31 to the two arcs where Δ and Δ' differ leads to a forbidden cycle of length n in Δ (contradiction Theorem 2). Indeed, if $i \notin \{1,2\}$ then we have $\ominus$ labels on arcs of the form $(j, j+1)$ for all $j \in [\![n]\!]$, and if $i = 2$ then the forbidden cycle contains the arc $(3,2)$ labeled $\oplus$. The case $i = 0$ is not a second difference.

(d) If $lab_\Delta((i,i+1)) = \ominus$ and $lab_{\Delta'}((i,i+1)) = \oplus$ for some $i \in [\![n]\!]$, then applying Lemma 31 to arc $(1,0)$ gives $lab_\Delta((j+1,j)) = \oplus$ for all $j \in [\![n]\!] \setminus \{0,1,2\}$, and applying Lemma 31 to arc $(i,i+1)$ gives $lab_{\Delta'}((j+1,j)) = \ominus$ for all $j \in [\![n]\!] \setminus \{i, i-1, i-2\}$. Starting from $n = 7$ we have $([\![n]\!] \setminus \{0,1,2\}) \cap ([\![n]\!] \setminus \{i, i-1, i-2\}) \neq \emptyset$, and as a consequence there is an arc $((j+1,j))$ in the case of Item (c).

(e) If $lab_\Delta((2,1)) = \oplus$ and $lab_{\Delta'}((2,1)) = \ominus$, then applying Lemma 31 to arc $(2,1)$ gives $lab_\Delta((0,1)) = \ominus$, however since by hypothesis $lab_{\Delta'}((1,0)) = \ominus$ we also have $lab_{\Delta'}((0,1)) = \oplus$ otherwise there is a forbidden cycle of length two in Δ' (Theorem 2). As a consequence, the arc $(0,1)$ is in the case of Item (d).

We conclude that in any case a second difference leads to a contradiction, either because an invalid cycle is created, or because repeated applications of Lemma 31 give contradictory labels (both $\oplus$ and $\ominus$) to some arc for some update schedule. □

Lemma 33. *For any $n > 6$, there exist $10n$ disjoint special pairs of schedules of size n for rule* 128*.*

Proof. Fix $n \geq 6$ and consider the set of special pairs $\Delta, \Delta' \in \mathcal{P}_n$ which have a difference between Δ and Δ' on the labeling of arc $(1,0)$, with $lab_\Delta((1,0)) = \oplus$ and $lab_{\Delta'}((1,0)) = \ominus$. Lemma 31 fixes the labels of many arcs of Δ, and from Lemma 32 the same labels hold for

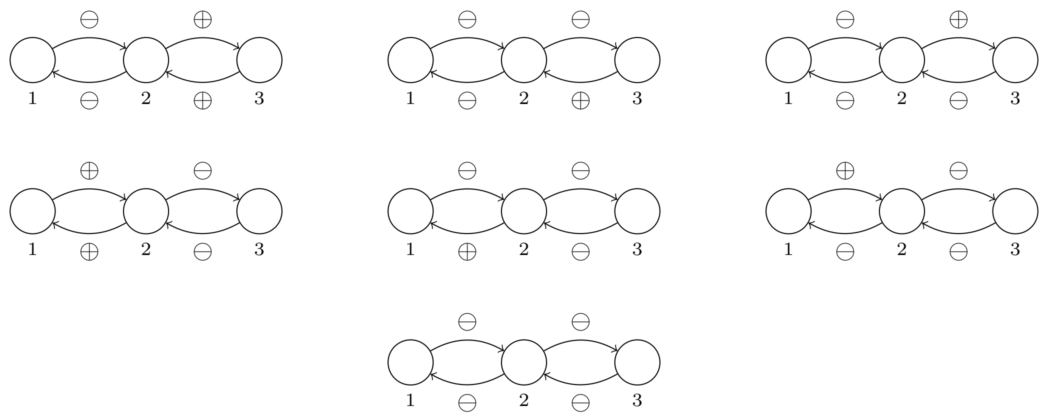


Figure 18: seven labelings of arcs $(1,2)$, $(2,3)$, $(2,1)$ and $(3,2)$ giving a forbidden cycle of length two, in the proof of Lemma 33.

Δ' since there is already a difference on arc $(1,0)$:

$$\text{for all } j \in [\![n]\!] \setminus \{0,1,2\} \text{ we have } lab_\Delta((j,j+1)) = lab_{\Delta'}((j,j+1)) = \ominus$$
$$\text{and } lab_\Delta((j+1,j)) = lab_{\Delta'}((j+1,j)) = \oplus.$$

Furthermore the labeling of arc $(1,0)$ is given by our hypothesis, and from Theorem 2 (to avoid a forbidden cycle of length two in Δ) and Lemma 32 (equality of lab_Δ and $lab_{\Delta'}$ except for the arc $(1,0)$) we also have $lab_\Delta((0,1)) = lab_{\Delta'}((0,1)) = \oplus$. As a consequence it remains to consider 2^4 possibilities for the labelings of arcs

$$(1,2), (2,3), (2,1) \text{ and } (3,2)$$

(which are equal on Δ and Δ', again by Lemma 32).

Among these, seven possibilities create a forbidden cycle of length two when the labels of the two arcs between cells 1 and 2, or 2 and 3, are both $\ominus$ (see Figure 18).

Among the remaining possibilities, four create a forbidden cycle of length n in Δ, when the labels of arcs $(2,1)$ and $(3,2)$ are set to $\oplus$ (see Figure 19).

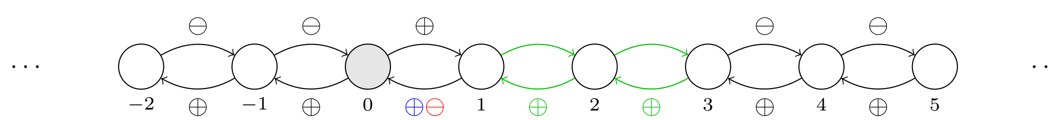


Figure 19: four labelings of arcs $(1,2)$, $(2,3)$, $(2,1)$ and $(3,2)$ giving a forbidden cycle of length n in Δ, in the proof of Lemma 33, with lab_Δ in blue, $lab_{\Delta'}$ in red, in black the labels on which they are equal, and in green are highlighted the arcs on which we consider the 2^4 possibilities. For any combination of $\oplus$ and $\ominus$ labels on arcs $(1,2)$ and $(2,3)$, the forbidden cycle is $0 \overset{\ominus}{\to} -1 \overset{\ominus}{\to} -2 \overset{\ominus}{\to} \ldots \overset{\ominus}{\to} 5 \overset{\ominus}{\to} 4 \overset{\ominus}{\to} 3 \overset{\oplus}{\to} 2 \overset{\oplus}{\to} 1 \overset{\oplus}{\to} 0$ (recall that the orientation of $\ominus$ arcs is reversed, Theorem 2).

The five remaining possibilities are presented on Figure 20, one can easily check that they indeed correspond to special pairs:

- neither Δ nor Δ' contain a forbidden cycle. Hence, they are pairs of non-equivalent update schedule,
- for any $i \in [\![n]\!] \setminus \{0\}$, we have $\overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_{\Delta'}(i)$ and $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i)$. Hence, $\mathtt{d}_\Delta(i) = \mathtt{d}_{\Delta'}(i)$, and for cell 0 we have $\mathtt{d}_\Delta(0) = \mathtt{d}_{\Delta'}(0) = [\![n]\!]$.

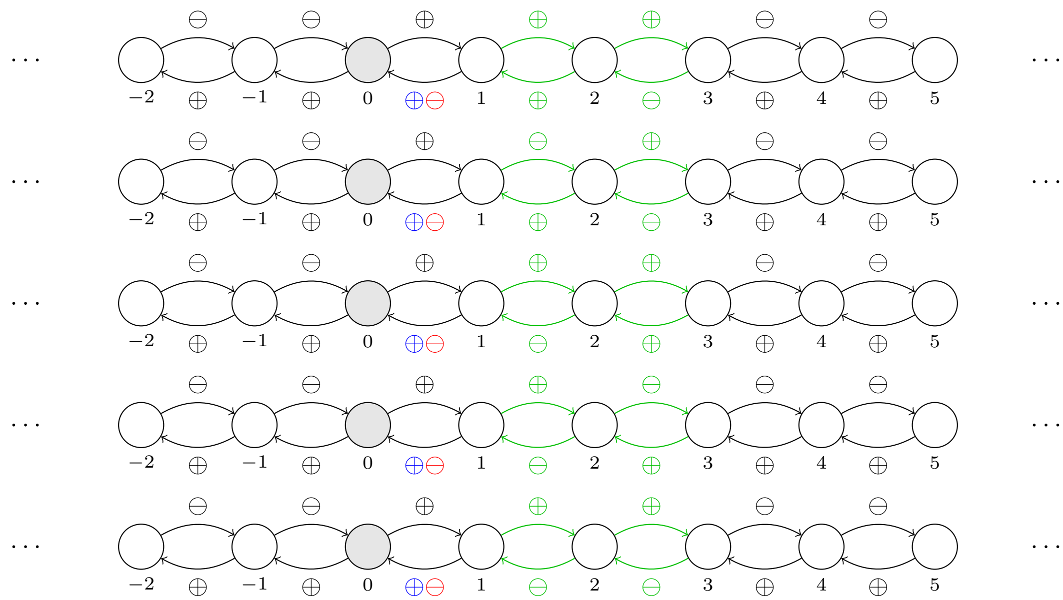


Figure 20: five base special pairs Δ, Δ' for rule 128 in the proof of Lemma 33, with lab_Δ in blue, $lab_{\Delta'}$ in red, in black the labels on which they are equal, and in green are highlighted the arcs on which we consider the five remaining possibilites.

We have seen so far that there are exactly five special pairs with their unique difference (Lemma 32) on arc $(1, 0)$. Let us call them the five *base* pairs and denote them as Δ^i, Δ'^i for $i \in [\![5]\!]$. When we consider the n rotations plus the left/right exchange (recall that rule 128 is symmetric), we obtain $10n$ pairs:

$$\rho^k(\sigma^j(\Delta^i)), \rho^k(\sigma^j(\Delta'^i)) \text{ for } i \in [\![5]\!], j \in [\![n]\!], k \in [\![2]\!]. \tag{9}$$

Let us finally argue that these pairs are disjoint, *i.e.* an update schedule belongs to at most one pair.

First, one can straightforwardly check on Figure 20 that the ten update schedules with a difference on arc $(1, 0)$ are all distinct, hence the five base pairs are disjoint.

Second, the n rotations of these ten update schedules are all distinct when $n > 6$, as can be noticed from n letter words on alphabet $\{\oplus, \ominus\}$ given by

$$\big(lab_\Delta((i, i+1))\big)_{i \in [\![n]\!]} \text{ for some } \Delta.$$

Indeed, each of these words contains a unique factor $\ominus\ominus\ominus\oplus$ which allows to identify the number of left rotations applied to some Δ^i or Δ'^i with $i \in [\![5]\!]$ in order to obtain Δ. As a consequence, two distinct base update schedules remain distinct when some rotation is applied to one of them.

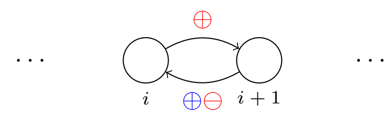


Figure 21: illustration of the setting for the contradiction in Lemma 35, with lab_Δ in blue and $lab_{\Delta'}$ in red.

Third, the left/right exchange of these $5n$ update schedules (base plus rotations) give $10n$ distinct update schedules, as can be noticed on the number of $\ominus$ labels on arcs of the form $(i, i+1)$ for $i \in [\![n]\!]$. Indeed, denoting

$$|\Delta|_\ominus = |\{(i, i+1) \mid i \in [\![n]\!] \text{ and } lab_\Delta((i, i+1)) = \ominus\}|,$$

we have for any $n > 6$ that $|\Delta|_\ominus > n-3$ when Δ is a base update schedule, the quantity is preserved by rotation, *i.e.* $|\sigma(\Delta)|_\ominus = |\Delta|_\ominus$, but it holds that $|\Delta|_\ominus > n-3$ if and only if $|\rho(\Delta)|_\ominus < n-3$. As a consequence, two distinct update schedules (among the $5n$ update schedules $\sigma^j(\Delta^i), \sigma^j(\Delta^i)$ for $i \in [\![5]\!]$ and $j \in [\![n]\!]$) remain distinct when the left/right exchange is applied to one of them. When the left/right exchange is applied to both of them then the situation is symmetric to the previous considerations.

We conclude that the $10n$ pairs given by Formula 9 are special and disjoint. □

As a consequence of Lemma 33 we have $|\{\mathtt{d}_\Delta \mid \Delta \in \mathcal{P}_n\}| = 3^n - 2^{n+1} - 10n + 2$ for any $n > 6$, and the result follows from Lemma 29.

Theorem 34. *$\mu_s(f_{128,n}) = \frac{3^n - 2^{n+1} - 10n + 2}{3^n - 2^{n+1} + 2}$ for any $n > 6$.*

4.3.2 ECA rule 162

The ECA rule 162 is based on the Boolean function $r_{162}(x_1, x_2, x_3) = (x_1 \vee \neg x_2) \wedge x_3$. Let f be a shorthand for $f_{162,n}$ when the context is clear.

The structure of the reasoning is to first prove that for any special pair $\Delta, \Delta' \in \mathcal{P}_n$ for rule 162, the labelings of arcs of the form $(i+1, i)$ for all $i \in [\![n]\!]$ are identical in Δ and Δ' (Lemma 35). Second, given a difference on the labels of some arc $(i, i+1)$, prove that it forces all other labels both in Δ and in Δ' (Lemma 36). Third, for the remaining case, prove that it is indeed a special pair, thus generating n disjoint special pairs by rotation (for any $n \geq 5$), leading to Theorem 37.

Lemma 35. *For any $n \geq 2$, if $\Delta, \Delta' \in \mathcal{P}_n$ is a special pair for rule 162, then for all $i \in [\![n]\!]$ we have $lab_\Delta((i+1, i)) = lab_{\Delta'}((i+1, i))$.*

Proof. By contradiction, assume that there exists $i \in [\![n]\!]$ such that, WLOG, $lab_\Delta((i+1, i)) = \oplus$ whereas $lab_{\Delta'}((i+1, i)) = \ominus$. This implies $lab_{\Delta'}((i, i+1)) = \oplus$ otherwise there is a forbidden cycle of length two in Δ' (Theorem 2). See Figure 21 for an illustration of the setting.

Consider some $x \in \{\mathtt{0}, \mathtt{1}\}^n$ with $x_i = \mathtt{0}$ and $x_{i+1} = \mathtt{1}$ (this require $n \geq 2$)). From our knowledge of Δ we have for some unknown $y_{i-1} \in \{\mathtt{0}, \mathtt{1}\}$ that

$$f^{(\Delta)}(x)_i = r_{162}(y_{i-1}, x_i, x_{i+1}) = r_{162}(y_{i-1}, \mathtt{0}, \mathtt{1}) = \mathtt{1}.$$

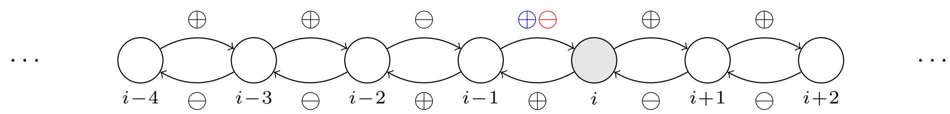


Figure 22: the special pair $\Delta, \Delta' \in \mathcal{P}_n$ for rule 162 in Lemma 36, with $lab_\Delta((i-1,i)) = \oplus$ and $lab_{\Delta'}((i-1,i)) = \ominus$ for some $i \in [\![n]\!]$. lab_Δ in blue, $lab_{\Delta'}$ in red, and in black the labels on which they are equal.

From our knowledge of Δ' we have for some unknown $y_{i-1}, y_{i+2} \in \{\mathtt{0}, \mathtt{1}\}$ that

$$\begin{aligned} f^{(\Delta')}(x)_i &= r_{162}(y_{i-1}, x_i, r_{162}(x_i, x_{i+1}, y_{i+2})) = r_{162}(y_{i-1}, \mathtt{0}, r_{162}(\mathtt{0}, \mathtt{1}, y_{i+1})) \\ &= r_{162}(y_{i-1}, \mathtt{0}, \mathtt{0}) = \mathtt{0}. \end{aligned}$$

Thus $f^{(\Delta)}(x)_i \neq f^{(\Delta')}(x)_i$, a contradiction to the fact that Δ, Δ' is a special pair. □

From Lemma 35 and the fact that $\Delta \not\equiv \Delta'$, we will now consider a special pair with a difference on some arc $(i, i+1)$ for $i \in [\![n]\!]$, and prove that this first difference enforces all the other labels both in Δ and in Δ'.

Lemma 36. *For any $n \geq 3$, there is a unique special pair $\Delta, \Delta' \in \mathcal{P}_n$ for rule 162 with $lab_\Delta((i-1,i)) \neq lab_{\Delta'}((i-1,i))$ for some $i \in [\![n]\!]$, and its labels are depicted on Figure 22.*

Proof. WLOG, as on Figure 22, assume that $lab_\Delta((i-1,i)) = \oplus$ and $lab_{\Delta'}((i-1,i)) = \ominus$. We deduce that $lab_{\Delta'}((i,i-1)) = \oplus$ otherwise there is a forbidden cycle of length two in Δ' (Theorem 2) and from Lemma 35 it follows that we also have $lab_\Delta((i,i-1)) = \oplus$.

We are going to prove that this forces all the other labels of Δ Δ', *i.e.* there is a unique such special pair. From the hypothesis that Δ, Δ' is a special pair, we will use the fact that for all $x \in \{\mathtt{0}, \mathtt{1}\}^n$ and for all $j \in [\![n]\!]$ we have $f^{(\Delta)}(x)_j = f^{(\Delta')}(x)_j$.

From Lemma 35 we deduce that $\overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_\Delta(i)$, meaning that at the time cell i is updated, the state of its right neighbor (cell $i+1$) are identical under update schedules Δ and Δ'. Let us denote $y_{i+1} \in \{\mathtt{0}, \mathtt{1}\}$ this state for the rest of this proof.

By contradiction assume that it is possible to have some configuration $x \in \{\mathtt{0}, \mathtt{1}\}^n$ such that

$$x_{i-1} = \mathtt{0} \text{ and } x_i = \mathtt{1} \text{ and } y_{i+1} = \mathtt{1} \tag{10}$$

(this requires $n \geq 3$). In this case we have

$$f^{(\Delta)}(x)_i = r_{162}(x_{i-1}, x_i, y_{i+1}) = r_{162}(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$$

but for some unknown $y_{i-2} \in \{\mathtt{0}, \mathtt{1}\}^n$ we always have

$$\begin{aligned} f^{(\Delta')}(x)_i &= r_{162}(r_{162}(y_{i-2}, x_{i-1}, x_i), x_i, y_{i+1}) = r_{162}(r_{162}(y_{i-2}, \mathtt{0}, \mathtt{1}), \mathtt{1}, \mathtt{1}) \\ &= r_{162}(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{1} \end{aligned}$$

i.e. $f^{(\Delta)}(x)_i \neq f^{(\Delta')}(x)_i$ which contradicts the hypothesis that Δ,Δ' is a special pair. We conclude that it must be impossible to have simultaneously $x_{i-1} = \mathtt{0}$, $x_i = \mathtt{1}$ and $y_{i+1} = \mathtt{1}$.

This hints at the fact that the value of $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i)$ must be close to n so that the constraints on x_{i-1} and x_i make it impossible to obtain $y_{i+1} = \mathtt{1}$ when updating the chain of influence to the right of cell i. This is what we are going to prove formally, via the following case disjunction.

- If $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i) < n-1$ then consider $x \in \{\mathtt{0},\mathtt{1}\}^n$ with $x_{i-1} = \mathtt{0}$ and $x_i = x_{i+1} = \cdots = x_{i+\overrightarrow{\mathtt{d}}_\Delta(i)} = \mathtt{1}$. From our current hypothesis on $\overrightarrow{\mathtt{d}}_\Delta(i)$ and for $n \geq 3$, such a configuration exists. We deduce from the definition of rule 162 that the updates (in this order, both in Δ and Δ') of cells $i + \overrightarrow{\mathtt{d}}_\Delta(i), i + \overrightarrow{\mathtt{d}}_\Delta(i) - 1, \ldots, i+1$ all give state $\mathtt{1}$, *i.e.* in particular $y_{i+1} = \mathtt{1}$, leading to a contradiction as developed from Equation 10.
- If $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i) \geq n$ then there is a forbidden cycle of length n in Δ:

$$i \overset{\ominus}{\to} i+1 \overset{\ominus}{\to} \ldots \overset{\ominus}{\to} i-2 \overset{\ominus}{\to} i-1 \overset{\oplus}{\to} i \tag{11}$$

  (recall that the orientation of $\ominus$ arcs is reversed, see Theorem 2). As a consequence we discard this case.
- If $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i) = n-1$ then it means that we have $lab_\Delta((j+1,j)) = lab_{\Delta'}((j+1,j)) = \ominus$ for all $j \in [\![n]\!] \setminus \{i-2, i-1\}$, and $lab_\Delta((i-1,i-2)) = lab_{\Delta'}((i-1,i-2)) = \ominus$, but also $lab_\Delta((j,j+1)) = lab_{\Delta'}((j,j+1)) = \oplus$ for all $j \in [\![n]\!] \setminus \{i-2, i-1\}$ from Theorem 2.
  Therefore it only remains to consider the labels of arc $(i-2, i-1)$ in schedules Δ and Δ'. To avoid a forbidden cycle of length n in δ, similar to Equation 11 with $i-2 \overset{\oplus}{\to} i-1$, we need to set $lab_\Delta((i-2,i-1)) = \ominus$. It only remains to consider $lab_{\Delta'}((i-2,i-1))$. Suppose for the contradiction that $lab_{\Delta'}((i-2,i-1)) = \oplus$, then similarly to our previous reasoning, for some $x \in \{\mathtt{0},\mathtt{1}\}^n$ with $x_{i-2} = \mathtt{0}$, $x_{i-1} = \mathtt{1}$ and $x_i = \mathtt{1}$, we have for some unknown $y_{i-3} \in \{\mathtt{0},\mathtt{1}\}$ that

$$\begin{aligned} f^{(\Delta)}(x)_{i-1} &= r_{162}(r_{162}(y_{i-3}, x_{i-2}, x_{i-1}), x_{i-1}, x_i) = r_{162}(r_{162}(y_{i-3}, \mathtt{0}, \mathtt{1}), \mathtt{1}, \mathtt{1}) \\ &= r_{162}(\mathtt{1}, \mathtt{1}, \mathtt{1}) = \mathtt{1} \end{aligned}$$

  whereas

$$f^{(\Delta)}(x)_{i-1} = r_{162}(x_{i-2}, x_{i-1}, x_i) = r_{162}(\mathtt{0}, \mathtt{1}, \mathtt{1}) = \mathtt{0}$$

  thus $f^{(\Delta)}(x)_{i-1} \neq f^{(\Delta')}(x)_{i-1}$, contradicting the fact that Δ, Δ' is a special pair.

We conclude that there is only one remaining possible special pair with a difference on the labelings of arc $(i-1, i)$, and that it is the one given on Figure 22.

Let us finally prove that this is indeed a special pair. One easily checks on Figure 22 that the update schedules of this pair have no forbidden cycle and are non-equivalent. For any $j \in [\![n]\!] \setminus \{i\}$ we have $\overleftarrow{\mathtt{d}}_\Delta(j) = \overleftarrow{\mathtt{d}}_{\Delta'}(j)$ and $\overrightarrow{\mathtt{d}}_\Delta(j) = \overrightarrow{\mathtt{d}}_{\Delta'}(j)$, *i.e.* the chain of influences are identical hence for all $x \in \{\mathtt{0},\mathtt{1}\}^n$ we have $f^{(\Delta)}(x)_j = f^{(\Delta')}(x)_j$.

Regarding cell i, we have $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i)$, meaning that at the time cell i is updated, its right neighbor (cell $i+1$) will be in the same state (denoted y_{i+1}) in both update schedules. Given some $x \in \{\mathtt{0},\mathtt{1}\}^n$, we proceed to a case disjunction.

- If $y_{i+1} = \mathtt{0}$ then

$$f^{(\Delta)}(x)_i = r_{162}(x_{i-1}, x_i, y_{i+1}) = r_{162}(x_{i-1}, x_i, \mathtt{0}) = \mathtt{0}$$

and for some unknown $y_{i-1} \in \{0, 1\}$ we have

$$f^{(\Delta')}(x)_i = r_{162}(y_{i-1}, x_i, y_{i+1}) = r_{162}(y_{i-1}, x_i, 0) = 0$$

therefore we conclude $f^{(\Delta)}(x) = f^{(\Delta')}(x)$.

- If $y_{i+1} = 1$ then, from the reasoning we have just made above and since cell $i + 2$ is updated prior to cell $i+1$, we deduce that cell $i+2$ is updated to state 1 otherwise cell $i+1$ would be updated to state 0 (in both Δ and Δ', contradicting our last hypothesis that $y_{i+1} = 1$). This applies to cell $i+3$, *etc*, until cell $i-2$ which must also be updated to state 1, and finally cell $i-1$ which must be in state 1, *i.e.* $x_{i-1} = 1$.
  We deduce from $y_{i+1} = 1$ and $x_{i-1} = 1$ that

  $$f^{(\Delta)}(x)_i = r_{162}(x_{i-1}, x_i, y_{i+1}) = r_{162}(1, x_i, 1) = 1.$$

  Regarding cell i in the update schedule Δ', we proceed to a last case disjunction.
  - If $x_i = 0$ then for some unknown $y_{i-1} \in \{0, 1\}$ we have

    $$f^{(\Delta')}(x)_i = r_{162}(y_{i-1}, x_i, y_{i+1}) = r_{162}(y_{i-1}, 0, 1) = 1$$

    and we conclude $f^{(\Delta)}(x) = f^{(\Delta')}(x)$.
  - If $x_i = 1$ then we can use our prior deduction that cell $i - 2$ is updated to state 1, therefore

    $$\begin{aligned} f^{(\Delta')}(x)_i &= r_{162}(r_{162}(1, x_{i-1}, x_i), x_i, y_{i+1}) = r_{162}(r_{162}(1, 1, 1), 1, 1) \\ &= r_{162}(1, 1, 1) = 1 \end{aligned}$$

    and we also conclude $f^{(\Delta)}(x) = f^{(\Delta')}(x)$ in this ultimate case.

We have seen that for any $x \in \{0, 1\}^n$, $f^{(\Delta)}(x) = f^{(\Delta')}(x)$. Thus, Δ, Δ' is a special pair, and, from the first part of this proof, it is unique. □

Theorem 37. $\mu_s(f_{162,n}) = \frac{3^n - 2^{n+1} - n + 2}{3^n - 2^{n+1} + 2}$ *for any* $n \geq 3$.

Proof. From Lemmas 35 and 36 there are n pairs of special pairs for rule 162 (no pair with a difference on the label of an arc of the form $(i + 1, i)$ for some $i \in [\![n]\!]$ by Lemma 35, and exactly one pair with a difference on arc $(i, i + 1)$ for each $i \in [\![n]\!]$ by Lemma 36). Denoting Δ, Δ' the special pair given by Lemma 36 with a difference on the arc $(0, 1)$, the n special pairs are $\sigma^j(\Delta), \sigma^j(\Delta')$ for $j \in [\![n]\!]$. Lemmas 35 and 36 hold for any $n \geq 3$, and for any such n one easily checks by considering the word formed by the labels of arcs $(i - 1, i)$ for $i \in [\![n]\!]$ (this word is identical for both schedules of each special pair, by Lemma 35) that these pairs are disjoint: these words contain exactly one factor $\oplus\oplus$ whose position differs for any rotation of Δ, Δ'. It follows that among the $3^n - 2^{n+1} + 2$ non-equivalent update schedules, we have $\mathcal{D}(f_{162,n}) = 3^n - 2^{n+1} + 2 - n$, as stated. □

4.3.3 ECA rule 160

The ECA rule 160 somewhat similar to ECA rule 128. Ideed, it is based on the Boolean function $r_{160}(x_1, x_2, x_3) = x_1 \wedge x_3$.

Remark 38. *It is clear from the definition of r_{160} that for any update schedule Δ and any configuration $x \in \{0,1\}^n$ such that $x_i = x_{i+1} = 0$ (or $x_{i-1} = x_i = 0$) for some $i \in [\![n]\!]$ it holds $f^{(\Delta)}(x)_i = 0$.*

We are going to adopt a adopt a reasoning analogous to rule 128 for the study of the sensitivity to synchronism of rule 160. Lemma 39 will be the first stone showing that as soon as two update schedules form a special pair for rule 160, the position of their difference enforces the labels of many other arcs. Then Lemma 40 will use applications of Lemma 39 according to some carefuly crafted case disjunction, in order to prove that a special pair with more than one difference among the two update schedules (*i.e.* differences on the labels of at least two arcs) is contradictory. Finally, Lemma 41 will use these previous results to characterize exactly the special pairs of update schedule for rule 160, which comes down to six disjoint base special pairs, leading to $12n$ special pairs when considering left/right exchange and rotations. This will give Theorem 42.

Lemma 39. *For any $n > 4$, consider a special pair $\Delta, \Delta' \in \mathcal{P}_n$ for rule 160 such that $lab_\Delta((i+1,i)) = \oplus$ and $lab_{\Delta'}((i+1,i)) = \ominus$ for some $i \in [\![n]\!]$. For all $j \in [\![n]\!] \setminus \{i, i+1, i+2, i+3\}$, it holds $lab_\Delta((j,j+1)) = \ominus$, $lab_\Delta((j+1,j)) = \oplus$ and also $lab_{\Delta'}((i+2,i+1)) = \ominus$, $lab_{\Delta'}((i+1,i+2)) = \oplus$.*

Proof. Let us prove that, with the hypothesis of the statement, we must have $\overleftarrow{\mathtt{d}}_\Delta(i) \geq n-4$ (which implies the $\ominus$ labels on Δ) and also $lab_{\Delta'}((i+2,i+1)) = \ominus$. The complete result follows by application of Theorem 2 to get the $\oplus$ labels (in order to avoid any forbidden cycle of length two).

For the first part, if $\overleftarrow{\mathtt{d}}_\Delta(i) < n-3$ then we can construct the following configuration $x \in \{0,1\}^n$ without a contradiction on the states of cells $i - \overleftarrow{\mathtt{d}}_\Delta(i)$ and $i+3$:

- $x_{i+2} = x_{i+3} = 0$
- $x_i = x_{i-1} = \cdots = x_{i-\overleftarrow{\mathtt{d}}_\Delta(i)} = 1$.

This requires $n \geq 5$, see Figure 23 for an illustration. Regarding Δ', it follows from Remark 38 that cell $i{+}2$ remains in state 0, and as a consequence, regardless of the label of arc $(i{+}2, i{+}1)$, cell $i+1$ is updated to state 0, then so is i. However in Δ, we have $x_j = 1$ for all $j \in \mathtt{d}_\Delta(i)$, *i.e.* cell i depends only on cells in state 1, and we deduce that it is updated to state 1. Thus $f^{(\Delta)}_{160,n}(x)_i \neq f^{(\Delta')}_{160,n}(x)_i$, a contradiction to the fact that Δ, Δ' is a special pair.

For the second part, suppose for the contradiction that $lab\Delta'((i+2,i+1) = \oplus$, and consider the configuration $x \in \{0,1\}^n$ with $x_{i+1} = 0$ and state 1 in all other cells. In Δ, at time cell 0 is updated it has a state 0 on its right (cell $i+1$, not yet updated), and $f^{(\Delta)}_{160,n}(x)_i = 0$. In Δ', cell $i{+}1$ is updated prior to its left and right neighbors (from Theorem 2 again we have $lab_{\Delta'}((i,i+1)) = \oplus$) thus it goes to state 1. We can deduce from this that all cells will go to state 1 because they all have two neighbors in state 1 at the time they are updated. Therefore in particular $f^{(\Delta')}_{160,n}(x)_i = 1$, again a contradiction. □

Let us recall that rule 160 is symmetric, therefore Lemma 39 also applies with a left/right exchange.

Lemma 40. *For any $n > 8$, if $\Delta, \Delta' \in \mathcal{P}_n$ is a special pair for rule 160 then Δ and Δ' differ on the labeling of exactly one arc.*

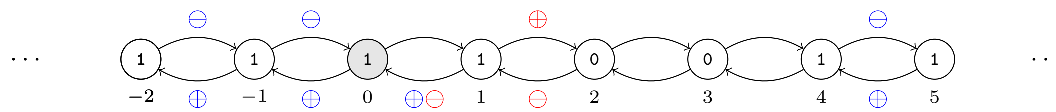


Figure 23: illustration of Lemma 39, with Δ in blue and Δ' in red: hypothesis on the labelings of arc $(1,0)$ imply many $\ominus$ labels on arcs of the form $(j,j+1)$, and $\oplus$ labels on arcs of the form $(j+1,j)$, for Δ, and labels on arcs $(2,1)$ and $(1,2)$ for Δ'. Inside the cells are depicted the states corresponding to a contradictory configuration (having different images on cell 0) when $\overleftarrow{\mathtt{d}}_\Delta(0) \leq n-4$.

Proof. Up to rotation and right/left exchange, let us suppose WLOG that $lab_\Delta((1,0)) = \oplus$ and $lab_{\Delta'}((1,0)) = \ominus$. Now, for the sake of contradiction, assume that they also differ on another arc, and consider the following cases disjunction (remark that the order of the case study is chosen so that cases make reference to previous cases).

(a) If $lab_\Delta((i,i+1)) = \oplus$ and $lab_{\Delta'}((i,i+1)) = \ominus$ for some $i \in [\![n]\!]$, then by applying Lemma 39 to the two arcs where Δ and Δ' differ leads to a contradiction on the labeling of some arc according to Δ. Indeed, Lemma 39 is applied to two arcs in different directions, one application leaves four arcs of the form $(j,j+1)$ not labeled $\ominus$ in Δ and three arcs of the form $(j+1,j)$ not labeled $\oplus$ in Δ, the converse for the other application, hence starting from $n=8$ these labelings overlap in a contradictory fashion.

(b) If $lab_\Delta((i+1,i)) = \ominus$ and $lab_{\Delta'}((i+1,i)) = \oplus$ for some $i \in [\![n]\!] \setminus \{0\}$, then $i \in \{1,2,3\}$ otherwise there is a forbidden cycle of length two in Δ with some $\ominus$ label given by the application of Lemma 39 to the arc $(1,0)$. However, for $i \in \{2,3,4\}$ the application of Lemma 39 to the arc $(i+1,i)$ gives $lab_{\Delta'}((0,1)) = \ominus$, creating a forbidden cycle of length two in Δ'.

(c) If $lab_\Delta((i+1,i)) = \oplus$ and $lab_{\Delta'}((i+1,i)) = \ominus$ for some $i \in [\![n]\!] \setminus \{1,2\}$, then applying Lemma 39 to the two arcs where Δ and Δ' differ leads to a forbidden cycle of length n in Δ (contradiction Theorem 2). Indeed, if $i \notin \{1,2,3\}$ then we have $\ominus$ labels on arcs of the form $(j,j+1)$ for all $j \in [\![n]\!]$, and if $i=3$ then the forbidden cycle contains the arc $(4,3)$ labeled $\oplus$. The case $i=0$ is not a second difference.

(d) If $lab_\Delta((i,i+1)) = \ominus$ and $lab_{\Delta'}((i,i+1)) = \oplus$ for some $i \in [\![n]\!]$, then applying Lemma 39 to arc $(1,0)$ gives $lab_\Delta((j+1,j)) = \oplus$ for all $j \in [\![n]\!] \setminus \{0,1,2,3\}$, and applying Lemma 39 to arc $(i,i+1)$ gives $lab_{\Delta'}((j+1,j)) = \ominus$ for all $j \in [\![n]\!] \setminus \{i,i-1,i-2\}$. Starting from $n=9$ we have $([\![n]\!] \setminus \{0,1,2,3\}) \cap ([\![n]\!] \setminus \{i,1-1,i-2\}) \neq \emptyset$, and as a consequence there is an arc $((j+1,j))$ in the case of Item (c).

(e) If $lab_\Delta((2,1)) = \oplus$ and $lab_{\Delta'}((2,1)) = \ominus$, then applying Lemma 39 to arc $(2,1)$ gives $lab_\Delta((0,1)) = \ominus$, however since by hypothesis $lab_{\Delta'}((1,0)) = \ominus$ we also have $lab_{\Delta'}((0,1)) = \oplus$ otherwise there is a forbidden cycle of length two in Δ' (Theorem 2). As a consequence, the arc $(0,1)$ is in the case of Item (d). The arc $(3,2)$ is involved in the same situation.

We conclude that in any case a second difference leads to a contradiction, either because an invalid cycle is created, or because repeated applications of Lemma 39 give contradictory labels (both $\oplus$ and $\ominus$) to some arc for some update schedule. $\square$

Lemma 41. *For any $n > 8$, there exist $12n$ disjoint special pairs of schedules of size n for rule* 160*.*

Proof. The structure of this proof is very similar to Lemma 33. Fix $n > 8$ and consider the set of special pairs $\Delta, \Delta' \in \mathcal{P}_n$ which have a difference between Δ and Δ' on the labeling of arc $(1,0)$, with $lab_\Delta((1,0)) = \oplus$ and $lab_{\Delta'}((1,0)) = \ominus$. Lemma 39 fixes the labels of many arcs of Δ, and from Lemma 40 the same labels hold for Δ' since there is already a difference on arc $(1,0)$:

$$\begin{aligned} \text{for all } j \in [\![n]\!] \setminus \{0,1,2,3\} \text{ we have } lab_\Delta((j,j+1)) = lab_{\Delta'}((j,j+1)) = \ominus \\ \text{and } lab_\Delta((j+1,j)) = lab_{\Delta'}((j+1,j)) = \oplus, \\ \text{and furthermore } lab_\Delta((1,2)) = lab_{\Delta'}((1,2)) = \oplus \\ \text{and } lab_\Delta((2,1)) = lab_{\Delta'}((2,1)) = \ominus, \end{aligned}$$

Furthermore the labeling of arc $(1,0)$ is given by our hypothesis, and from Theorem 2 (to avoid a forbidden cycle of length two in Δ) and Lemma 40 (equality of lab_Δ and $lab_{\Delta'}$ except for the arc $(1,0)$) we also have $lab_\Delta((0,1)) = lab_{\Delta'}((0,1)) = \oplus$. As a consequence it remains to consider 2^4 possibilities for the labelings of arcs

$$(2,3), (3,4), (3,2) \text{ and } (4,3)$$

(which are equal on Δ and Δ', again by Lemma 40).

Among these, seven possibilities create a forbidden cycle of length two when the labels of the two arcs between cells 1 and 2, or 2 and 3, are both $\ominus$ (see Figure 18 relative to rule 128, the seven possibilities for rule 160 are analogous with the four respective arcs we are now considering).

Among the nine remaining possibilities, three do not correspond to special pairs, as we will prove now by exhibit for each of them a configuration $x \in \{\mathtt{0},\mathtt{1}\}^n$ such that the images at cell 0 differ in Δ and Δ'. These three possibilities are depicted on Figure 24, let us denote them $\hat{\Delta}^i, \hat{\Delta}'^i$ for $i \in [\![3]\!]$.

- For $\hat{\Delta}^0, \hat{\Delta}'^0$ we have $x \in \{\mathtt{0},\mathtt{1}\}^n$ with $x_2 = \mathtt{0}$ and all other cells in state $\mathtt{1}$,
- For $\hat{\Delta}^1, \hat{\Delta}'^1$ we have $x \in \{\mathtt{0},\mathtt{1}\}^n$ with $x_2 = \mathtt{0}$ and all other cells in state $\mathtt{1}$,
- For $\hat{\Delta}^2, \hat{\Delta}'^2$ we have $x \in \{\mathtt{0},\mathtt{1}\}^n$ with $x_3 = \mathtt{0}$ and all other cells in state $\mathtt{1}$.

One can check that in these three cases $i \in [\![n]\!]$ with these three respective configurations, we have $f^{(\hat{\Delta}^i)}(x)_0 = \mathtt{1}$ but $f^{(\hat{\Delta}'^i)}(x)_0 = \mathtt{0}$, because in both update schedules of each pair the left neighbor of cell 0 (cell -1) will be updated to state $\mathtt{1}$, and the right neighbor of cell 0 (cell 1) will be updated to state $\mathtt{0}$ before the update of cell 0 in $\hat{\Delta}'^i$ whereas it is still in state $x_1 = \mathtt{1}$ when cell 0 is updated in $\hat{\Delta}^i$.

The six remaining possibilities are presented on Figure 25. Let us argue that they indeed correspond to special pairs:

- neither Δ nor Δ' contain a forbidden cycle. Hence, they are pairs of non-equivalent update schedule,
- for any $i \in [\![n]\!] \setminus \{0\}$, we have $\overleftarrow{\mathtt{d}}_\Delta(i) = \overleftarrow{\mathtt{d}}_{\Delta'}(i)$ and $\overrightarrow{\mathtt{d}}_\Delta(i) = \overrightarrow{\mathtt{d}}_{\Delta'}(i)$. Hence, $f^{(\Delta)}(x)_i = f^{(\Delta')}(x)_i$ for any $x \in \{\mathtt{0},\mathtt{1}\}^n$ (Lemma 4). For cell 0 let us show that $f^{(\Delta)}(x)_0 = f^{(\Delta')}(x)_0$ for any $x \in \{\mathtt{0},\mathtt{1}\}^n$. In order to have a difference in the update of cell 0, one of the two

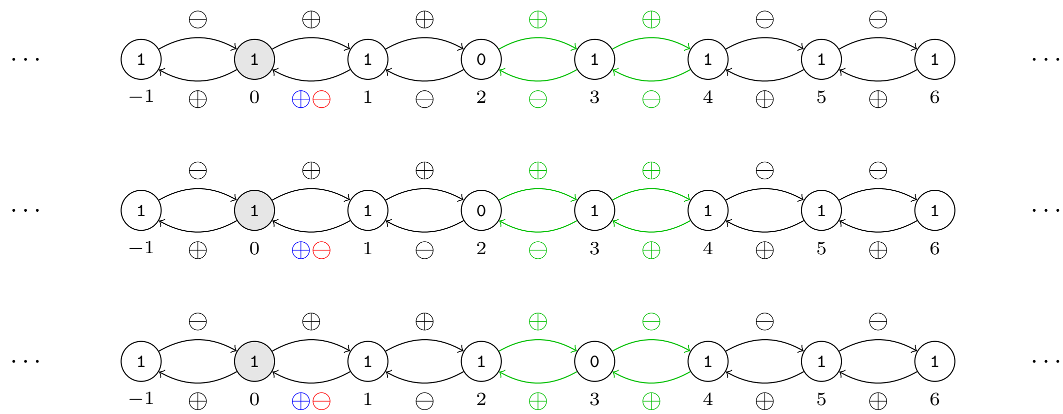


Figure 24: three pairs $\hat{\Delta}^i, \hat{\Delta}'^i$ for $i \in [\![3]\!]$ for rule 160 ($\hat{\Delta}^0, \hat{\Delta}'^0$ on the top; $\hat{\Delta}^1, \hat{\Delta}'^1$ in the middle; $\hat{\Delta}^2, \hat{\Delta}'^2$ on the bottom) not corresponding to special pairs because for each of them there exists a configuration $x \in \{\mathtt{0},\mathtt{1}\}^n$ such that $f^{(\hat{\Delta}^i)}(x)_0 = \mathtt{1} \neq \mathtt{0} = f^{(\hat{\Delta}'^i)}(x)_0$. The states of configuration x are given inside the cells.

update schedules must update it to state $\mathtt{1}$. Now remark that, given the definition of rule 160, the only possibility for cell 0 to be updated to state $\mathtt{1}$ in some update schedule (recall that $\overleftarrow{\mathtt{d}}_\Delta(0) = \overleftarrow{\mathtt{d}}_{\Delta'}(0)$) is that $x_0 = x_{-1} = x_{-2} = \cdots = x_{-\overleftarrow{\mathtt{d}}_\Delta(0)-2} = \mathtt{1}$, and $x_{-\overleftarrow{\mathtt{d}}_\Delta(0)} = \mathtt{1}$. Indeed, if any of these cells is in state $\mathtt{0}$, then at some point in the update of the chain of influence to the left of cell 0 (in this order: cell $-\overleftarrow{\mathtt{d}}_\Delta$ then $-\overleftarrow{\mathtt{d}}_\Delta+1$ then ... then -1 and finally 0) some cell will be updated to state $\mathtt{0}$, and then all subsequent cells will be updated to state $\mathtt{0}$ as well. Given that $\overleftarrow{\mathtt{d}}_\Delta(0) = \overleftarrow{\mathtt{d}}_{\Delta'}(0) \geq n-3$, this would enforce the states of all cells in x except (in the order of Figure 25):

- cells 1 and 3 for the first and third pairs,
- cells $1, 2$ and 4 for the second, fourth and fifth pairs,
- cell 2 for the sixth pair.

A straightfoward exhaustive analysis of these $2\times 2^2+3\times 3^2+2$ cases would convince the reader that, for any configuration $x \in \{\mathtt{0},\mathtt{1}\}^n$ where cell 0 may be updated to state $\mathtt{1}$ in Δ or in Δ' (otherwise $f^{(\Delta)}(x)_0 = f^{(\Delta')}(x)_0 = \mathtt{0}$), it turns out that $f^{(\Delta)}(x)_0 = f^{(\Delta')}(x)_0$ (this is tedious but reveals the nice combinatorics of green labels on Figure 25).

We have seen so far that there are exactly six special pairs with their unique difference (Lemma 40) on arc $(1,0)$. Let us finally argue that these six *base* pairs for rule 160 give $12n$ distinct pairs when considering their rotations and left/right exchange, *i.e.* an update schedule belongs to at most one pair.

It is clear from Figure 25 that all the base pairs are all disjoint. Moreover, considering the pattern $\ominus\ominus\ominus\ominus\oplus\oplus$ and any of the $24n$ update schedules, for any $n > 8$ either it appears exactly once on arcs of the form $(i, i+1)$, or its mirror appears exaclty once on arcs of the form $(i+1, i)$, but not both. This allows to uniquely determine the left/right exchange and rotations applied to some base pair, and the remaining labelings straighforwardly allow to determine one of the six base special pair, and one of Δ or Δ'. Therefore all special pairs are disjoint. $\square$

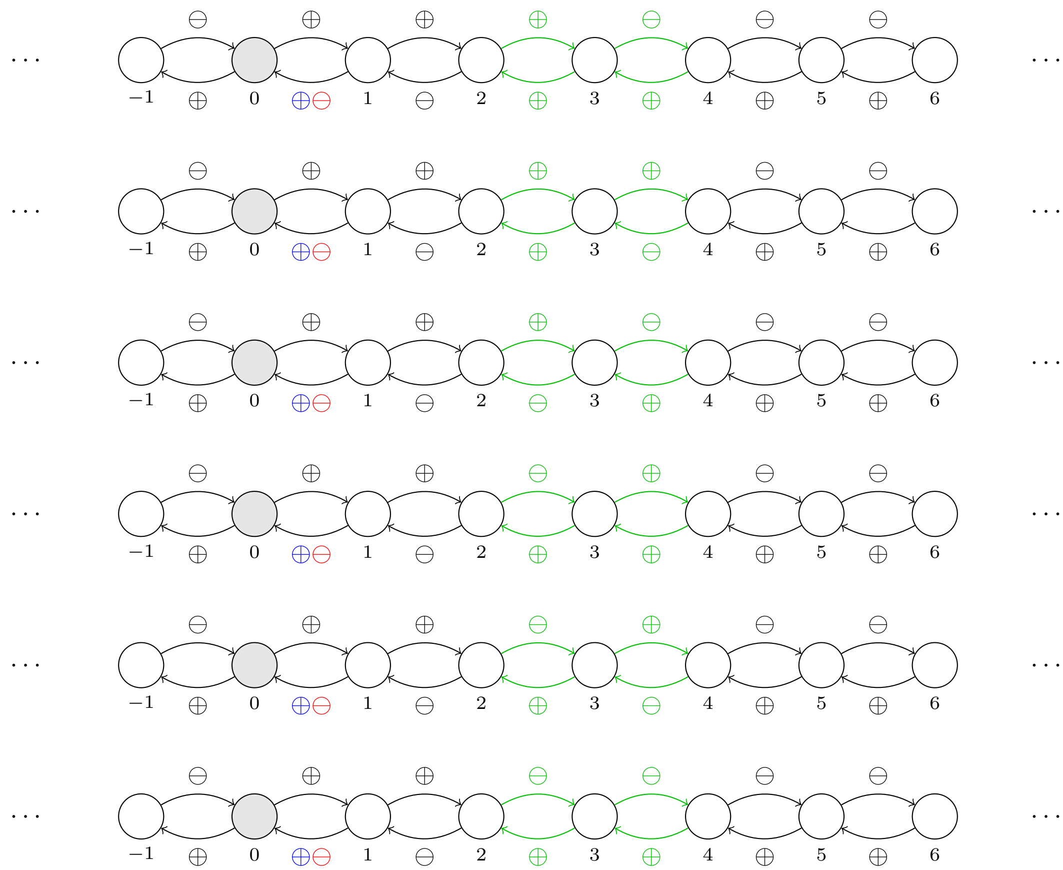


Figure 25: six base special pairs Δ, Δ' for rule 160 in the proof of Lemma 41, with lab_Δ in blue, $lab_{\Delta'}$ in red, in black the labels on which they are equal, and in green are highlighted the arcs on which we consider the six remaining possibilities.

As a consequence of Lemma 41 we have the following result.

Theorem 42. $\mu_s(f_{160,n}) = \frac{3^n - 2^{n+1} - 12n + 2}{3^n - 2^{n+1} + 2}$ *for any* $n > 8$.

5 Conclusion and perspectives

Asynchrony highly impacts the dynamics of CA and new original dynamical behaviors are introduced. In this new model the dynamics become dependent from the update schedule of cells. However, not all schedules produce original dynamics. For this reason, a measure to quantify the sensitivity of ECA *w.r.t* to changes of the update schedule has been introduced in [13]. All ECA were hence classified into two classes: max-sensitive and non-max sensitive.

This paper provides a finer study of the sensitivity measure *w.r.t* the size of the configurations. Indeed, we found that there are four classes (see Table 2). In particular, it is interesting to remark that the asymptotic behavior is not dichotomic *i.e.* the sensitivity function does not always either go to 0 or to 1 when the size of configurations grows. The ECA rule 8 when considered as a classical ECA (*i.e.* when all cells are updated synchronously) has a very simple dynamical behavior but its asynchronous version has a sensitivity to asynchronism function which tends to $\frac{1+\phi}{3}$ when n tends to infinity (ϕ is the golden ratio). Remark that in the classical case, the limit set of the ECA rule 8 is the same as ECA rule 0 after just two steps. It would be interesting to understand which are the relations between the limit set (both in the classical and in the asynchronous cases) and the sensitivity to asynchronism.

Indeed, remark that in our study the sensitivity is defined on one step of the dynamics. It would be interesting to compare how changes the sensitivity function of an ECA when the limit set is considered. This idea has been investigated in works on *block-invariance* [9, 10], with the difference that it concentrates only on the set of configurations in attractors, and discards the transitions within these sets.

Remark also that this study focus on block-sequential updating schemes. However, block-parallel updating schedules are gaining growing interest [6]. It is a promising research direction to investigate how the sensitivity functions change when block-parallel schedules are considered.

Another interesting research direction would consider the generalization of our study to arbitrary CA in order to verify if a finer grained set of classes appear or not. Maybe, the set of possible functions is tightly related to the structure of the neighborhood.

Finally, another possible generalization would consider infinite configurations in the spirit of [12]. However, it seems much more difficult to come out with precise asymptotic results in this last case.

Acknowledgments

The work of Kévin Perrot was funded mainly by his salary as a French State agent and therefore by French taxpayers' taxes, affiliated to Aix-Marseille Univ, Univ. de Toulon, CNRS, LIS, France, and to Univ. Côte d'Azur, CNRS, I3S, France, and secondarily by ANR-18-CE40-0002 FANs project, ECOS-Sud C16E01 project, STIC AmSud CoDANet 19-STIC-03 (Campus France 43478PD) project.

References

[1] OEIS A005248. The online encyclopedia of integer sequences. `https://oeis.org/A005248`.

[2] Julio Aracena, Eric Fanchon, Marco Montalva, and Mathilde Noual. Combinatorics on update digraphs in boolean networks. *Discrete Applied Mathematics*, 159(6):401–409, 2011.

[3] Julio Aracena, Eric Goles, Andrès Moreira, and Lilian Salinas. On the robustness of update schedules in boolean networks. *Biosystems*, 97:1–8, 2009.

[4] Gianpiero Cattaneo, Enrico Formenti, Luciano Margara, and Giancarlo Mauri. Transformations of the one-dimensional cellular automata rule space. *Parallel Comput.*, 23(11):1593–1611, 1997.

[5] Matthew Cook. Universality in elementary cellular automata. *Complex Systems*, 15(1):1–40, 2004.

[6] Jacques Demongeot and Sylvain Sené. About block-parallel boolean networks: a position paper. *Nat. Comput.*, 19(1):5–13, 2020.

[7] Nazim Fatès. A guided tour of asynchronous cellular automata. *Journal of Cellular Automata*, 9(5-6):387–416, 2014.

[8] Martin Gardner. Mathematical games: The fantastic combinations of John Conway's new solitaire game "Life". *Scientific American*, 223(4):120–123, 1970.

[9] Eric Goles, Marco Montalva-Medel, Stéphanie Maclean, and Henning S. Mortveit. Block invariance in a family of elementary cellular automata. *J. Cellular Automata*, 13(1-2):15–32, 2018.

[10] Eric Goles Chacc, Marco Montalva-Medel, Henning S. Mortveit, and Salvador Ramírez-Flandes. Block invariance in elementary cellular automata. *J. Cellular Automata*, 10(1-2):119–135, 2015.

[11] Kevin Perrot, Marco Montalva-Medel, Pedro Paulo Balbi de Oliveira, and Eurico L. P. Ruivo. Maximum sensitivity to update schedule of elementary cellular automata over periodic configurations. *Natural Computing*, 2019.

[12] Eurico L. P. Ruivo, Pedro Paulo Balbi de Oliveira, Marco Montalva-Medel, and Kevin Perrot. Maximum sensitivity to update schedules of elementary cellular automata over infinite configurations. *Information and Computation*, 2019. to appear.

[13] Eurico L. P. Ruivo, Marco Montalva-Medel, Pedro Paulo Balbi de Oliveira, and Kevin Perrot. Characterisation of the elementary cellular automata in terms of their maximum sensitivity to all possible asynchronous updates. *Chaos, Solitons & Fractals*, 113:209–220, 2018.

[14] Stephen Wolfram. *A New Kind of Science*. Wolfram Media, 2002.