Two pairs of congruences concerning sums of central binomial coefficients

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Abstract. In this paper, we prove the following two pairs of congruences: if $p \equiv 1 \pmod 3$ then

$$\sum_{k=1}^{\lfloor \frac{2p}{3} \rfloor} \binom{2k}{k}(-2)^k \equiv 0 \quad \text{and} \quad \sum_{k=0}^{\lfloor \frac{5p}{6} \rfloor} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p}\right) \pmod{p^2}$$

and, if $p \equiv 1 \pmod 5$ then

$$\sum_{k=1}^{\lfloor \frac{4p}{5} \rfloor} \binom{2k}{k}(-1)^k \equiv 0 \quad \text{and} \quad \sum_{k=1}^{\lfloor \frac{7p}{10} \rfloor} \frac{\binom{2k}{k}}{(-16)^k} \equiv 0 \pmod{p^2}$$

where p is a prime and $\left(\frac{\cdot}{p}\right)$ stands for the Legendre symbol.

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1. Introduction

In the past decades, many people studied congruences modulo a power of a prime p for sums of binomial coefficients (see, for instance, [2, 6, 7, 10, 13, 14, 16, 24, 25]). A certain number concern the central binomial coefficients and have the form

$$\sum_{k=0}^{n} \binom{2k}{k} x^k$$

where upper limit n is $p-1$ or $\frac{p-1}{2}$. We would like to mention two of them, taken from [17] and [23] respectively, that we will need later: for any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} \binom{2k}{k}(-2)^k \equiv 1 - \frac{4pq_p(2)}{3} \pmod{p^3} \tag{1.1}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p}\right)\left(1+\frac{pq_p(2)}{6}-\frac{p^2q_p(2)}{8}\right) \pmod{p^3}, \tag{1.2}$$

where $q_p(a) = (a^{p-1}-1)/p$ is the so-called *Fermat quotient*.

Much rarer are the examples where the upper limit of the sum is strictly between $\frac{p-1}{2}$ and $p-1$. In 2014, Pan and Sun [21] proved that for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{\lfloor\frac{3p}{4}\rfloor} \frac{\binom{2k}{k}}{(-4)^k} \equiv \left(\frac{2}{p}\right) \pmod{p^2}.$$

Recently Mao [11, 12] proved that for any prime $p \equiv 1 \pmod 3$, we have

$$\sum_{k=1}^{\lfloor\frac{2p}{3}\rfloor} \binom{2k}{k} \equiv 0 \quad \text{and} \quad \sum_{k=0}^{\lfloor\frac{5p}{6}\rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p}\right) \pmod{p^2}.$$

where the first one has been conjectured in 2006 by Adamchuk [1].

The main purpose of this paper is to show more congruences of the same flavour.

Theorem 1.1. *Let p be a prime. If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=1}^{\lfloor\frac{2p}{3}\rfloor} \binom{2k}{k}(-2)^k \equiv 0 \pmod{p^2} \tag{1.3}$$

and

$$\sum_{k=0}^{\lfloor\frac{5p}{6}\rfloor} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p}\right) \pmod{p^2}. \tag{1.4}$$

Theorem 1.2. *Let p be a prime. If $p \equiv 1 \pmod 5$, then*

$$\sum_{k=1}^{\lfloor\frac{4p}{5}\rfloor} \binom{2k}{k}(-1)^k \equiv 0 \pmod{p^2} \tag{1.5}$$

and

$$\sum_{k=1}^{\lfloor\frac{7p}{10}\rfloor} \frac{\binom{2k}{k}}{(-16)^k} \equiv 0 \pmod{p^2}. \tag{1.6}$$

The first pair of congruences seems to have (so far) slipped the attention of the mathematical community whereas the second pair appeared as a conjecture in [23].

We will give a proof of Theorem 1.1 and Theorem 1.2 in Section 4. The key ingredients are the following congruences which are interesting in their own right. We shall prove them in Section 2 and 3 respectively.

Theorem 1.3. *For any prime $p \equiv 1 \pmod 3$, we have*

$$\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}8^k}{3k+1} \equiv -\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}} q_p(2) \pmod p. \tag{1.7}$$

Theorem 1.4. *For any prime $p \equiv 1 \pmod 5$, we have*

$$\sum_{\substack{k=0\\k\neq\frac{p-1}{5}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}4^k}{5k+1} \equiv -4^{\frac{p-1}{5}}\binom{\frac{p-1}{2}}{\frac{p-1}{5}} f_p \pmod p. \tag{1.8}$$

where $f_p := F_{p-(\frac{p}{5})}/p$ is the Fibonacci quotient *and F_n denotes the nth Fibonacci number.*

2. Proof of Theorem 1.3

Define the *hypergeometric series*

$$ {}_{m+1}F_m\left[\begin{matrix}\alpha_0 & \alpha_1 & \ldots & \alpha_m\\ & \beta_1 & \ldots & \beta_m\end{matrix}\middle| z\right] := \sum_{k=0}^{\infty} \frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_m)_k}{(\beta_1)_k\cdots(\beta_m)_k}\cdot\frac{z^k}{k!}, \tag{2.1}$$

where $\alpha_0,\ldots,\alpha_m,\beta_1,\ldots,\beta_m,z\in\mathbb{C}$ and

$$(\alpha)_k = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+k-1), & \text{if } k\geq 1,\\ 1, & \text{if } k=0.\end{cases}$$

For a prime p, let $\mathbb{Z}_p$ denote the ring of all p-adic integers and let

$$\mathbb{Z}_p^\times := \{a\in\mathbb{Z}_p:\ a \text{ is prime to } p\}.$$

For each $\alpha\in\mathbb{Z}_p$, define the p-adic order $\nu_p(\alpha) := \max\{n\in\mathbb{N}:\ p^n \mid \alpha\}$ and the p-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the p-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1\leq j<n\\(k,p)=1}} k, \qquad n=1,2,3,\ldots,$$

and

$$\Gamma_p(\alpha) = \lim_{\substack{|\alpha-n|_p\to 0\\ n\in\mathbb{N}}} \Gamma_p(n), \qquad \alpha\in\mathbb{Z}_p.$$

In particular, we set $\Gamma_p(0)=1$. Throughout the whole paper, we only need to use the most basic properties of Γ_p, and all of them can be found in [18, 20]. For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases}-x, & \text{if } |x|_p = 1,\\ -1, & \text{if } |x|_p > 1.\end{cases} \tag{2.2}$$

We also know that for any $\alpha \in \mathbb{Z}_p$,

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)} \equiv \Gamma_p'(0) + H_{p-\langle -\alpha\rangle_p - 1} \pmod p, \tag{2.3}$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ is the nth classic harmonic number and $\langle x\rangle_p$ is the least nonnegative residue of x modulo p.

Lemma 2.1. ([8]). *For any prime $p > 3$, we have the following congruences modulo p*

$$H_{\lfloor p/2\rfloor} \equiv -2q_p(2),\ H_{\lfloor p/3\rfloor} \equiv -\frac{3}{2}q_p(3),\ H_{\lfloor p/6\rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3).$$

Proof of Theorem 1.3. For any $\alpha, s \in \mathbb{Z}_p$, we have

$$\frac{\binom{2k}{k}}{4^k} = \frac{\left(\frac12\right)_k}{(1)_k}, \quad \frac{\left(\frac13\right)_k}{\left(\frac43\right)_k} = \frac{1}{3k+1} \text{ and } (\alpha + sp)_k \equiv (\alpha)_k \pmod p.$$

For each $\frac{p+2}{3} \le k \le \frac{p-1}{2}$, we have

$$\begin{aligned}\frac{\left(\frac13 - \frac{p}{6}\right)_k}{\left(\frac43 - \frac{2p}{3}\right)_k} &= \frac{\frac{p}{6}\left(\frac13 - \frac{p}{6}\right)_{\frac{p-1}{3}}\left(\frac{p}{6}+1\right)_{k-\frac{p+2}{3}}}{\frac{-p}{3}\left(\frac43 - \frac{2p}{3}\right)_{(p-4)/3}\left(-\frac{p}{3}+1\right)_{k-\frac{p-1}{3}}} \equiv -\frac{1}{2}\frac{\left(\frac13\right)_{\frac{p-1}{3}}(1)_{k-\frac{p+2}{3}}}{\left(\frac43\right)_{(p-4)/3}(1)_{k-\frac{p-1}{3}}} \\ &= -\frac12 \frac{\left(\frac13\right)_{\frac{p-1}{3}}}{\left(\frac43\right)_{(p-4)/3}}\frac{1}{k-\frac{p-1}{3}} \equiv -\frac32 \frac{\left(\frac13\right)_{\frac{p-1}{3}}}{\left(\frac43\right)_{(p-4)/3}}\frac{1}{3k+1} \pmod p.\end{aligned}$$

And

$$\frac{\left(\frac13\right)_{\frac{p-1}{3}}}{\left(\frac43\right)_{(p-4)/3}} = \frac{p-1}{3}\frac{\left(\frac13\right)_{\frac{p-1}{3}}(p-4)/3!}{\left(\frac43\right)_{(p-4)/3}\frac{p-1}{3}!} \equiv -\frac13(-1)^{\frac{p-1}{3}}(-1)^{(p-4)/3} = \frac13 \pmod p.$$

Hence for each $\frac{p+2}{3} \le k \le \frac{p-1}{2}$,

$$\frac{\left(\frac13 - \frac{p}{6}\right)_k}{\left(\frac43 - \frac{2p}{3}\right)_k} \equiv -\frac12\frac{1}{3k+1} \pmod p.$$

That means that

$$\sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{\left(\frac{1-p}{2}\right)_k\left(\frac13 - \frac{p}{6}\right)_k}{(1)_k\left(\frac43 - \frac{2p}{3}\right)_k}(-8)^k \equiv -\frac12 \sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{\left(\frac12\right)_k\left(\frac13\right)_k}{(1)_k\left(\frac43\right)_k}(-8)^k \pmod p.$$

Hence

$$\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\binom{\frac{p-1}{2}}{k}8^k}{3k+1}\equiv\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}(-8)^k$$

$$\equiv\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}(-8)^k-3\sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}(-8)^k$$

$$\equiv\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}(-8)^k+\frac{3}{2}\sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}(-8)^k \pmod p.$$

Thus, (1.7) is equivalent to

$$\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}(-8)^k\equiv-\frac{3}{2}\sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}(-8)^k-\frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}q_p(2) \pmod p. \tag{2.4}$$

Set

$$\sum_{\substack{k=0\\k\neq\frac{p-1}{3}}}^{\frac{p-1}{2}}\frac{\left(\frac{1-p}{2}\right)_k\left(\frac{1}{3}-\frac{p}{6}\right)_k}{(1)_k\left(\frac{4}{3}-\frac{2p}{3}\right)_k}(-8)^k=\mathfrak{A}-\mathfrak{F},$$

where

$$\mathfrak{A}={}_2F_1\left[\begin{matrix}\frac{1-p}{2} & \frac{1}{3}-\frac{p}{6}\\ & \frac{4}{3}-\frac{2p}{3}\end{matrix}\middle| -8\right]$$

and

$$\mathfrak{F}=\frac{\left(\frac{1-p}{2}\right)_{\frac{p-1}{3}}\left(\frac{1}{3}-\frac{p}{6}\right)_{\frac{p-1}{3}}}{(1)_{\frac{p-1}{3}}\left(\frac{4}{3}-\frac{2p}{3}\right)_{\frac{p-1}{3}}}8^{\frac{p-1}{3}}. \tag{2.5}$$

In view of [19, 15.8.6], we have

$${}_2F_1\left[\begin{matrix}-m & b\\ & c\end{matrix}\middle| z\right]=\frac{(b)_m}{(c)_m}(1-z)^m{}_2F_1\left[\begin{matrix}-m & c-b\\ & 1-b-m\end{matrix}\middle| \frac{1}{1-z}\right].$$

Setting $a=\frac{1-p}{2}, b=\frac{1}{3}-\frac{p}{6}, c=\frac{4}{3}-\frac{2p}{3}$, $z=-8$, we have

$$\mathfrak{A}=\frac{\left(\frac{1}{3}-\frac{p}{6}\right)_{\frac{p-1}{2}}}{\left(\frac{4}{3}-\frac{2p}{3}\right)_{\frac{p-1}{2}}}9^{\frac{p-1}{2}}{}_2F_1\left[\begin{matrix}\frac{1-p}{2} & 1-\frac{p}{2}\\ & \frac{7}{6}-\frac{p}{3}\end{matrix}\middle| \frac{1}{9}\right].$$

In the light of [19, 15.4.32], we have

$$ {}_2F_1\left[\begin{matrix} a & \frac{1}{2}+a \\ & \frac{5}{6}+\frac{2}{3}a \end{matrix}\middle| \frac{1}{9}\right] = \left(\frac{3}{4}\right)^a \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}-\frac{p}{3}\right)}{\Gamma\left(\frac{2}{3}-\frac{p}{6}\right)\Gamma\left(1-\frac{p}{6}\right)}. $$

Substituting $a=\frac{1-p}{2}$ in this identity, we have

$$ \mathfrak{A} = \frac{\left(\frac{1}{3}-\frac{p}{6}\right)_{\frac{p-1}{2}}}{\left(\frac{4}{3}-\frac{2p}{3}\right)_{\frac{p-1}{2}}} 12^{\frac{p-1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{6}-\frac{p}{3}\right)}{\Gamma\left(\frac{2}{3}-\frac{p}{6}\right)\Gamma\left(1-\frac{p}{6}\right)}. \tag{2.6} $$

On the other hand, we have

$$ \sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}(-8)^k \equiv \sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}8^k}{3k+1} = -2\left(\frac{2}{p}\right)\sum_{k=0}^{(p-7)/6}\frac{\binom{\frac{p-1}{2}}{k}}{(6k+1)8^k} \pmod p. $$

It is easy to see that

$$ \sum_{k=0}^{(p-7)/6}\frac{\binom{\frac{p-1}{2}}{k}}{(6k+1)8^k} \equiv \sum_{k=0}^{(p-7)/6}\frac{\left(\frac{1-p}{6}\right)_k\left(\frac{1+p}{2}\right)_k}{(1)_k\left(\frac{7}{6}+\frac{p}{3}\right)_k(-8)^k} \pmod p. $$

Set

$$ \mathfrak{D} = \sum_{k=0}^{(p-1)/6}\frac{\left(\frac{1-p}{6}\right)_k\left(\frac{1+p}{2}\right)_k}{(1)_k\left(\frac{7}{6}+\frac{p}{3}\right)_k(-8)^k} = {}_2F_1\left[\begin{matrix} \frac{1-p}{6} & \frac{1+p}{2} \\ & \frac{7}{6}+\frac{p}{3} \end{matrix}\middle| -\frac{1}{8}\right], $$

and

$$ \mathfrak{G} = \frac{\left(\frac{1-p}{6}\right)_{(p-1)/6}\left(\frac{1+p}{2}\right)_{(p-1)/6}}{(1)_{(p-1)/6}\left(\frac{7}{6}+\frac{p}{3}\right)_{(p-1)/6}(-8)^{(p-1)/6}}. \tag{2.7} $$

Hence

$$ \sum_{k=\frac{p+2}{3}}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{3}\right)_k}{(1)_k\left(\frac{4}{3}\right)_k}(-8)^k \equiv -2\left(\frac{2}{p}\right)(\mathfrak{D}-\mathfrak{G}) \pmod p. $$

Substituting $a=\frac{1-p}{3}, b=\frac{2}{3}+\frac{p}{3}, z=-1$ into [19, 15.8.14], we have

$$ \mathfrak{D} = {}_2F_1\left[\begin{matrix} \frac{1-p}{6} & \frac{1}{2}+\frac{p}{2} \\ & \frac{7}{6}+\frac{p}{3} \end{matrix}\middle| -\frac{1}{8}\right] = 2^{(1-p)/6}{}_2F_1\left[\begin{matrix} \frac{1-p}{3} & \frac{2}{3}+\frac{p}{3} \\ & \frac{4}{3}+\frac{2p}{3} \end{matrix}\middle| -1\right]. $$

And then substituting $a=\frac{1-p}{3}, b=\frac{2}{3}+\frac{p}{3}, c=\frac{4}{3}+\frac{2p}{3}, z=-1$ into [19, 15.8.1], we have

$$ \mathfrak{D} = {}_2F_1\left[\begin{matrix} \frac{1-p}{6} & \frac{1}{2}+\frac{p}{2} \\ & \frac{7}{6}+\frac{p}{3} \end{matrix}\middle| -\frac{1}{8}\right] = 2^{(p-1)/6}{}_2F_1\left[\begin{matrix} \frac{1-p}{3} & \frac{2}{3}+\frac{p}{3} \\ & \frac{4}{3}+\frac{2p}{3} \end{matrix}\middle| \frac{1}{2}\right]. $$

By using [19, 15.4.30] with $a = \frac{1-p}{3}, b = \frac{4}{3} + \frac{2p}{3}$, we have

$$\mathfrak{D} = {}_2F_1\left[\begin{matrix} \frac{1-p}{6} & \frac{1}{2} + \frac{p}{2} \\ & \frac{7}{6} + \frac{p}{3} \end{matrix}\bigg| -\frac{1}{8}\right] = 2^{(p-1)/6} \frac{2^{-\frac{2p+1}{3}}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{3}+\frac{2p}{3}\right)}{\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(\frac{5}{6}+\frac{p}{6}\right)}$$
$$= 2^{-\frac{p+1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{3}+\frac{2p}{3}\right)}{\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(\frac{5}{6}+\frac{p}{6}\right)}. \tag{2.8}$$

It follows that (2.4) is equivalent to

$$\mathfrak{A} - \mathfrak{F} \equiv 3\left(\frac{2}{p}\right)(\mathfrak{D} - \mathfrak{G}) - \frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}} q_p(2) \pmod{p}. \tag{2.9}$$

By (2.6), (2.5), and [9, Lemma 17]

$$\mathfrak{A} = \frac{2^{p-1}3^{\frac{p+1}{2}}}{p} \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{4}{3}-\frac{2p}{3}\right)\Gamma_p\left(-\frac{1}{6}+\frac{p}{3}\right)\Gamma_p\left(\frac{7}{6}-\frac{p}{3}\right)}{\Gamma_p\left(\frac{1}{3}-\frac{p}{6}\right)\Gamma_p\left(\frac{2}{3}-\frac{p}{6}\right)\Gamma_p\left(\frac{5}{6}-\frac{p}{6}\right)\Gamma_p\left(1-\frac{p}{6}\right)}$$

and

$$\mathfrak{F} = \frac{2^{p-1}\,3}{p} \frac{\Gamma_p\left(\frac{4}{3}-\frac{2p}{3}\right)\Gamma_p\left(\frac{p}{6}\right)\Gamma_p\left(\frac{1}{6}-\frac{p}{6}\right)}{\Gamma_p\left(\frac{1}{2}-\frac{p}{2}\right)\Gamma_p\left(\frac{2}{3}+\frac{p}{3}\right)\Gamma_p\left(\frac{1}{3}-\frac{p}{6}\right)\Gamma_p\left(1-\frac{p}{3}\right)}.$$

By (2.3) and by Lemma 2.1,

$$p\mathfrak{A} \equiv 2^{p-1}3^{\frac{p+1}{2}} \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{4}{3}\right)\Gamma_p\left(-\frac{1}{6}\right)\Gamma_p\left(\frac{7}{6}\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)\Gamma_p\left(\frac{5}{6}\right)\Gamma_p(1)}\left(1 - \frac{p}{3}H_{\frac{p-1}{3}} + \frac{p}{6}H_{\frac{p-1}{6}} - 2p\right)$$
$$\equiv 2^{p-1}3^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(1 - \frac{pq_p(2)}{3} + \frac{pq_p(3)}{4} - 2p\right) \pmod{p^2}$$

and

$$p\mathfrak{F} \equiv 2^{p-1}\,3 \frac{\Gamma_p\left(\frac{4}{3}\right)\Gamma_p\left(\frac{1}{6}\right)}{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{2}{3}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p(1)}\left(1 + pH_{\frac{p-1}{2}} - \frac{p}{3}H_{\frac{p-1}{3}} - \frac{2p}{3}H_{\frac{p-1}{6}} - 2p\right)$$
$$\equiv 2^{p-1}(-1)^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(1 - \frac{2pq_p(2)}{3} + \frac{3pq_p(3)}{2} - 2p\right) \pmod{p^2}.$$

Hence, since $3^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}\left(1 + \frac{pq_p(3)}{2}\right)$ when $p \equiv 1 \pmod 3$, and $2^{p-1} \equiv 1 \pmod p$ we find

$$\mathfrak{A} - \mathfrak{F} \equiv (-1)^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(\frac{q_p(2)}{3} - \frac{3q_p(3)}{4}\right) \pmod{p}. \tag{2.10}$$

Similarly, by (2.8) and (2.7), we have

$$\mathfrak{D} = -\frac{1}{p2^{\frac{p-1}{2}}} \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{4}{3}+\frac{2p}{3}\right)}{\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{5}{6}+\frac{p}{6}\right)}$$

and

$$\mathfrak{G} = \frac{2\,(-1)^{\frac{p-1}{6}}}{p2^{\frac{p-1}{2}}} \frac{\Gamma_p\left(\frac{1}{3}+\frac{2p}{3}\right)\Gamma_p\left(\frac{7}{6}+\frac{p}{3}\right)}{\Gamma_p\left(\frac{1}{2}+\frac{p}{2}\right)\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{5}{6}+\frac{p}{6}\right)\Gamma_p\left(\frac{1}{6}-\frac{p}{6}\right)}.$$

Hence

$$\begin{aligned} p\mathfrak{D} &\equiv -\frac{1}{2^{\frac{p-1}{2}}} \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{4}{3}\right)}{\Gamma_p\left(\frac{5}{6}\right)} \left(1+\frac{2p}{3}H_{\frac{p-1}{3}}-\frac{p}{6}H_{\frac{p-1}{6}}+2p\right) \\ &\equiv -\frac{(-1)^{\frac{p-1}{2}}}{3\,2^{\frac{p-1}{2}}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(1+\frac{pq_p(2)}{3}-\frac{3pq_p(3)}{4}+2p\right) \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} p\mathfrak{G} &\equiv -\frac{2\,(-1)^{\frac{p-1}{2}}\left(\frac{1}{6}+\frac{p}{3}\right)}{2^{\frac{p-1}{2}}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(1-\frac{p}{2}H_{\frac{p-1}{2}}+\frac{2p}{3}H_{\frac{p-1}{3}}+\frac{p}{3}H_{\frac{p-1}{6}}\right) \\ &\equiv -\frac{(-1)^{\frac{p-1}{2}}}{3\,2^{\frac{p-1}{2}}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(1+\frac{pq_p(2)}{3}-\frac{3pq_p(3)}{2}+2p\right) \pmod{p^2}. \end{aligned}$$

Therefore

$$\mathfrak{D}-\mathfrak{G} = -\frac{(-1)^{\frac{p-1}{2}}}{4\,2^{\frac{p-1}{2}}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)q_p(3) \pmod{p}.$$

Moreover, since

$$\begin{aligned} \binom{\frac{p-1}{2}}{\frac{p-1}{3}} &= \frac{(1)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{3}}(1)_{\frac{p-1}{6}}} = -\frac{\Gamma_p\left(\frac{1}{2}+\frac{p}{2}\right)}{\Gamma_p\left(\frac{2}{3}+\frac{p}{3}\right)\Gamma_p\left(\frac{5}{6}+\frac{p}{6}\right)} \\ &\equiv -(-1)^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right) \pmod{p} \end{aligned}$$

and $\left(\frac{2}{p}\right) \equiv 2^{\frac{p-1}{2}} \pmod{p}$, it follows

$$\begin{aligned} &3\left(\frac{2}{p}\right)(\mathfrak{D}-\mathfrak{G}) - \frac{1}{3}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}q_p(2) \\ &\qquad \equiv (-1)^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{1}{6}\right)\left(\frac{q_p(2)}{3}-\frac{3q_p(3)}{4}\right) \pmod{p}. \qquad (2.11) \end{aligned}$$

By comparing (2.10) and (2.11), we may conclude that the proof of (2.9) is complete. □

3. Proof of Theorem 1.4

Lemma 3.1. *For any prime p such that $p \equiv 1 \pmod 5$ we have that*

$$H_{\frac{p-1}{2}} + H_{\frac{p-1}{5}} - H_{\frac{3(p-1)}{10}} \equiv -5f_p \pmod{p}. \qquad (3.1)$$

Proof. By [26], for any prime $p > 5$,

$$\sum_{k=1}^{\lfloor \frac{4p}{5} \rfloor} \frac{(-1)^k}{k} \equiv \frac{5f_p}{2} \pmod{p}. \tag{3.2}$$

Hence, if $p \equiv 1 \pmod 5$, $n = \frac{p-1}{2}$ and $m = \frac{p-1}{5}$ then $n - m = \frac{3(p-1)}{10}$ and, by (3.2),

$$\begin{aligned} H_n + H_m - H_{n-m} &\equiv H_n - 2\sum_{k=1}^{m} \frac{1}{p-k} - H_m + 2\sum_{k=1}^{n-m} \frac{1}{p-2k} \\ &= -2\sum_{k=1}^{\frac{4(p-1)}{5}} \frac{(-1)^k}{k} \equiv -5f_p \pmod{p}. \end{aligned}$$

□

Proof of Theorem 1.4. Let p be a prime $p \equiv 1 \pmod 5$, then

$$\begin{aligned} \sum_{\substack{k=0 \\ k \neq \frac{p-1}{5}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k} 4^k}{5k+1} &\equiv \sum_{k=0}^{\frac{p-1}{5}-1} \frac{\binom{\frac{p-1}{2}}{k} 4^k}{5k+1} + \sum_{k=\frac{p-1}{5}+1}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k} 4^k}{5k+1} \\ &\equiv \sum_{k=0}^{\frac{p-1}{5}-1} \frac{\binom{\frac{p-1}{2}}{k} 4^k}{5k+1} - 2 \sum_{k=0}^{\frac{3(p-1)}{10}-1} \frac{\binom{\frac{p-1}{2}}{k} 4^{-k}}{10k+3} \\ &\equiv \sum_{k=0}^{\frac{p-1}{5}-1} \frac{\left(\frac{1-p}{5}\right)_k \left(\frac{1}{2}-p\right)_k}{(1)_k \left(\frac{6}{5}-\frac{6p}{5}\right)_k} (-4)^k - \frac{2}{3} \sum_{k=0}^{\frac{3(p-1)}{10}-1} \frac{\left(\frac{3(1-p)}{10}\right)_k \left(\frac{1}{2}+\frac{3p}{2}\right)_k}{(1)_k \left(\frac{13}{10}+\frac{6p}{5}\right)_k} (-4)^{-k} \\ &\equiv (\mathfrak{A} - \mathfrak{F}) - \frac{2}{3}(\mathfrak{D} - \mathfrak{G}) \pmod{p} \end{aligned}$$

where

$$\mathfrak{A} = {}_2F_1\left[\begin{matrix} \frac{1-p}{5} & \frac{1}{2}-p \\ & \frac{6}{5}-\frac{6p}{5} \end{matrix} \middle| -4\right], \quad \mathfrak{F} = \frac{(-1)^{(p-1)/5} \left(\frac{1}{2}-p\right)_{(p-1)/5}}{\left(\frac{6}{5}-\frac{6p}{5}\right)_{(p-1)/5}} (-4)^{(p-1)/5},$$

and

$$\mathfrak{D} = {}_2F_1\left[\begin{matrix} \frac{3(1-p)}{10} & \frac{1}{2}+\frac{3p}{2} \\ & \frac{13}{10}+\frac{6p}{5} \end{matrix} \middle| -\frac{1}{4}\right], \quad \mathfrak{G} = \frac{(-1)^{3(p-1)/10} \left(\frac{1}{2}+\frac{3p}{2}\right)_{3(p-1)/10}}{\left(\frac{13}{10}+\frac{6p}{5}\right)_{3(p-1)/10}} (-4)^{-3(p-1)/10}.$$

In view of [19, 15.8.6], and by [4, Theorem 20] we have

$$\begin{aligned} \mathfrak{A} &= {}_2F_1\left[\begin{matrix} -n & \frac{1}{2}-p \\ & \frac{6}{5}-\frac{6p}{5} \end{matrix} \middle| -4\right] = 5^n \left(\frac{1}{2}-p\right)_n \left(\frac{6}{5}-\frac{6p}{5}\right)_n {}_2F_1\left[\begin{matrix} -n & -n+\frac{1}{2} \\ & 4n+\frac{3}{2} \end{matrix} \middle| \frac{1}{5}\right] \\ &= 5^n \left(\frac{1}{2}-p\right)_n \left(\frac{6}{5}-\frac{6p}{5}\right)_n \frac{2^{10n}\Gamma(4/5)\Gamma(6/5)\Gamma(3/2+4n)}{5^{6n}\Gamma(3/2)\Gamma(4/5+2n)\Gamma(6/5+2n)} \end{aligned}$$

where $n=\frac{p-1}{5}$. Moreover, by [19, 15.8.1], and by a variation of [4, Theorem 20] we have

$$\begin{aligned}\mathfrak{D} &= {}_2F_1\left[\begin{matrix}-n & \frac{1}{2}+\frac{3p}{2}\\ & \frac{13}{10}+\frac{6p}{5}\end{matrix}\middle| -\frac{1}{4}\right] = \frac{5^n}{4^n}\,{}_2F_1\left[\begin{matrix}-n & -n+\frac{1}{2}\\ & 4n+\frac{5}{2}\end{matrix}\middle| \frac{1}{5}\right]\\ &= \frac{5^n}{4^n}\,\frac{2^{10n}\Gamma(7/5)\Gamma(8/5)\Gamma(5/2+4n)}{5^{6n}\Gamma(5/2)\Gamma(7/5+2n)\Gamma(8/5+2n)}\end{aligned}$$

where $n=\frac{3(p-1)}{10}$.

In a similar way as we did in the previous section, we find that

$$\mathfrak{A}-\mathfrak{F} \equiv \frac{4^{\frac{p-1}{5}}}{25}\binom{\frac{p-1}{2}}{\frac{p-1}{5}}\left(4H_{\frac{p-1}{2}}-4H_{\frac{3(p-1)}{10}}\right) \pmod p,$$

and

$$\frac{2}{3}(\mathfrak{D}-\mathfrak{G}) \equiv -\frac{4^{\frac{p-1}{5}}}{25}\binom{\frac{p-1}{2}}{\frac{p-1}{5}}\left(H_{\frac{p-1}{2}}+5H_{\frac{p-1}{5}}-H_{\frac{3(p-1)}{10}}\right) \pmod p$$

which together imply

$$\begin{aligned}\sum_{\substack{k=0\\ k\neq\frac{p-1}{5}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}4^k}{5k+1} &\equiv (\mathfrak{A}-\mathfrak{F})-\frac{2}{3}(\mathfrak{D}-\mathfrak{G})\\ &\equiv \frac{4^{\frac{p-1}{5}}}{5}\binom{\frac{p-1}{2}}{\frac{p-1}{5}}\left(H_{\frac{p-1}{2}}+H_{\frac{p-1}{5}}-H_{\frac{3(p-1)}{10}}\right)\\ &\equiv -4^{\frac{p-1}{5}}\binom{\frac{p-1}{2}}{\frac{p-1}{5}}f_p \pmod p.\end{aligned}$$

where at the last step we applied (3.1). □

4. Proofs of Theorem 1.1 and 1.2

We first collect a couple of lemmas which are needed to prove the main theorem.

Lemma 4.1. ([17, (41) with $t=-1/2$]). *For any prime $p>3$,*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}(-2)^k}{k} \equiv -4q_p(2)+4pq_p(2) \pmod{p^2}.$$

Lemma 4.2. *For any $1\le m\le n$, and for any $z\in\mathbb{C}$,*

$$\frac{(z+1)^{n+1}}{n}\sum_{k=0}^{m-1}\frac{z^k}{\binom{n-1}{k}} = \sum_{k=1}^{n}\binom{n}{k}\frac{z^{n-k}}{k}+(H_n+H_m-H_{n-m})z^n-\frac{z^m}{\binom{n}{m}}\sum_{\substack{k=0\\ k\neq m}}^{n}\binom{n}{k}\frac{z^{n-k}}{k-m} \quad (4.1)$$

and

$$\frac{(z+1)^{n+1}}{n}\sum_{k=m}^{n-1}\frac{z^k}{\binom{n-1}{k}}=\sum_{k=1}^{n}\binom{n}{k}\frac{z^{n+k}}{k}+(H_n-H_m+H_{n-m})z^n+\frac{z^m}{\binom{n}{m}}\sum_{\substack{k=0\\k\neq m}}^{n}\binom{n}{k}\frac{z^{n-k}}{k-m} \quad (4.2)$$

Proof. Both sides of (4.1) are polynomials of degree $n+m$, so it suffices to compare the coefficients of z^d for each d such that $0\le d\le n+m$:

$$\frac{1}{n}\sum_{k=0}^{m-1}\frac{\binom{n+1}{d-k}}{\binom{n-1}{k}}=\begin{cases}\dfrac{1}{n-d}\left(\dbinom{n}{n-d}-\dfrac{\binom{n}{n+m-d}}{\binom{n}{m}}\right) & \text{if } d\neq n,\\[2ex] H_n+H_m-H_{n-m} & \text{if } d=n,\end{cases}$$

which can be easily verified by induction with respect to m. As regards (4.1), just subtract (4.1) from (4.1) with $m=n$. □

Proof of (1.3). It is easy to verify that

$$\binom{2k}{k}\equiv\binom{n}{k}(-4)^k \pmod p \text{ for all } k=0,\ldots,n,$$

and in view of [22, Lemma 2.1], we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv -2p \pmod{p^2} \text{ for all } k=n,\ldots,p-1.$$

If $1\le m\le n$ then

$$\begin{aligned}\sum_{k=p-m}^{p-1}\binom{2k}{k}x^k&=\sum_{k=1}^{m}\binom{2(p-k)}{p-k}x^{p-k}\equiv -2p\sum_{k=1}^{m}\frac{x^{p-k}}{(p-k)\binom{2k}{k}}\\&\equiv 2p\sum_{k=1}^{m}\frac{x^{p-k}}{k\binom{n}{k}(-4)^k}\equiv -p\sum_{k=1}^{m}\frac{x^{1-k}}{\binom{n-1}{k-1}(-4)^{k-1}}\\&\equiv -p\sum_{k=0}^{m-1}\frac{z^k}{\binom{n-1}{k}} \pmod{p^2}\end{aligned}$$

where $z=-1/(4x)$. Hence, by (4.1), the following congruence holds modulo p^2,

$$\sum_{k=1}^{p-1-m}\binom{2k}{k}x^k\equiv\sum_{k=1}^{p-1}\binom{2k}{k}x^k+\frac{npz^n}{(z+1)^{n+1}}\left(\sum_{k=1}^{n}\frac{\binom{2k}{k}x^k}{k}+H_n+H_m-H_{n-m}\right.$$
$$\left.-\frac{z^m}{\binom{n}{m}}\sum_{\substack{k=0\\k\neq m}}^{n}\binom{n}{k}\frac{z^{-k}}{k-m}\right). \quad (4.3)$$

Finally, let $x = -2$ and $m = \frac{p-1}{3}$. Then $z = 1/8$, and by (1.1), Lemma 4.1, Lemma 2.1, Theorem 1.3,

$$\sum_{k=1}^{\lfloor \frac{2p}{3} \rfloor} \binom{2k}{k}(-2)^k \equiv -\frac{4pq_p(2)}{3} - \frac{4p}{9^{n+1}} \left(-4q_p(2) + 0 - \frac{3}{8^m \binom{n}{m}} \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{\binom{n}{k} 8^k}{3k+1} \right) \equiv 0 \pmod{p^2}$$

□

Proof of (1.4). Similarly as before, for $1 \leq m \leq n$ and $z = -1/(4x)$ we find

$$\sum_{k=n+1}^{p-1-m} \binom{2k}{k} x^k \equiv -p \sum_{k=m}^{n-1} \frac{z^k}{\binom{n-1}{k}} \pmod{p^2}.$$

Therefore, by (4.2), the next congruence holds modulo p^2,

$$\sum_{k=0}^{p-1-m} \binom{2k}{k} x^k \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k} x^k - \frac{npz^n}{(z+1)^{n+1}} \left(\sum_{k=1}^{n} \frac{\binom{2k}{k}}{k(16x)^k} + H_n - H_m + H_{n-m} \right.$$
$$\left. - \frac{z^{m-n}}{\binom{n}{m}} \sum_{\substack{k=0 \\ k \neq n-m}}^{n} \binom{n}{k} \frac{z^k}{k-(n-m)} \right). \quad (4.4)$$

Finally, let $x = -1/32$ and $m = \frac{p-1}{6}$. Then $z = 8$, and by (1.2), Lemma 4.1, Lemma 2.1, Theorem 1.3,

$$\sum_{k=0}^{\lfloor \frac{5p}{6} \rfloor} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p}\right) \left(1 + \frac{pq_p(2)}{6}\right) - \frac{p8^n}{2\,9^{n+1}} \left(-4q_p(2) + 0 - \frac{3\,8^{m-n}}{\binom{n}{m}} \sum_{\substack{k=0 \\ k \neq n-m}}^{n} \frac{\binom{n}{k} 8^k}{3k+1} \right)$$
$$\equiv \left(\frac{2}{p}\right) \left(1 + \frac{pq_p(2)}{6}\right) - \frac{p8^n q_p(2)}{6} \equiv \left(\frac{2}{p}\right) \pmod{p^2}$$

□

Proofs of (1.4) *and* (1.5). This same approach can be applied also for Theorem 1.2. In [21] and [23] respectively, we find that

$$\sum_{k=0}^{p-1} \binom{2k}{k}(-1)^k \equiv \left(\frac{p}{5}\right)(1 - 2pf_p) \pmod{p^3}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p}{5}\right)\left(1 + \frac{pf_p}{2}\right).$$

Moreover in [17], it is showed that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}(-1)^k}{k} \equiv -5f_p + 5pf_p^2 \pmod{p^2}.$$

Let $p \equiv 1 \pmod 5$ and $m = \frac{p-1}{5}$ then $n - m = \frac{3(p-1)}{10}$ and, by (3.1),

$$H_n + H_m - H_{n-m} \equiv -5f_p \pmod p.$$

Then after letting $x = -1$, $m = \frac{p-1}{5}$ in (4.3) and $x = -1/16$, $m = \frac{3(p-1)}{10}$ in (4.4), we get that (1.5) and (1.6) are established as soon as

$$\sum_{\substack{k=0 \\ k \neq \frac{p-1}{5}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k} 4^k}{5k+1} \equiv -4^{\frac{p-1}{5}} \binom{\frac{p-1}{2}}{\frac{p-1}{5}} f_p \pmod p$$

which have been shown in Section 3. □

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