Automorphisms of shift spaces and the Higman–Thomspon groups: the one-sided case

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Abstract

Let $1 \leq r < n$ be integers. We give a proof that the group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ of automorphisms of the one-sided shift on n letters embeds naturally as a subgroup $\mathcal{H}_n$ of the outer automorphism group $\mathrm{Out}(G_{n,r})$ of the Higman-Thompson group $G_{n,r}$. From this, we can represent the elements of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ by finite state non-initial transducers admitting a very strong synchronizing condition.

Let $H \in \mathcal{H}_n$ and write $|H|$ for the number of states of the minimal transducer representing H. We show that H can be written as a product of at most $|H|$ torsion elements. This result strengthens a similar result of Boyle, Franks and Kitchens, where the decomposition involves more complex torsion elements and also does not support practical *a priori* estimates of the length of the resulting product.

We also give new proofs of some known results about $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$.

1 Introduction

In this article, we prove that the group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ of automorphisms of the one-sided full shift is isomorphic to a subgroup $\mathcal{H}_n$ of the group of outer automorphisms of the Higman–Thomspon groups $G_{n,r}$. Using this embedding we are able to study $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ from a new perspective.

Fix an alphabet $X_n := \{0, 1, 2 \ldots, n-1\}$ of size $n \geq 2$. The shift map σ_n on the Cantor space of infinite sequences $X_n^{\mathbb{N}}$ is the map which shifts a sequence to the left; i.e., a point that was formerly at index $i+1$ now occupies the index i. An automorphism of the dynamical system $(X_n^{\mathbb{N}}, \sigma_n)$, is a homeomorphism of $X_n^{\mathbb{N}}$ that commutes with the map σ_n. The collection of all such automorphisms forms a group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$. We refer to this group as the group of automorphisms of the shift dynamical system.

The group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ has been well studied (although, many questions about it remain). For instance, the seminal paper of Hedlund [8] shows that elements of this group can be represented by *sliding block codes* requiring no future information. In the same paper, as mentioned above, it is shown that if $n = 2$, this group is isomorphic to the cyclic group of order 2; in the paper [5] the finite subgroups of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ are characterised, and a full description of the numbers which arise as the order of some torsion element is also given.

The paper [3] gives a description of $\mathrm{Out}(G_{n,r})$ as a particular group of non-initial *transducers*. Note that here a transducer is a finite state machine whereby each state reads an element from an input alphabet, possibly changes state, and writes a string from an output alphabet. Let T be such a transducer. We call the number of states of T the size of T and denote this by $|T|$.

While realising elements of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ by transducers has been seen before (see [5, 7]), our realisation takes advantage of extra structure arising from a small category of "folded" de Bruijn graphs. These are a special set of labeled directed graphs each admitting a synchronizing condition stronger than that appearing in the literature around the Road Colouring Problem and the Černý Conjecture. We refer to these as *strongly synchronizing automata*, below. Using this structure, we give a combinatorial proof of the following theorem (see Theorem 5.4 for the more detailed statement).

Theorem 1.1. *An element $T \in \mathcal{H}_n \cong \mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ can be written as a product of at most $|T|$ elements of $\mathcal{H}_n$ arising from automorphisms of directed graphs which are quotients of the underlying graph of T.*

This result is an improvement on a similar result in [5]. There, in order to decompose an element T of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ as a product of torsion elements, one first needs to construct, in the best case, a graph with vertex size of the order of $n^{|T|}$, and it is unclear at the end how many torsion elements one ends up with in the decomposition. Our decomposition on the other hand begins with the transducer T and at each step i, produces a torsion factor H_i of T with strictly fewer states than T.

We also give new combinatorial arguments for the following two results (see Section 4).

- Any finite subgroup of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ is isomorphic to a subgroup of automorphisms of a folded de Bruijn graph. For any such strongly synchronizing automaton the group of label ignoring automorphisms embeds as a subgroup of $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$. For the full one-sided shift, the directed graphs arising from *state splitting* as described in [5] and [2] are actually unlabeled directed graphs of strongly synchronizing automata when the directions of the arrows are reversed. Thus, this embedding result is implicit in [5, 2].

- When $n = 2$, the unlabeled directed graph corresponding to a strongly synchronizing automaton over a 2 letter alphabet either has trivial automorphism group or its automorphism group is isomorphic to the cyclic group of order 2. This gives a new proof of a classic result of Hedlund [8] that $\mathrm{Aut}(X_2^{\mathbb{N}}, \sigma_2) \cong C_2$. (Note that in [5] it is shown that when $n > 2$ that $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ contains a non-abelian free group.)

Our next result is the promised embedding of $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ in $\mathrm{Out}(G_{n,r})$ (given in Section 3). Recall that the Higman–Thompson groups $G_{n,r}$, for $1 \leq r < n$, are amongst the first examples of finitely presented infinite simple groups (when n is even $G_{n,r}$ is simple otherwise its derived subgroup is simple, see [9]).

Theorem 1.2. *Let $n \in \mathbb{N}$ be a natural number with $2 \leq n$, then $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ embeds as a subgroup of Out($G_{n,r}$).*

We briefly discuss the strategy of the proof.

A synchronous transducer which satisfies the strong synchronizing condition induces in a natural way a shift commuting map on $X_n^{\mathbb{N}}$. The subgroup of $\mathrm{Out}(G_{n,r})$ consisting of synchronous transducers which induce automorphisms of $(X_n^{\mathbb{N}}, \sigma_n)$ is what is denoted in the paper [3] as $\mathcal{H}_n$. (A result of [3] asserts that $\mathcal{H}_n$ does not depend on r.) The action of $\mathcal{H}_n$ on $X_n^{\mathbb{N}}$ yields an injective homomorphism to the group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$. In order to show that this map is onto, we use the characterisation by Hedlund of automorphisms of $(X_n^{\mathbb{N}}, \sigma_n)$ as sliding block codes which require no future information; we show that a sliding block code with no future information can be simulated by a strongly synchronizing transducer. Thus, we show that $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ is isomorphic to the group $\mathcal{H}_n$ of bi-synchronizing, synchronous, transducers. It is in the framework of this group $\mathcal{H}_n$, that we prove the results stated above.

As mentioned above, in the discussion of the group $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$, there arises an interesting family of small categories of automata and foldings between them. The automata in any such category are what we call strongly synchronizing automata, below, and are a finite set of natural quotients of some particular de Bruijn graph ([6]). The categories are organised in a two-parameter family, and our final result (given in Section 6) is to count the number of elements in any such category when one of the parameters is less than or equal to 2, extending earlier results from [4]. The *Bell number* $B(a)$, the number of partitions of a set of size a, naturally occurs in the obtained formula.

Theorem 1.3. *The number of foldings of the de Bruijn graph with word length* 2 *over an alphabet of cardinality* n *is*

$$\sum_{\pi} \prod_{i=1}^{|\pi|} R(|\pi|, |A_i|),$$

where π *runs over partitions of the alphabet,* A_i *is the* i*th part, and*

$$R(s,t) = \sum_{\rho} (-1)^{|\rho|-1} (|\rho| - 1)! \prod_{i=1}^{|\rho|} B(|C_i| s),$$

where ρ *runs over all partitions of* $\{1, \ldots, t\}$*, and* C_i *is the* i*th part.*

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2 The Curtis, Hedlund, Lyndon Theorem

In this paper, unlike the paper of Hedlund [8], operators will be on the right of their arguments; but sequences will be indexed from left to right in the usual way.

We begin with some basic definitions and notation.

We denote by X_n the n-element set $\{0, 1, \ldots, n-1\}$. Then X_n^* denotes the set of all finite strings (including the empty string ε) consisting of elements of X_n. For an element $w \in X_n^*$, we let $|w|$ denote the length of w (so that $|\varepsilon| = 0$). We further define

$$X_n^+ = X_n^* \setminus \{\varepsilon\}, \quad X_n^k = \{w \in X_n^* : |W| = k\}, \quad X_n^{\leq k} = \bigcup_{1 \leq i \leq k} X_n^i.$$

We denote the concatenation of strings $x, y \in X_n^*$ by xy; in this notation we do not distinguish between an element of X_n and the corresponding element of X_n^1.

For $x, x_1, x_2 \in X_n^*$, if x is the concatenation x_1x_2 of x_1 and x_2, we write $x_2 = x - x_1$. One can think of the minus operator as "subtracting off a prefix".

A bi-infinite sequence is a map $x : \mathbb{Z} \to X_n$. We sometimes write this sequence as $\ldots x_{-1}x_0x_1x_2\ldots$, where $x_i = x(i) \in X_n$ (we use left actions for determining sequences). We denote the set of such sequences by $X_n^{\mathbb{Z}}$. In a similar way, we define a *(positive) singly-infinite sequence* as a map $x : \mathbb{N} \to X_n$ (where, by convention, $0 \in \mathbb{N}$). We write such a sequence as $x_0x_1x_2\ldots$ and denote the set of all such maps as $X_n^{\mathbb{N}}$. Finally, we also set $X_n^{-\mathbb{N}}$ for the set of all maps $x : X_n^{-\mathbb{N}} \to X_n$ (the (negative) singly infinite sequences). Such a map will be written as a sequence $\ldots x_{-2}x_{-1}x_0$.

Normally, one thinks of a full one-sided shift as $(X_n^{\mathbb{N}}, \sigma)$, where the shift operator σ operates as $y = x\sigma$, where $y_i = x_{i+1}$ for all $i \in \mathbb{N}$. However, in our context it will be much more natural to think of the one-sided shift space as $(X_n^{-\mathbb{N}}, \sigma_n)$, where the shift operator σ_n operates as $y = x\sigma_n$, where $y_i = x_{i-1}$ for all $i \in -\mathbb{N}$. This will ease many notational difficulties later on.

We can concatenate a string $y \in X_n^*$ with an singly infinite string $x \in X_n^{-\mathbb{N}}$, by adding y as a suffix to x. We may can also subtract a finite string y from a singly infinite string x which has y as a suffix by deleting the suffix y.

For a string $\nu \in X_n^*$ we write $[\nu]$ for the set of all elements of $X_n^{-\mathbb{N}}$ with ν as a suffix. Clearly $[\varepsilon] = X_n^{-\mathbb{N}}$.

Let $F(X_n, m)$ denote the set of functions from X_n^m to X_n. Then, for all $m, r > 0$, and all $f \in F(X_n, m)$, we define a map $f_r : X_n^{m+r-1} \to X_n^r$ as follows.

> Let $x = x_{-m-r+2}\ldots x_0$. For $-r+1 \leq i \leq 0$, set $y_i = (x_{i-m+1}x_{i-m+2}\ldots x_i)f$. Then $xf_r = y$, where $y = y_{-r+1}\ldots y_0$.

In other words, we take a "window" of length m which slides along the sequence x, and at the ith step we apply f to the symbols visible in the window. (One may think of the map as acting on the rightmost letter in the viewing window, with $m-1$ digits of history.) This procedure can be extended to define a map $f_\infty : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$, by setting $xf_\infty = y$ where $y_i = (x_{i-m+1} \dots x_i)f$ for all $i \in \mathbb{Z}$; and similarly for $X_n^{-\mathbb{N}}$.

A function $f \in F(X_n, m)$ is called *right permutive* if, for distinct $x, y \in X_n$ and any fixed block $a \in X_n^{m-1}$, we have $(ax)f \neq (ay)f$. Alternatively, the map from X_n to itself given by $x \mapsto (ax)f$ is a permutation for all $a \in X_n^{m-1}$. Analogously, a function $f \in F(X_n, m)$ is called *left permutive* if the map from X_n to itself given by $x \mapsto (xa)f$ is a permutation for all $a \in X_n^{m-1}$.

We note that, if f is not right permutive, then the induced map f_∞ from X_n to itself is not injective. The preceding sentence is false if we replace 'right' with 'left'. For example, take the map $g \in F(X_3, 2)$ defined by $ax \mapsto x$ for all $x \in \{0, 1, 2\}$ and all $a \in \{0, 1\}$; $20 \mapsto 1$, $21 \mapsto 0$ and $22 \mapsto 2$. Then g is right permutive but not left permutive and g_∞ is a bijection. It is not always the case that a right permutive map $f \in F(X_n, m)$ induces a bijective map $f_\infty : X_n^{\mathbb{N}} \to X_n^{\mathbb{N}}$. For example the map $f \in F(X_3, 2)$ defined by $a0 \mapsto 0$, $a1 \mapsto 2$, $a2 \mapsto 1$ for all $a \in \{0, 1\}$; $20 \mapsto 1$, $21 \mapsto 0$, $22 \mapsto 2$ is a right permutive map such that $(\dots 111 \dots)f_\infty = (\dots 222 \dots)f_\infty$. We note that a right permutive map always induces a surjective map from $X_n^{\mathbb{N}}$ to itself.

Remark 2.1. Observe that, if $f \in F(X_n, m)$ and $k \geq 1$, then the map $g \in F(X_n, m+k)$ given by $(x_{-m-k+1} \dots x_0)g = (x_{-m+1} \dots x_0)f$, satisfies $g_\infty = f_\infty$.

The sets $X_n^{\mathbb{Z}}$, $X_n^{\mathbb{N}}$ and $X_n^{-\mathbb{N}}$ are topological spaces, equipped with the Tychonoff product topology derived from the discrete topology on X_n. Each is homeomorphic to Cantor space. The set $\{[\nu] \mid \nu \in X_n^*\}$ is a basis of clopen sets for the topology on $X_n^{-\mathbb{N}}$.

In this paper the *shift map* σ_n is the map which sends a sequence x in $X_n^{\mathbb{Z}}$ or $X_n^{\mathbb{N}}$ to the sequence y given by $y(i) = x(i-1)$ for all i in $\mathbb{Z}$ or $-\mathbb{N}$ respectively.

The following result is due to Curtis, Hedlund and Lyndon [8, Theorem 3.1]:

Theorem 2.2. *Let $f \in F(X_n, m)$. Then f_∞ is continuous on $X_n^{-\mathbb{N}}$ and $X_n^{\mathbb{Z}}$ and commutes with the shift map on $X_n^{\mathbb{Z}}$ and $X_n^{-\mathbb{N}}$.*

A continuous function from $X_n^{\mathbb{Z}}$ to itself which commutes with the shift map is called an *endomorphism* of the shift dynamical system $(X_n^{\mathbb{Z}}, \sigma_n)$. If the function is invertible, since $X_n^{\mathbb{Z}}$ is compact and Hausdorff, its inverse is continuous: it is an *automorphism* of the shift system. The sets of endomorphisms and of automorphisms are denoted by $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ respectively. Under composition, the first is a monoid, and the second a group.

Analogously, a continuous function from $X_n^{-\mathbb{N}}$ to itself which commutes with the shift map on this space is an *endomorphism of the one-sided shift* $(X_n^{-\mathbb{N}}, \sigma_n)$; if it is invertible, it is an *automorphism* of this shift system. The sets of such maps are denoted by $\mathrm{End}(X_n^{-\mathbb{N}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{-\mathbb{N}}, \sigma_n)$; again the first is a monoid and the second a group.

Note that $\sigma_n \in \mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$, whereas $\sigma_n \in \mathrm{End}(X_n^{-\mathbb{N}}, \sigma_n) \setminus \mathrm{Aut}(X_n^{-\mathbb{N}}, \sigma_n)$. More generally, the inclusions $\mathrm{End}(X_n^{-\mathbb{N}}, \sigma_n) \subseteq \mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{-\mathbb{N}}, \sigma_n) \subsetneq \mathrm{Aut}(X_n^{\mathbb{Z}}, \sigma_n)$ are valid.

Define

$$
\begin{aligned}
F_\infty(X_n) &= \bigcup_{m \geq 0} \{f_\infty : f \in F(X_n, m)\}, \\
RF_\infty(X_n) &= \bigcup_{m \geq 0} \{f_\infty : f \in F(X_n, m), f \text{ is right permutive}\}.
\end{aligned}
$$

Theorem 2.2 shows that $F_\infty(X_n) \subseteq \mathrm{End}(X_n^{\mathbb{Z}})$. In fact $F_\infty(X_n)$ and RF_∞ are submonoids of $\mathrm{End}(X_n^{\mathbb{Z}})$. For given natural numbers l and m, $f \in F(X_n, l)$ and $g \in F(X_n, m)$, the function $h \in F(X_n, l+m-1)$ defined by $(a_{-l-m+2} \dots a_{-1}a_0)h = ((a_{-l-m+2} \dots a_{-1}a_0)f_{l+m-1})g$ satisfies $h_\infty = f_\infty \circ g_\infty$. If f and g are both right permutive, then so also is h. Note that $\sigma_n \in F_\infty(X_n)$ since the function $f \in X_n^2$ defined by

$$(x_{-1}x_0)f = x_{-1},$$

satisfies $f_\infty = \sigma_n$. However σ_n^{-1} is not an element of $F_\infty(X_n)$. Now, [8, Theorem 3.4] shows:

Theorem 2.3. $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n) = \{\sigma_n^i \phi \mid i \in \mathbb{Z}, \phi \in F_\infty(X_n)\}$.

The following result is a corollary:

Theorem 2.4. *RF_∞ is a submonoid of $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ and $\mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ is the largest inverse closed subset of $RF_\infty(X_n)$.*

3 Connections to transducers

In this section we shall describe how the elements of $F_\infty(X_n)$ can be described by a certain class of finite synchronous transducers.

3.1 Automata and transducers

An *automaton*, in our context, is a triple $A = (X_A, Q_A, \pi_A)$, where

(a) X_A is a finite set called the *alphabet* of A (we assume that this has cardinality n, and identify it with X_n, for some n);

(b) Q_A is a finite set called the *set of states* of A;

(c) π_A is a function $X_A \times Q_A \to Q_A$, called the *transition function*.

We regard an automaton A as operating as follows. If it is in state q and reads symbol a (which we suppose to be written on an input tape), it moves into state $\pi_A(a, q)$ before reading the next symbol. As this suggests, we can imagine that the automaton A is in the middle of an input word, reads the next letter and moves to the right, possibly changing state in the process.

We can extend the notation as follows. For $w \in X_n^m$, let $\pi_A(w, q)$ be the final state of the automaton which reads the word w from initial state q. Thus, if $w = x_0 x_1 \ldots x_{m-1}$, then

$$\pi_A(w, q) = \pi_A(x_{m-1}, \pi_A(x_{m-2}, \ldots, \pi_A(x_0, q) \ldots)).$$

By convention, we take $\pi_A(\varepsilon, q) = q$.

For a given state $q \in Q_A$, we call the automaton A which starts in state q an *initial automaton*, denoted by A_q, and say that it is *initialised* at q.

An automaton A can be represented by a labeled directed graph, whose vertex set is Q_A; there is a directed edge labeled by $a \in X_a$ from q to r if $\pi_A(a, q) = r$.

A *transducer* is a quadruple $T = (X_T, Q_T, \pi_T, \lambda_T)$, where

(a) (X_T, Q_T, π_T) is an automaton;

(b) $\lambda_T : X_T \times Q_T \to X_T^*$ is the *output function*.

Such a transducer is an automaton which can write as well as read; after reading symbol a in state q, it writes the string $\lambda_T(a, q)$ on an output tape, and makes a transition into state $\pi_T(a, q)$. An *initial transducer* T_q is simply a transducer which starts in state q. Transducers which are *synchronous* (i.e., which always write one letter whenever they read one letter) are also known as *Mealy machines* (see [7]), although we generally will not use that language here. Transducers which are not synchronous are described as *asynchronous* when this aspect of the transducer is being highlighted. In this paper, we will only work with synchronous transducers without an initial state, and, below, **we will simply call these transducers**.

In the same manner as for automata, we can extend the notation to allow transducers to act on finite strings: we let $\pi_T(w, q)$ and $\lambda_T(w, q)$ be, respectively, the final state and the concatenation of all the outputs obtained when a transducer T reads a string w from a state q.

A transducer T can also be represented as an edge-labeled directed graph. Again the vertex set is Q_T; now, if $\pi_T(a, q) = r$, we put an edge with label $a|\lambda_T(a, q)$ from q to r. In other words, the edge label describes both the input and the output associated with that edge. We call a the *input label* of the edge and $\lambda_T(a, q)$ the *output label* of the edge.

For example, Figure 1 describes a synchronous transducer over the alphabet X_2.

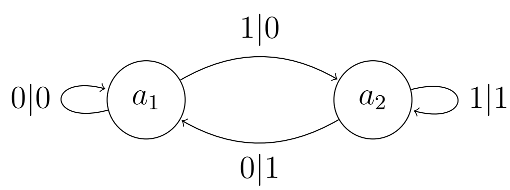


Figure 1: A transducer over X_2

We can regard an automaton, or a transducer, as acting on an infinite string from $X_n^{\mathbb{N}}$ where X_n is the alphabet. This action is given by iterating the action on a single symbol; so the output string is given by

$$\lambda_T(xw, q) = \lambda_T(x, q)\lambda_T(w, \pi_T(x, q)).$$

Thus T_q induces a map $w \mapsto \lambda_T(w, q)$ from $X_n^{\mathbb{N}}$ to itself; it is easy to see that this map is continuous. If it is a homeomorphism, then we call the state q a *homeomorphism state*. We write image(q) for the image of the map induced by T_q.

Two states q_1 and q_2 are said to be ω*-equivalent* if the transducers T_{q_1} and T_{q_2} induce the same continuous map. (This can be checked in finite time, see [7].) More generally, we say that two initial transducers T_q and $T'_{q'}$ are ω*-equivalent* if they induce the same continuous map on $X_n^{\mathbb{N}}$.

A transducer is said to be *weakly minimal* if no two states are ω-equivalent. For a synchronous transducer T, two states q_1 and q_2 are ω-equivalent if $\lambda_T(a, q_1) = \lambda_T(a, q_2)$ for any finite word $a \in X_n^*$. Moreover, if q_1 and q_2 are ω-equivalent states of a synchronous transducer, then for any finite word $a \in X_n^p$, $\pi_T(a, q_1)$ and $\pi_T(a, q_2)$ are also ω-equivalent states.

There is a stronger notion of minimality which appears in [7] and applies also to asynchronous transducer, hence our use of the adjective *weakly*.

Two weakly minimal non-initial transducers T and U are said to be ω-*equal* if there is a bijection $f : Q_T \to Q_U$, such that for any $q \in Q_T$, T_q is ω-equivalent to $U_{(q)f}$. Two weakly minimal initial transducers T_p and U_q are said to be ω-equal if there is a bijection $f : Q_T \to Q_U$, such that $(p)f = q$ and for any $t \in Q_T$, T_t is ω-equivalent to $U_{(t)f}$. We shall use the symbol '=' to represent ω-equality of initial and non-initial transducers. Two non-initial transducers are said to be ω-equivalent if they have ω-equal minimal representatives.

In the class of synchronous transducers, the ω-equivalence class of any transducer has a unique weakly minimal representative. In the general case, if one permits infinite outputs from finite inputs, Grigorchuk *et al.* [7] prove that the ω-equivalence class of an initialised transducer T_q has a unique minimal representative and give an algorithm for computing this representative.

Throughout this article, as a matter of convenience, we shall not distinguish between ω-equivalent transducers. Thus, for example, we introduce various groups as if the elements of those groups are transducers, whereas the elements of these groups are in fact ω-equivalence classes of transducers.

Given two transducers $T = (X_n, Q_T, \pi_T, \lambda_T)$ and $U = (X_n, Q_U, \pi_U, \lambda_U)$ with the same alphabet X_n, we define their product $T * U$. The intuition is that the output for T will become the input for U. Thus we take the alphabet of $T * U$ to be X_n, the set of states to be $Q_{T*U} = Q_T \times Q_U$, and define the transition and rewrite functions by the rules

$$\begin{aligned}
\pi_{T*U}(x, (p, q)) &= (\pi_T(x, p), \pi_U(\lambda_T(x, p), q)), \\
\lambda_{T*U}(x, (p, q)) &= \lambda_U(\lambda_T(x, p), q),
\end{aligned}$$

for $x \in X_n$, $p \in Q_T$ and $q \in Q_U$. Here we use the earlier convention about extending λ and π to the case when the transducer reads a finite string. If T and U are initial with initial states q and p respectively then the state (q, p) is considered the initial state of the product transducer $T * U$.

In automata theory a synchronous (not necessarily initial) transducer $T = (X_n, Q_T, \pi_T, \lambda_T)$ is *invertible* if for any state q of T, the map $\rho_q := \lambda_T(\centerdot, q) : X_n \to X_n$ is a bijection. In this case the inverse of T is the transducer T^{-1} with state set $Q_{T^{-1}} := \{q^{-1} \mid q \in Q_T\}$, transition function $\pi_{T^{-1}} : X_n \times Q_{T^{-1}} \to Q_{T^{-1}}$ defined by $(x, p^{-1}) \mapsto q^{-1}$ if and only if $\pi_T((x)\rho_p^{-1}, p) = q$, and output function $\lambda_{T^{-1}} : X_n \times Q_{T^{-1}} \to X_n$ defined by $(x, p) \mapsto (x)\rho_p^{-1}$.

One can the interpret the previous paragraph as follows: For the invertible synchronous transducer T, the inverse transducer T^{-1} is the result of switching inputs and outputs on all transitions of T. In particular, we can think of a synchronous transducer as an ordered pair of automata, each with the same structure as directed graphs. Inversion then corresponds to swapping the ordering on this ordered pair, much as we do in constructing inverses for non-zero fractions by switching the numerator and denominator in a non-zero fraction of integers. In the transducer T depicted in Figure 2 below, the input automaton corresponds to the directed graph with the input labels on the edges and the output automaton corresponds to the directed graph with the output labels on the edges. Henceforth, we will refer to the input automaton as the *domain automaton* and the output automaton as the *range automaton*.

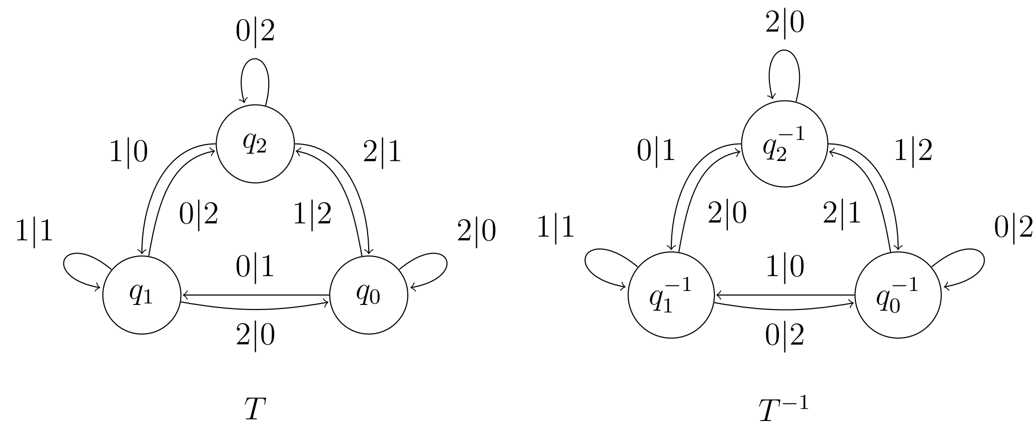


Figure 2: Inverting a synchronous transducer T.

In this article, we will come across synchronous transducers which are not invertible in the automata theoretic sense but which nevertheless induce self-homeomorphisms of the spaces $X_n^{\mathbb{Z}}$ and $X_n^{-\mathbb{N}}$. Consequently it will important to distinguish between an automaton theoretic inverse and the inverse of the induced action on the various spaces we consider.

3.2 Synchronizing automata and bisynchronizing transducers

Given a natural number k, we say that an automaton A with alphabet X_n is *synchronizing at level* k if there is a map $\mathfrak{s}_k : X_n^k \mapsto Q_A$ such that, for all q and any word $w \in X_n^k$, we have $\pi_A(w, q) = \mathfrak{s}_k(w)$. In other words, A is synchronizing at level k if, after reading a word w of length k from a state q, the final state depends only on w and not on q. (Again we use the extension of π_A to allow the reading of an input string rather than a single symbol.) We call $\mathfrak{s}_k(w)$ the state of A *forced* by w; the map $\mathfrak{s}_k$ is called the *synchronizing map at level* k. An automaton A is called *strongly synchronizing* if it is synchronizing at level k for some k.

We remark here that the notion of synchronization occurs in automata theory in considerations around the *Černý conjecture*, in a weaker sense. A word w is said to be a *reset word* for A if $\pi_A(w, q)$ is independent of q; an automaton is called *synchronizing* if it has a reset word [12, 1]. Our definition of "synchonizing at level k"/"strongly synchronizing" requires every word of length k to be a reset word for the automaton.

If the automaton A is synchronizing at level k, we define the *core* of A to be the set of states forming the image of the map $\mathfrak{s}$. It is an easy observation that, if A is synchronizing at level k, then its core is an automaton in its own right, and is also synchronizing at level k. We denote this automaton by core(A). We say that an automaton or transducer is *core* if it is equal to its core. Moreover, if T is a transducer which (regarded as an automaton) is synchronizing at level k, then the core of T (similarly denoted core(T)) induces a continuous map $f_T : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$.

Clearly, if A is synchronizing at level k, then it is synchronizing at level l for all $l \geq k$; but the map f_T is independent of the level chosen to define it.

Let T_q be an initial transducer which is invertible with inverse T_q^{-1}. If T_q is synchronizing at level k, and T_q^{-1} is synchronizing at level l, we say that T_q is *bisynchronizing* at level (k, l). If T_q is invertible and is synchronizing at level k but not bisynchronizing, we say that it is *one-way synchronizing* at level k.

For a non-initial invertible transducer T we also say T is bi-synchronizing (at level (k,l)) if both T and its inverse T^{-1} are synchronizing at levels k and l respectively.

Notation 3.1. Let T be a transducer which is synchronizing at level k and let $l \geq k$ be any natural number. Then for any word $w \in X_n^l$, we write q_w for the state $\mathfrak{s}_l(w)$, where $\mathfrak{s}_l : X_n^l \to Q_T$ is the synchronizing map at level l.

The following result was proved in Bleak *et al.* [3] .

Proposition 3.2. *Let T_q, U_p be initial transducers which (as automata) are synchronizing at levels j, k respectively, Then $T * U$ is synchronizing at level at most $j + k$.*

In what follows we give a formula specifying how strongly synchronizing transducers act by continuous functions on $X_n^{\mathbb{Z}}$. The formula induces a natural action on $X_n^{-\mathbb{N}}$ which immediately commutes with the shift. We recall that in our context the shift map σ_n is the map which sends a sequence $x \in X_n^{-\mathbb{N}} \sqcup X_n^{\mathbb{Z}}$ to the sequence $y \in X_n^{-\mathbb{N}} \sqcup X_n^{\mathbb{Z}}$ given by $y_i = x_{i-1}$ for all valid $i \in -\mathbb{N} \sqcup \mathbb{Z}$. This represents a deviation from the way the shift map conventionally operates, however, in this point of view, as we will become clear, synchronizing transducers can locally process inputs in a manner consistent with the definition given in Subsection 3.1. The formula is as follows:

Let T be a transducer which is core, and is synchronizing at level k. The map $f_T : X_n^{\mathbb{Z}} \to X_n^{\mathbb{Z}}$ maps an element $x \in X_n^{\mathbb{Z}}$ to the sequence y defined by $y_i = \lambda_T(x_i, q_{x_{i-k}x_{i-k+1}\ldots x_{i-1}})$. We also write f_T for the continuous map from $X_n^{-\mathbb{N}}$ to itself defined by $y_i = \lambda_T(x_i, q_{x_{i-k}x_{i-k+1}\ldots x_{i-1}})$ for all $i \in -\mathbb{N}$. We note that the induced map on $X_n^{-\mathbb{N}}$ is simply the restriction of the map on $X_n^{\mathbb{Z}}$ to the subsequence indexed by the negative integers.

We note that given an element $x \in X_n^{\mathbb{Z}}$ such that $(x)f_T = y$, then $y_0 y_1 \ldots = (x_0 x_1 \ldots)T_{q_{x_{-k}x_{-k+1}\ldots x_{-1}}}$. This is what was meant by the transducer T acts locally in a manner consistent with the definitions of Subsection 3.1.

Now strongly synchronizing transducers may induce endomorphisms of the shift:

Proposition 3.3. *Let T be a minimal transducer which is synchronizing at level k and which is core. Then $f_T \in \mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$ and $f_T \in \mathrm{End}\, X_n^{-\mathbb{N}}, \sigma_n$.*

Proof. It is clear from the assumptions that f_T is continuous and by definiton induces a map from $X_n^{\mathbb{Z}}$ to itself and from $X_n^{-\mathbb{N}}$ to itself. Now let $x \in X_n^{\mathbb{Z}} \sqcup X_n^{-\mathbb{N}}$ and $i \in \mathbb{Z}$ an appropriate index for x. Let $y = (x)f_T$. Observe that $y_i = \lambda(x_i, q)$, where $q = \mathfrak{s}(x_{i-k} \dots x_{i-1})$ is the state forced by $x_{i-k} \dots x_{i-1}$.

Now let $u = (x)\sigma_n$ and $v = (u)f_T$. Then

$$v_{i-1} = \lambda(u_{i-1}, q'),$$

where q' is the state of T forced by $u_{i-k-1} \dots u_{i-2}$. But by assumption, $u_{i-k-1} \dots u_{i-2} = x_{i-k} \dots x_{i-1}$, and this string forces state q; so $q' = q$, and hence $v_{i-1} = y_i$.

It now follows that $(x)f_T\sigma_n = (y)\sigma_n = v = (u)f_T = (x)\sigma_n f_T$. □

The transducer in Figure 1 induces the shift map on $X_n^{\mathbb{Z}}$. More generally, let $\mathfrak{S}_n = (X_n, Q_{\mathfrak{S}_n}, \pi_{\mathfrak{S}_n}, \lambda_{\mathfrak{S}_n})$ be the transducer defined as follows. Let $Q_{\mathfrak{S}_n} := \{0, 1, 2, \dots, n-1\}$, and let $\pi_{\mathfrak{S}_n} : X_n \times Q_{\mathfrak{S}_n} \to Q_{\mathfrak{S}_n}$ and $\lambda_{\mathfrak{S}_n} : X_n \times Q_{\mathfrak{S}_n} \to X_n$ be defined by $\pi_{\mathfrak{S}_n}(x, i) = x$ and $\lambda_{\mathfrak{S}_n}(x, i) = i$ for all $x \in X_n$ $i \in Q_{\sigma_n}$. Then $f_{\mathfrak{S}_n} = \sigma_n$.

In [3], the authors show that the set $\widetilde{\mathcal{P}_n}$ of weakly minimal finite synchronous core transducers is a monoid; the monoid operation consists of taking the product of transducers and reducing it by removing non-core states and identifying ω-equivalent states to obtain a weakly minimal and synchronous representative. Let $\mathcal{P}_n$ be the subset of $\widetilde{\mathcal{P}_n}$ consisting of transducers which induce automorphisms of the shift. (Note that these may not be minimal.) Clearly $\mathfrak{S}_n \in \mathcal{P}_n$.

3.3 De Bruijn graphs and $\mathrm{End}(X_n^{\mathbb{Z}}, \sigma_n)$

The *de Bruijn graph* $G(n, m)$ can be defined as follows, for integers $m \geq 1$ and $n \geq 2$. The vertex set is X_n^m, where X_n is the alphabet $\{0, \dots, n-1\}$ of cardinality n. There is a directed arc from $a_1 \dots a_m$ to $a_2 \dots a_m a_0$, with label a_0.

Note that, in the literature, the directed edge is from $a_0 a_1 \dots a_{m-1}$ to $a_1 \dots a_{m-1} a_m$ and the label on this edge is often given as the $(m+1)$-tuple $a_0 a_1 \dots a_{m-1} a_m$. However, to fit with the notation already defined, the equivalent definition given above is more apt.

Figure 3 shows the de Bruijn graph $G(3, 2)$.

Observe that the de Bruijn graph $G(n, m)$ describes an automaton over the alphabet X_n. Moreover, this automaton is synchronizing at level m:

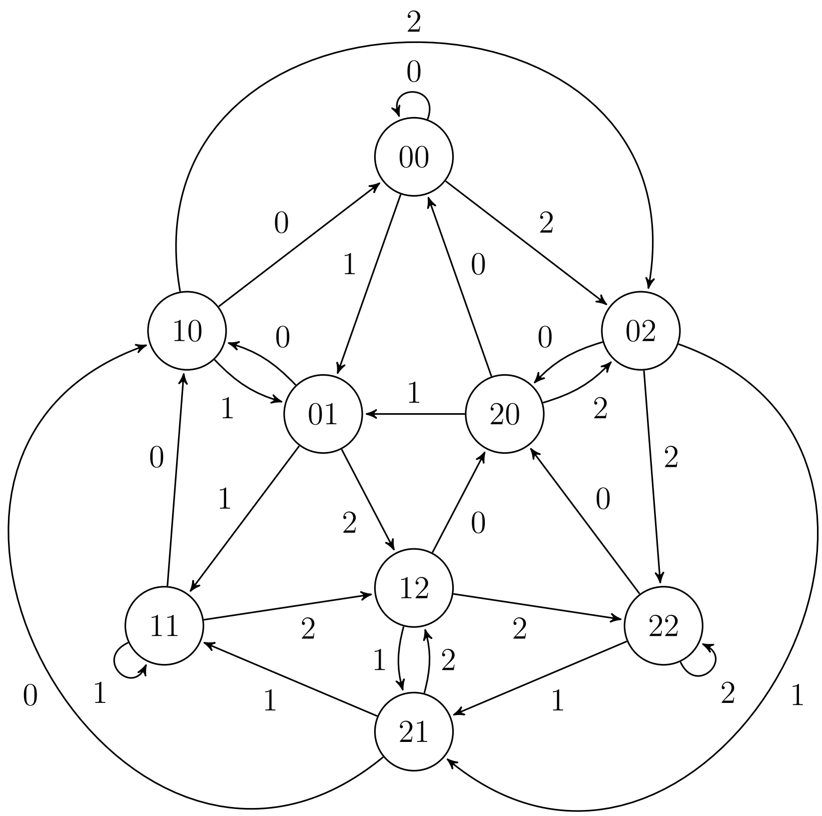


Figure 3: The de Bruijn graph $G(3, 2)$.

when it reads the string $b_0 b_1 \ldots b_{m-1}$ from any initial state, it moves into the state labeled $b_0 b_1 \ldots b_{m-1}$.

The de Bruijn graph is, in a sense we now describe, the universal automaton over X_n which is synchronizing at level m.

We define a *folding* of an automaton A over the alphabet X_n to be an equivalence relation $\equiv$ on the state set of A with the property that, if $a \equiv a'$ and $\pi_A(x, a) = b$, $\pi_A(x, a') = b'$, then $b \equiv b'$. That is, reading the same letter from equivalent states takes the automaton to equivalent states. If $\equiv$ is a folding of A, then we can uniquely define the *folded automaton* $A/{\equiv}$: the state set is the set of $\equiv$-classes of states of A; and, denoting the $\equiv$-class of a by $[a]$, we have $\pi_{A/\equiv}(x, [a]) = [\pi_A(x, a)]$ (note that this is well-defined).

Proposition 3.4. *The following are equivalent for an automaton A on the alphabet X_n:*

- *A is synchronizing at level m, and is core;*
- *A is the folded automaton from a folding of the de Bruijn graph $G(n, m)$.*

Proof. The "if" statement is clear. So suppose that A is synchronizing at level m. Define a relation $\equiv$ on the vertex set X_n^m of $G(n, m)$ by the rule that $a \equiv b$ if the states of A after reading a and b respectively are equal. (These states are independent of the initial state, by assumption.) It is readily seen that $\equiv$ is a folding of $G(n, m)$, and the $\equiv$-classes are bijective with the states of A. (The fact that A is core shows that the map which takes the state q of A to the set of $\equiv$-classes of m-tuples which bring A to state q is well-defined and injective by definition of $\equiv$, and is onto since A is core.) Moreover, this bijection is clearly an isomorphism. $\square$

Remark 3.5. An automaton A over an alphabet X_n can be regarded, in terms of universal algebra, as an algebra with unary operators ν_x for $x \in X_n$, where the elements of the algebra are the states, and $a\nu_x = \pi(x, a)$. A folding is precisely the kernel of an algebra homomorphism, and the folded automaton is isomorophic to the image of the homomorphism. The automata which are synchronizing at level m form a variety, defined by the identities

$$a\nu_{x_0}\nu_{x_1} \cdots \nu_{x_{m-1}} = b\nu_{x_0}\nu_{x_1} \cdots \nu_{x_{m-1}}$$

for all elements a, b of the algebra and all choices of $x_0, \ldots, x_{m-1}$.

We now describe how to make the de Bruijn automaton into a transducer by specifying outputs. Let $f \in F(X_n, m+1)$ be a function from X_n^{m+1} to X_n. The output function of the transducer T_f will be given by

$$\lambda_T(x, a_{m-1}a_{m-2}\dots a_0) = (a_{m-1}a_{m-2}\dots a_0 x)f.$$

In other words, if the transducer reads $m+1$ symbols, then its output is obtained by applying f to the sequence of symbols read. Note that this transducer is synchronous; it writes one symbol for each symbol read. When applied to $x \in X_n^{\mathbb{Z}}$, it produces $y = (x)f_\infty \in X_n^{\mathbb{Z}}$. Recall that the function $f \in F(X_n, 2)$ given by $(x_{-1}x_0)f = x_{-1}$ for all $x_{-1}, x_0 \in X_n$, induces the shift map σ_n on $X_n^{\mathbb{Z}}$ and $X_n^{-\mathbb{N}}$. For this map we have $T_f = \mathfrak{S}_n$.

Remark 3.6. Given any de Bruijn graph $G(n,m)$, and any transducer T with underlying directed graph $G(n,m)$ there is a function $f \in F(X_n, m+1)$ such that $T_f = T$.

Clearly the transducer T_f is synchronizing at level m. This remains true if we minimise it or identify its ω-equivalent states; so by Proposition 3.4, the resulting minimal or weakly minimal transducer is a folding of the de Brujin graph $G(n,m)$. Let $T \in \widetilde{\mathcal{P}_n}$ be the weakly-minimal representative of T_f, then $f_T = f_{T_f} = f_\infty$ holds since identifying ω-equivalent states does not affect the map f_T.

Remark 3.7. The preceding paragraph together with Remarks 2.1 and 3.6 show that there is a bijection from $F_\infty(X_n)$ to $\widetilde{\mathcal{P}_n}$. The next result demonstrates that this bijection is a monoid homomorphism.

Proposition 3.8. *Let $A, B \in \widetilde{\mathcal{P}_n}$. Then $f_A \circ f_B = f_{A*B}$.*

Proof. Let j, k be natural numbers such that A is synchronizing at level j and B is synchronizing at level k. By Proposition 3.2, $A * B$ is synchronizing at level $k + j$.

Let $x \in X_n^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ be arbitrary and $y, z, t \in X_n^{\mathbb{Z}}$ be such that $y = (x)f_A$, $z = (y)f_B$ and $t = (x)f_{A*B}$. Set $a := x_{i-j-k}\dots x_{i-1} \in X_n^{j+k}$, $b := x_{i-k}\dots x_{i-1} \in X_n^k$, $b' = x_{i-j}\dots x_{i-1} \in X_n^j$ and $c := x_{i-j-k}\dots x_{i-k-1} \in X_n^j$.

By definition of the function f_A, the block $d := y_{i-k}y_{i-k+1}\dots y_{i-1}$ of y satisfies dy_i is precisely equal to $\lambda_A(\overleftarrow{bx_i}, q_c)$. Once more, by definition, $z_i = \lambda_B(y_i, p_d)$ and since $y_i = \lambda_A(x_i, q_{b'})$, $z_i = \lambda_{A*B}(x_i, (q_{b'}, p_d))$ as well. However, the state of $A * B$ forced by a is precisely $(q_{b'}, p_d)$, and so we conclude that $t_i = \lambda_{A*B}(x_i, (q_{b'}, p_d)) = z_i$. Since i and x were arbitrarily chosen, $t = z$ and $f_A \circ f_B = f_{A*B}$. □

Corollary 3.9. *The monoid $F_\infty(X_n)$ is isomorphic to $\widetilde{\mathcal{P}_n}$.*

Let $\widetilde{\mathcal{H}_n}$ be the submonoid of $\widetilde{\mathcal{P}_n}$ consisting of those elements $P \in \mathcal{P}_n$ all of whose states are homeomorphism states. Set $\mathcal{H}_n$ to be the largest inverse closed subset of $\widetilde{\mathcal{H}_n}$ (where we take the automata theoretic inverse in this case). Observe that $\mathcal{H}_n$ is a group and by Proposition 3.8 the automata theoretic inverse of $\mathcal{H}_n$ coincides with its inverse in $\widetilde{\mathcal{P}_n}$ as a map of $X_n^{\mathbb{Z}}$. Thus, $\mathcal{H}_n$, as a set, is precisely the set of (ω-equivalence classes of) core, synchronous, invertible, bi-synchronizing transducers. It is a result in [3] that $\mathcal{H}_n$ is isomorphic to a subgroup of $\mathrm{Out}(G_{n,r})$. We further remark that right permutive maps $f \in F(X_n, m)$ give rise to transducers T_f which are elements of $\widetilde{\mathcal{H}_n}$. By Theorem 2.3 we therefore have the following corollary:

Theorem 3.10. *$RF_\infty \cong \widetilde{\mathcal{H}_n}$ and $Aut(X_n^{-\mathbb{N}}, \sigma_n) \cong \mathcal{H}_n$. Thus $Aut(X_n^{-\mathbb{N}}, \sigma_n)$ is isomorphic to a subgroup of $Out(G_{n,r})$.*

4 Automorphisms of de Bruijn graphs and $\mathcal{H}_n$

In this section we show that a finite subgroup G of $\mathcal{H}_n \cong \mathrm{Aut}(X_n^{-\mathbb{N}}, \sigma_n)$ is isomorphic to the automorphism group $\mathrm{Aut}(\Gamma)$ of the underlying directed graph Γ of an automaton A arising from a folding of a de Bruijn graph. Moreover, for any directed graph Γ underlying an automaton A arising from a folding of a de Bruijn graph, there is a subgroup G of $\mathcal{H}_n$ isomorphic to $\mathrm{Aut}(\Gamma)$. We make use of this result and results [5] to characterise the group $\mathrm{Aut}(\Gamma)$ for Γ the underlying directed graph of an automaton A arising from a folding of a de Bruijn graph. In particular we show that the automorphism group of a de Bruijn $G(n, m)$ is precisely the symmetric group on a set of size n.

4.1 Elements of $\mathcal{H}_n$ from automorphisms of directed graphs underlying folded automata

We use the connection to de Bruijn graphs to construct elements of $\mathcal{H}_n$. Recall that an automaton A may be regarded as labeled directed graph with vertex set Q_A, and edge set $E_A \subset Q_A \times X_n \times Q_A$. In this view, for vertices or states $p, q \in Q_A$, and a letter $x \in X_n$, $(p, x, q) \in E_A$ is an edge from p

to q with label x if and only if $\pi_A(x,p) = q$. Let G_A denote the unlabeled directed graph corresponding to an automaton A. We may therefore consider the automorphisms of the directed graph G_A underlying an automaton A. We construct elements of $\mathcal{H}_n$ from automorphisms of G_A where A is a folded automata arising from foldings of de Bruijn graphs. Though, we do not distinguish between an automaton and the labeled directed graph it generates, we shall distinguish between an automaton A and its unlabeled directed graph G_A.

It turns out that all elements of $\mathcal{H}_n$ arising from an automorphism of the underlying graph of a folded automaton have finite order. In the paper [5] Boyle *et al.* show that $\mathcal{H}_n$ is generated by elements of finite order, and give a generating set: the 'vertex' and 'simple' automorphisms. The elements in this generating set are in fact a subset of those elements of $\mathcal{H}_n$ constructed from automorphisms of folded automata that are considered here.

Let $G = (V, E, \iota, \tau)$ be a directed graph where V is the set of vertices of G, E is its set of edges, $\iota : E \to V$ is a map which returns the origin of an edge, and τ is a map that returns the terminus of an edge. An automorphism of G is a map $\phi := (\phi_V, \phi_E)$ such that:

(a) $\phi_V : V \to V$ is a bijection,

(b) $\phi_E : E \to E$ is a bijection, and,

(c) for an edge $e \in E$, $((e)\iota)\phi_V = ((e)\phi_E)\iota$ and $((e)\tau)\phi_V = ((e)\phi_E)\tau$.

In general usage, we shall suppress subscripts in the maps ϕ_E and ϕ_V, the arguments determining which is meant in each case. Thus for an edge e we write $(e)\phi$ for $(e)\phi_E$ and $((e)\iota)\phi$ for $((e)\iota)\phi_V$.

Let A be a folded automaton arising from a folding of a de Bruijn graph and let ϕ be an automorphism of the directed graph G_A corresponding to A. Let $H(A,\phi)$ be a transducer with state set $Q_{H(A,\phi)} := Q_A$ transition function $\pi_{H(A,\phi)} := \pi_A$ and output function $\lambda_{H(A,\phi)} : X_n \times Q_{H(A,\phi)} \to X_n$ defined as follows. For $x \in X_n$ and $p \in Q_A$, let $q = \pi_A(x,p)$ so that (p,x,q) is an edge of G_A, let (r,y,s) be the image of (p,x,q) under ϕ, noting that $(p)\phi = r$ and $(q)\phi = s$, then set $\lambda_{H(A,\phi)}(x,p) = y$.

The transducer $H(A,\phi)$ can be thought of as the result of gluing the automata A to itself along the map $\phi : Q_A \to Q_A$. That is, if $p, q \in Q_A$ and (p,x,q) is an edge from p to q with label x in A, and if y is the label of the edge $((p,x,q))\phi$ in A, then the vertex p is identified with the vertex $(p)\phi$, the vertex q with the vertex $(q)\phi$, the input label is x and the output label is the label y. Note that this fits in our view of a transducer as an ordered pair of automata where there is an isomorphism of the underlying graphs which associates to each edge of that graph a domain and range label.

We make a few observations:

(a) For each state $q \in Q_{H(A,\phi)}$, the map $\lambda_{H(A,\phi)}(\centerdot, q) : X_n \to X_n$ is a bijection. This follows from the definition of G_A: for each $x \in X_n$ there is precisely one edge of the form $((q)\phi, x, p)$ based at the vertex $(q)\phi$. It follows that the transducer $H(A, \phi)$ is invertible.

(b) If A is synchronizing at level k (and so a folding of $G(n, k)$ by Proposition 3.4) then both $H(A, \phi)$ and $H(A, \phi)^{-1}$ are synchronizing at level k hence the minimal $\overline{H(A, \phi)}$ representative of $H(A, \phi)$ is an element of $\mathcal{H}_n$.

(c) In fact, for a state $q \in Q_A$, if $W_{k,q}$ is the set of words of length k, that force the state q, i.e.,

$$W_{k,q} := \{a \in X_n^k : \pi_{H(A,\phi)}(a, q) = q\},$$

then $\{\lambda_{H(A,\phi)}(a, p) \mid a \in Q_{k,q}, p \in Q_{H(A,\phi)}\}$ is equal to $W_{k,(q)\phi}$.

(d) Let $A(H(A, \phi)) = (X_n, Q_{H(A,\phi)}, \pi_{H(A,\phi)})$ and

$$A(H(A, \phi)^{-1}) = (X_n, Q_{H(A,\phi)^{-1}}, \pi_{H(A,\phi)^{-1}})$$

be the automata corresponding to $H(A, \phi)$ and $H(A, \phi)^{-1}$ when outputs are ignored. By construction $A(H(A, \phi)) = A$, and the previous two points indicate that $A(H(A, \phi)^{-1})$ is also isomorphic as an automaton to A (by the map sending a state q^{-1} of $H(A, \phi)^{-1}$ to the state $(q)\phi$ of A).

The third point above and results of the paper [10] show that an element of $\mathcal{H}_n$ obtained from an automorphism of a folded automaton must have finite order. This result, which also follows from Theorem 4.2 below, means that not all elements of $\mathcal{H}_n$ for $n \geq 3$ arise from automorphisms of the directed graph underlying some folded automaton.

4.2 Automorphisms of folded automata and permutations of the alphabet.

Consider the de Bruijn graph $G(n, m)$. Any permutation ρ of the set X_n induces an automorphism, which we again denote by ρ, of $G(n, m)$ as follows. A vertex $a = a_1 a_2 \ldots a_n$ of the graph $G(n, m)$ is mapped to the vertex $b = (a_1)\rho(a_2)\rho \ldots (a_n)\rho := (a)\rho$. An edge $e = (a, x, b)$ is mapped to the edge $((a)\rho, (x)\rho, (b)\rho)$. In this case, the transducer $H := H(G(n, m), \rho)$ arising from the pair $(G(n, m), \rho)$ has the property that for any state $q \in Q_H$, the bijection $\lambda_H(\centerdot, q) : X_n \to X_n$ is the permutation ρ. Therefore, the minimal transducer $\overline{H}$ representing H has exactly one state, and this state induces the permutation ρ on the alphabet X_n. We show below that these are the only automorphisms of the automaton $G(n, m)$.

Let A be a folded automaton arising from a folding of $G(n, m)$ and let ρ, as above, be a permutation of X_n. By the definition of a folding the individual states of A correspond to subsets of the vertices of $G(n, m)$ and the set of states of A forms a partition of the vertices of $G(n, m)$. As vertices of $G(n, m)$ are words of length m in X_n, we may define a map ϕ_{V_A} on the vertices of G_A to the set of subsets of X_n^m, by mapping a vertex q to the set $\{(a)\rho \mid a \in X_n^m \cap q\}$. If the image of ϕ_{V_A} in the set of subsets of X_n^m is again precisely the partition V_A, then we may define an edge map $\phi_{E_A} : E_A \to E_A$ by mapping an edge (a, x, b) to the edge $((a)\rho, (x)\rho, (b)\rho)$ and this will be well defined for the folding A by the definition of a folding. In this case, the map (ϕ_{V_A}, ϕ_{E_A}) is an automorphism of G_A which we once again denote by ρ.

The example below indicates that, in general, not all automorphisms of the directed graph underlying a folded automaton arise from a permutation of the symbol set.

The automorphism group of the underlying directed graph of the automaton A in Figure 4 is the group S_3 as all three vertices may be permuted and any permutation of the three vertices forces a bijection on the edges. The automaton A is a folded automaton arising from a folding of $G(3, 2)$; the vertex q_0 corresponds to the set $\{00, 21, 10\}$, the vertex q_1 corresponds to the set $\{01, 11, 20\}$ and the vertex q_2 to the set $\{02, 12, 22\}$. The automorphism ϕ which swaps the vertex q_0 with q_2 but fixes the vertex q_1 is not induced by a permutation of the set X_3. (If ϕ were induced by a permutation ρ of X_n, then $\{(00)\rho, (21)\rho, (10)\rho\} = \{02, 12, 22\}$ and $\{(01)\rho, (11)\rho, (20)\rho\} = \{01, 11, 20\}$, which is not possible.)

The result below characterises when an automorphism of the directed

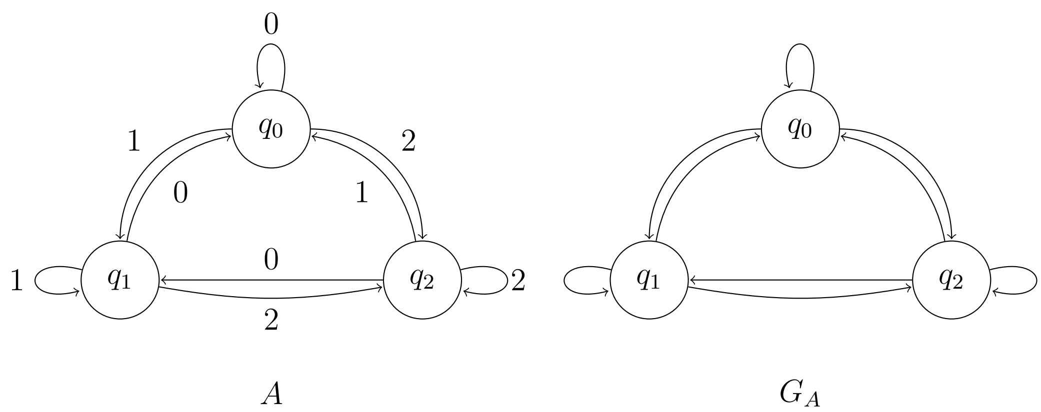


Figure 4: A folded automaton with an automorphism not induced by a permutation.

graph of a folded automaton is induced by a permutation of the alphabet set X_n.

Proposition 4.1. *Let A be a folded automaton arising from a folding of $G(n,m)$. An automorphism ϕ of the graph G_A arises from a permutation ρ of the set X_n if and only if the minimal representative of the transducer $H := H(A,\phi)$ has exactly one state, and this state induces the permutation ρ on X_n.*

Proof. If the automorphism ϕ arises from a permutation ρ of X_n, then as in the $G(n,m)$ case, all state of the transducer H induce the permutation ρ on the set X_n. Therefore, the minimal representative $\overline{H}$ of H has exactly one state, and this state induces the permutation ρ on X_n.

Therefore, suppose that the minimal representative $\overline{H}$ of the transducer $H = H(A,\phi)$ has exactly one state, and this state induces the permutation ρ on X_n. It must be the case that all states of H induce the permutation ρ on X_n. It follows that for an edge $e = (p,x,q)$ of G_A, $(e)\phi = ((p)\phi, (x)\rho, (q)\phi)$. Let q be state of H, then as, by definition, q is a state of A q corresponds to a subset of X_n^m. In particular, q corresponds to the subset $W_{m,q}$ of X_n^m consisting of all elements of X_n^m which force the state q when read from any state of A. Now as all states of H induce the permutation ρ on X_n, it follows that the state q^{-1} of the automaton H^{-1} corresponds to the subset

$\{(a)\rho \mid a \in W_{m,q}\}$. Therefore as $(q)\phi = q^{-1}$, we see that ϕ must arise from the permutation ρ. $\square$

Returning to the automaton A in Figure 4, the automorphism ϕ of G_A which swaps the vertices q_0 and q_1 yields the automaton $(A)\phi$ and the transducer $H(A, \phi)$ depicted in Figure 5. The transducer $H(A, \phi)$ is minimal.

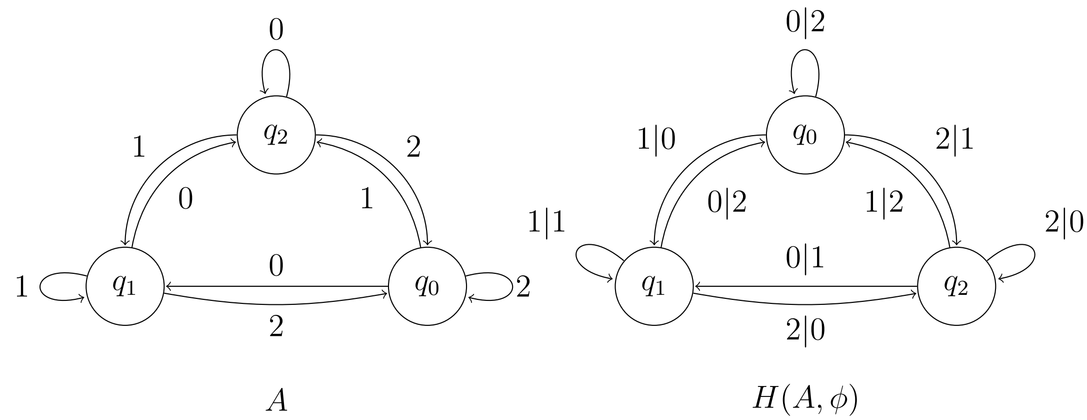


Figure 5: The transducer arising from the automorphism swapping vertices q_0 and q_1.

Theorem 4.2. *Let A be a folded automaton arising from a folding of $G(n, m)$ for m minimal. The map from the group* $\mathrm{Aut}(G_A)$ *of automorphisms of the directed graph G_A to $\mathcal{H}_n$ which maps an automorphism ϕ to the minimal representative of the transducer $H(A, \phi)$, is a monomorphism.*

Proof. If $|A| = 1$ then the result is a consequence of Proposition 4.10. Thus we may assume that $|A| > 1$.

Let ϕ be a non-trivial automorphism of G_A. Then as ϕ is not trivial, either it moves some state or fixes every state and move some edges.

Suppose firstly that ϕ moves some state. Let $p, q \in Q_A$ be distinct states that $(p)\phi = q$. Since, A is a folding of $G(n, m)$, p and q correspond to distinct subsets of X_n^m consisting of all words $W_{m,p}$ and $W_{m,q}$ that force the states p and q respectively. Now, by an observation above, the state p of $H(A, \phi)$ is such that $\lambda_{H(A,\phi)}(\centerdot, p)$ induces a bijection from $W_{m,p}$ to $W_{m,q}$. Therefore, we see that $H(A, \phi)$ is not the identity transducer.

In the case that ϕ fixes every state and moves some edges, let $e = (p, x, q)$ be an edge move by ϕ. Since ϕ fixes all vertices, there must be an edge (p, y, q) from p to q, for $x \neq y$ such that $((p, x, q))\phi = (p, y, q)$. In this case, we have that the state p of $H(A, \phi)$ satisfies $\lambda_A(x, p) = y$. We once again conclude that $H(A, \phi)$ is not the identity transducer.

Therefore, the only element of $\mathrm{Aut}(G_A)$ that maps to the identity transducer, is the identity element. This means that it suffices to show that the map from $\mathrm{Aut}(G_A) \to \mathcal{H}_n$ which sends an automorphism ϕ to the minimal representative of $H(A, \phi)$ is a homomorphism to conclude that it is a monomorphism.

Let ϕ, ψ be two automorphisms of G_A and let $H(A, \phi)$ and $H(A, \psi)$ be the corresponding transducers. Notice that the trio $H(A, \phi)$, $H(A, \psi)$ and $H(A, \phi\psi)$, all by definition, have state set Q_A. This should not cause confusion below, as whenever we write a pair (p, q) $H(A, \phi) * H(A, \psi)$, the first coordinate corresponds to the state of $H(A, \phi)$ and the second to the state of $H(A, \psi)$ and for a single state $p \in Q_A$ it will be clear below which of the three transducers $H(A, \phi)$, $H(A, \psi)$ and $H(A, \phi\psi)$ it is being regarded as a state of. On the other hand, the set $W_{m,q}$ for a state $q \in Q_A$, depends only on the automaton A. That is the set of words in X_n^m which force the state q in $H(A, \phi)$, $H(A, \psi)$ or $H(A, \phi\psi)$ are all equal to $W_{m,q}$.

A state (p, q) of the product $H(A, \phi) * H(A, \psi)$ is a state of the core if and only if $\{a \in X_n^m \mid a = \lambda_{H(A,\phi)}(b, q) \text{ for some } b \in W_{m,p}, q \in Q_A\} = W_{m,q}$. This is because, by an observation above,

$$\{a \in X_n^m \mid a = \lambda_{H(A,\phi)}(b, q) \text{ for some } b \in W_{m,p}, q \in Q_A\} = W_{m,(p)\phi}$$

and this set depends only on A. Thus a state (p, q) is a state of the core$(H(A, \phi) * H(A, \psi))$ if and only if it is of the form $(p, (p)\phi)$.

Let (p, x, q) be an edge of G_A, $((p)\phi, y, (q)\phi)$ be its image under ϕ and $((p)\phi\psi, z, (q)\phi\psi)$ its image under $\phi\psi$. This means that the state p of $H(A, \phi)$ satisfies, $\lambda_{H(A,\phi)}(x, p) = y$ and $\pi_{H(A,\phi)}(x, p) = q$. The state $(p)\phi$ of $H(A, \psi)$ satisfies, $\lambda_{H(A,\psi)}(y, (p)\phi) = z$ and $\pi_{H(A,\psi)}(y, (p)\phi) = (q)\phi$. Thus

$$\lambda_{H(A,\phi\psi)}(x, (p, (p)\phi)) = z$$

and

$$\pi_{H(A,\phi\psi)}(x, (p, (p)\phi)) = (q, (q)\phi).$$

The above calculation demonstrates that the map from $H(A, \phi\psi)$ to core$(H(A, \phi) * H(A, \psi))$ which sends a state p of $H(A, \phi\psi)$ to the state

$(p,(p)\phi)$ of $\mathrm{core}(H(A,\phi)*H(A,\psi))$ is an automaton isomorphism. This concludes the proof. $\square$

4.3 Finite subgroups of $\mathcal{H}_n$

We observe that a converse of Theorem 4.2 is valid, namely, every finite subgroup of $\mathcal{H}_n \cong \mathrm{Aut}(X_n^{\mathbb{N}}, \sigma_n)$ arises from the automorphism group of a folded de Bruijn graph. This follows from work in the paper [5], however we give a proof below.

The proof we give below is more automata theoretic and is based on the following result from [10].

Proposition 4.3. *Let $G \leq \mathcal{H}_n$ be a finite subgroup. Let $k \in \mathbb{N}$ the largest minimal synchronizing level of any element of G. Then for for any $H \in G$, and for any word $\Gamma \in X_n^k$, there is a word $W(\Gamma, H) \in Q_H^+$ such that for any word $P \in Q_H^+$, $\lambda_H(\Gamma, P) = W(\Gamma,H)^i\overline{W(\Gamma,H)}_r$, where, $i \in \mathbb{N}$, satisfies, $|P| = i|W(\Gamma,H)| + r$, for $1 \leq r < |W(\Gamma,H)|$ and $\overline{W(\Gamma,H)}_r$ is the length r prefix of $W(\Gamma,H)$.*

Theorem 4.4. *Let $G \leq \mathcal{H}_n$ be a finite subgroup, then G is isomorphic to a subgroup of the automorphism group of the underlying digraph of a strongly synchronizing automaton $A(G)$. Moreover, every element of G is the minimal representative of a transducer $H(A(G), \phi)$ for an automorphism ϕ of the underlying di-graph of $A(G)$.*

Proof. Let $k \in \mathbb{N}$ be the such that any element of G has minimal synchronizing level at most k. Define an equivalence relation $\sim$ on X_n^k as follows: $\Gamma \sim \Delta$ if and only if $W(\Gamma,H) = W(\Delta,H)$ for all $H \in G$.

Observe that, for $\Gamma = a\gamma$ and $\Delta = d\delta$, for $a, d \in X_n$, in the same equivalence class, then for $x \in X_n$, γx and δx are also in the same equivalence class. This is because for any $H \in G$ and any word $P \in Q_H^+$ we have, $\lambda_{H^{|P|}}(a\gamma, P) = \lambda_{H^{|P|}}(a\delta, P)$, and so $\lambda_{H^{|P|}}(a\gamma x, P) = \lambda_{H^{|P|}}(a\delta x, P)$. From this we deduce that $W(\gamma x, H) = W(\delta x, H)$.

Thus, writing $[\gamma]$ for the equivalence class of an element γ of X_n^k, we may form an automaton $A(G)$ with state set $X_n^k/\sim$, and transitions $\pi_{A(G)}(x, [\gamma]) = [\overline{\gamma}x]$ where $\overline{\gamma}$ is the length $|\gamma| - 1$ suffix of γ. By the previous a paragraph the automaton $A(G)$ is well defined; by construction the automaton $A(G)$ is strongly synchronizing.

We now show that G acts by automorphisms on the underlying digraph of $A(G)$.

We begin by proving the following observation. Let $\gamma, \delta \in X_n^k$ belong to the same equivalence class, and let $H \in G$ be arbitrary. Then for any $p, q \in Q_H$, the elements of the set $\{\lambda_H(\xi, t) \mid (\xi, t) \in \{(\gamma, p), (\delta, q)\}\}$ belong to the same equivalence class.

First observe that by Proposition 4.3, there is a word $W_H \in Q_H^+$ such that $W_H = W(\lambda_H(\xi, t), H)$ for all $(\xi, t) \in \{(\gamma, p), (\delta, q)\}$. Since γ and δ are in the same equivalence class, let s_0 be the state of H forced by both γ and δ. Let $I \in G$, $I \neq H$ be arbitrary, we show that there is a word $W_I \in Q_I^+$ such that $W_I = W(\lambda_H(\xi, t), I)$ for all $\xi \in \{\gamma, \delta\}$ and all $t \in \{p, q\}$. We prove this inductively.

Let us establish the base case. Observe that since $HI \in G$ and since γ and δ are in the same equivalence class, there is a unique state, s_1 of HI such that for any state $s \in Q_{HI}$, the state of HI forced by $\lambda_{HI}(\gamma, s)$ and $\lambda_{HI}(\delta, s)$ are equal and are equal to s_1. Notice that HI is the minimal representative of $\mathrm{core}(H * I)$. There are state $s, s' \in I$ such that $(p, s), (q, s')$ are states of $\mathrm{core}(H * I)$; let $t, t' \in Q_I$ be such that $\pi_{H*I}(\gamma, (p, s)) = (s_0, t)$ and $\pi_{H*I}(\delta, (q, s')) = (s_0, t')$. Since the state of HI forced by γ and δ is s_1, we have (s_0, t) and (s_0, t') are ω-equivalent to the state s_1, and so $t = t'$. Set $t_1 = t = t'$. Therefore we have shown that the state of I forced by $\lambda_H(\gamma, p)$ is equal to the state of I forced by $\lambda_H(\delta, q)$ and that state is t_1.

Inductively assume that there is an $m \in \mathbb{N}$ such that for any word $u \in Q_I^+$ of length m, $\pi_{I^m}(\lambda_H(\gamma, p), u) = \pi_{I^m}(\lambda_H(\delta, q), u) = t_1 t_2 \ldots t_m$. We now prove the inductive step.

As before, HI^{m+1} is an element of G and, as γ and δ are in the same equivalence class, they both force the same state s_{m+1} of HI^{m+1}. There are words $s, s' \in Q_I^{m+1}$ such that ps and qs' are states of $\mathrm{core}(H * \underbrace{I * I \ldots * I}_{m+1 \text{ times}})$. Since HI^{m+1} is the minimal representative of $\mathrm{core}(H * \underbrace{I * I \ldots * I}_{m+1 \text{ times}})$, it follows that, if $T_{m+1}, T'_{m+1} \in Q_I^{m+1}$ satisfy, $\pi_{H*I}(\gamma, ps) = s_0 T_{m+1}$ and $\pi_{H*I}(\delta, qs')) = s_0 T'_{m+1}$, then $s_0 T_{m+1}$ and $s_0 T'_{m+1}$ are both ω-equivalent to the state s_{m+1} of HI^{m+1}. By the inductive assumption, we have that that the first m letters of T_{m+1} and T'_{m+1} coincide, the preceding sentence now implies that $T_{m+1} = T'_{m+1}$. Set t_{m+1} to the final letter of $T_{m+1} = T'_{m+1}$. By Proposition 4.3 it now follows that for any word for any word $u \in Q_I^+$ of length $m+1$, $\pi_{I^{m+1}}(\lambda_H(\gamma, p), u) = \pi_{I^{m+1}}(\lambda_H(\delta, q), u) = t_1 t_2 \ldots t_m t_{m+1}$. We therefore conclude that there is a word $W_I \in Q_I^+$ such that $W_I = W(\lambda_H(\xi, t), I)$ for all $\xi \in \{\gamma, \delta\}$ and all $t \in \{p, q\}$.

Since $I \in G$, $I \neq H$, was chosen arbitrarily, it follows that $\lambda_H(\gamma, p)$ and $\lambda_H(\delta, q)$ are in the same equivalence class.

Let $[\gamma]$ be a vertex of $A(G)$, let $\overline{\gamma}$ be the length $k-1$ suffix of γ and let $x \in X_n$ be the label of the edge from $[\gamma]$ to $[\overline{\gamma}x]$. Let $H \in G$ be arbitrary and let $y = \lambda_H(x, q_\gamma)$, then by the preceding paragraph for any pair of states $p, q \in Q_H$ and any $I \in G$, $W(\lambda_H(\gamma, p)y, I) = W(\lambda_H(\gamma, q)y, I)$. From this it follows that setting μ, ν to be the length $k-1$ suffices of $\lambda_H(\gamma, p)$ and $\lambda_H(\gamma, q)$ respectively, $[\mu y] = [\nu y]$. Now as there is a state s of H such that $\lambda_H(\overline{\gamma}x, s) = \mu y$, it follows, by the preceding paragraphs once more, that for any state $t \in Q_H$, $[\lambda_H(\overline{\gamma}x, t)] = [\mu y]$. Since μ is a length $k-1$ suffix of an element of $[\lambda_H(\gamma, p)]$, there is an edge labeled y from $[\lambda_H(\gamma, p)]$ to $[\mu y]$.

For $H \in G$, define a map ϕ_H as follows. For a vertex $[\gamma]$, and edge labeled x from $[\gamma]$ to $[\overline{\gamma}x]$ of the digraph $A(G)$ (where $\overline{\gamma}$ is the length $k-1$ suffix of γ) of $A(G)$, $([\gamma])\phi_H = [\lambda_H(\gamma, p)]$, $([\overline{\gamma}x)\phi_H = [\lambda_H(\overline{\gamma}x, p)]$, for some state $p \in Q_H$, and the edge x maps to the edge labeled $\lambda_H(x, q_\gamma)$ from the state $[\lambda_H(\gamma, p)]$ to the state $[\lambda_H(\overline{\gamma}x, p)]$. By the preceding a paragraphs this map is well defined. It is easily verified that for $H, I \in G$, $\phi_{HI} = \phi_H \phi_I$. Thus the map $H \mapsto \phi_H$ is an embedding of G into the automorphism group of the underlying digraph of $A(G)$. Moreover, it is not hard to see that the minimal representative of the tranducer $H(A(G), \phi_H)$ is H. $\square$

In light of Theorem 4.2 above, Theorem 3.8 of [5] can be states as follows:

Corollary 4.5. *Let A be a folded automaton arising from a folding of $G(n, m)$ for m minimal. For the group $\mathrm{Aut}(G_A)$ of automorphisms of the directed graph G_A, one of the following holds:*

(i) $\mathrm{Aut}(G_A)$ *isomorphic to a subgroup of* $\mathrm{Sym}(X_n)$ *that has a composition factor that cannot be embedded in* $\mathrm{Sym}(X_{n-1})$*. In this case all automorphisms of G_A arise as permutations of the symbol set X_n.*

(ii) All the composition factors of $\mathrm{Aut}(G_A)$ *are isomorphic to subgroups of* $\mathrm{Sym}(X_{n-1})$.

Corollary 4.6. *Let A be a folded automaton arising from a folding of $G(3, m)$ for some $m \in \mathbb{N}$. The group* $\mathrm{Aut}(G_A)$ *is either* $\mathrm{Sym}(X_3)$ *or a 2-group.*

It is a result of Hedlund [8] that $\mathrm{Aut}(X_2, \sigma_2)$ is isomorphic to the cyclic group of order 2. Below we give a new proof of this result by identifying conditions on (non-minimal) strongly synchronizing transducers to have a minimal representative in $\mathcal{H}_n$ with exactly one state. From this we also derive implications (via Proposition 4.1) for folded automata: more precisely we show that certain folded automata, including the graphs $G(n,m)$, admit only automorphisms arising from permutations of the symbol set X_n.

4.4 Synchronizing sequences

We require an algorithm given in [3] for detecting when an automaton is strongly synchronizing. We state a version below.

Let $A = (X_n, Q_A, \pi_A)$ be an automaton. Define an equivalence relation $\sim_A$ on the states of A by $p \sim_A q$ if and only if the maps $\pi_A(\cdot, p) : Q_A \to Q_A$ and $\pi_A(\cdot, q) : Q_A \to Q_A$ are equal. For a state $q \in Q_A$ let q represent the equivalence class of q under $\sim_A$. Further set $\mathsf{Q_A} := \{\mathsf{q} \mid q \in Q_A\}$ and let $\pi_{\mathsf{A}} : \mathsf{Q_A} \to \mathsf{Q_A}$ be defined by $\pi_{\mathsf{A}}(x, \mathsf{q}) = \mathsf{p}$ where $p = \pi_A(x, q)$. Observe that π_{A} is a well defined map. Define a new automaton $\mathsf{A} = (X_n, \mathsf{Q_A}, \pi_{\mathsf{A}})$ noting that $|\mathsf{Q_A}| \le |Q_A|$ and $|\mathsf{Q_A}| = |Q_A|$ implies that A is isomorphic to A.

Given an automaton A, let $A_0 := A, A_1, A_2, \ldots$ be the sequence of automata such that $A_i = \mathsf{A}_{i-1}$ for all $i \ge 1$. We call the sequence $(A_i)_{i \in \mathbb{N}}$ the *synchronizing sequence of* A. We make a few observations.

By definition each term in the synchronizing sequence is a folding of the automaton which precedes it, therefore there is a $j \in \mathbb{N}$ such that all the A_i for $i \ge j$ are isomorphic to one another. By a simple induction argument, for each i, the states of A_i corresponds to a partition of Q_A. We identify the states of A_i with this partition. For two states $q, p \in Q_A$ that belong to a state P of A_i, $\pi_A(x, q)$ and $\pi_A(x, p)$ belong to the same state of Q_A for all $x \in X_n$. We will use the language 'two states of A are identified at level i' if the two named states belong to the same element of Q_{A_i}.

If the automaton A is strongly synchronizing and core, then an easy induction argument shows that all the terms in its synchronizing sequence are core and strongly synchronizing as well (since they are all foldings of A). For example if $A = G(n,m)$, then the first m terms of the synchronizing sequence of A are $(G(n,m), G(n,m-1), G(n,m-2), \ldots, G(n,1)$, after this all the terms in the sequence are the single state automaton on X_n.

The result below is from [3].

Theorem 4.7. *Let A be an automaton and $A_0 := A, A_1, A_2, \ldots$ be the sequence of automata such that $A_i = \mathsf{A}_{i-1}$ for all $i > 1$. Then*

(a) a pair of states $p, q \in Q_A$, belong to the same element $t \in Q_{A_i}$ if and only if for all words $a \in X_n^i$, $\pi_A(a, p) = \pi_A(a, q)$, and

(b) A is strongly synchronizing if and only if there is a $j \in \mathbb{N}$ such that $|Q_{A_j}| = 1$. The minimal j for which $|A_j| = 1$ is the minimal synchronizing level of A.

4.5 Applying synchronizing sequences to understand automorphisms of de Bruijn graphs

Lemma 4.8. *Let A be a core strongly synchronizing automaton, $A_0 := A, A_1, \ldots$ be its synchronizing sequence and $j \in \mathbb{N}$ be minimal such that $A_j = 1$. If A_{j-1} is isomorphic as an automaton to $G(n, 1)$ then the sets $Q_{A,x} := \{\pi_A(x, p) \mid p \in Q_A\}$ for $x \in X_n$ form a partition of the set Q_A the states of A.*

Proof. This follows from the identification of the states of A_i with partitions of states of A. For if there were distinct $x, y \in X_n$ and states $p_1, p_2 \in Q_A$ such that $\pi_A(x, p_1) = \pi_A(y, p_2)$, then the states P_1 and P_2 of A_{j-1} containing p_1 and p_2 respectively satisfy, $\pi_{A_i}(x, P_1) = \pi_{A_i}(y, P_2)$. However, since A_{j-1} is isomorphic as an automaton to $G(n, 1)$ this is not possible (A_{j-1} has n distinct states, is synchronizing at level 1 and core). $\square$

A consequence of Lemma 4.8 is the following result.

Lemma 4.9. *There is no minimal, core, invertible transducer T which is bi-synchronizing at minimal level (j, k) and satisfies the following: if A and B are the automata obtained from T and T^{-1} respectively by forgetting outputs, then the terms A_{j-1} and B_{k-1} in the synchronizing sequences of A and B are isomorphic to $G(n, 1)$.*

Proof. Since T is minimal and strongly synchronizing, there is a pair $p, q \in Q_T$ and $x \in X_n$ such that $\pi_T(x, p = \pi_T(x, q)$ but $y := \lambda_T(x, p) \neq \lambda_T(x, q) =: z$. However, we therefore have that in T^{-1}, and so in B, $\pi_{T^{-1}}(y, p^{-1}) = \pi_T(z, q^{-1})$ with $z \neq q$. This contradicts Lemma 4.8. $\square$

We note the lack of the minimality hypothesis in the statement of the proposition below. We require the non-minimality hypothesis in order to deduce results about elements of $\mathcal{H}_n$ arising from automorphisms of de Bruijn graphs $G(n, m)$. In particular as a consequence of the following Proposition, we show that $\mathrm{Aut}(G_{n,m})$ is isomorphic to the symmetric group on n symbols.

Proposition 4.10. *Let T be a core, invertible bi-synchronizing transducer of size at least 2 with automata theoretic inverse T^{-1}. Let (j, k) be the minimal bi-synchronizing level of T, and let A and B be the automata obtained from T and T^{-1} respectively by forgetting outputs. Suppose that the terms A_{j-1} and B_{k-1} in the synchronizing sequence $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ of A and B are both isomorphic, as automata, to $G(n, 1)$. Then $j = k$ and the minimal transducer representing T has only one state.*

Proof. We proceed by induction on the number of states of T.

Note that as $j, k \geq 1$, it follows that the base case occurs when $|T| = n$. In this case, both A and B are isomorphic to $G(n, 1)$ and $j = k = 1$. If all the states of T induce the same permutation ϕ on the set X_n, then the minimal transducer representing T has one state, and that state also induces the permutation ρ on X_n. Therefore, suppose there are two states $p, q \in Q_T$ and $x \in X_n$ such that $t := \lambda_T(x, p) \neq \lambda_T(x, q) =: z$. Since $\pi_T(x, p) = \pi_T(x, q)$, it follows that in T^{-1}, the state p^{-1}, q^{-1} satisfy, $\pi_{T^{-1}}(t, p^{-1}) = \pi_{T^{-1}}(z, q^{-1})$. This yields the desired contradiction by Lemma 4.8, since B is isomorphic to $G(n, 1)$.

Now suppose the conclusion of the proposition holds for all transducers T with $n \leq |T| < m$ and which satisfy the hypothesis of the proposition.

Let T be a transducer with size $|T| = m$ satisfying the hypothesis. Let (j, k) be the minimal bi-synchronizing level of T. Since $|T| > n$, it follows that both j and k are strictly greater than 1. As, because T is core, if j or k were 1, T or T^{-1} would be a folding of $G(n, 1)$ and so, T and T^{-1} would have size less than or equal to n.

Let A and B the automata obtained from T and T^{-1} respectively by forgetting outputs and let $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ be their respective synchronizing sequences.

Let $p, q \in Q_T$ be any pair of states satisfying $\pi_T(x, p) = \pi_T(x, q)$ for all $x \in X_n$. Then, by the argument given in the base case, we must also have $\lambda_T(x, p) = \lambda_T(x, q)$ for all $x \in X_n$, otherwise we obtain the contradiction that T does not satisfy the hypothesis of the proposition. By the same argument, if $p^{-1}, q^{-1} \in Q_{T^{-1}}$ are any pair of states satisfying $\pi_{T^{-1}}(x, p^{-1}) = \pi_{T^{-1}}(x, q^{-1})$ for all $x \in X_n$, then $\lambda_{T^{-1}}(x, p^{-1}) = \lambda_{T^{-1}}(x, q^{-1})$ for all $x \in X_n$ as well.

Let $\sim$ be the equivalence relation on the states of T given by $p \sim q$ if $\pi_T(x, p) = \pi_T(x, q)$ for all $x \in X_n$. By an abuse of notation we also use $\sim$ for the same equivalence relation on the states of T^{-1}. For $q \in Q_T$, let q be its equivalence class and let $\mathsf{Q}_T := \{\mathsf{q} \mid q \in Q_T\}$. Notice that, by the preceding paragraph, for states $p, q \in Q_T$, $p \sim q$ if and only if $\pi_T(\centerdot, p) = \pi_T(\centerdot, q)$ and $\lambda_T(\centerdot, p) = \lambda_T(\centerdot, q)$ if and only if $p^{-1} \sim q^{-1}$. Moreover, by hypothesis, $\sim$ is not the trivial equivalence relation i.e its equivalence classes do not all consist of singleton sets and it also does not consist of one equivalence class.

Form a new transducer T as follows. Let $Q_{\mathsf{T}} := \mathsf{Q}_T$. Define the transition function $\pi_{\mathsf{T}} : X_n \times Q_{\mathsf{T}} \to Q_{\mathsf{T}}$ by $\pi_{\mathsf{T}}(x, \mathsf{q}) = \mathsf{p}$ where $p = \pi_T(x, q)$ for some $q \in \mathsf{q}$. The output function $\lambda_{\mathsf{T}} : X_n \times Q_{\mathsf{T}} \to X_n$ is defined by $\lambda_{\mathsf{T}}(x, \mathsf{q}) = \lambda_T(x, q)$ for some $q \in \mathsf{q}$. The preceding paragraph implies that T is well-defined.

Observe, that if C is the automaton obtained from T by forgetting outputs and D is the automaton obtained from T^{-1} by forgetting outputs, then C is isomorphic to A_1 and D is isomorphic to B_1, by definition of the synchronizing sequence. This means that the minimal bi-synchronizing level of T is $(j-1, k-1)$. Moreover, as $k-1$ and $j-1$ are at least 1, in the synchronizing sequence of C and D, the terms C_{k-2} and D_{k-2} are isomorphic to $G(n, 1)$. This means that T satisfies the hypothesis of the proposition. Furthermore, as $\sim$ is not the trivial relation, we have $|\mathsf{T}| < |T|$. Thus, we conclude that the minimal transducer representing T has only one state and $j - 1 = k - 1$. However, by construction of T, the minimal transducer representing T is also the minimal transducer representing T. This concludes the proof. $\square$

We have some corollaries of the result above.

Corollary 4.11. *Let A be a folded automaton arising from a folding of $G(n, m)$. If an element of the synchronizing sequence of A is isomorphic to $G(n, 1)$, then any automorphism of G_A is induced by a permutation of the symbol set X_n.*

Proof. Let ϕ be any automorphism of G_A, and let $H := H(A, \phi)$. Let $A(H)$ and $A(H^{-1})$ be the automata obtained from H and H^{-1} by forgetting outputs. Note that since $A(H)$ and $A(H^{-1})$ are isomorphic as automata to A, it follows that H satisfies the hypothesis of Proposition 4.10. This means that the minimal representative of H has exactly one state. Proposition 4.1 now implies that ϕ is induced by a permutation of the symbol set X_n. $\square$

Corollary 4.12. *Let A be the de Bruijn automaton $G(n, m)$. Then* $\mathrm{Aut}(G_A)$ *is isomorphic to the symmetric group on n points.*

Proof. $G(n, m)$ is clearly a folding of itself, thus Corollary 4.11 implies that the automorphism group of its underlying directed graph is isomorphic to a subgroup of the symmetric group on n points. However, we have seen above that any permutation of X_n induces an automorphism of $G(n, m)$. $\square$

The corollaries of Proposition 4.10 below require the following straightforward lemma.

Lemma 4.13. *Let A be any strongly synchronizing, core automaton over the 2-letter alphabet. Let $(A_i)_{i\in\mathbb{N}}$ be the synchronizing sequence of A. if $|A| > 1$, then the minimal synchronizing level k of A is at least 1 and A_{k-1} is isomorphic to $G(2, 1)$.*

Proof. If $|A| > 1$ then it is not the single state automaton (which is the only automaton strongly synchronizing at level 0). Thus let $k \geq 1$ be the minimal synchronizing level of A. Now, since A is core, it follows that the automaton A_{k-1} is isomorphic to $G(2, 1)$. This is because the only core, level 1 synchronizing automaton over the 2 letter alphabet is $G(2, 1)$. $\square$

Corollary 4.14. *Let A be an folded automaton over the 2 letter alphabet, then* $\mathrm{Aut}(G_A)$ *is either trivial or the cyclic group of order 2. Moreover any automorphism of G_A is induced by a permutation of X_2.*

Proof. This is a direct consequence of Lemma 4.13 and Corollary 4.11. $\square$

Theorem 4.15. *The group $\mathcal{H}_2$ is isomorphic to the cyclic group of order 2.*

Proof. Let A be a minimal, core, bi-synchronizing transducer over the 2 letter alphabet. Suppose for a contradiction that $|A| > 1$. By Lemma 4.13, A satisfies the hypothesis of Proposition 4.10. However, this yields a contradiction as the size of A must then be 1.

Thus, every element of $\mathcal{H}_2$ has exactly one state yielding the result. $\square$

5 Decomposing elements of $\mathcal{H}_n$

In this section we give an algorithm for decomposing an arbitrary element of $\mathcal{H}_n$ as a product of elements arising from automorphisms of the directed graphs underlying the folded automata arising from foldings of $G(n, m)$. Our method can be thought of as an interpretation of the approach in [5] in the language of strongly synchronizing automata. However, we are able to simplify that approach a great deal. In particular, we show that an element $T \in \mathcal{H}_n$ of size l for some $l \in \mathbb{N}$ can be written as a product of at most l elements of $\mathcal{H}_n$ arising from automorphisms of directed graphs underlying foldings of the underlying automaton of A^{-1}. Note that an element $T \in \mathcal{H}_n$ of size l is strongly synchronizing at level at most $l-1$, thus f_T (by Remark 3.6) corresponds to a map f_∞ for some $f \in F(X_n, l)$. To decompose the element f_∞ using the approach given in [5], one would first have to construct a graph (isomorphic to the underlying graph of some folded automaton) with at least n^l vertices.

5.1 Collapse equivalence and amalgamation

We introduce some terminology. Let A and B be automata. Then B is said to be *collapse equivalent* to A, if there is a sequence

$$A = A_0, A_1, \ldots, A_m = B$$

where, for $i \geq 1$, A_i is obtained from A_{i-1} by identifying two states $p, q \in Q_{A_{i-1}}$ such that $\pi_{A_{i-1}}(\cdot, p) = \pi_{A_{i-1}}(\cdot, q)$. We stress that each term in the sequence is obtained from the previous one by identifying exactly two states. Observe that if A is strongly synchronizing and B is an automaton which is collapse equivalent to A, then B is synchronizing at the minimal synchronizing level of A. More generally, let A and B be collapse equivalent automata, with synchronizing sequences $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$, and suppose that $k, l \in \mathbb{N}$ are minimal such that $A_j = A_k$ for all $j \geq k$ and $B_j = B_l$ for all $j \geq l$, then $A_k = B_l$. This is a consequence of Theorem 4.7.

The following terminology, which is for the underlying graphs of an automaton, should be compared with the similarly named terminology in the paper [5] (recall the direction of edges will be reversed in our context). Let A and B be automata. Then G_B is called an *amalgamation* of G_A if there is a sequence $G_A = G_0, G_1, \ldots, G_m = G_B$ where, for $i \geq 1$, G_i is obtained from G_{i-1} by identifying two vertices v_1 and v_2 of G_{i-1} having the property that for all vertices v of G_{i-1}, if there are precisely k outgoing edges from v_1 to v (for some $1 \leq k \leq n$) then there are also precisely k outgoing edges from v_2 to v. That is, we replace the vertices v_1 and v_2 with a single vertex

$v_{1,2}$ and, for every vertex v of G_{i-1} if there are k edges from v_1 to v (and hence, from v_2 to v), then there are k edges from $V_{1,2}$ to v (and of course, we retain all other vertices and edges of G_{i-1}). Also, if v is a vertex of G_{i-1} then there will be t edges in G_i from the vertex corresponding to v to $v_{1,2}$ if the cardinality of the set of edges from v to v_1 is r while the cardinality of the set of edges from v to v_2 is s, where $r+s=t$. In particular, if there are m loops based as v_1 and m' loops based at v_2 in G_{i-1}, there are exactly $m+m'$ loops based at $v_{1,2}$ in G_i. In this context, the vertices v_1 and v_2 of G_{i-1} are called *amalgamable*.

Let T be an invertible transducer. Let A and B be the underlying automata of T and T^{-1} respectively. Let $(B_i)_{i\in\mathbb{N}}$ be the synchronizing sequence of B. Then, by definition of the inverse transducer, G_{B_i} is an amalgamation of G_A for all $i \in \mathbb{N}$. The condition "for two states $p^{-1}, q^{-1} \in Q_{T^{-1}}$, $\pi_B(\cdot, p^{-1})$ and $\pi_B(\cdot, q^{-1})$ are equal" is equivalent to the condition "the vertices p and q of G_A are amalgamable". Further observe that for collapse equivalent automata the underlying directed graph of one is an amalgamation of the other.

5.2 Description of the decomposition algorithm

Here we give a short description of the algorithm for decomposing an element T of $\mathcal{H}_n$ as a product of torsion elements as described in Theorem 1.1. The proof that our various steps can be carried out is given in full detail in Subsection 5.3. The algorithm allows the user some choices, so decomposition is not unique, but our upper bound on the decomposition length still holds.

We conclude with an example decomposition and statements of choices we made so the reader can verify by following the algorithm.

A1 Let $T_0 \in \mathcal{H}_n$. Let A and B be the underlying automata of T_0 and T_0^{-1} respectively.

A2 If T_0 has only one state, then it represents a permutation, and so there is a finite order single state transducer that we can multiply against T_0 to produce the identity element (in this case, go to the final step of the algorithm with this finite order factor in hand). Otherwise, proceed to the next step.

A3 Compute the synchronizing sequence $(B_i)_{i\in\mathbb{N}}$ for $B = B_0$.

A4 Compute the first step A_1 of the synchronizing sequence of $A = A_0$.

A5 Find a pair (p, q) of distinct states of A which belong to the same state of A_1.

A6 Find the non-identity permutation α of the output labels such that $\lambda(\cdot, q) \circ \alpha : X_n \to X_n$ is precisely $\lambda(\cdot, p) : X_n \to X_n$. Determine the disjoint cycle decomposition of α.

A7 There is a smallest index i so that the state $[q]$ of the automaton B_i has the following properties:

- The states $[q]$ and $[p]$ remain distinct states of B_i, and
- For all $x, y \in X_n$ belonging to the same disjoint cycle in the cycle decomposition of α, the edges labelled x and y from $[q]$ are parallel edges.

Now determine the isomorphism τ_α of B_i which fixes all vertices and induces the permutation α on the edges leaving $[q]$.)

A8 Build the transducer $H(B_i, \tau_\alpha)$. This is a finite factor in a product sequence that will eventually trivialize T_0.

A9 Compute the product $R = \text{core}(T * H(B_i, \tau_\alpha))$. This product has the same underlying graph as T but is not minimal. The states corresponding to p and q in this product are ω-equivalent, and will be identified by minimising the result R to produce a new element T_1 with fewer states than T_0.

A10 Repeat this process from the beginning, remembering the list of finite factors found so far.

A11 The transducer T now factors as the inverse product of the finite order factors found above.

We give an example. Consider the element $T := H(A, \phi)$ from Figure 5. Working through the algorithm, with $p = q_1$, and $q = q_0$ in the first instance, one obtains the following decomposition below (up to changing the final single state transducer; different choices for p and q in building the second factor result in different single-state third-factor transducers):

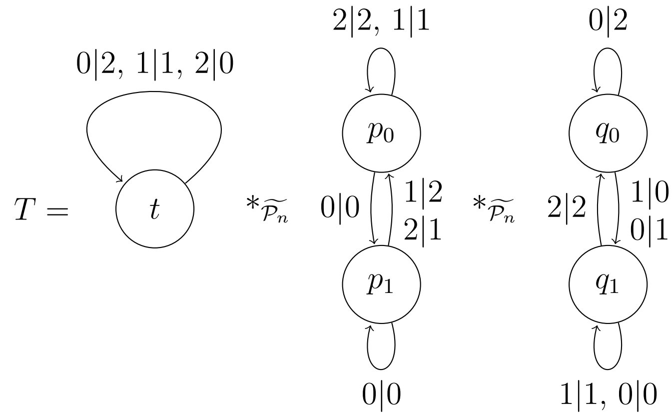


Figure 6: Decomposing an element of $\mathcal{H}_3$ as a product of involutions.

5.3 Proof of Theorem 1.1

Here we prove that the algorithm above works.

Recall that $\widetilde{\mathcal{H}_n}$ consist of those transducers which are strongly synchronizing and have an automata-theoretic inverse but which do not necessarily induce homeomorphisms of $X_n^{\mathbb{Z}}$. Further recall that for $T, U \in \widetilde{\mathcal{P}_n}$ the product, in $\mathcal{P}_n$, of T and U is obtained by identifying the ω-equivalent states of $\mathrm{core}(T * U)$, write $T *_{\widetilde{\mathcal{P}_n}} U$ for this transducer.

Lemma 5.1. *Let A be a minimal transducer in $\mathcal{H}_n$. Let B be the underlying automaton of A^{-1} and $(B_i)_{i \in \mathbb{N}}$ be the synchronizing sequence of B. Let $H \in \widetilde{\mathcal{H}_n}$ be any transducer such that the underlying automaton of H is B_j for some $j \in \mathbb{N}$. For a state p^{-1} of A^{-1} write $[p^{-1}]$ for the state of B_j containing p^{-1}. Then*

(a) the set of states of $\mathrm{core}(A * H)$ *is precisely the set $\{(p, [p^{-1}]) \mid p \in Q_A\}$. Consequently,*

(b) $|A *_{\widetilde{\mathcal{P}_n}} H| \leq |A|$, and,

(c) the underlying automaton of $A *_{\widetilde{\mathcal{P}_n}} H$ is collapse equivalent to the underlying automaton of A.

Proof. Let $p \in Q_A$ and $x \in X_n$ and consider the transition $\pi_{A*H}(x, (p, [p^{-1}]))$. Let $y = \lambda_A(x, p)$ and $q = \pi_A(x, p)$. Then, in A^{-1}, we have $\pi_{A^{-1}}(y, p^{-1}) = q^{-1}$, therefore, in B_j, $\pi_{B_j}(y, [p^{-1}]) = [q^{-1}]$. Hence, we conclude that

$$\pi_{A*H}(x, (p, [p^{-1}])) = (q, [q^{-1}]).$$

To see that $(p, [p^{-1}])$ is a state core$(A*H)$, let $\gamma \in X_n^+$ be a word such that $\pi_A(\gamma, p) = p$. The preceding paragraph now shows that $\pi_{A*H}(\gamma, (p, [p^{-1}])) = (p, [p^{-1}])$.

Thus we see that $|\operatorname{core}(A * H)| = |A|$. In particular the underlying automaton of core$(A * H)$ is isomorphic as an automaton to the underlying automaton of A via the map sending $(p, [p^{-1}])$ to p.

Now, $A *_{\widetilde{\mathcal{P}_n}} H$ is obtained by identifying ω-equivalent states of core$(A*H)$. Therefore the underlying automaton of $A *_{\widetilde{\mathcal{P}_n}} H$ is collapse equivalent to the underlying automaton of core$(A * H)$ as required. □

Lemma 5.2. *Let $A \in \mathcal{H}_n$ be a minimal transducer, let B be the underlying automaton of A^{-1} and $(B_i)_{i\in\mathbb{N}}$ be the synchronizing sequence of B. Suppose there are distinct states $q_1, q_2 \in Q_A$ such that the maps $\pi_A(\cdot, q_1)$ and $\pi_A(\cdot, q_2)$ are equal. Then there is a transducer H with the following properties:*

(a) there is a $j \in \mathbb{N}$ such that $H = H(B_j, \phi)$ for an automorphism ϕ of B_j and,

(b) writing $[q^{-1}]$ for the state of B_j containing q^{-1}, $q \in Q_A$, we have

$$\lambda_A(\cdot, q_2) \circ \lambda_{H(B_j,\phi)}(\cdot, [q_2^{-1}]) : X_n \to X_n$$

is precisely the map $\lambda_A(\cdot, q_1) : X_n \to X_n$.

Proof. Since q_1, q_2 are distinct states of A and since A is minimal, q_1 and q_2 are not ω-equivalent. Therefore, there are $x \neq y$ and $z \in X_n$ such that $\lambda_A(x, q_1) = \lambda_A(y, q_2) = z$. Let $p_1 = \pi_A(x, q_1)$ and $p_2 = \pi_A(y, q_2)$. In A^{-1}, we have $\pi_{A^{-1}}(z, q_1^{-1}) = p_1^{-1}$ and $\pi_{A^{-1}}(z, q_2^{-1}) = p_2^{-1}$. Since A^{-1} has minimal synchronizing level k, it therefore follows that either $k = 1$ and $p_1 = p_2$ or $k \geq 2$ and the maps $\pi_{A^{-1}}(\cdot, p_1^{-1}) : X_n^{k-1} \to Q_{A^{-1}}$ and $\pi_{A^{-1}}(\cdot, p_2^{-1}) : X_n^{k-1} \to Q_{A^{-1}}$ are equal. Therefore, by Theorem 4.7, the minimal $j \in \mathbb{N}$ for which p_1^{-1} and p_2^{-1} belong to the same state of B_j is at most $k - 1$.

Define a relation $\mathscr{R}$ on the set of states

$$Q_{q_1^{-1}, q_2^{-1}} := \{p^{-1} \in Q_{A^{-1}} \mid \exists x \in X_n, a \in \{1, 2\} : \pi_{A^{-1}}(x, q_a^{-1}) = p^{-1}\}$$

by setting $p^{-1}\mathscr{R}q^{-1}$ if and only if there is a letter $z \in X_n$ such that

$$\pi_{A^{-1}}(z, q_1^{-1}) = p^{-1} \text{ and } \pi_{A^{-1}}(z, q_2^{-1}) = q^{-1}.$$

Let $\overline{\mathscr{R}}$ be the transitive closure of $\mathscr{R}$, so that $\overline{\mathscr{R}}$ is an equivalence relation on $Q_{q_1^{-1},q_2^{-1}}$. By the preceding paragraph, for a state $p^{-1} \in Q_{q_1^{-1},q_2^{-1}}$, there is a minimal $j \in \mathbb{N}$, $j \leq k-1$, and all elements of $[p^{-1}]_{\overline{\mathscr{R}}}$, the equivalence class of p^{-1}, belong to the same state of B_j.

Let $J \in \mathbb{N}$, $J \leq k-1$, be minimal such that for any $p^{-1} \in Q_{A^{-1}}$ there is a state of B_J such that all elements of $[p^{-1}]$ belong to the same state of B_J. Observe that if $\mathscr{R}$ is the diagonal relation, that is, if $\mathscr{R}$ is precisely the set $\{(p^{-1}, p^{-1}) \mid p^{-1} \in Q_{q_1^{-1},q_2^{-1}}\}$, then $B_j = B_0$. Further observe that $\mathscr{R}$ is the diagonal relation precisely when for all $x \in X_n$, $\pi_{A^{-1}}(x, q_1^{-1}) = \pi_{A^{-1}}(x, q_2^{-1})$. If there is $z \in X_n$, such that $\pi_{A^{-1}}(z, q_1^{-1}) \neq \pi_{A^{-1}}(z, q_2^{-1})$, then minimality of J forces that the states q_1^{-1} and q_2^{-1} do not belong to the same state of B_J. Therefore, as q_1 and q_2 are distinct states of A, they are contained in distinct states of B_J.

Let t_1 and t_2 be the distinct states of B_J containing q_1^{-1} and q_2^{-1} respectively. Observe that the maps $\pi_{B_J}(\cdot, t_1)$ and $\pi_{B_J}(\cdot, t_2)$ are equal by choice of J and definition of the relation $\overline{\mathscr{R}}$. Define a map $\lambda_{B_J}(\cdot, t_2) : X_n \to X_n$ as follows. Let $x, \in X_n$ and let $z = \lambda_A(x, q_1)$ and $y = \lambda_A(x, q_2)$ then set $\lambda_{B_J}(y, t_2) := z$. Since $\lambda_A(\cdot, q_1)$ and $\lambda_A(\cdot, q_2)$ are permutations of X_n, then $\lambda_A(\cdot, t_2)$ is a bijection as well. Moreover we note that

$$\lambda_A(\cdot, q_2) \circ \lambda_A(\cdot, t_2) : X_n \to X_n$$

is precisely the map $\lambda_A(\cdot, q_1) : X_n \to X_n$.

Let $a, b, c \in X_n$ be arbitrary such that $\lambda_{B_J}(a, t_2) = b$ and $\lambda_{B_J}(b, t_2) = c$. By definition, there are $x, y \in X_n$ such that $\lambda_A(x, q_1) = a$, $\lambda_A(x, q_2) = b$, $\lambda_A(y, q_1) = b$ and $\lambda_A(y, q_2) = c$. By the assumption that $\pi_A(\cdot, q_1)$ and $\pi_A(\cdot, q_2)$ are equal, we have $p^{-1} := \pi_{A^{-1}}(a, q_1^{-1}) = \pi_{A^{-1}}(b, q_2^{-1})$ and $q^{-1} := \pi_{A^{-1}}(b, q_1^{-1}) = \pi_{A^{-1}}(c, q_2^{-1})$. Thus, p^{-1} is $\mathscr{R}$ related to q^{-1}. Therefore, it is the case that $\pi_{B_J}(b, t_2) = \pi_{B_J}(c, t_2)$.

Let $(x_1\ x_2\ x_3\ \ldots\ x_m)$ be a sequence of elements of X_n such that for $1 \leq i \leq m-1$, $\lambda_{B_J}(x_i, t_2) = x_{i+1}$ and $\lambda_{B_J}(x_m, t_2) = x_1$. By an induction argument making use of the previous paragraph we see that there is a state $t \in Q_{B_J}$ such that $\pi_{B_J}(x_i, t_2) = t$ for all $1 \leq i \leq m$. Thus, it follows that given $a, b \in X_n$ such that $\lambda_{B_J}(a, t_2) = b$ then, $\pi_{B_J}(a, t_2) = \pi_{B_J}(b, t_2)$.

Let t be any state of B_J not equal to t_2, we set $\lambda_{B_J}(\cdot, t) : X_n \to X_n$ to be the identity permutation. Set $H(B_j) := (X_n, Q_{B_J}, \pi_{B_J}, \lambda_{B_J})$.

Let ϕ be the automorphism of G_{B_J} which fixes all vertices of G_{B_j} and whose action on the edges of G_{B_j} is as follows. For an edge (t_2, x, t) of G_{B_J} with initial vertex t_2, set $(t_2, x, t)\phi := (t_2, \lambda_{B_J}(x, t_2), t)$; ϕ fixes every other edge. It is clear from the preceding paragraphs that $H(B_J, \phi) = H(B_J)$. Thus we may take $H = H(B_J)$ concluding the proof. $\square$

Proposition 5.3. *Let $A \in \mathcal{H}_n$ be a minimal transducer, B be the underlying automaton of A^{-1}, $(B_i)_{i\in\mathbb{N}}$ be the synchronizing sequence of B and $k \in \mathbb{N}$ be minimal such that $|B_k| = 1$. Suppose there are distinct states $q_1, q_2 \in Q_A$ such that the maps $\pi_A(\cdot, q_1)$ and $\pi_A(\cdot, q_2)$ are equal. Then, there is an $i \in \mathbb{N}$, and an automorphism ϕ of G_{B_i} fixing vertices and such that $|A *_{\widetilde{\mathcal{P}_n}} H(B_i, \phi)| < |A|$. Thus, $G_{H(B_i,\phi)} = G_{B_i}$ is an amalgamation of G_A. Moreover, the underlying automaton of $A *_{\widetilde{\mathcal{P}_n}} H(B_i, \phi)$ is collapse equivalent to the underlying automaton of A. Therefore, $(A *_{\widetilde{\mathcal{P}_n}} H(B_i, \phi))$ has minimal synchronizing level at most the minimal synchronizing level of A.*

Proof. By Lemma 5.2 there is a transducer H with the following properties:

- there is a $j \in \mathbb{N}$ such that $H = H(B_j, \phi)$ for an automorphism ϕ of B_j and,
- writing $[q^{-1}]$ for the state of B_j containing q^{-1}, $q \in Q_A$, we have

  $$\lambda_A(\cdot, q_2) \circ \lambda_{H(B_j,\phi)}(\cdot, [q_2^{-1}]) : X_n \to X_n$$

  is precisely the map $\lambda_A(\cdot, q_1) : X_n \to X_n$.

The result follows by applying Lemma 5.1 to the product $A *_{\widetilde{\mathcal{P}_n}} H$. $\square$

Theorem 5.4. *Let $T \in \mathcal{H}_n$, A the underlying automaton of T, $(A_i)_{i\in\mathbb{N}}$ the synchronizing sequence of A and k be minimal such that $A_j = A_k$ for all $j \geq k$. Note that since T is strongly synchronizing, $A_k = 1$. Then T can be written as a product of a single state transducer U and at most $|A| - 1$ elements of $\mathcal{H}_n$ which arise from vertex-fixing automorphisms of directed graphs which are amalgamations of G_A.*

Proof. The proof follows by repeatedly applying Proposition 5.3. $\square$

We note that Theorem 1.1 is a corollary of Theorem 5.4 above.

Lemma 5.5. *Let A be a strongly synchronizing core automaton with more that one state. Then for any pair $p, q \in Q_A$ there are is a least one element $x \in X_n$ such that $\pi_A(x, p) \neq q$. In other words, there are at most $n-1$ edges in G_A from the vertex p to the vertex q.*

Proof. Suppose for a contradiction that there are states $p, q \in Q_A$ such that $\pi_A(x, p) = q$ for all $x \in X_n$. Let $(A_i)_{i \in \mathbb{N}}$ be the synchronizing sequence of A, and let k be minimal such that $A_k = 1$. Notice that A_{k-1} is synchronizing at level 1 and core and has more than one state by assumption on k. Let t be the state of A_{k-1} which contains p, and t' be the state of A_{k-1} containing q. It follows that $\pi_A(x_n, t) = t'$ for all $x \in X_n$. Since A_{k-1} is synchronizing at level 1, this forces, $|A_{k-1}| = 1$ which yields the desired contradiction. $\square$

Corollary 5.6. *Let A be a strongly synchronizing core automaton over the alphabet X_3 with more than one state. Let ϕ be any automorphism of G_A that fixes vertices, then ϕ has order at most 2.*

Corollary 5.7. *Let $T \in \mathcal{H}_3$, A be the underlying automaton of T and $(A_i)_{i \in \mathbb{N}}$ be the synchronizing sequence of A. Let $k \in \mathbb{N}$ be minimal such that $|A_k| = 1$. Then T can be written as a product of a single state transducer U and at most $|A| - 1$ elements of $\mathcal{H}_n$ of order 2 which arise from vertex-fixing automorphisms of directed graphs which are amalgamations of G_A.*

Proof. The proof follows by repeated applications of Proposition 5.3 and Corollary 5.6. $\square$

We generalise Corollary 5.7 to all n. However, the number of elements of order 2 required is bigger than the number of states in general. We require first the following straight-forward observation.

Lemma 5.8. *Let G be a directed graph and ϕ be an automorphism of G that fixes vertices. Then ϕ can be written as a product of vertex-fixing automorphisms of G of order 2.*

Corollary 5.9. *Let $T \in \mathcal{H}_n$, $(A_i)_{i \in \mathbb{N}}$ be the synchronizing sequence of A and k be minimal such that $A_j = A_k$ for all $j \geq k$. Then T can be written as a product of a single state transducer U with underlying automaton A_k, and elements of $\mathcal{H}_n$ of order 2 arising from vertex-fixing automorphisms of directed graphs which are amalgamations of G_A.*

It is possible to bound the number of involutions appearing in Corollary 5.9 in terms of A (i.e the number of vertices and edges of G_A) but we have not attempted to do so.

6 Counting foldings

Counting foldings of the de Bruijn graph $G(n,k)$ is an important and challenging problem. We give here the solution for $k = 1$ (which is trivial) and for $k = 2$.

The *Bell number* $B(n)$ is the number of partitions of an n-set. This well-studied combinatorial sequence is given by the recurrence relation

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k)$$

for $n > 0$, with $B(0) = 1$.

Proposition 6.1. *The number of foldings of $G(n,1)$ is the Bell number $B(n)$.*

Proof. The vertex set is identified with X_n, so any folding is a partition of X_n; and clearly any partition of X_n is a folding. $\Box$

Theorem 6.2. *The number of foldings of the de Bruijn graph with word length 2 over an alphabet of cardinality n is*

$$\sum_{\pi} \prod_{i=1}^{|\pi|} R(|\pi|, |A_i|),$$

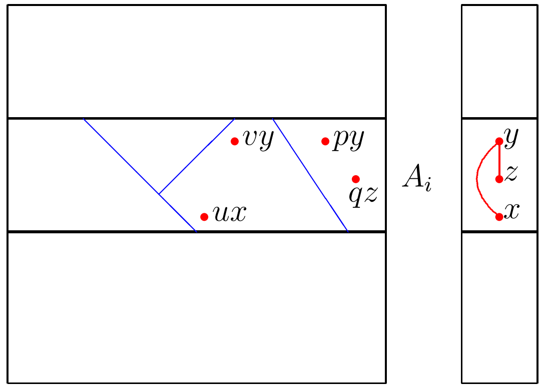
where π runs over partitions of the alphabet, A_i is the ith part, and

$$R(s,t) = \sum_{\rho} (-1)^{|\rho|-1} (|\rho|-1)! \prod_{i=1}^{|\rho|} B(|C_i|s),$$

where ρ runs over all partitions of $\{1,\ldots,t\}$, and C_i is the ith part.

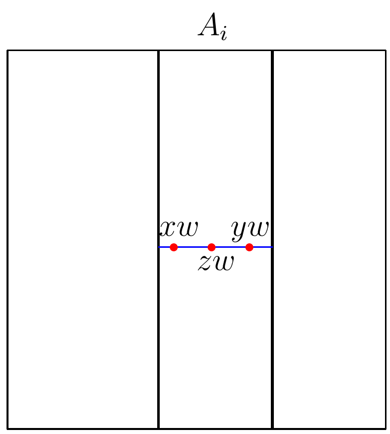
The formula is somewhat complicated, but values are easily computed (and rapidly growing): the numbers for $n = 1,\ldots,7$ are 1, 5, 192, 78721, 519338423, 82833228599906, 429768478195109381814.

Proof. We define a graph Γ associated with a folding: the vertex set is the alphabet X_n, and two vertices x and y are joined if there exist u and v such that $ux \equiv vy$.



Let π be the partition of X_n into connected components of the graph Γ. If A_i is a part of Γ, then the set $X_n \times A_i$ (the horizontal stripe in the figure) is a union of parts of the folding: no part can cross into a different horizontal stripe.

Moreover, by the definition of a folding, we see that if $x, y \in A_i$, then xw and yw lie in the same part of the folding.



The sets $X_n \times A_i$ can be treated independently, so we have to count the number of good partitions of each and multiply them. Moreover, by the last remark, we can shrink each horizontal interval $A_j \times \{v\}$ to a point, so we have to partition $\pi \times A_i$.

There are $B(|\pi| \cdot |A_i|)$ partitions of $\pi \times A_i$. We have to filter out the ones which do not induce partitions of $\pi \times B$ for any proper subset B of A_i. By Möbius inversion [11, Section 3.7] over the lattice of partitions of A_i, we find that the number of these is $R(|\pi|, |A_i|)$, where R is as defined earlier.

Putting all this together gives the result. □

Apart from this result, only a few values of the function counting foldings are known: $G(2,3)$ has 30 foldings, while $G(2,4)$ has 1247. (These numbers were obtained by brute-force computation.)

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