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Phase transitions from $\exp(n^{1/2})$ to $\exp(n^{2/3})$ in the asymptotics of banded plane partitions[1]

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Abstract

We examine the asymptotics of a class of banded plane partitions under a varying bandwidth parameter m, and clarify the transitional behavior for large size n and increasing $m = m(n)$ to be from $c_1 n^{-1}\exp\big(c_2 n^{1/2}\big)$ to $c_3 n^{-49/72}\exp\big(c_4 n^{2/3} + c_5 n^{1/3}\big)$ for some explicit coefficients $c_1, \dots, c_5$. The method of proof, which is a unified saddle-point analysis for all phases, is general and can be extended to other classes of plane partitions.

1 Introduction

Partition asymptotics of generating functions with unit circle as natural boundary has been the subject of study since Hardy and Ramanujan's 1918 epoch-making paper [12], marking already the first centennial and finding their use in various scientific disciplines. In particular, it is known that the number of partitions of n into positive integers is asymptotic to

$$p_n := [z^n]\prod_{k\geqslant 1}\frac{1}{1-z^k} \sim cn^{-1}e^{\beta n^{1/2}}, \quad \text{with} \quad (c,\beta) = \left(\frac{1}{4\sqrt{3}}, \frac{\sqrt{2}\,\pi}{\sqrt{3}}\right), \tag{1}$$

(see [1, 12] or [15, A000041]), and that of *plane partitions* of n satisfies

$$\mathbb{p}_n = [z^n]\prod_{k\geqslant 1}\frac{1}{\left(1-z^k\right)^k} \sim cn^{-25/36}e^{\beta n^{2/3}}, \quad \text{with} \quad (c,\beta) = \left(\frac{\zeta(3)^{7/36}e^{-\zeta'(-1)}}{2^{11/36}\sqrt{3\pi}}, \frac{3\zeta(3)^{1/3}}{2^{2/3}}\right), \tag{2}$$

(see [1, 20] or [15, A000219]). Here the symbol $[z^n]f(z)$ denotes the coefficient of z^n in the Taylor expansion of f and $\zeta(s)$ the Riemann zeta function [2, 19]. *Throughout this paper, the values of the*

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generic (or local) symbols c, β or c_j, β_j may differ from one occurrence to the other, and will always be locally specified.

The increase of the sub-exponential (or stretched exponential) term from $e^{\beta n^{1/2}}$ in the case of ordinary partitions to $e^{\beta n^{2/3}}$ in the case of plane partitions is noticeable, and marks the essential difference in the respective asymptotic enumeration. As integer partitions are also encountered in statistical physics, astronomy, and other engineering applications, one naturally wonders if there is a tractable combinatorial model that interpolates between the two different orders $e^{n^{1/2}}$ and $e^{n^{2/3}}$ when some structural parameter varies. This paper aims to address this aspect of partition asymptotics and examines in detail a class of plane partitions with a natural notion of bandwidth m whose variation yields a model in which we can fully clarify the transitional behavior from being of order $e^{\beta n^{1/2}}$ for bounded m to $e^{\beta n^{2/3}}$ when $m \gg n^{1/3}$, providing more modeling flexibility of these partitions. Our study constitutes the first asymptotic realization of such phase transitions in the analytic theory of partitions.

Intuitively, if we impose a constraint to one or two of the dimensions of plane partitions, then by suitably varying the constraint, we can generate families of objects whose asymptotic behaviors interpolate between $e^{n^{1/2}}$ and $e^{n^{2/3}}$. An initial attempt can be found, *e.g.*, in [8], where Gordon and Houten computed the asymptotic counting formula for "k-rowed partitions" whose nonzero parts decrease strictly along each row of size n. However, they studied only the situations when k is bounded and when $k \to \infty$, and do not consider how exactly the asymptotic behavior changes with respect to varying k (depending on n). See Section 6 for the phase transitions in plane partitions with a given number of rows.

The plane partitions of $n \geqslant 0$ may be viewed as a matrix with nonincreasing entries along rows and columns and with the entry-sum equal to n. The class of plane partitions we work on in this paper is the *double shifted plane partitions* studied by Han and Xiong in [10] with an explicit notion of width, which for simplicity will be referred to as the *banded plane partitions* (or *BPPs*) in this paper. These are plane partitions arranged on the *stair-shaped region* $\mathbb{T}_m = \{(i,j) \in \mathbb{N}^2 \mid j \leqslant i \leqslant j+m-1\}$, $m \in \mathbb{Z}^+$. Formally, a banded plane partition of width m is a function $f : \mathbb{T}_m \to \mathbb{N}$ with finite support such that, for any $(i,j) \in \mathbb{T}_m$, we have $f(i,j) \geqslant f(i,j+1)$ when $(i,j+1) \in \mathbb{T}_m$, and $f(i,j) \geqslant f(i+1,j)$ when $(i+1,j) \in \mathbb{T}_m$. Figure 1 illustrates two instances of BPPs.

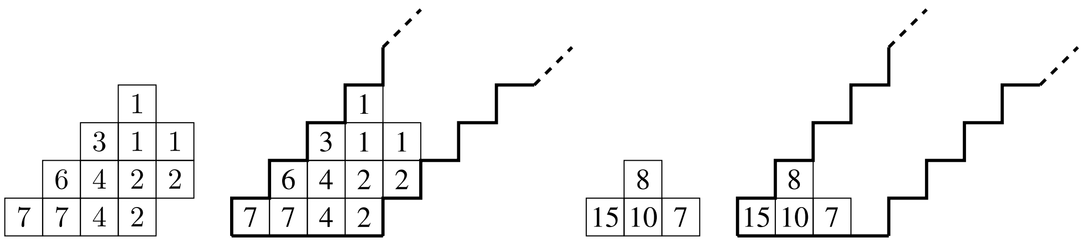


Figure 1: Two instances of banded plane partition of size 40 *and width* 4 *(with and without the outer banded staircase).*

The *size* of a BPP is the sum $\sum_{(i,j)\in\mathbb{T}_m} f(i,j)$. We denote by $G_{n,m}$ the number of BPPs of size n and of width m. A closed-form expression for the generating function $G_m(z) := \sum_{n\geqslant 0} G_{n,m} z^n$ is given in [10, Theorem 1.1] as $G_m(z) = P(z)Q_m(z)$, where

$$P(z) = \prod_{k\geqslant 1} \frac{1}{1-z^k}, \quad \text{and} \quad Q_m(z) = \prod_{k\geqslant 0} \prod_{1\leqslant h<j<m} \frac{1}{1-z^{2mk+h+j}}. \tag{3}$$

In particular,

$$Q_3(z)=\prod_{k\geqslant 0}\frac{1}{1-z^{6k+3}},\quad Q_4(z)=\prod_{k\geqslant 0}\frac{1}{\big(1-z^{8k+3}\big)\big(1-z^{8k+4}\big)\big(1-z^{8k+5}\big)},$$
$$Q_5(z)=\prod_{k\geqslant 0}\frac{1}{\big(1-z^{10k+3}\big)\big(1-z^{10k+4}\big)\big(1-z^{10k+5}\big)^2\big(1-z^{10k+6}\big)\big(1-z^{10k+7}\big)}.$$

For a BPP f with $m\geqslant n$, the function g on $\mathbb{N}^2$ defined by $g(i,j)=f(i-j,j)$ is a plane partition, and by replacing each row of g (which is an integer partition) by its conjugate partition, we obtain a column-strict plane partition (weakly decreasing in each row but strictly decreasing in each column). This transformation is clearly bijective. The generating function of column-strict plane partitions is known:

$$\prod_{k\geqslant 1}\frac{1}{(1-z^k)^{\lfloor (k+1)/2\rfloor}};$$

see [7, 18] or [15, A003293].

Based on the generating function (3), Han and Xiong showed in [10], by an elementary convolution approach developed in [9], that the number $G_{n,m}$ of banded plane partitions of size n and of width m satisfies

$$G_{n,m}\sim c(m)n^{-1}e^{\beta(m)\sqrt{n}}, \tag{4}$$

for large n and bounded $m\geqslant 1$, where

$$(c(m),\beta(m)):=\left(\frac{\sqrt{m^2+m+2}}{2^{(m^2-3m+14)/4}\sqrt{3m}}\prod_{3\leqslant j<m}\sin\Big(\frac{j\pi}{2m}\Big)^{-\lfloor (j-1)/2\rfloor},\sqrt{\frac{m^2+m+2}{6m}}\,\pi\right).$$

Thus $\log G_{n,m}$ is still of asymptotic order $\sqrt{n}$ when m is bounded. Note that $c(1)=c(2)=1/(4\sqrt{3})$ and $\beta(1)=\beta(2)=\sqrt{2}\,\pi/\sqrt{3}$, the same as c and β in (1), respectively.

Now if we pretend that the formula (4) holds also for increasing m, then since $\beta(m)\sim\sqrt{m/6}\,\pi$ for large m, we see that $\beta(m)\sqrt{n}\asymp\sqrt{mn}\asymp n^{2/3}$ when $m\asymp n^{1/3}$ (the symbol $a_n\asymp b_n$ standing for *equivalence of growth order* for large n). Furthermore, we will show in Proposition 3.1 that $\log c(m)\sim -\frac{7\zeta(3)}{8\pi^2}m^2$ for large m. Then equating $m^2\asymp\sqrt{mn}$ also gives $m\asymp n^{1/3}$. Thus we would expect that (4) remains valid for $m=o\big(n^{1/3}\big)$ and the "phase transition" occurs around $m\asymp n^{1/3}$. However, while the latter is true by such a heuristic reasoning, the former is not as we will show that (4) holds indeed only when $m=o\big(n^{1/7}\big)$, although the weaker asymptotic estimate $\log G_{n,m}\sim\beta(m)\sqrt{n}$ does hold uniformly for $1\leqslant m=o\big(n^{1/3}\big)$ (see (77) and (81)). This implies particularly the estimate

$$\log G_{n,m}\sim\frac{\pi}{\sqrt{6}}\sqrt{mn}, \tag{5}$$

uniformly when $m\to\infty$, $m=o\big(n^{1/3}\big)$.

On the other hand, Gordon and Houten [8] showed that

$$G_{n,n}=[z^n]\prod_{k\geqslant 1}\frac{1}{(1-z^k)^{\lfloor (k+1)/2\rfloor}}\sim cn^{-49/72}e^{\beta_1 n^{2/3}+\beta_2 n^{1/3}}, \tag{6}$$

where

$$(c,\beta_1,\beta_2)=\left(\frac{e^{\zeta'(-1)/2-\pi^4/(3456\zeta(3))}\zeta(3)^{13/72}}{2^{3/4}(3\pi)^{1/2}},\frac{3\zeta(3)^{1/3}}{2},\frac{\pi^2}{24\zeta(3)^{1/3}}\right). \tag{7}$$

This implies particularly the weak asymptotic estimate

$$\log G_{n,n} \sim \frac{3\zeta(3)^{1/3}}{2}\, n^{2/3}. \tag{8}$$

We will derive in Section 5 stronger asymptotic approximations to $G_{n,m}$ for all possible values of m, $1 \leqslant m \leqslant n$, covering as special cases (4) and (6). In particular, as far as log-asymptotics is concerned, we derive a uniform estimate, covering also the most interesting critical range when $m \asymp n^{1/3}$; see Proposition 5.3. Define

$$\eta_d(z) := \sum_{\ell\geqslant 1} \frac{e^{-\ell z}}{\ell^{2d-1}(1+e^{-\ell z})} \qquad (d = 0, 1, \dots). \tag{9}$$

Theorem 1.1. *Let $\alpha := mn^{-1/3}$. Then*

$$\frac{\log G_{n,m}}{n^{2/3}} \sim G(\alpha) := r + \frac{\zeta(3) - 2\eta_2(\alpha r)}{2r^2}, \tag{10}$$

uniformly as $\alpha \gg n^{-1/3}$ (or $m \to \infty$), where $r = r(\alpha) > 0$ solves the equation

$$r^3 - \zeta(3) + 2\eta_2(\alpha r) - \alpha r \eta_2'(\alpha r) = 0. \tag{11}$$

In particular,

$$G(\alpha) \sim \begin{cases} \dfrac{\pi}{\sqrt{6}}\sqrt{\alpha}, & \text{if } \alpha \to 0; \\ \dfrac{3}{2}\zeta(3)^{1/3}, & \text{if } \alpha \to \infty. \end{cases} \tag{12}$$

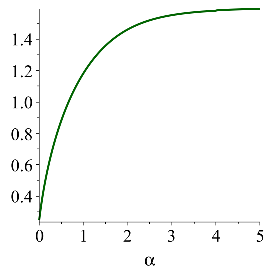


Figure 2: A plot of the increasing function $G(\alpha)$.

We thus have a combinatorial model that interpolates nicely between integer partitions and column-strict plane partitions, in the sense of asymptotic behavior. A very similar looking expression will be derived in Section 6 for m-rowed plane partitions, which bridges ordinary partitions and plane partitions.

The BPPs we study here can be connected to ordinary plane partitions through the following decomposition. Given a plane partition g of size n, denote by $t = \sum_{i\geqslant 0} g(i,i)$ its trace. We separate g by the diagonal $x = y$, obtaining an integer partition on the diagonal and two BPPs f_1, f_2 of sizes n_1, n_2 respectively, such that $n = n_1 + n_2 + t$. The weak asymptotics of such a triple (n_1, n_2, t) is bounded above by

$$\begin{aligned} &\log G_{n_1,n_1} + \log G_{n_2,n_2} + \log p_t \\ &\quad \leqslant \beta_1(n_1^{2/3} + n_2^{2/3}) + \beta_2(n_1^{1/3} + n_2^{1/3}) + O(\sqrt{t} + \log n) \\ &\quad \leqslant 2^{1/3}\beta_1 n^{2/3} - 2^{4/3}\beta_1 n^{-1/3} t(1 + o(1)) + 2^{2/3}\beta_2 n^{1/3} + O(\sqrt{t} + \log n), \end{aligned}$$

with β_1, β_2 defined in (7) and β in (1). Since $t = O(n)$, the dominant term of the last upper bound matches that in (2). If $t = \omega(n^{2/3})$, the subdominant term will be negative and of order $\Theta(n^{-1/3}t)$, making the bound exponentially smaller than (2). The main contribution thus comes from $t = O(n^{2/3})$. This is consistent with the results in [13] on the asymptotic normality of t, with mean asymptotic to $c_1 n^{2/3}$ and variance to $c_2 n^{2/3} \log n$ for some explicit constants c_1 and c_2.

For the method of proofs, we will employ a more classical approach based on Mellin transforms (see [5]) and saddle-point method (see [1, 6]), instead of the elementary approach used in [9, 10], which becomes cumbersome when finer asymptotic expansions are required. The approach we adopted is, although standard, becomes more delicate as far as uniformity of error terms with varying m is concerned.

Of additional interest here is that, similar to the functional equation satisfied by the generating function of p_n

$$P(e^{-\tau}) := \sum_{n\geqslant 0} p_n e^{-n\tau} = \sqrt{\frac{\tau}{2\pi}}\,\exp\Big(\frac{\pi^2}{6\tau} - \frac{\tau}{24}\Big) P\big(e^{-4\pi^2/\tau}\big) \qquad (\mathrm{Re}(\tau) > 0), \tag{13}$$

(see [3]), we also have the following (non-modular) relation satisfied by the generating function of $G_{n,m}$.

Theorem 1.2. *For* $\mathrm{Re}(\tau) > 0$*, the function* $G_m(e^{-\tau})$ *satisfies the identity*

$$G_m(e^{-\tau}) = g_m\sqrt{\tau}\,\exp\Big(\frac{\varpi_m}{\tau} + \phi_m\tau\Big) K_m\big(e^{-4\pi^2/\tau}\big) L_m\big(e^{-4\pi^2/\tau}\big), \tag{14}$$

where the constants depending on m *are given by*

$$\begin{cases} g_m := (2\pi)^{-(m^2-3m+4)/4} \displaystyle\prod_{1\leqslant k<j<m} \Gamma\Big(\frac{k+j}{2m}\Big), \\ \varpi_m := \dfrac{\pi^2}{24}\Big(m+1+\dfrac{2}{m}\Big), \quad \phi_m := \dfrac{m^3-7m+2}{96}, \end{cases} \tag{15}$$

and the two functions K_m *and* L_m *by*

$$\begin{cases} K_m(z) := \sqrt{\dfrac{P\big(z^{1/m}\big)}{P\big(z^{1/2}\big)}}\, P\big(z\big)^{(m+2)/4}, \\ L_m(z) := \exp\Bigg(-\dfrac{1}{2m} \displaystyle\sum_{1\leqslant \ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \sum_{j\geqslant 0} \frac{z^{j+\frac{2\ell-1}{2m}}}{\big(j+\frac{2\ell-1}{2m}\big)\big(1-z^{j+\frac{2\ell-1}{2m}}\big)}\Bigg). \end{cases} \tag{16}$$

Both $K_m(z)$ *and* $L_m(z)$ *are analytic in* $|z| < 1$*,* $z \notin [-1, 0]$*.*

The expression (14) is complicated but exact, and is the basis of our saddle-point analysis for characterizing the asymptotic behaviors of $G_{n,m}$. It is derived by Mellin transforms and the functional equation for the Hurwitz zeta function; see [2, §12.9]. Note that

$$Q_3(z) = \prod_{k\geqslant 0} \frac{1}{1-z^{6k+3}} = \frac{P(z^3)}{P(z^6)} = \prod_{k\geqslant 1}(1+z^{3k}),$$

so we also have, by (13), the functional equation

$$Q_3(e^{-\tau}) = \frac{e^{\pi^2/(36\tau)+\tau/8}}{\sqrt{2}\,Q_3\big(e^{-2\pi^2/(9\tau)}\big)}.$$

No such equation is available for higher $Q_m(z)$ with $m \geqslant 4$. On the other hand, the sequence $G_{n,3}$ coincides with A266648 in OEIS [15].

The rest of this paper is structured as follows. The exact expression of G_m in Theorem 1.2 is first proved in the next section. Then we turn to the asymptotics of G_m in Section 3. A uniform asymptotic approximation to $G_{n,m}$ is then derived in Section 4, which is used to characterize in Section 5 the more precise behaviors of $G_{n,m}$ in each of the three phases: sub-critical, critical and super-critical. We then extend the same approach in Section 6 to m-rowed plane partitions, together with a few other similar variants.

Notations. Since $Q_m(z) = 1$ for $m \leqslant 2$, we assume throughout this paper $m \geqslant 3$. The symbols c, c', β and c_j, β_j are generic whose values will always be locally specified. Other symbols are global except otherwise defined (e.g., in Section 6).

2 Exact expression for $G_m(e^{-\tau})$: proof of Theorem 1.2

We prove Theorem 1.2 for the exact expression (14) for $G_m(e^{-\tau})$ in this section by Mellin transforms, starting with rewriting $Q_m(z)$ in (3) as

$$Q_m(z) = \prod_{k\geqslant 0} \prod_{1\leqslant j<2m} \left(\frac{1}{1-z^{2mk+j}}\right)^{w_m(j)}, \tag{17}$$

where

$$w_m(j) := \left\lfloor \frac{m-1-|m-j|}{2} \right\rfloor \qquad (1 \leqslant j < 2m). \tag{18}$$

For convenience, the kth moment of w_m is denoted by $\mu_k(w_m)$:

$$\mu_k = \mu_k(w_m) := \sum_{1\leqslant j<2m} j^k w_m(j) \qquad (k = 0, 1, \dots).$$

By considering the parity of j and m, we deduce that

$$W_m(z) := \sum_{1\leqslant j<2m} w_m(j) z^j = \frac{z^3(1-z^{m-1})(1-z^{m-2})}{(1+z)(1-z)^2} \qquad (m \geqslant 3). \tag{19}$$

From this expression, it is straightforward to compute the first few moments $\mu_k = k![s^k]W_m(e^s)$, as given explicitly in Table 1.

μ_0	μ_1	μ_2	μ_3
$\frac{(m-1)(m-2)}{2}$	$\frac{m(m-1)(m-2)}{2}$	$\frac{m(m-1)(m-2)(7m-3)}{12}$	$\frac{3m^2(m-1)^2(m-2)}{4}$

Table 1: The exact expressions of μ_k for $0 \leqslant k \leqslant 3$.

Since all singularities of $G_m(z)$ lie on the unit circle, we consider the change of variables $z = e^{-\tau}$ and examine the asymptotic behavior of $G_m(e^{-\tau})$ as $|\tau| \to 0$ from the right half-plane.

Let

$$\zeta(s,b) := \sum_{k\geqslant 0} (k+b)^{-s} \qquad (\mathrm{Re}(s) > 1, b > 0)$$

denote the Hurwitz zeta function. In addition to Mellin transforms, some properties we need for the Gamma function $\Gamma(s)$ and $\zeta(s,b)$ can be found in, for example, [2, Ch. 12], [4, Ch. 1] or [19, Chs. XII & XIII]. Since $P(e^{-\tau})$ satisfies (13), we need only derive a similar expression for $Q_m(e^{-\tau})$ in order to prove (14).

Proposition 2.1. *For* $\mathrm{Re}(\tau) > 0$, $q_m(e^{-\tau}) := \log Q_m(e^{-\tau})$ *satisfies*

$$\begin{aligned} q_m(e^{-\tau}) = \frac{(m-1)(m-2)\pi^2}{24m\tau} + \sum_{1\leqslant j<2m} w_m(j) \log\Gamma\left(\frac{j}{2m}\right) \\ - \frac{(m-1)(m-2)}{4}\log(2\pi) + \frac{(m-1)(m-2)(m+3)}{96}\tau + E(\tau), \end{aligned} \tag{20}$$

where $E(\tau)$ *is given by*

$$E(\tau) = E(m;\tau) := \frac{1}{2\pi i}\int_{(-2)} \Gamma(s)\zeta(s+1)\mathscr{M}_m(s)\tau^{-s}\,\mathrm{d}s, \tag{21}$$

with $\int_{(c)}$ *representing* $\int_{c-i\infty}^{c+i\infty}$ *and*

$$\mathscr{M}_m(s) := (2m)^{-s} \sum_{1\leqslant j\leqslant 2m} w_m(j)\zeta\Big(s, \frac{j}{2m}\Big). \tag{22}$$

Proof. Let $\mathscr{M}_m^{[q]}(s)$ be the Mellin transform of $q_m(e^{-\tau})$. Then $\mathscr{M}_m^{[q]}(s) = \Gamma(s)\zeta(s+1)\mathscr{M}_m(s)$ for $\mathrm{Re}(s) > 1$, where $\mathscr{M}_m(s)$ is defined in (22). By the inverse Mellin transform, we have

$$q_m(e^{-\tau}) = \frac{1}{2\pi i}\int_{(c)} \mathscr{M}_m^{[q]}(s)\tau^{-s}\,\mathrm{d}s \qquad (c > 1). \tag{23}$$

We will move the line of integration to the left and collect all the residues of the poles encountered. For that purpose, we need the growth properties of the integrand at $c\pm i\infty$ to ensure the absolute convergence of the integral.

By the known estimate for Gamma function (see [4, §1.18])

$$|\Gamma(c+it)| = O\big(|t|^{c-1/2}e^{-\pi|t|/2}\big), \qquad (c \in \mathbb{R}, |t| > 1),$$

and that for Hurwitz zeta function (see [19, §13.51, p. 276])

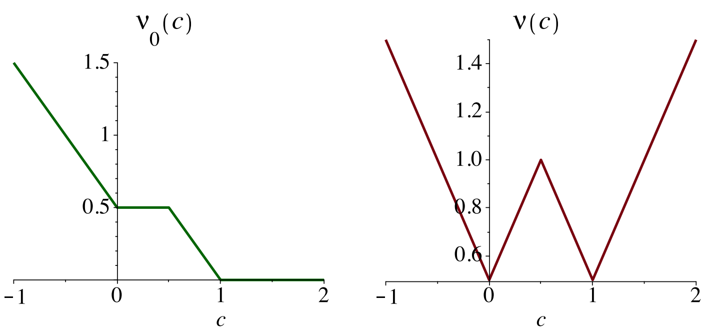
$$|\zeta(c+it, b)| = O\big(|t|^{\nu_0(c)}\log|t|\big), \quad \text{with} \quad \nu_0(c) := \begin{cases} \frac{1}{2} - c, & \text{if } c < 0; \\ \frac{1}{2}, & \text{if } c \in [0, \frac{1}{2}]; \\ 1 - c, & \text{if } c \in [\frac{1}{2}, 1]; \\ 0, & \text{if } c > 1, \end{cases} \tag{24}$$

for $|t| > 1$, we have

$$|\mathscr{M}_m^{[q]}(c+it)| = O\big(m^{2-c}|t|^{\nu(c)}(\log|t|)^2 e^{-\frac{\pi}{2}|t| + t\arg(\tau)}\big), \tag{25}$$

for $c \in \mathbb{R}, |t| > 1$, where

$$\nu(c) := \begin{cases} \frac{1}{2} + |c|, & \text{if } |c - \frac{1}{2}| \geqslant \frac{1}{2}; \\ \min\{\frac{1}{2} + c, \frac{3}{2} - c\}, & \text{if } |c - \frac{1}{2}| \leqslant \frac{1}{2}. \end{cases}$$



$\nu_0(c)$ $\qquad$ $\nu(c)$

Thus the integral in (23) is absolutely convergent as long as $|\arg(\tau)| \leqslant \pi/2 - \varepsilon$ and $|\tau| \to 0$, and this justifies the analytic properties we need for summing the residues, which we now compute. Since $w_m(j) = w_m(2m - j)$ (see (18)), we can rewrite (22) as

$$\mathscr{M}_m(s) = \sum_{1\leqslant j<m} w_m(j)\Big(\zeta\Big(s, \frac{j}{2m}\Big) + \zeta\Big(s, 1 - \frac{j}{2m}\Big)\Big) + w_m(m)\zeta\Big(s, \frac{1}{2}\Big). \tag{26}$$

Observe that $\mathscr{M}_m(-2j) = 0$ for $j \in \mathbb{Z}^+$ because $\zeta(-2j, x) = -B_{2j+1}(x)/(2j+1)$, where $B_{2j+1}(x)$ is the Bernoulli polynomial of order $2j+1$, which satsfies $B_{2j+1}(x) = -B_{2j+1}(1-x)$; see [4, § 1.13]. On the other hand, $\zeta(s+1) = 0$ when $s < -1$ is odd. Thus the only poles of the integrand in (23) are $s = -1$ (simple), $s = 0$ (double) and $s = -1$ (simple); this similarity to that of $\log P(e^{-\tau})$ suggests the possibility of the identity (14).

From these properties, it follows that

$$q_m(e^{-\tau}) = \sum_{-1 \leqslant j \leqslant 1} \mathrm{Res}_{s=j}\big(\Gamma(s)\zeta(s+1)\mathscr{M}_m(s)\tau^{-s}\big) + E(\tau), \tag{27}$$

where $E(\tau)$ is as defined in (21). By the local expansions of $\Gamma(s)$, $\zeta(s+1)$ and $\zeta(s,b)$ for $s \sim 0$ (see [4]):

$$\Gamma(s) = \frac{1}{s} - \gamma + O(|s|), \quad \zeta(s+1) = \frac{1}{s} + \gamma + O(|s|),$$
$$\zeta(s,b) = \frac{1}{2} - b + \Big(\log\Gamma(b) - \frac{1}{2}\log(2\pi)\Big)s + O(|s|^2),$$

where γ is the Euler-Mascheroni constant, we then have

$$q_m(e^{-\tau}) = \frac{\pi^2\mu_0}{12m\tau} + \sum_{1\leqslant j<2m} w_m(j)\log\Gamma\Big(\frac{j}{2m}\Big) - \frac{\mu_0}{2}\log(2\pi) + \Big(-\frac{\mu_2}{8m} + \frac{\mu_1}{4} - \frac{m\mu_0}{12}\Big)\tau + E(\tau).$$

This, together with the expressions in Table 1, proves (23). □

We now evaluate $E(\tau)$, beginning with a simple lemma.

Lemma 2.2. *For integers $m > 1$, $1 \leqslant \ell \leqslant 2m$ and real number θ, we have*

$$\sum_{1\leqslant k<j<m} \sin\Big(\theta + \frac{\ell(k+j)\pi}{m}\Big) = \sin\theta \times \begin{cases} \binom{m-1}{2}, & \text{for } \ell = 2m; \\ -\Big\lfloor \frac{m-1}{2} \Big\rfloor, & \text{for } \ell = m; \\ 1, & \text{for } 1 \leqslant \ell < 2m; \ell \neq m \text{ and } \ell \text{ even;} \\ -\frac{\cos(\ell\pi/m)}{1-\cos(\ell\pi/m)}, & \text{for } 1 \leqslant \ell < 2m, \ell \neq m \text{ and } \ell \text{ odd.} \end{cases}$$

Proof. (Sketch) From the identity

$$\Big(\sum_{1\leqslant k<m} \exp\Big(\frac{k\ell\pi i}{m}\Big)\Big)^2 = 2\sum_{1\leqslant k<j<m} \exp\Big(\frac{(k+j)\ell\pi i}{m}\Big) \times \sum_{1\leqslant k<m} \exp\Big(\frac{2k\ell\pi i}{m}\Big),$$

and straightforward simplifications, the lemma follows. □

We now compute the error term $E(\tau)$. Let $p(z) := \log P(z)$.

Proposition 2.3. *The error term $E(\tau)$ defined in (21) satisfies*

$$E(\tau) = \kappa_m\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big),$$

for $\mathrm{Re}(\tau) > 0$*, where (K_m, L_m defined in* (16)*)*

$$\kappa_m(z) := \log K_m(z) - p(z) = \frac{m-2}{4}p(z) + \frac{1}{2}p\big(z^{1/m}\big) - \frac{1}{2}p\big(z^{1/2}\big), \tag{28}$$

$$\lambda_m(z) := \log L_m(z) = -\frac{1}{2m}\sum_{1\leqslant\ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \sum_{k\geqslant 0} \frac{z^{k+\frac{2\ell-1}{2m}}}{\big(k+\frac{2\ell-1}{2m}\big)\big(1-z^{k+\frac{2\ell-1}{2m}}\big)}. \tag{29}$$

Proof. We first rewrite the single-sum relation (22) for $\mathscr{M}_m(s)$ as a double sum:

$$\mathscr{M}_m(s) = (2m)^{-s} \sum_{1\leqslant h<j<m} \zeta\Big(s, \frac{h+j}{2m}\Big).$$

Combining with the functional equation for the Hurwitz zeta function (see [2, §12.9])

$$\zeta\Big(s, \frac{j}{d}\Big) = \frac{2\Gamma(1-s)}{(2d\pi)^{1-s}} \sum_{1\leqslant \ell\leqslant d} \sin\Big(\frac{\pi s}{2} + \frac{2\ell j\pi}{d}\Big)\zeta\Big(1-s, \frac{\ell}{d}\Big) \qquad (d = 1, 2, \dots), \tag{30}$$

we then have

$$\mathscr{M}_m(s) = \frac{\Gamma(1-s)}{m(2\pi)^{1-s}} \sum_{0\leqslant \ell\leqslant 2m} \zeta\Big(1-s, \frac{\ell}{2m}\Big) \sum_{1\leqslant k<j<m} \sin\Big(\frac{\pi s}{2} + \frac{\ell(k+j)\pi}{m}\Big).$$

Now, by Lemma 2.2, the sum above can be reduced to

$$\begin{aligned}\mathscr{M}_m(s) = \frac{\Gamma(1-s)}{m(2\pi)^{1-s}} \sin\Big(\frac{\pi s}{2}\Big) \Bigg[&\binom{m-1}{2}\zeta(1-s) - \Big\lfloor\frac{m-1}{2}\Big\rfloor\zeta\Big(1-s, \frac{1}{2}\Big) \\ &+ \sum_{1\leqslant \ell<m, 2\ell\neq m} \zeta\Big(1-s, \frac{\ell}{m}\Big) - \sum_{1\leqslant \ell<m, 2\ell-1\neq m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \zeta\Big(1-s, \frac{2\ell-1}{2m}\Big)\Bigg].\end{aligned}$$

Then, by the relation

$$\sum_{1\leqslant \ell\leqslant d} \zeta\Big(s, \frac{\ell}{d}\Big) = d^s\zeta(s) \qquad (d = 2, 3, \dots), \tag{31}$$

which implies, in particular, $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$, we deduce that

$$\mathscr{M}_m(s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \sin\Big(\frac{\pi s}{2}\Big) \Bigg[c(m, s)\zeta(1-s) - \frac{1}{m} \sum_{1\leqslant \ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \zeta\Big(1-s, \frac{2\ell-1}{2m}\Big)\Bigg],$$

where $c(m, s) := (m-2)/2 + m^{-s} - 2^{-s}$.

By applying the change of variables $s \mapsto -s$ in the integral representation in (21) of $E(\tau)$, we obtain

$$E(\tau) = -\frac{1}{2\pi i} \int_{(2)} \Gamma(-s)\zeta(1-s)\mathscr{M}_m(-s)\tau^s \,\mathrm{d}s. \tag{32}$$

Note that the functional equation (30) with $d = j = 1$ becomes that for the Riemann zeta function:

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s) \sin\Big(\frac{\pi s}{2}\Big). \tag{33}$$

By this and Euler's reflection formula for Gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{34}$$

we then get

$$\Gamma(-s)\zeta(1-s) = -\frac{(2\pi)^{1-s}}{s\sin(\pi s)} \zeta(s) \cos\Big(\frac{\pi s}{2}\Big).$$

Consequently, the integrand in (32) can be written as

$$\Gamma(-s)\zeta(1-s)\mathscr{M}_m(-s)\tau^s = -\frac{1}{2}\left(\frac{4\pi^2}{\tau}\right)^{-s}\Gamma(s)\zeta(s)$$
$$\times\left[c(m,-s)\zeta(1+s) - m^{-1}\sum_{1\leqslant \ell<m}\frac{\cos\left(\frac{(2\ell-1)\pi}{m}\right)}{1-\cos\left(\frac{(2\ell-1)\pi}{m}\right)}\zeta\left(1+s,\frac{2\ell-1}{2m}\right)\right].$$

The two expressions (28) (contributed by terms involving $c(m,s)$) and (29) (contributed by terms involving the partial sum with the cosine functions) then follow from inverting the Mellin transform using the relation

$$J(b,\tau) := \frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta(1+s,b)\tau^s\,\mathrm{d}s = \sum_{k\geqslant 0}\frac{e^{-(k+b)/\tau}}{(k+b)\left(1-e^{-(k+b)/\tau}\right)}, \tag{35}$$

for $\mathrm{Re}(\tau) > 0$ and $b > 0$, where $c > 1$. In particular, the right-hand side equals $p\left(e^{-1/\tau}\right)$ when $a = b = 1$. This completes the proof. □

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of combining Proposition 2.1 with Proposition 2.3 and (13). □

3 Asymptotics of $\log G_m(e^{-\tau})$

We derive the asymptotic behaviors of $\log G_m(e^{-\tau})$ as $m\to\infty$ and $|\tau|\to 0$. From Theorem 1.2 and Proposition 2.3, we have

$$\log G_m(e^{-\tau}) = \frac{\varpi_m}{\tau} + \frac{1}{2}\log\tau + \log g_m + \phi_m\tau + \kappa_m\left(e^{-4\pi^2/\tau}\right) + \lambda_m\left(e^{-4\pi^2/\tau}\right), \tag{36}$$

for $\mathrm{Re}(\tau) > 0$. Since $\kappa_m(z)$ depends only on $p(z)$ (see (28)), which, by (13), satisfies

$$p(e^{-\tau}) = \frac{\pi^2}{6\tau} - \frac{\tau}{24} + \frac{1}{2}\log\tau - \frac{1}{2}\log(2\pi) + p\left(e^{-4\pi^2/\tau}\right) \qquad (\mathrm{Re}(\tau) > 0), \tag{37}$$

so we need only to examine more closely the asymptotics of $\log g_m$ and λ_m when m is large and $|\tau|\to 0$. Complications arise when τ may depend also on m.

3.1 Asymptotics of $\log g_m$

We now derive an asymptotic expansion for $\log g_m$ by the Euler-Maclaurin formula (see [11, Ch. VIII]).

Proposition 3.1. *When $m\to\infty$, $\log g_m$ satisfies the asymptotic expansion*

$$\log g_m \sim -\frac{7\zeta(3)}{8\pi^2}m^2 + \frac{11}{24}\log m + c_1 - \sum_{j\geqslant 1}\frac{B_{2j}B_{2j+2}(-\pi^2)^j}{8j(j+1)(2j)!}m^{-2j}, \tag{38}$$

where $c_1 := \frac{1}{2}\zeta'(-1) - \frac{11}{24}\log\pi - \frac{7}{24}\log 2$ and $B_j = B_j(0)$ denote the Bernoulli numbers.

Proof. We begin with

$$\log g_m = -\frac{m^2-3m+4}{4}\log(2\pi) + S_m,$$

where

$$S_m := \sum_{1\leqslant j<2m} w_m(j)\log\Gamma\Big(\frac{j}{2m}\Big).$$

Since $w_m(j) = w_m(2m-j)$, we have, by Euler's reflection formula (34),

$$\begin{aligned} S_m &= \frac{\mu_0}{2}\log\pi - \sum_{1\leqslant j<m}\Big\lfloor\frac{j-1}{2}\Big\rfloor\log\Big(\sin\frac{j\pi}{2m}\Big) \\ &= \frac{(m-1)(m-2)}{4}\log\pi - \sum_{1\leqslant j<m}\frac{j-1}{2}\log\Big(\sin\frac{j\pi}{2m}\Big) + \frac{1}{2}\sum_{1\leqslant j\leqslant\lfloor m/2\rfloor}\log\Big(\sin\frac{j\pi}{m}\Big) \\ &= \frac{(m-1)(m-2)}{4}\log\pi - \frac{S_{m,1}}{2} + \frac{S_{m,2}}{2} + \frac{S_{m,3}}{2}, \end{aligned}$$

where

$$S_{m,1} := \sum_{1\leqslant j\leqslant m} j\log\Big(\sin\frac{j\pi}{2m}\Big),\quad S_{m,2} := \sum_{1\leqslant j\leqslant m}\log\Big(\sin\frac{j\pi}{2m}\Big),\quad S_{m,3} := \sum_{1\leqslant j\leqslant\lfloor m/2\rfloor}\log\Big(\sin\frac{j\pi}{m}\Big).$$

The last two sums are easily simplified by the elementary identity

$$\prod_{1\leqslant j<k}\sin\Big(\frac{\pi j}{k}\Big) = \frac{k}{2^{k-1}} \qquad (k=1,2,\dots),$$

giving

$$S_{m,2} = -(m-1)\log 2 + \frac{\log m}{2} \quad\text{and}\quad S_{m,3} = -\frac{m-1}{2}\log 2 + \frac{\log m}{2}. \tag{39}$$

We now evaluate $S_{m,1}$. By the local expansion $\log(\sin x) = \log x + O(x^2)$, we decompose first the sum into two parts:

$$S_{m,1} = \sum_{1\leqslant j\leqslant m} j\Big(\log\Big(\sin\frac{j\pi}{2m}\Big) - \log\Big(\frac{j\pi}{2m}\Big)\Big) + \sum_{1\leqslant j\leqslant m} j\log\Big(\frac{j\pi}{2m}\Big),$$

and then apply Euler-Maclaurin formula (see [11, Ch. VIII]) to each sum, yielding

$$\sum_{1\leqslant j\leqslant m} j\Big(\log\Big(\sin\frac{j\pi}{2m}\Big) - \log\Big(\frac{j\pi}{2m}\Big)\Big) = c_2m^2 - \frac{m}{2}\log\frac{\pi}{2} - \frac{1}{12}\Big(1+\log\frac{\pi}{2}\Big) + O(m^{-2}),$$

where

$$c_2 := \frac{1}{m^2}\int_0^m x\Big(\log\Big(\sin\frac{x\pi}{2m}\Big) - \log\Big(\frac{x\pi}{2m}\Big)\Big)\,\mathrm{d}x = \frac{7\zeta(3)}{4\pi^2} - \frac{\log\pi}{2} + \frac{1}{4},$$

and

$$\sum_{1\leqslant j\leqslant m} j\log\Big(\frac{j\pi}{2m}\Big) = \Big(\frac{1}{2}\log\frac{\pi}{2} - \frac{1}{4}\Big)m^2 + \frac{m}{2}\log\frac{\pi}{2} + \frac{\log m}{12} + \frac{1}{12} - \zeta'(-1) + O(m^{-2}).$$

Summing up these two parts, we have

$$S_{m,1} = \Big(\frac{7\zeta(3)}{4\pi^2} - \frac{\log 2}{2}\Big)m^2 + \frac{\log m}{12} - \Big(\zeta'(-1) + \frac{1}{12}\log\frac{\pi}{2}\Big) + O(m^{-2}). \tag{40}$$

By substituting (39) and (40) into

$$\log g_m = -\frac{1}{2}\log\pi - \frac{m^2-3m+4}{4}\log 2 - \frac{S_{m,1}}{2} + \frac{S_{m,2}}{2} + \frac{S_{m,3}}{2}, \tag{41}$$

we obtain the expansion (38) up to an error of order m^{-2}. Further terms in (38) are computed by refining the expansion for $S_{m,1}$ following the same procedure and using the relation

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\log(\sin(x))\Big|_{x=\pi/2} = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\tan(x)\Big|_{x=\pi/2} = \frac{(2i)^k}{k}(2^k-1)B_k \qquad (k \geqslant 2),$$

where $i=\sqrt{-1}$; see [15, A155585]. □

3.2 Asymptotics of $E(\tau)$

We now consider the asymptotic behavior of the key "calibrating" term $E(\tau)$ defined in (21) as $\tau \to 0$. This term is asymptotically negligible when $m = o(n^{1/3})$, but plays a role for larger m, notably in the transitional zone when $m \asymp n^{1/3}$. We then need finer asymptotic approximations to $E(\tau)$, which equals, by Proposition 2.3, $E(\tau) = \kappa_m\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big)$. We begin with the asymptotics of the first term, which is simpler.

Corollary 3.2. *Assume* $\mathrm{Re}(\tau) \to 0$ *in the half-plane* $\mathrm{Re}(\tau) > 0$*. Then the function* κ_m *satisfies*

$$\begin{aligned}\kappa_m\big(e^{-4\pi^2/\tau}\big) &= \frac{1}{2}\,p\big(e^{-4\pi^2/(m\tau)}\big) + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\\ &= \begin{cases} O\big(e^{-\mathrm{Re}(4\pi^2/(m\tau))}\big), & \text{if } m|\tau| \to 0,\\ \dfrac{m\tau}{48} + \dfrac{1}{4}\log\dfrac{2\pi}{m\tau} - \dfrac{\pi^2}{12m\tau} + \dfrac{1}{2}p(e^{-m\tau}) + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big), & \text{if } m|\tau| \to \infty.\end{cases}\end{aligned} \tag{42}$$

Proof. By (28), we obtain the first relation in (42). On the other hand, the series

$$p\big(e^{-4\pi^2/\tau}\big) = \sum_{j\geqslant 1}\frac{e^{-4j\pi^2/\tau}}{j\big(1-e^{-4j\pi^2/\tau}\big)}$$

is itself an asymptotic expansion when $|\tau| \to 0$. The other estimate in (42) when $m|\tau| \to \infty$ follows from the functional equation (13). □

We now examine the other term $\lambda_m\big(e^{-4\pi^2/\tau}\big)$, beginning with the asymptotics of the integral $J(b,w)$ defined in (35).

Lemma 3.3. *If* $b > 0$*, then*

$$J(b,\tau) = \begin{cases} b^{-1}e^{-b/\tau}\big(1 + O\big(e^{-\mathrm{Re}(b/\tau)} + e^{-\mathrm{Re}(1/\tau)}\big)\big), & \text{as } |\tau| \to 0;\\ \zeta(2,b)\tau - \dfrac{1}{2}\log\tau + \dfrac{1}{2}\psi(b) + O(1), & \text{as } |\tau| \to \infty,\end{cases} \tag{43}$$

uniformly in the half-plane $\mathrm{Re}(\tau) > 0$*, where* ψ *is the digamma function. These estimates hold also when* $b/|\tau| \to 0$ *and* $b/|\tau| \to \infty$*, respectively.*

Proof. The estimate in the small $|\tau|$ case follows from the series representation in (35), while that in the large $|\tau|$ case from moving the line of integration in the integral representation in (35) to the left, adding the residues at $s=1$ and $s=0$. Note that $\zeta(2,b) = b^{-2} + \pi^2/6 + O(b)$ and $\psi(b) \to b^{-1}$ when $b \to 0$. □

Define

$$\varphi_d(z) := \sum_{\ell\geqslant 1}(2\ell-1)^{1-2d}\frac{e^{-2(2\ell-1)\pi^2/z}}{1-e^{-2(2\ell-1)\pi^2/z}} \qquad (d\in\mathbb{Z}, \mathrm{Re}(z)>0). \tag{44}$$

Proposition 3.4. *Uniformly for* $|\tau|\to 0$ *in the half-plane* $\mathrm{Re}(\tau)>0$,

$$\lambda_m\big(e^{-4\pi^2/\tau}\big) = \big(1+O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\big)\big(m^2\xi_2(m\tau)+\xi_1(m\tau)+O\big(m^{-2}|\xi_0(m\tau)|\big)\big), \tag{45}$$

where

$$\xi_2(z) := -\frac{2}{\pi^2}\,\varphi_2(z), \quad \xi_1(z) := \frac{5}{6}\,\varphi_1(z), \quad \xi_0(z) := \varphi_0(z). \tag{46}$$

Note that when $m=O(1)$, the O-term is of the same order as $\xi_1(m\tau)\asymp e^{-\mathrm{Re}(2\pi^2/m\tau)}$.

Proof. In the defining series (29), we observe that the inner sum with $z=e^{-4\pi^2/\tau}$ is itself an asymptotic expansion when $|\tau|\to 0$, namely, the term with $k=0$ is dominant and all others with $k\geqslant 1$ are exponentially smaller. Thus

$$\begin{aligned}\lambda_m\big(e^{-4\pi^2/\tau}\big) &= -\big(1+O\big(e^{-\mathrm{Re}(4\pi^2/\tau)}\big)\big)\sum_{1\leqslant\ell<m}\frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)}\cdot\frac{e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}}{(2\ell-1)\left(1-e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}\right)}\\ &= -\big(1+O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\big)\sum_{1\leqslant\ell\leqslant\lfloor m/2\rfloor}\frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)}\cdot\frac{e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}}{(2\ell-1)\left(1-e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}\right)},\end{aligned} \tag{47}$$

where in the second approximation we truncate terms with $\ell>\lfloor m/2\rfloor$ whose total contribution is bounded above by $O\big(m^2e^{-\mathrm{Re}(2\pi^2/\tau)}\big)$.

By expanding the ratio of cosines in (47) using the inequalities

$$-x^2\leqslant\frac{\cos x}{1-\cos x}-\frac{2}{x^2}+\frac{5}{6}\leqslant x^2 \qquad (0\leqslant x\leqslant 1/2),$$

we then get (45) by summing the resulting terms and extending then the summation range to infinity. The error terms introduced are bounded above by

$$O\bigg(\sum_{\ell>\lfloor m/2\rfloor}\Big(\frac{m^2}{(2\ell-1)^3}+\frac{1}{2\ell-1}+\frac{2\ell-1}{m^2}\Big)e^{-\mathrm{Re}(2(2\ell-1)\pi^2/(m\tau))}\bigg) = O\big(m^{-1}e^{-\mathrm{Re}(2\pi^2/\tau)}\big).$$

This proves the proposition. □

When $z\to 0$, we see that $\varphi_d(z)$ is itself an asymptotic expansion. However, when $z\to\infty$, the asymptotic behavior of ξ_2,ξ_1,ξ_0 cannot be read directly from their defining equations. We now consider this range of z. Recall the functions $\eta_d(z)$ defined in (9), which are themselves asymptotic expansions for large $|z|$ in the right half-plane.

Lemma 3.5. *The functions* $\xi_d(z)$ *(*$j=0,1,2$*) satisfy the identities:*

$$\xi_0(z) = \frac{z^2}{48\pi^2}-\frac{1}{24}+\frac{z^2}{2\pi^2}\,\eta_0(z), \tag{48}$$

$$\xi_1(z) = \frac{5z}{96}+\frac{5}{24}\log\Big(\frac{\pi}{2z}\Big)-\frac{5}{12}\,\eta_1(z), \tag{49}$$

$$\xi_2(z) = -\frac{z}{96}+\frac{7\zeta(3)}{8\pi^2}-\frac{\pi^2}{24z}+\frac{\zeta(3)}{2z^2}-\frac{\eta_2(z)}{z^2}, \tag{50}$$

which are also asymptotic expansions for large $|z|$ *in* $\mathrm{Re}(z)>0$.

Proof. We apply the same Mellin transform techniques, together with the functional equation (33) for the Riemann zeta function, as in the previous section.

Consider first $\xi_2(z)$. By direct calculations using (31), we have

$$\xi_2(z) = -\frac{2}{\pi^2}\cdot\frac{1}{2\pi i}\int_{(3/2)} X_2(s)z^s\,\mathrm{d}s,$$

where

$$X_2(s) = \Gamma(s)\zeta(s)(1-2^{-3-s})\zeta(3+s)(2\pi^2)^{-s}.$$

By a similar analysis as in the proof of Proposition 20, we deduce that

$$\xi_2(z) = -\frac{2}{\pi^2}\bigg(\sum_{-2\leqslant k\leqslant 1}\mathrm{Res}_{s=k}(X_2(s)z^s) + \frac{1}{2\pi i}\int_{(-5/2)} X_2(s)z^s\,\mathrm{d}s\bigg).$$

The sum of the residues yields the first four terms on the right-hand side of (50). We then simplify the integral

$$\int_{(-5/2)} X_2(s)z^s\,\mathrm{d}s = \int_{(1/2)} X_2(-2-s)z^{-s-2}\,\mathrm{d}s.$$

By (33),

$$\begin{aligned} X_2(-2-s) &= \Gamma(-2-s)\zeta(-2-s)(1-2^{s-1})\zeta(1-s)\big(2\pi^2\big)^{s+2}\\ &= \frac{\pi^2}{2}(1-2^{1-s})\zeta(s+3)\Gamma(s)\zeta(s),\end{aligned}$$

which is nothing but the Mellin transform of $\frac{\pi^2}{2}\eta_2(z)$. This proves (50).

The proofs of the other two identities (48) and (49) are similar, and omitted. □

Corollary 3.6. *Assume* $|\tau|\to 0$ *in the half-plane* $\mathrm{Re}(\tau)>0$*. Then the function* $\lambda_m\big(e^{-4\pi^2/\tau}\big)$ *satisfies:*
(i) *if* $m|\tau|\leqslant 1$*, then*

$$\lambda_m\big(e^{-4\pi^2/\tau}\big) = m^2\xi_2(m\tau)+\xi_1(m\tau)+O\big(m^{-2}e^{-\mathrm{Re}(2\pi^2/(m\tau))}\big); \tag{51}$$

(ii) *if* $m|\tau|\geqslant 1$*, then*

$$\begin{aligned}\lambda_m\big(e^{-4\pi^2/\tau}\big) &= m^2\xi_2(m\tau)+\xi_1(m\tau)+O(|\tau|^2)\\ &= \frac{\zeta(3)-2\eta_2(m\tau)}{2\tau^2}-\frac{\pi^2 m}{24\tau}+\frac{7\zeta(3)}{8\pi^2}m^2-\frac{m^3\tau}{96}+\frac{5m\tau}{96}\\ &\quad -\frac{5}{24}\log\Big(\frac{2m\tau}{\pi}\Big)-\frac{5\eta_1(m\tau)}{12}+O(|\tau|^2).\end{aligned} \tag{52}$$

4 Asymptotics of $G_{n,m}$

Our analytic approach to the asymptotics of $G_{n,m}$ relies on the Cauchy integral formula

$$G_{n,m} = [z^n]G_m(z) = \frac{1}{2\pi i}\oint_{|z|=e^{-\rho}} z^{-n-1}G_m(z)\,\mathrm{d}z \qquad (\rho>0).$$

Since $G_m(e^\tau)$ grows very fast near the singularity $\tau=0$ (see (14)), we will apply the saddle-point method to the integral on the right-hand side. We derive first crude (but effective) approximations to $G_{n,m}$ and then sketch our approach to refining them, more details being given in the next sections.

4.1 Crude bounds

By the nonnegativity of the coefficients, we have the simple inequality

$$\begin{aligned} G_{n,m} &\leqslant e^{n\rho} G_m(e^{-\rho}) \\ &= \exp\Big((n+\phi_m)\rho + \frac{\varpi_m}{\rho} + \kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big)\Big) \qquad (n,m \geqslant 1), \end{aligned}$$

where $\rho = \rho(n,m) > 0$ is taken to be the saddle-point, namely, it satisfies the equation

$$nG_m(e^{-\rho}) = e^{-\rho} G_m'(e^{-\rho}), \quad \text{or} \quad n + \phi_m = \frac{\varpi_m}{\rho^2} - \partial_\rho \Big(\kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big)\Big).$$

By (42) and (51), if m is not too large, or, more precisely, if

$$\kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big) = O\big(m^2 e^{-2\pi^2/(m\rho)}\big) = o\Big(\frac{\phi_m}{\rho}\Big) \asymp \frac{m}{\rho}, \quad \text{or} \quad m\rho \to 0,$$

then the saddle-point satisfies

$$n + \phi_m \sim \frac{\varpi_m}{\rho^2}, \quad \text{or} \quad \rho \sim \sqrt{\frac{\varpi_m}{n+\phi_m}}.$$

Thus ρ is of order $\sqrt{m/n}$, which in turn specifies the range of m: $m\rho \asymp m^{3/2}/n^{1/2} \to 0$, or $m = o(n^{1/3})$. In this range of m, we see that

$$\log G_{n,m} \leqslant 2\sqrt{(n+\phi_m)\varpi_m}(1+o(1)) \sim \frac{\pi}{\sqrt{6}}\sqrt{mn},$$

which is tight when compared with the asymptotic estimate in (5). Note that $\kappa_m(e^{-4\pi^2/\rho})$ is not uniformly $o(1)$ in this range, although it is of a smaller order than m/ρ; indeed, if

$$m \leqslant \frac{6\pi^{2/3} n^{1/3}}{(\log n - 2\log\log n + \log \omega_n)^{2/3}}, \tag{53}$$

for any sequence ω_n tending to infinity, then

$$\kappa_m(e^{-4\pi^2/\rho}) \asymp m^2 e^{-2\pi^2/(m\rho)} \asymp \omega_n^{-2/3} \to 0.$$

For larger m with $m\rho \geqslant \varepsilon > 0$, we use (45) and Lemma 3.5, giving

$$\log G_m(e^{-\rho}) = \frac{\zeta(3)}{2\rho^2} + \frac{\pi^2}{24\rho} + \frac{\log \rho}{24} + O(1),$$

as $\rho \to 0$ and $m\rho \to \infty$. Thus the saddle-point ρ satisfies

$$\rho \sim \Big(\frac{\zeta(3)}{2}\Big)^{1/3} n^{-1/3},$$

implying that

$$\log G_{n,m} \leqslant \frac{3\zeta(3)^{1/3}}{2} n^{2/3}(1+o(1)),$$

which is also tight in view of (8).

4.2 The uniform saddle-point approximation

The tightness of the crude bounds in the previous subsections is well-known. We now refine these bounds and derive a uniform asymptotic approximate for $G_{n,m}$.

For convenience, let $\Lambda(z) := \log G_m(z)$ and write the Taylor expansion

$$\Lambda(e^{-\rho(1+it)}) = \sum_{k\geqslant 0} \frac{\Lambda_k(\rho)}{k!}(-it)^k, \quad \text{with} \quad \Lambda_k(\rho) := \rho^k \sum_{0\leqslant j\leqslant k} \left\{ {k \atop j} \right\} e^{-j\rho}\Lambda^{(j)}(e^{-\rho}). \tag{54}$$

In particular,

$$\Lambda_1(\rho) = \rho e^{-\rho}\Lambda'(e^{-\rho}), \quad \text{and} \quad \Lambda_2(\rho) = \rho^2\big(e^{-\rho}\Lambda'(e^{-\rho}) + e^{-2\rho}\Lambda''(e^{-\rho})\big).$$

As we will see below, each $\Lambda_k(\rho)$ is of the same order as $\Lambda(e^{-\rho}) = \log G_m(e^{-\rho})$.

Theorem 4.1. *Uniformly for $m \geqslant 1$*

$$G_{n,m} = \frac{\rho e^{n\rho} G_m(e^{-\rho})}{\sqrt{2\pi\Lambda_2(\rho)}}\big(1 + O\big(\Lambda_2(\rho)^{-1}\big)\big), \tag{55}$$

where $\rho > 0$ solves the equation

$$n\rho - \Lambda_1(\rho) = 0, \quad \textit{or} \quad n = \partial_\tau \log G_m(e^{-\tau})\big|_{\tau=\rho}. \tag{56}$$

The extra factor ρ in (55) is cancelled with a factor ρ^2 in $\sqrt{\Lambda_2(\rho)}$.

We will prove Theorem 4.1 in Section 4.5. The justification of the finer saddle-point approximation (55) consists of the following two propositions, which will be proved in Sections 4.3 and 4.4, respectively.

Proposition 4.2. *Let $\delta := (n\rho)^{-2/5} > 0$. Then*

$$\int_{\delta\rho\leqslant|t|\leqslant\pi} e^{n(\rho+it)} G_m(e^{-\rho-it})\,\mathrm{d}t = O\big(e^{n\rho}G_m(e^{-\rho})e^{-c'(n\rho)^{1/5}}\big). \tag{57}$$

Proposition 4.3. *Let $\delta := (n\rho)^{-2/5} > 0$. Then, uniformly for $|t| \leqslant \delta$, the Taylor expansion (54) is itself an asymptotic expansion as $|t| \to 0$.*

Note that $\delta = (n\rho)^{-2/5} > 0$ is a specially tuned parameter, chosen in the standard way such that $(n\rho)\delta^2 \to \infty$ and $(n\rho)\delta^3 \to 0$.

4.3 Justification of the saddle-point method: proof of Proposition 4.2

Before proving Proposition 4.2, we derive a few useful expressions.

Lemma 4.4. *For $|z| < 1$,*

$$G_m(z) = \exp\left(\sum_{\ell\geqslant 1}\frac{U_m(z^\ell)}{\ell}\right), \quad \textit{with} \quad U_m(z) := \frac{z}{1-z} + \frac{z^3(1-z^{m-2})(1-z^{m-1})}{(1-z^{2m})(1-z)(1-z^2)}. \tag{58}$$

Proof. By (17), we have, for $|z| < 1$,

$$\begin{aligned}\log G_m(z) &= -\sum_{k\geqslant 1}\log(1-z^k) - \sum_{1\leqslant j<2m} w_m(j)\sum_{k\geqslant 0}\log(1-z^{2mk+j}) \\ &= \sum_{\ell\geqslant 1}\frac{z^\ell}{\ell(1-z^\ell)} + \sum_{1\leqslant j<2m} w_m(j)\sum_{\ell\geqslant 1}\frac{z^{j\ell}}{\ell(1-z^{2m\ell})}.\end{aligned}$$

Thus

$$U_m(z) = \frac{z}{1-z} + \frac{1}{1-z^{2m}} \sum_{1\leqslant j<2m} w_m(j) z^j.$$

Then (58) follows from (19). □

Lemma 4.5. *For $\rho > 0$*

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(|V_m(e^{-\rho+it})| - V_m(e^{-\rho})\bigr) \qquad (-\pi \leqslant t \leqslant \pi), \tag{59}$$

where

$$V_m(z) := \frac{z(1-z^m)}{2(1-z)^2(1+z^m)}. \tag{60}$$

Proof. Since each $U_m(z^\ell)$ contains only nonnegative Taylor coefficients, we have, by (58),

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(-U_m(e^{-\rho}) + \mathrm{Re}(U_m(e^{-\rho+it}))\bigr) \qquad (-\pi \leqslant t \leqslant \pi). \tag{61}$$

From (58), we have the decomposition

$$U_m(z) = V_m(z) + \frac{z^{2m}}{1-z^{2m}} + \frac{z}{2(1-z^2)}, \tag{62}$$

where each term contains only nonnegative Taylor coefficients; this implies that we also have

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(-V_m(e^{-\rho}) + \mathrm{Re}(V_m(e^{-\rho+it}))\bigr),$$

from which (59) follows. □

Another interesting use of (58) is the following very effective way of computing $G_{n,m}$, with only weak dependence on m.

Corollary 4.6. *For $m \geqslant 1$, $G_{n,m}$ satisfies $G_{0,m} = 1$ and for $n \geqslant 1$*

$$G_{n,m} = \frac{1}{n} \sum_{1\leqslant k\leqslant n} G_{n-k,m} \sum_{d\,|\,k} [z^d] z U_m'(z),$$

where

$$[z^d]zU_m'(z) = \begin{cases} \dfrac{d}{2} + \dfrac{dm}{4}\Bigl(1 + (-1)^{\lfloor d/m\rfloor}\Bigl(2\Bigl\{\dfrac{d}{m}\Bigr\} - 1\Bigr)\Bigr), & \text{if } d \text{ is odd;} \\ \dfrac{dm}{4}\Bigl(1 + (-1)^{\lfloor d/m\rfloor}\Bigl(2\Bigl\{\dfrac{d}{m}\Bigr\} - 1\Bigr)\Bigr), & \text{if } d \text{ is even}, d \nmid 2m; \\ d, & \text{if } d \,|\, 2m. \end{cases} \tag{63}$$

Proof. Since $(1-x)/(1+x) = 1 - 2x/(1+x)$, we have, by a direct expansion,

$$V_m(z) = \frac{m}{4} \sum_{d\geqslant 1} \Bigl(1 + (-1)^{\lfloor d/m\rfloor}\Bigl(2\Bigl\{\frac{d}{m}\Bigr\} - 1\Bigr)\Bigr) z^d. \tag{64}$$

Now taking derivative with respect to z and then multiplying by z on both sides of (58) give

$$zG_m'(z) = G_m(z)\sum_{\ell\geqslant 1} z^\ell U_m'(z^\ell),$$

or, taking coefficient of z^n on both sides yields

$$G_{n,m} = \frac{1}{n}\sum_{1\leqslant k\leqslant n} G_{n-k,m}[z^k]\sum_{\ell\geqslant 1} z^\ell U_m'(z^\ell) = \frac{1}{n}\sum_{1\leqslant k\leqslant n} G_{n-k,m}\sum_{d\,|\,k}[z^d]zU_m'(z).$$

By (62) and (64), we then deduce (63). □

We now focus on uniform bounds for $|V_m(e^{-\rho-it})|$.

Proposition 4.7. *For any* $3 \leqslant m \leqslant n$ *and* $\rho \to 0^+$,

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \begin{cases} 1 - c\rho^{-2}t^2, & \text{if } |t| \leqslant \rho; \\ \dfrac{7}{8}, & \text{if } \rho \leqslant |t| \leqslant \pi. \end{cases} \tag{65}$$

Before the proof, we observe that $V_m(z)$ admits the partial fraction expansion,

$$V_m(z) = \frac{m}{4(1-z)} + \sum_{1\leqslant j\leqslant m} \frac{e_{m,j}^2}{m(1-e_{m,j})^2(e_{m,j}-z)}, \quad \text{with} \quad e_{m,j} := e^{(2j+1)\pi i/m},$$

which shows the subtlety of estimating

$$\big|V_m(e^{-\rho-it})\big| = \frac{e^{-\rho}}{2(1-2e^{-\rho}\cos t + e^{-2\rho})}\sqrt{\frac{1-2e^{-m\rho}\cos(mt)+e^{-2m\rho}}{1+2e^{-m\rho}\cos(mt)+e^{-2m\rho}}}. \tag{66}$$

Proof. Our proof of (65) is long and divided into several parts.

Growth order of $V_m(e^{-\rho})$. By the definition (60) of $V_m(z)$, we easily obtain the estimates

$$V_m(e^{-\rho}) \sim \begin{cases} \dfrac{m}{4\rho}, & \text{if } m\rho \to 0; \\ \dfrac{1-e^{-m\rho}}{2\rho^2(1+e^{-m\rho})}, & \text{if } m\rho \asymp 1; \\ \dfrac{1}{2\rho^2}, & \text{if } m\rho \to \infty. \end{cases}$$

In all cases, we have $V_m(e^{-\rho}) \asymp n\rho$.

Uniform bounds for $|z/(1-z)^2|$. We consider first the modulus of $|z/(1-z)^2|$, which is independent of m and simpler. Observe that

$$\frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} = \frac{(1-e^{-\rho})^2}{1-2e^{-\rho}\cos t + e^{-2\rho}} = \frac{(1-e^{-\rho})^2}{(1-e^{-\rho})^2 + 2e^{-\rho}(1-\cos t)},$$

for $-\pi \leqslant t \leqslant \pi$. Now if $|t| = O(\rho)$, then we have the uniform expansion

$$\frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} = \frac{1}{1+\rho^{-2}t^2}\Big(1 + \frac{t^2}{12} + \frac{t^2(t^2-\rho^2)}{240} + O(t^6+\rho^4t^2)\Big), \tag{67}$$

while if $\rho \leqslant |t| \leqslant \pi$, then, by monotonicity,

$$\max_{\rho\leqslant|t|\leqslant\pi} \frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} \leqslant \frac{(1-e^{-\rho})^2}{1-2e^{-\rho}\cos\rho + e^{-2\rho}} \sim \frac{1}{2}. \tag{68}$$

A uniform bound when $|t| \leqslant \rho$. The other factor in (66) is more complicated. For convenience, write

$$v(w) := \frac{1 - e^{-w}}{2(1 + e^{-w})}.$$

Consider first the range $|t| \leqslant \rho$, beginning with the expression

$$\frac{|v(m(\rho + it))|}{v(m\rho)} = \sqrt{\frac{1 + \frac{2e^{-m\rho}}{(1-e^{-m\rho})^2}(1 - \cos(mt))}{1 - \frac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1 - \cos(mt))}}.$$

When $|t| \leqslant \rho$, we have the inequality

$$\begin{aligned} \frac{2e^{-m\rho}}{(1 + e^{-m\rho})^2}(1 - \cos(mt)) &\leqslant \begin{cases} \dfrac{2e^{-m\rho}}{(1 + e^{-m\rho})^2}(1 - \cos(m\rho)), & \text{if } m\rho \leqslant \pi \\ \dfrac{4e^{-m\rho}}{(1 + e^{-m\rho})^2}, & \text{if } m\rho > \pi \end{cases} \\ &< 0.3. \end{aligned} \tag{69}$$

Then, by the inequalities

$$\begin{cases} (1 + x)^{1/2} \leqslant 1 + x/2, & \text{for } x \geqslant 0; \\ (1 - x)^{-1/2} \leqslant 1 + 2x/3, & \text{for } 0 \leqslant x \leqslant 0.3, \end{cases}$$

we obtain

$$\frac{|v(m(\rho + it))|}{v(m\rho)} \leqslant 1 + e^{-m\rho}(1 - \cos(mt))\Big(\frac{4}{3(1 + e^{-m\rho})^2} + \frac{1}{(1 - e^{-m\rho})^2}\Big),$$

and then, by (67),

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \frac{1 + \Upsilon\rho^{-2}t^2}{1 + \rho^{-2}t^2}(1 + O(t^2)),$$

where $\Upsilon = \Upsilon(\rho, t)$ is defined as

$$\begin{aligned} \Upsilon(\rho, t) &:= \rho^2 t^{-2} e^{-m\rho}(1 - \cos(mt))\Big(\frac{4}{3(1 + e^{-m\rho})^2} + \frac{1}{(1 - e^{-m\rho})^2}\Big) \\ &= \frac{1 - \cos(mt)}{(mt)^2/2} \cdot e^{-m\rho}\Big(\frac{2(m\rho)^2}{3(1 + e^{-m\rho})^2} + \frac{(m\rho)^2}{2(1 - e^{-m\rho})^2}\Big). \end{aligned}$$

Since $(1 - \cos t)/(t^2/2) \leqslant 1$ for all $t \in \mathbb{R}$ and

$$\max_{x \geqslant 0} e^{-x}\Big(\frac{2x^2}{3(1 + e^{-x})^2} + \frac{x^2}{2(1 - e^{-x})^2}\Big) < 0.65,$$

we have

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \frac{1 + 0.65\rho^{-2}t^2}{1 + \rho^{-2}t^2}(1 + O(t^2)) \leqslant 1 - c\rho^{-2}t^2, \tag{70}$$

for $|t| \leqslant \rho$, where $0 < c < 0.35$.

A uniform bound when $\rho \leqslant |t| \leqslant \pi$ and $m\rho > \pi$. In this case, we follow the same procedure as above, noting that

$$\frac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1-\cos(mt)) \leqslant \frac{4e^{-m\rho}}{(1+e^{-m\rho})^2} < 0.19 < 0.3,$$

when $m\rho > \pi$ and $|t| \leqslant \pi$. Then

$$\frac{|\upsilon(m(\rho+it))|}{\upsilon(m\rho)} \leqslant 1 + 2e^{-m\rho}\Big(\frac{4}{3(1+e^{-m\rho})^2} + \frac{1}{(1-e^{-m\rho})^2}\Big) < 1.25.$$

This, together with (68), gives

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} < \frac{1.25}{2} = \frac{5}{8}, \tag{71}$$

when $m\rho > \pi$ and $\rho \leqslant |t| \leqslant \pi$.

A uniform bound when $\rho \leqslant |t| \leqslant \pi$ and $m\rho \leqslant \pi$. In this case, $1/(1-z)^2$ has a double pole at $z = 1$, while $(1-z^m)/(1+z^m)$ has simple poles at $z = e^{t_j i}$ for $-\lfloor m/2 \rfloor \leqslant j \leqslant \lceil m/2 \rceil$, where $t_j := (2j-1)\pi/m$. Since $1/|1-e^{-\rho-it}|^2$ is monotonically decreasing in $|t|$ when $|t| \leqslant \pi$ and $|\upsilon(m(\rho+it))|$ reaches the same maximum at $t = t_j$, we then deduce that

$$\max_{\rho \leqslant |t| \leqslant \pi} |V_m(e^{-\rho-it})| \leqslant \max\{|V_m(e^{-\rho-i\rho})|, |V_m(e^{-\rho-it_1})|\},$$

where $t_1 = \pi/m \geqslant \rho$ when $m\rho \leqslant \pi$. By (70), we have

$$\frac{|V_m(e^{-\rho-i\rho})|}{V_m(e^{-\rho})} \leqslant \frac{1.65}{2}(1+O(\rho^2)) < \frac{7}{8}.$$

On the other hand, when $t = t_1$,

$$\frac{|\upsilon(m(\rho+it_1))|}{\upsilon(m\rho)} = \frac{(1+e^{-m\rho})^2}{(1-e^{-m\rho})^2}.$$

It follows, by (67), that

$$\frac{|V_m(e^{-\rho-it_1})|}{V_m(e^{-\rho})} = \frac{(1+e^{-m\rho})^2}{(1+\pi^2(m\rho)^{-2})(1-e^{-m\rho})^2}(1+O(t^2)) < \frac{7}{8},$$

when $m\rho \leqslant \pi$, since the value of the monotonic function

$$x \mapsto \frac{(1+e^{-x})^2}{(1+\pi^2 x^{-2})(1-e^{-x})^2},$$

lies between $4/\pi^2$ and 0.6 when $x \in [0, \pi]$. Summarizing, we proved that, for $\rho \leqslant |t| \leqslant \pi$,

$$\frac{|V_m(e^{-\rho-it_1})|}{V_m(e^{-\rho})} \leqslant \frac{7}{8}, \tag{72}$$

whether $m\rho \leqslant \pi$ or $m\rho > \pi$.

By collecting the estimates (70), (71), and (72), we obtain (65) and complete the proof of the lemma. □

Proof. (Proposition 4.2: smallness of the integral over $\delta\rho \leqslant |t| \leqslant \pi$) By (61), we obtain

$$\begin{aligned}\int_{\delta\rho\leqslant|t|\leqslant\pi} & e^{n(\rho+it)} G_m(e^{-\rho-it})\,\mathrm{d}t \\ &= O\Big(e^{n\rho}G_m(e^{-\rho})\Big(\int_{\delta\rho}^{\rho} + \int_{\rho}^{\pi}\Big)\exp\big(-V_m(e^{-\rho}) + \big|V_m(e^{-\rho-it})\big|\big)\,\mathrm{d}t\Big) \\ &=: O\big(e^{n\rho}G_m(e^{-\rho})(J_1+J_2)\big).\end{aligned}$$

By (70), we have

$$J_1 = O\Big(\int_{\delta\rho}^{\rho} e^{-cV_m(e^{-\rho})t^2/\rho^2}\,\mathrm{d}t\Big) = O\big(\rho e^{-cV_m(e^{-\rho})\delta^2}\big) = O\big(\rho e^{-c'(n\rho)^{1/5}}\big).$$

On the other hand, by (72), J_2 is bounded above by

$$J_2 = O\big(e^{-cV_m(e^{-\rho})}\big) = O\big(e^{-c'n\rho}\big). \tag{73}$$

This completes the proof of Proposition 4.2. □

4.4 Asymptotic nature of the expansion (54): proof of Proposition 4.3

We now prove Proposition 4.3 from which the asymptotic approximation (55) will then follow.

We begin with the following uniform estimates for $\log G_m(e^{-\tau})$.

Lemma 4.8. *Let $\tau = \rho + it$. Then, uniformly for $\rho \to 0$ and $|t| = O(\rho)$ in the half-plane $\rho > 0$,*

$$\log G_m(e^{-\tau}) = \begin{cases} O(m/|\tau|), & \text{if } m\rho \leqslant 1, \\ O(|\tau|^{-2}), & \text{if } m\rho \geqslant 1. \end{cases} \tag{74}$$

Proof. If $m\rho \leqslant 1$, then, by (42) and (51), we obtain

$$\kappa_m\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big) = O\big(m^2 e^{-\operatorname{Re}(2\pi^2/(m\tau))}\big) = O\big(m^2 e^{-c/(m\rho)}\big),$$

which is obviously $O(m/|\tau|)$. Now, by (36) and the asymptotic expansion (38), we have

$$\log G_m(e^{-\tau}) = O(m/|\tau| + m^2 + m^3|\tau|) = O(m/|\tau|),$$

since $m|\tau| = O(1)$.

On the other hand, if $m\rho \geqslant 1$, then, by (36) using the expressions in (15), (38), (42) and (52), we deduce that

$$\begin{aligned}\log G_m(e^{-\tau}) = \frac{\zeta(3) - 2\eta_2(m\tau)}{2\tau^2} + \frac{\pi^2}{24\tau} + \frac{\log\tau}{24} + \frac{\zeta'(1)}{2} - \frac{\log 2}{4} + \frac{\tau}{48} \\ - \frac{5\eta_1(m\tau)}{12} + \frac{1}{2}p(e^{-m\tau}) + O\big(|\tau|^2 + m^{-2}\big),\end{aligned} \tag{75}$$

where many terms in $\varpi_m/\tau + \log g_m + \phi_m\tau$ are cancelled with the corresponding ones in (52). Thus, by (9), we have $\log G_m(e^{-\tau}) = O(|\tau|^{-2})$. □

Lemma 4.9. *For $k \geqslant 0$, we have, uniformly for $|t| = O(\rho)$,*

$$|\Lambda^{(k)}(e^{-\rho-it})| = O\big(\rho^{-k}\Lambda(e^{-\rho})\big).$$

Proof. We apply a standard argument (or Ritt's Lemma; see [16, § 4.3]) for the asymptotics of the derivatives of an analytic function in a compact domain, starting from the integral representation

$$\Lambda^{(k)}(e^{-\rho-it}) = \frac{k!}{2\pi i}\oint_{|w-e^{-\rho-it}|=c\rho e^{-\rho}} \frac{\Lambda(w)}{(w-e^{-\rho-it})^{k+1}}\,\mathrm{d}w,$$

where $c > 0$ is a suitably chosen small number. Then, since $\rho \to 0$, we see that

$$\Lambda^{(k)}(e^{-\rho}) = O\Big(\rho^{-k}\max_{|\theta|\leqslant\pi}|\Lambda(e^{-\rho-it}(1+c\rho e^{i\theta}))|\Big) = O\Big(\rho^{-k}\max_{|\theta|\leqslant\pi}|\Lambda(e^{-\rho-it+c\rho e^{i\theta}})|\Big).$$

By choosing c sufficiently small, the circular range specified by $\rho+it-c\rho e^{i\theta}$ for $|\theta|\leqslant\pi$ is covered in the cone $|t| = O(\rho)$, and we can then apply the bounds for Λ given in (74). □

Proof. (Proposition 4.3) Lemma 4.9 implies, by the definition (54), that

$$\Lambda_k(\rho) \asymp \Lambda(e^{-\rho}) = \log G_m(e^{-\rho}), \qquad (k = 1, 2, \dots).$$

Thus the Taylor expansion (54) is also an asymptotic expansion when $|t| \to 0$. □

4.5 The saddle-point approximation.

Theorem 4.1 is a direct consequence of Propositions 4.2 and 4.3.

Proof. (Theorem 4.1) By (57), we obtain

$$G_{n,m} = \frac{1}{2\pi}\int_{-\delta\rho}^{\delta\rho} e^{n(\rho+it)}G_m(e^{-\rho-it})\,\mathrm{d}t + O\big(e^{n\rho}G_m(e^{-\rho})e^{-c'(n\rho)^{1/5}}\big).$$

Then by the expansion (54), Proposition 4.3 and the estimate in Lemma 4.9, we have

$$\begin{aligned}\frac{1}{2\pi}&\int_{-\delta\rho}^{\delta\rho} e^{n(\rho+it)}G_m(e^{-\rho-it})\,\mathrm{d}t\\ &= \frac{\rho e^{n\rho}G_m(e^{-\rho})}{2\pi}\int_{-\delta}^{\delta}\exp\Big(it(n\rho-\Lambda_1(\rho)) - \frac{\Lambda_2(\rho)}{2}t^2 + \frac{\Lambda_3(\rho)}{6}(-it)^3 + O(\Lambda(e^{-\rho})t^4)\Big)\,\mathrm{d}t.\end{aligned}$$

Choose $\rho > 0$ to be the solution of the equation (56), which exists by the estimates in (74). Then take δ as we described above, namely, $\Lambda_2(\rho)\delta^2 \to \infty$ and $\Lambda_2(\rho)\delta^3 \to 0$. The evaluation of the integral is then straightforward, and omitted. □

Remark 1. The same calculations lead indeed to an asymptotic expansion of the form

$$G_{n,m} \sim \frac{e^{n\rho}G_m(e^{-\rho})}{\sqrt{2\pi\Lambda_2(\rho)}}\Big(1 + \sum_{j\geqslant 1}\gamma_j(\rho)\Lambda_2(\rho)^{-j}\Big),$$

for some (messy) coefficients $\gamma_j(\rho)$ depending on ρ. In particular (for simplicity, $\Lambda_j = \Lambda_j(\rho)$),

$$\gamma_1(\rho) = \frac{3}{16}\cdot\frac{4\Lambda_2\Lambda_4 - 5\Lambda_3^2}{\Lambda_2^3},$$

and

$$\gamma_2(\rho) = -\frac{15}{512}\cdot\frac{64\Lambda_2^3\Lambda_6 - 224\Lambda_2^2\Lambda_3\Lambda_5 - 112\Lambda_2^2\Lambda_4^2 + 504\Lambda_3^2\Lambda_4\Lambda_2 - 231\Lambda_3^4}{\Lambda_2^6}.$$

5 Phase transitions

Based on the less explicit saddle-point approximation (55), we now derive more precise asymptotic estimates according to the relative growth rate of m with $n^{1/3}$.

5.1 Subcritical phase: $m = o(n^{1/3}(\log n)^{-2/3})$

We consider here m in the range

$$3 \leqslant m \leqslant \frac{6\pi^{2/3}n^{1/3}}{(\log n - \frac{1}{2}\log\log n + \log\omega_n)^{2/3}}, \tag{76}$$

for any sequence ω_n tending to infinity; compare (53).

Proposition 5.1. *If m lies in (76), then*

$$G_{n,m} \sim \frac{g_m\sqrt{\varpi_m}}{\sqrt{2\pi}\,n}\, e^{2\sqrt{\varpi_m(n+\phi_m)}} \sim \frac{g_m\sqrt{\pi m}}{4\sqrt{3}\,n}\, e^{2\sqrt{\varpi_m(n+m^3/96)}}, \tag{77}$$

where g_m, ϖ_m and ϕ_m are defined in (15). If $m \to \infty$ and still lies in the interval (76), then

$$G_{n,m} \sim c_1 n^{-1} m^{23/24} e^{-c_2 m^2 + 2\sqrt{\varpi_m(n+m^3/96)}}, \quad \textit{with} \quad (c_1, c_2) := \left(\frac{e^{\zeta'(-1)/2}\pi^{1/24}}{2^{55/24}\sqrt{3}}, -\frac{7\zeta(3)}{8\pi^2}\right).$$

Proof. In this range of m, $\log G_m(e^{-\rho})$ satisfies, by (36) together with the expressions in (15), (42) and (51),

$$\log G_m(e^{-\rho}) = \frac{\varpi_m}{\rho} + \frac{1}{2}\log\rho + \log g_m + \phi_m\rho + O\big(m^2\xi_2(m\rho)\big), \tag{78}$$

where $m^2\xi_2(m\rho) \asymp m^2 e^{-2\pi^2/(m\rho)}$, and the saddle-point equation has the form (by an argument similar to the proof of Lemma 4.9 using (36))

$$n + \phi_m = \frac{\varpi_m}{\rho^2} - \frac{1}{2\rho} + O\big(m^3\xi_2'(m\rho)\big). \tag{79}$$

Asymptotically, we have, by a direct bootstrapping argument,

$$\rho = \sqrt{\frac{\varpi_m}{n+\phi_m}} + O\big(n^{-1} + m^{5/2}n^{-3/2}e^{-4\sqrt{6}\pi n^{1/2}/m^{3/2}}\big). \tag{80}$$

Then the upper limit of m in (76) implies that the O-terms in the above three equations are all of order $o(1)$; in particular,

$$\begin{cases} m^3\rho\xi_2'(m\rho) \asymp m\rho^{-1}e^{-2\pi^2/(m\rho)} \leqslant \omega_n^{-2/3} \to 0, \\ m^2\xi_2(m\rho) \asymp m^2 e^{-2\pi^2/(m\rho)} = o\big(m\rho^{-1}e^{-2\pi^2/(m\rho)}\big) = o\big(\omega_n^{-2/3}\big). \end{cases}$$

[This range is slightly smaller than (53) because we need an expansion for $n\rho$ up to $o(1)$ error, or $(n+\phi_m)\rho = \varpi_m/\rho - 1/2 + o(1)$.] Substituting this choice of ρ and using (79) into (78), we have

$$\begin{aligned} n\rho + \log G_m(e^{-\rho}) &= \frac{\varpi_m}{\rho} + \frac{1}{2}\log\rho + \log g_m + (n+\phi_m)\rho + o(1) \\ &= 2\sqrt{\varpi_m(n+\phi_m)} + \frac{1}{2}\log\rho + \log g_m + o(1). \end{aligned}$$

On the other hand, we also have

$$\frac{\rho}{\sqrt{2\pi\Lambda_2(\rho)}} \sim \frac{\rho^{3/2}}{\sqrt{2\pi\varpi_m}};$$

thus

$$G_{n,m} \sim \frac{g_m\rho^2}{\sqrt{2\pi\varpi_m}}\, e^{2\sqrt{\varpi_m(n+\phi_m)}},$$

proving (77) by (80). □

From this estimate, it is straightforward to show that (4) holds only when $m = o\big(n^{1/7}\big)$:

$$e^{2\sqrt{\varpi_m(n+m^3/96)}} = e^{2\sqrt{\varpi_m n}+O(m^{7/2}n^{-1/2})}; \tag{81}$$

and when $n^{1/7} \ll m = o(n^{3/13})$,

$$e^{2\sqrt{\varpi_m(n+m^3/96)}} = e^{2\sqrt{\varpi_m n}+\sqrt{\varpi_m}\, m^3 n^{-1/2}/192+O(m^{13/2}n^{-3/2})}.$$

A connection to the modified Bessel functions. By the same analysis used in the proof of Proposition 4.2 (see (73)), we have

$$G_{n,m} = \frac{1}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho} e^{n\tau}G_m(e^{-\tau})\,\mathrm{d}\tau + O\big(e^{n\rho}G_m(e^{-\rho})e^{-c'n\rho}\big).$$

The integral on the right-hand side is indeed well-approximated by the modified Bessel function when m lies in the interval (76). By (14) and (51),

$$\begin{aligned}\frac{1}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho} e^{n\tau}G_m(e^{-\tau})\,\mathrm{d}\tau &= \frac{g_m}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho}\sqrt{\tau}e^{(n+\phi_m)\tau+\varpi_m/\tau}\Big(1+O\big(me^{-\operatorname{Re}(2\pi^2/(m\tau))}\big)\Big)\,\mathrm{d}\tau\\ &= \frac{g_m}{2\pi i}\int_{\mathscr{H}}\sqrt{\tau}e^{(n+\phi_m)\tau+\varpi_m/\tau}\,\mathrm{d}\tau + O\big(me^{-\operatorname{Re}(2\pi^2/(m\tau))}+e^{-cn\rho}\big),\end{aligned}$$

where $\mathscr{H}$ denotes a Hankel contour, which starts from $-\infty$, encircles around the origin counterclockwise, and then returns to ∞ (the exact shape being immaterial). The last integral over $\mathscr{H}$ is nothing but the modified Bessel function:

$$\begin{aligned}G_{n,m} &\sim \frac{g_m}{2\pi i}\int_{\mathscr{H}}\sqrt{\tau}e^{(n+\phi_m)\tau+\varpi_m/\tau}\,\mathrm{d}\tau\\ &= g_m\sum_{j\geqslant 0}\frac{\varpi_m^j(n+\phi_m)^{j+3/2}}{j!\Gamma(j-1/2)}\\ &= \frac{g_m(n+\phi_m)^{-3/2}}{4\sqrt{\pi}}\Big(\big(2\sqrt{\varpi_m(n+\phi_m)}-1\big)e^{2\sqrt{\varpi_m(n+\phi_m)}}\\ &\qquad -\big(2\sqrt{\varpi_m(n+\phi_m)}+1\big)e^{-2\sqrt{\varpi_m(n+\phi_m)}}\Big),\end{aligned}$$

which holds as long as m lies in the range (76). Numerical fit of the last expression is very satisfactory.

5.2 Supercritical phase: $m \gg n^{1/3}\log n$

We now consider m in the following stationary range

$$m \geqslant \Big(\frac{n}{\zeta(3)}\Big)^{1/3}\Big(\frac{2}{3}\log n + \log\log n + \omega_n\Big), \tag{82}$$

for any sequence ω_n tending to infinity with n.

Proposition 5.2. *If m satisfies (82), then*

$$G_{n,m} \sim G_{n,n} \sim cn^{-49/72}e^{\beta_1 n^{2/3}+\beta_2 n^{1/3}}, \tag{83}$$

where the constants (c,β_1,β_2) are defined in (7).

Proof. For this range of m, we have, by (75),

$$\log G_m(e^{-\rho}) = \frac{\zeta(3)}{2\rho^2} + \frac{\pi^2}{24\rho} + \frac{\log\rho}{24} + \frac{\zeta'(1)}{2} - \frac{\log 2}{4} + \frac{\rho}{48} + O\big(\rho^{-2}\eta_2(m\rho) + e^{-m\rho} + \rho^2\big),$$

and the saddle-point equation

$$n + \frac{1}{48} = \frac{\zeta(3)}{\rho^3} + \frac{\pi^2}{24\rho^2} - \frac{1}{24\rho} + O\big(\partial_\rho(\eta_2(m\rho)/\rho^2) + me^{-m\rho} + \rho\big). \tag{84}$$

Solving asymptotically the saddle-point equation (84) gives, with $N := n + \frac{1}{48}$,

$$\rho = \zeta(3)^{1/3}N^{-1/3} + \frac{\pi^2}{72\zeta(3)^{1/3}}N^{-2/3} - \frac{1}{72}N^{-1} + O\big(N^{-2/3}me^{-\zeta(3)^{1/3}m/n^{1/3}}\big).$$

Then we obtain

$$\begin{cases} \rho\partial_\rho(\eta_2(m\rho)/\rho^2) \asymp m\rho^{-1}e^{-m\rho} \leqslant e^{-\omega_n} \to 0, \\ \rho^{-2}\eta_2(m\rho) \asymp \rho^{-2}e^{-m\rho} = o\big(m\rho^{-1}e^{-m\rho}\big) = o\big(e^{-\omega_n}\big), \\ n^{-2/3}me^{-\zeta(3)^{1/3}m/n^{1/3}} = o\big(n^{-1}e^{-\omega_n}\big). \end{cases}$$

Thus we have expansions for $n\rho + \log G_m(e^{-\rho})$ and ρ to within an error of order $o(1)$, which, together with the relation $\Lambda_2(\rho) \sim 3\zeta(3)\rho^{-2}$, gives the same asymptotic approximation as in (6). □

5.3 Critical phase: $\log m \sim \frac{1}{3}\log n$

In this range, we begin with the expansion (75) and the approximate saddle-point equation

$$\begin{aligned} n = \frac{\zeta(3) - 2\eta_2(m\rho) + m\rho\eta_2'(m\rho)}{\rho^3} + \frac{\pi^2}{24\rho^2} + \frac{1}{24\rho} - \frac{1}{48} \\ + \frac{5m\eta_1(m\rho)}{12} + \frac{me^{-m\rho}p(e^{-m\rho})}{2} + O(\rho). \end{aligned} \tag{85}$$

Define

$$R(\alpha, r) := r^3 - \zeta(3) + 2\eta_2(\alpha r) - \alpha r\eta_2'(\alpha r),$$

and

$$\sigma(x) := 3\zeta(3) - 6\eta_2(x) + 4x\eta_2'(x) - x^2\eta_2''(x),$$

where the $\eta_d(x)$ are defined in (9).

Proposition 5.3. *For* $\log m \sim \frac{1}{3}\log n$*, we have the uniform asymptotic approximation*

$$G_{n,m} \sim c(\alpha, r) n^{-49/72} e^{\beta_1(\alpha,r)n^{2/3}+\beta_2(\alpha,r)n^{1/3}}, \tag{86}$$

where

$$\beta_1(\alpha, r) := r + \frac{\zeta(3) - 2\eta_2(\alpha r)}{2r^2}, \qquad \beta_2(\alpha, r) := \frac{\pi^2}{24r},$$

and

$$c(\alpha, r) := \frac{r^{49/24}}{2^{3/4}\sqrt{\pi\sigma(\alpha r)}} \exp\Big(-\frac{r_1\pi^2}{24r} + \frac{\zeta'(-1)}{2} + \frac{r_1^2\sigma(\alpha r)}{2} - \frac{5}{12}\eta_1(\alpha r) + \frac{p(e^{-\alpha r})}{2}\Big),$$

the coefficients r_1 *and* r_2 *being given in* (87)*, and* r *is the unique positive solution of* $R(\alpha, r) = 0$.

Proof. We need first a few simple lemmas. Write first $m = \alpha n^{1/3}$ and $\rho = r/n^{1/3}$. Then the equation (85) can be written as

$$R(\alpha, r) = \frac{\pi^2 r}{24n^{1/3}} - \frac{(1 - 10\alpha r\eta_1(\alpha r) - 12\alpha e^{-\alpha r}p'(e^{-\alpha r}))r^2}{24n^{2/3}} - \frac{r^3}{48n} + O\big(n^{-1}r^4\big);$$

also $\Lambda_2(\rho) \sim \rho^{-2}\sigma(\alpha r)$.

Lemma 5.4. *The function* $\sigma(x)$ *is positive for* $x > 0$.

Proof. Note that $\sigma(x) \sim 3\zeta(3)$ as $x \to \infty$, and $\sigma(x) \sim \zeta(2)x/2$ as $x \to 0$. So we prove the monotonicity of $\sigma(x)$ for $x \geqslant 0$:

$$\sigma'(x) = \sum_{j\geqslant 1} \frac{e^{-jx}\tilde{\sigma}(jx)}{j^2(1 + e^{-jx})^4},$$

where $\tilde{\sigma}(x) := 2(1 + e^{-x})^2 + 2(1 - e^{-x})x + (1 - 4e^{-x} + e^{-2x})x^2 > 2 + x^2 + 4e^{-x}(1 - x^2) > 2.9$ for $x \geqslant 0$. ☐

Once m is given, α is fixed and then r can be solved from the equation $R(\alpha, r) = 0$, which is nothing but (11).

Lemma 5.5. *For any* $\alpha > 0$*, the equation* $R(\alpha, r) = 0$ *has a unique solution* $r > 0$.

Proof. Consider the function $\tilde{R}(x) := \zeta(3) - 2\eta_2(x) + x\eta_2'(x)$, which has the explicit series form

$$\tilde{R}(x) = \sum_{j\geqslant 1} \frac{1 - jxe^{-jx} - e^{-2jx}}{j^3(1 + e^{-jx})^2}.$$

For large x, $\tilde{R}(x) \sim \zeta(3)$, while, for small x, $\tilde{R}(x) \sim \zeta(2)x/4$. Also

$$\tilde{R}'(x) = \sum_{j\geqslant 1} \frac{je^{-jx}(1 + e^{-jx} + jx(1 - e^{-jx}))}{j^3(1 + e^{-jx})^3} > 0,$$

for $x > 0$. Thus for each fixed $\alpha > 0$, the equation $r^3 = \tilde{R}(\alpha r)$ has a unique positive solution. ☐

The general expansion for ρ can be further refined, which is of the form

$$\rho = \frac{r}{n^{1/3}}\Big(1 + \frac{r_1}{n^{1/3}} + \frac{r_2}{n^{2/3}} + \cdots\Big),$$

where the coefficients $r_j = r_j(\rho, \eta_1, \eta_2)$ are generally messy; in particular,

$$\begin{aligned} r_1 &= \frac{\pi^2 r}{24\sigma(\alpha r)}, \\ r_2 &= \frac{r^2}{\sigma(\alpha r)}\Big(\frac{5}{12}\,\alpha r\eta_1'(\alpha r) + \frac{\alpha r e^{-\alpha r}p'(e^{-\alpha r})}{2} - \frac{1}{24} + \frac{\pi^4}{576\,\sigma(x)} \\ &\qquad - \frac{\pi^4 r^3}{192\,\sigma(x)^2} + \frac{\pi^4\alpha^3 r^3\eta_2'''(\alpha r)}{1152\,\sigma(x)^2}\Big). \end{aligned} \tag{87}$$

Thus (85) follows from applying (55) and straightforward expansions. □

In particular, the growth of the number of BPPs when their widths get close to the typical length behaves asymptotically like a Gumbel distribution.

Corollary 5.6. *Assume that m satisfies*

$$\alpha = \frac{m}{n^{1/3}} = \frac{1}{\zeta(3)^{1/3}}\Big(\frac{2}{3}\log\Big(\frac{n}{\zeta(3)}\Big) + x\Big). \tag{88}$$

Then

$$\frac{G_{n,m}}{G_{n,n}} = \exp\left(-e^{-x}(1+o(1))\right),$$

uniformly for $x = o(\log n)$.

Proof. By Proposition 5.3, when $\alpha \to \infty$,

$$r = \zeta(3)^{1/3}\left(1 - \frac{\alpha}{3\zeta(3)}\,e^{-\zeta(3)^{1/3}\alpha}(1+o(1))\right).$$

The ratio between $G_{n,m}$ and $G_{n,n}$ thus has the following form with α given in (88):

$$\frac{G_{n,m}}{G_{n,n}} = \exp\left(-\frac{n^{2/3}}{\zeta(3)^{2/3}}\,e^{-\zeta(3)^{1/3}\alpha}(1+o(1))\right) = \exp(-e^{-x}(1+o(1))).$$ □

Similar to Theorem 1.1 in [17], we may conclude that there is an exponential decay of the number of BPPs of size n and width m when m is close to the typical width, which is of order $\Theta(n\log n)$. See also [14] for a similar Gumbel limiting distribution for the largest size in random ordinary plane partitions.

6 Phase transitions in m-rowed plane partitions

Our method of proof extends to some other classes of plane partitions. For simplicity, we only consider briefly in this section plane partitions with m rows, which has the known generating function (see [1])

$$\sum_{n\geqslant 0} H_{n,m}z^n = \prod_{k\geqslant 1}(1-z^k)^{-\min\{k,m\}} = P(z)^m\tilde{Q}_m(z) = \exp\Big(\sum_{\ell\geqslant 1}\frac{\tilde{U}_m(z^\ell)}{\ell}\Big),$$

where $H_{n,m}$ denotes the number of m-rowed plane partitions of n, P is given in (3), and

$$\tilde{Q}_m(z) := \prod_{1\leqslant k<m}(1-z^k)^{m-k}, \quad \text{and} \quad \tilde{U}_m(z) := \frac{z(1-z^m)}{(1-z)^2}.$$

For $2 \leqslant m \leqslant 9$, these partitions appear in OEIS with the following identities.

m	2	3	4	5
OEIS	A000990	A000991	A002799	A001452
m	6	7	8	9
OEIS	A225196	A225197	A225198	A225199

For simplicity, we only describe the transitional behavior of $\log H_{n,m}$. Define

$$\eta(t) := \sum_{j\geqslant 1} \frac{1-e^{-jt}}{j^3}. \tag{89}$$

Theorem 6.1. *Let $\alpha := m/n^{1/3}$. Then*

$$\frac{\log H_{n,m}}{n^{2/3}} \sim H(\alpha) := r + r^{-2}\eta(\alpha r), \tag{90}$$

uniformly as $m \to \infty$ and $m \leqslant n$, where $r = r(\alpha) > 0$ solves the equation

$$r^3 - 2\eta(\alpha r) + \alpha r \eta'(\alpha r) = 0.$$

In particular,

$$H(\alpha) \sim \begin{cases} \frac{2\pi}{\sqrt{6}}\sqrt{\alpha}, & \text{if } \alpha \to 0; \\ 3\cdot 2^{-2/3}\zeta(3)^{1/3}, & \text{if } \alpha \to \infty. \end{cases} \tag{91}$$

Proof. (Sketch) By Euler-Maclaurin summation formula, we obtain

$$\begin{aligned}\log \tilde{Q}_m(e^{-\tau}) = \frac{\eta(m\tau)}{\tau^2} + \frac{m}{2}\log\Big(\frac{2\pi}{\tau}\Big) - \frac{\pi^2 m}{6\tau} - \frac{\log m}{12} + \frac{m\tau}{8} + \zeta'(-1) \\ - \frac{1}{12}\log\Big(\frac{1-e^{-m\tau}}{\tau}\Big) - \frac{\tau^2(1+10e^{-m\tau}+e^{-2m\tau})}{2880(1-e^{-m\tau})^2} + O\Big(\frac{|\tau|^4}{|1-e^{-m\tau}|^4}\Big),\end{aligned}$$

which holds uniformly as long as $\tau \to 0$ and $m \to \infty$. Then in this range

$$\begin{aligned}m\log P(e^{-\tau}) + \log \tilde{Q}_m(e^{-\tau}) = \frac{\eta(m\tau)}{\tau^2} - \frac{\log m}{12} + \frac{m\tau}{12} + \zeta'(-1) - \frac{1}{12}\log\Big(\frac{1-e^{-m\tau}}{\tau}\Big) \\ - \frac{\tau^2(1+10e^{-m\tau}+e^{-2m\tau})}{2880(1-e^{-m\tau})^2} \\ + O\Big(\frac{|\tau|^4}{|1-e^{-m\tau}|^4} + me^{-\operatorname{Re}(4\pi^2/\tau)}\Big).\end{aligned}$$

In particular, when $m/n^{1/3} \to \infty$, then $\eta(t) \sim \zeta(3)$ and $\eta'(t) = o(1)$. Thus $r \sim (2\zeta(3))^{1/3}$, and

$$\log\big([z^n]P(z)^m\tilde{Q}_m(z)\big) \sim 3\zeta(3)^{1/3}(n/2)^{2/3},$$

consistent with (2). On the other hand, when $m = o(n^{1/3})$, we use the asymptotic expansion

$$\eta(z) = \frac{\pi^2 z}{6} + \frac{z^2}{4}\Big(2\log z - 3\Big) + \sum_{j\geqslant 1}\frac{B_j z^{j+2}}{j\cdot(j+2)!},$$

the series being convergent when $|z| < 2\pi$. Thus in this case

$$\log\big([z^n]P(z)^m\tilde{Q}_m(z)\big) \sim \frac{2\pi}{\sqrt{6}}\sqrt{\alpha}n^{2/3} = \frac{2\pi}{\sqrt{6}}\sqrt{nm}.$$

The theorem is proved by examining the error terms in each case. We omit the details. □

When $m\rho = o(1)$, we can write down more precise expansions, similar to (81), beginning with

$$\log \tilde{Q}_m(e^{-\tau}) \sim \sum_{1\leqslant k<m} (m-k)\log(k\tau) + \sum_{j\geqslant 1} \frac{B_j \varsigma_j(m)}{j\cdot j!}\,\tau^j,$$

while in the case of BPPs the corresponding expansion is a finite one (with exponentially smaller error in $1/\tau$). Here the B_j's are Bernoulli numbers and $\varsigma_j(m) := \sum_{1\leqslant k<m}(m-k)k^j$ is a polynomial in m of degree $j+2$ and divisible by $m(m-1)$. The series is divergent when $m|\tau| \geqslant 2\pi$. In particular,

$$\varsigma_0(m) = \frac{m(m-1)}{2},\ \varsigma_1(m) = \frac{m(m^2-1)}{6},\ \varsigma_2(m) = \frac{m^2(m^2-1)}{12}.$$

The saddle-point equation is now of the form

$$N := n - \frac{m(2m^2-1)}{24} \sim \frac{m\pi^2}{6\rho^2} - \frac{m^2}{2\rho} - \sum_{j\geqslant 2}\frac{B_j\varsigma_j(m)}{j!}\,\rho^{j-1}.$$

Then, writing $\varsigma_j(m) = m\bar{\varsigma}_j(m)$,

$$\rho = \sqrt{\frac{m}{n}}\left(\frac{\pi^2}{6} - \frac{m}{2}\,\rho + \frac{2m^2-1}{24}\,\rho^2 - \sum_{j\geqslant 2}\frac{B_j\bar{\varsigma}_j(m)}{j!}\,\rho^{j+1}\right)^{1/2} =: r\Psi(\rho),$$

where $r := \pi\sqrt{m/(6n)}$ and

$$\Psi(\rho) := \left(1 - \frac{3m}{\pi^2}\,\rho + \frac{2m^2-1}{4\pi^2}\,\rho^2 - \frac{6}{\pi^2}\sum_{j\geqslant 2}\frac{B_j\bar{\varsigma}_j(m)}{j!}\,\rho^{j+1}\right)^{1/2}.$$

Thus, by Lagrange Inversion Formula,

$$\rho \sim \sum_{j\geqslant 1} d_j r^j, \quad \text{with } d_j = \frac{1}{j}[t^{j-1}]\Psi(t)^j.$$

Since each $d_j = d_j(m)$ is a polynomial in m of degree $m-1$, we see that the general term in the expansion of ρ is of the form $m^{(3j-2)/2}/n^{j/2}$, which, after substituting such ρ into the corresponding saddle-point approximation gives an expansion in terms of r as follows:

$$[z^n]P(z)^m\tilde{Q}_m(z) \sim \sqrt{2}\,\pi N^{-(m+5)/4}(m/24)^{(m+3)/4}\exp\left(\pi\sqrt{\frac{Nm}{6}} + \frac{m^2}{4} + \sum_{j\geqslant 1}\frac{e_j(m)}{N^{j/2}}\right),$$

where $e_j(m)$ is a polynomial of degree $(3j+4)/2$. In general, if $n^{j_0/(3j_0+4)} \asymp m = o(n^{(j_0+1)/(3j_0+7)})$, we have the asymptotic approximation

$$[z^n]P(z)^m\tilde{Q}_m(z) \sim \sqrt{2}\,\pi N^{-(m+5)/4}(m/24)^{(m+3)/4}\exp\left(\pi\sqrt{\frac{Nm}{6}} + \frac{m^2}{4} + \sum_{1\leqslant j<j_0}\frac{e_j(m)}{N^{j/2}}\right).$$

In particular, if $m = o(N^{1/7})$, then $j_0 = 0$, while if $m = o(N^{1/5})$, then retaining the term $e_1(m)/\sqrt{N}$ and dropping the remaining terms yields an error of order $o(1)$, etc.

Remark 2. (m-rowed plane partitions whose non-zero parts decrease strictly along each row) The generating function now has the form (see [8])

$$\begin{aligned} F_m(z) &:= \prod_{k\geqslant 1}(1-z^k)^{-\lfloor m/2\rfloor} \times \prod_{k\geqslant 1}(1-z^{2k-1})^{-2\lfloor m/2\rfloor} \times \prod_{1\leqslant k\leqslant m-2}(1-z^k)^{\lfloor (m-k)/2\rfloor} \\ &= \frac{P(z)^{3\lfloor m/2\rfloor}}{P(z^2)^{2\lfloor m/2\rfloor}}\,\bar{Q}_m(z), \end{aligned}$$

where $P(z)$ is as in (3) and $\bar{Q}_m(z) := \prod_{1\leqslant k\leqslant m-2}\big(1-z^k\big)^{\lfloor (m-k)/2\rfloor}$. Note that

$$F_m(z) = \left(\frac{P(z)}{P(z^2)}\right)^{2\lfloor m/2\rfloor} \exp\left(\sum_{\ell\geqslant 1}\frac{\bar{U}_m(z^\ell)}{\ell}\right), \quad \text{with } \bar{U}_m(z) := \frac{z^{1+\mathbf{1}_{m\text{ odd}}} - z^{m+1}}{(1-z)(1-z^2)}.$$

We then deduce the same type of transitional behavior as that of m-rowed plane partitions:

$$\log\big([z^n]P(z)^m\bar{Q}_m(z)\big) \sim \Big(r + \frac{\eta(\alpha r)}{2r^2}\Big)n^{2/3},$$

where η is defined in (89) and $r > 0$ solves the equation $2r^3 - 2\eta(\alpha r) + \alpha r\eta'(\alpha r) = 0$.

Remark 3. In a very similar manner, we can derive the phase transitions in the asymptotics of

$$[z^n]\prod_{1\leqslant k\leqslant m}\big(1-z^k\big)^{-k},$$

the difference here being that for small $m = O(1)$ the saddle-point method fails and one needs instead the singularity analysis [6] for the corresponding asymptotic approximation. Indeed, singularity analysis applies when $1 \leqslant m = o(n^{1/3})$:

$$[z^n]\prod_{1\leqslant k\leqslant m}\big(1-z^k\big)^{-k} \sim \frac{[z^n](1-z)^{-m(m+1)/2}}{\prod_{1\leqslant k\leqslant m}k^k} \sim \frac{n^{m(m+1)/2-1}}{\Gamma(m(m+1)/2)\prod_{1\leqslant k\leqslant m}k^k},$$

while our saddle-point analysis applies when $m \to \infty$.

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