A COMBINATORIAL EXPANSION OF VERTICAL-STRIP LLT POLYNOMIALS IN THE BASIS OF ELEMENTARY SYMMETRIC FUNCTIONS

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Abstract. We give a new characterization of the vertical-strip LLT polynomials $\mathrm{G}_P(\mathbf{x};q)$ as the unique family of symmetric functions that satisfy certain combinatorial relations. This characterization is then used to prove an explicit combinatorial expansion of vertical-strip LLT polynomials in terms of elementary symmetric functions. Such formulas were conjectured independently by A. Garsia et al. and the first named author, and are governed by the combinatorics of orientations of unit-interval graphs. The obtained expansion is manifestly positive if q is replaced by $q+1$, thus recovering a recent result of M. D'Adderio. Our results are based on linear relations among LLT polynomials that arise in the work of D'Adderio, and of E. Carlsson and A. Mellit. To some extent these relations are given new bijective proofs using colorings of unit-interval graphs. As a bonus we obtain a new characterization of chromatic quasisymmetric functions of unit-interval graphs.

Contents

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1. Overview of main results

LLT polynomials form a large class of symmetric functions that can be viewed as q-deformations of products of (skew) Schur functions. They appear in many different contexts in representation theory and algebraic combinatorics. In this paper we are concerned with a particular subclass of LLT polynomials, namely the vertical-strip LLT polynomials. They contain information about the equivariant cohomology ring of regular semisimple Hessenberg varieties, and are integral to the study of diagonal harmonics and the proof of the Shuffle Theorem.

Vertical-strip LLT polynomials can be indexed by certain lattice paths called Schröder paths. We view the vertical-strip LLT polynomial $\mathrm{G}_P(\mathbf{x};q)$ associate to a Schröder path P of size n as the generating function of certain colorings of a unit-interval graph Γ_P on n vertices. The main result of this paper is an explicit expansion of $\mathrm{G}_P(\mathbf{x};q+1)$ in the basis of elementary symmetric functions as a weighted sum over orientations of Γ_P. This expansion is given by

$$\sum_{\kappa\in\mathcal{C}(P)}(q+1)^{\mathrm{asc}(\kappa)}x_{\kappa(1)}x_{\kappa(2)}\cdots x_{\kappa(n)},=\sum_{\theta\in\mathcal{O}(P)}q^{\mathrm{asc}(\theta)}\mathrm{e}_{\lambda(\theta)}(\mathbf{x}), \tag{1}$$

where $\mathcal{C}(P)$ is a set of colorings, $\mathcal{O}(P)$ is a set of orientations, and asc denotes certain combinatorial statistics on these objects. The expression $\lambda(\theta)$ is an integer partition determined by the orientation θ. In particular, the above identity proves that $\mathrm{G}_P(\mathbf{x};q+1)$ is e-positive. This resolves several open conjecture stated in [AP18, Ale20, GHQR19]. Algebraically the e-positivity phenomenon implies that the symmetric function in question is — up to a twist by the sign representation — the Frobenius character of a permutation representation in which each point stabilizer is a Young subgroup. Note that our approach to vertical-strip LLT polynomials highlights their connection to chromatic quasisymmetric functions. The expansion in (1) is of particular interest because it serves as an analog of the Shareshian–Wachs conjecture regarding the e-positivity of chromatic quasisymmetric functions.

We now briefly outline the organisation of this paper. Section 2 contains background on LLT polynomials, and all definitions necessary to state our main results. In Theorem 2.1 and Theorem 2.5 we provide sets of relations which uniquely determine the vertical-strip LLT polynomials. Theorem 2.9 states that the right-hand side in (1) satisfies the same set of relations, which implies equality in (1). Section 3 treats linear relations among LLT polynomials and contains the proof of Theorem 2.1.

Moreover we give a new characterisation of the (smaller) class of unicellular LLT polynomials in Theorem 3.5. Section 4 contains bijective results on colorings and the proof of Theorem 2.5. Section 5 deals with the combinatorics of orientations, culminating in a proof of Theorem 2.9.

In Section 6 we give an overview of related areas and open problems. We only mention a few here. In Corollary 6.2 we state a new signed Schur expansion of vertical-strip LLT polynomials. We obtain new expressions for quantities in diagonal harmonics. To be more precise, we consider $\nabla \mathrm{e}_n$ and $\nabla \omega \mathrm{p}_n$, which appear in the Shuffle theorem and the Square Paths theorem, respectively. Both these expressions can be expressed using vertical-strip LLT polynomials. As a consequence they are both e-positive after the substitution $q \to q+1$, see Corollary 6.11 and Theorem 6.14. Finally we discuss chromatic quasisymmetric functions and the representation theory of regular semisimple Hessenberg varieties. In particular Corollary 6.16 provides a new characterisation of chromatic quasisymmetric functions of unit-interval graphs.

2. Background and main results

We first give additional history and references which lead up to this paper. Then we go into more detail and give all necessary definitions needed in order to state the main results.

2.1. **Brief history of LLT polynomials.** Below is a very brief history of LLT polynomials — there are several more references which are not included with strong results.

- (1997) Motivated by the study certain Fock space representations, and plethysm coefficients, A. Lascoux, B. Leclerc and J. Y. Thibon introduce the LLT polynomials in [LLT97] under the name *ribbon Schur functions*. They are defined as a sum over semi-standard border-strip tableaux of straight shape.
- (2000) Three years later B. Leclerc and J. Y. Thibon [LT00] show that the LLT polynomials are Schur positive by means of representation theory.
- (2000+) M. Bylund and M. Haiman use the so called *Littlewood map* (see history in [Hag07, p. 92]) which is a bijection between semi-standard border-strip tableaux and k-tuples of semi-standard Young tableaux of straight shape. They extend their model to include tuples of skew shapes, and they conjecture that LLT polynomials in this family are also Schur positive. Note that the proof of Schur positivity from 2000 does not extend to the Bylund–Haiman model. We remark that a similar model was introduced independently in [SSW03], but they use a more complicated statistic for the q-weight.
- (2005) J. Haglund, M. Haiman and M. Loehr [HHL05a] give a combinatorial formula for the *modified Macdonald polynomials*. They show that these Macdonald polynomials can be expressed as positive linear combinations of LLT polynomials indexed by k-tuples of ribbon shapes (these are skew shapes). Hence a deeper understanding of LLT polynomials has consequences for the modified Macdonald polynomials.
- (2006) I. Grojnowski and M. Haiman post a preprint [GH06], giving an argument for Schur positivity of LLT polynomials by using Kazhdan–Luztig theory and geometric representation theory.

(2007+) J. Haglund [Hag07, Ch. 6] gives an overview of the appearance of LLT polynomials in the study of *diagonal harmonics*. In particular, the *path symmetric functions* — which are LLT polynomials indexed by k-tuples of vertical strips — appear in the *Shuffle Conjecture* [HHL+05b] and later in the *Compositional Shuffle Conjecture* [HMZ12], *Square paths theorem* [Ser17] and *Delta Conjecture* [HRW18] .

(2016) In his work on chromatic quasisymmetric functions M. Guay-Paquet [GP16] uses a Hopf algebra approach to show that unicellular LLT polynomials are graded Frobenius series derived from the equivariant cohomology rings of regular semisimple Hessenberg varieties. This yields a second proof that unicellular LLT polynomials are Schur positive, but no combinatorial formula for the coefficients.

(2017) A. Carlsson and A. Mellit [CM17] use the so called zeta map on the path symmetric functions to obtain a more convenient model for vertical-strip LLT polynomials. They introduce the *Dyck path algebra* and prove the Compositional Shuffle Conjecture — henceforth Shuffle Theorem.

A. Carlson and A. Mellit briefly use a plethystic identity originating in [HHL05a]. In the language of Haglund et al. it essentially states that there is a simple plethystic relationship between general fillings and non-attacking fillings. In terms of vertex colorings of graphs this relates a sum over arbitrary colorings to a sum over proper colorings. Consequently there is a plethystic relationship between certain (unicellular) LLT polynomials and the chromatic quasisymmetric functions of J. Shareshian and M. Wachs [SW12, SW16].

(2018) The Carlsson–Mellit model is used by G. Panova and the first named author [AP18] to emphasize the connection with the chromatic quasisymmetric functions. A conjecture regarding e-positivity of certain LLT polynomials is stated in [AP18, Conj. 4.5]. This conjecture serves as an analog of the e-positivity conjecture of Shareshian–Wachs, and thus the Stanley–Stembridge conjecture for chromatic symmetric functions [Sta95, SS93].

(2019) The first named author posts a preprint [Ale20] with the explicit formula (1) for the expansion of vertical-strip LLT polynomials. Shortly after, A. Garsia, J. Haglund, D. Qiu and M. Romero [GHQR19] independently publish an equivalent conjecture in the language of diagonal harmonics.

(2019) M. D'Adderio [D'A19] proves e-positivity by using the Dyck path algebra and recursions developed by Carlsson–Mellit. However, this does not prove the conjectured formula (1).

2.2. **Symmetric functions.** Let $K = \mathbb{Q}(q)$ be the field of rational functions over $\mathbb{Q}$, and let Λ denote the algebra of symmetric functions over K. For an introduction to symmetric functions the reader is referred to [Mac96, Sta01]. Elements of Λ may be viewed as formal power series over K in infinitely many variables $\mathbf{x} := (x_1, x_2, \dots)$ that have finite degree and are invariant under permutation of the variables. Alternatively we may think of elements of Λ as K-linear combinations of the *elementary symmetric functions* $\mathrm{e}_\lambda = \mathrm{e}_{\lambda_1} \cdots \mathrm{e}_{\lambda_r}$ where λ ranges over all integer partitions.

2.3. **Schröder paths.** Let $\mathtt{n} := (0, 1)$ be a *north step*, $\mathtt{e} := (1, 0)$ be an *east step*, and $\mathtt{d} := (1, 1)$ be a *diagonal step*. A *Schröder path* P of *size* n is a lattice path from

$(0,0)$ to (n,n) using steps from $\{\mathtt{n},\mathtt{e},\mathtt{d}\}$ that never goes below the main diagonal, and has no diagonal step on the main diagonal. That is, every east step and every diagonal step of P is preceded by more north steps than east steps. Let $\mathcal{S}_n$ denote the set of Schröder paths of size n, and let $\mathcal{S}$ be the set of all Schröder paths. We describe Schröder paths as words in $\{\mathtt{n},\mathtt{d},\mathtt{e}\}^*$, see Figure 1 for examples. A Schröder path without diagonal steps is called a *Dyck path*.

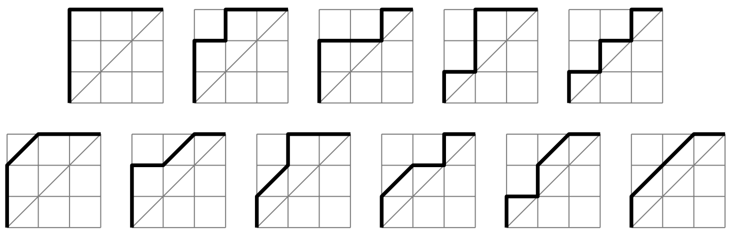


FIGURE 1. The set $\mathcal{S}_3 = \{\mathtt{nnneee}, \mathtt{nnenee}, \dots, \mathtt{ndde}\}$ consisting of the eleven Schröder paths of size 3. The first row consists of all Dyck paths. The number of Schröder paths of size n is given by the sequence A001003, and starts with 1, 3, 11, 45, 197, $\dots$.

2.4. **Bounce paths.** An important tool in our analysis is a type of bounce path that is similar to bounce paths that appear for example in [GH02, AKOP02].

Given a Schröder path $P \in \mathcal{S}$ and a point $(u_1, u_0) \in \mathbb{Z}^2$ that lies on P define the *(partial reverse) bounce path* of P at (u_1, u_0) as follows: Starting at (u_1, u_0) move south until you reach the point (u_1, u_1) on the main diagonal. Now move west until you reach a point (u_2, u_1) on P. If this point lies between two diagonal steps of P then continue south until the point (u_2, u_2) on the diagonal. Then move west until you reach a point (u_3, u_2) on P. Continue in this fashion until the bounce path ends at a point incident to a north step or east step of P. An example of a bounce path is shown in Figure 2. The points $(u_1, u_1), \dots, (u_k, u_k)$ where the bounce path touches the main diagonal are called *bounce points*. The coordinates of the peaks $(u_1, u_0), (u_2, u_1), \dots, (u_{k+1}, u_k)$ of the bounce path are recorded in the *bounce partition* $(u_0, u_1, \dots, u_{k+1})$ of P at (u_1, u_0). There is a unique decomposition

$$P = U s_1 . s_2 V s_3 . s_4 W$$

where $s_1, s_2, s_3, s_4 \in \{\mathtt{n},\mathtt{d},\mathtt{e}\}$ and $U, V, W \in \{\mathtt{n},\mathtt{d},\mathtt{e}\}^*$ such that Us_1 is a path from $(0,0)$ to the endpoint (u_{k+1}, u_k) of the bounce path, and $s_4 W$ is a path from the starting point (u_1, u_0) of the bounce path to (n,n). We call this the *bounce decomposition* of P at (u_1, u_0). Note that we use a small dot to indicate the starting point and endpoint of the bounce path. In this paper our focus is mainly on the special case where $s_1 s_2 \in \{\mathtt{nn}, \mathtt{dn}\}$, $V \in \{\mathtt{n},\mathtt{d}\}^*$ and $s_3 s_4 = \mathtt{de}$. The pieces that such bounce paths are made of are found in Figure 3.

2.5. **The main recursion.** An essential step in M. D'Adderio's proof of the e-positivity of vertical-strip LLT polynomials is an inductive argument that relies on certain linear relations among them. The starting point of our work is to simplify these relations, and to make them more explicit. This leads to our first main result, namely, that the following initial conditions and relations determine a unique symmetric-function valued statistic on Schröder paths.

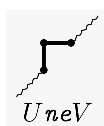
Theorem 2.1. *Let A be a K-algebra that contains Λ. Then there exists a unique function $F : \mathcal{S} \to A$, $P \mapsto F_P$ that satisfies the following four conditions:*

(i) *For all $k \in \mathbb{N}$ the initial condition $F_{\mathtt{n}\mathtt{d}^k\mathtt{e}} = \mathrm{e}_{k+1}$ is satisfied.*
(ii) *The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{S}$.*
(iii) *For all $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$ we have $F_{U\mathtt{ne}V} - F_{U\mathtt{en}V} = (q-1)F_{U\mathtt{d}V}$.*
(iv) *Let $P \in \mathcal{S}$ be a Schröder path, and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x+1 < z$ such that the bounce path of P at (x, z) has* only one *bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = Us.tV\,\mathtt{d.e}W$ for some $V \in \{\mathtt{n}, \mathtt{d}\}^*$ and some $st \in \{\mathtt{nn}, \mathtt{dn}\}$. Then*

$$F_P = \begin{cases} qF_{U\mathtt{nn}V\,\mathtt{ed}W} & \text{if } st = \mathtt{nn}, \\ F_{U\mathtt{nd}V\,\mathtt{ed}W} & \text{if } st = \mathtt{dn}. \end{cases}$$

Parts of this recursion are not new and have shown up in various forms and shapes in previous research. Due to the plethystic relationship between unicellular LLT polynomials and chromatic quasisymmetric function, some of them have analogs on the chromatic quasisymmetric function side.

Remark 2.2. *We call the relation in Theorem 2.1* (iii) *the* unicellular relation*. It can be illustrated by*



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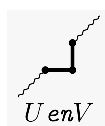
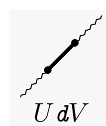


$= (q-1) \times$


(2)

and has appeared for example in [AP18, Ale20, D'A19]*. It can be used to express the function F_P indexed by a Schröder path as a linear combination such functions indexed by Dyck paths. Equivalently it can be used to express vertical-strip LLT polynomials as linear combinations of unicellular LLT polynomials. Moreover it is reflected in the definition of the operator φ in terms of the operators d_+ and d_- in the Dyck path algebra of E. Carlsson and A. Mellit* [CM17].

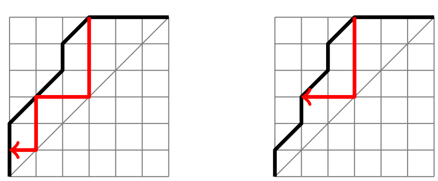


FIGURE 2. The bounce paths of two Schröder paths P (left) and Q (right) at the point $(3, 6)$. The bounce decompositions are given by $P = U\mathtt{n.n}V\mathtt{d.e}W$ where $U = \emptyset, V = \mathtt{ddn}$, and $W = \mathtt{ee}$, and $Q = U'\mathtt{n.d}V'\mathtt{d.e}W'$ where $U' = \mathtt{nd}, V' = \mathtt{n}$, and $W' = \mathtt{ee}$ respectively. The left bounce path has two bounce points $(3, 3)$ and $(1, 1)$, and bounce partition $(6, 3, 1, 0)$.

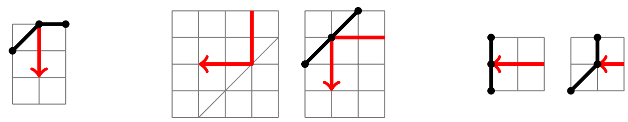
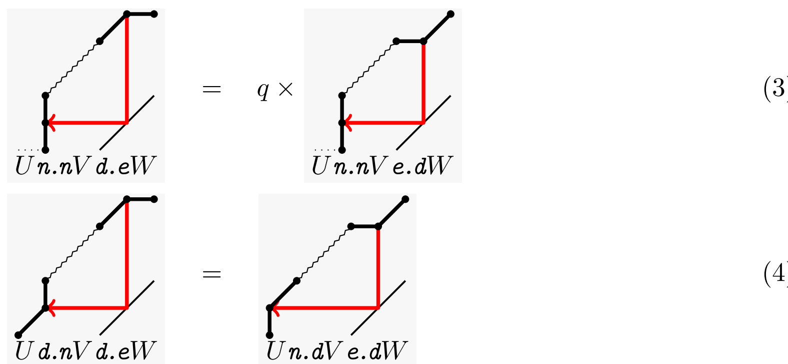


FIGURE 3. The start (left), middle (middle), and end (right) of the bounce paths we are mainly interested in.

Remark 2.3. *We call relations of the type of Theorem 2.1* (iv) *bounce relations. The two bounce relations in* (iv) *can be illustrated by*



Both of these relations are implicit in the work of M. D'Adderio [D'A19, Lem. 5.2] *who derives them from commutation relations between operators in the Dyck path algebra. The relation in* (3) *is equivalent to* (12) *further down. To the best of the authors' knowledge,* (12) *was first described in the context of chromatic quasisymmetric functions by M. Guay-Paquet* [GP13, Prop. 3.1] *where it is called the modular relation. It was independently found on the LLT side under the name local linear relation by S. J. Lee* [Lee18, Thm. 3.4] *who worked in the setting of abelian Dyck paths. Subsequently it appears in* [Ale20, Mil19]. *Moreover in* [HNY20, Thm. 3.1] *Lee's linear relation on LLT polynomials is translated to chromatic symmetric functions.*

Theorem 2.1 is formulated in such a way that it only assumes the weakest set bounce relations that are necessary to imply uniqueness. In Section 3 we will see that these relations together with the unicellular relation are strong enough to imply more general bounce relations with arbitrarily many bounce points and bounce paths that may end in different patterns. Moreover Theorem 2.1 remains valid if we allow $V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$.

We prove the uniqueness statement in Theorem 2.1 in Section 3 along with other results of a similar flavour. The existence statement is taken care of in Theorems 2.5 and 2.9. There we show by combinatorial means that there exist *two* statistics on Schröder paths that satisfy the conditions of Theorem 2.1. We then conclude that they must be equal. It turns out that the unique function F in Theorem 2.1 assigns to each Schröder path the corresponding vertical-strip LLT polynomial.

2.6. **Unit-interval graphs.** A graph $\Gamma = (V, E)$ with vertex set $V = [n]$ is a *unit-interval graph* if $xz \in E$ implies $xy, yz \in E$ for all $x, y, z \in [n]$ with $x < y < z$. unit-interval graphs are the imcomparability graphs of unit-interval orders. They are in bijection with Dyck paths and of great interest in the context of chromatic (quasi)symmetric functions. The seminal work on this topic is due to J. Shareshian and M. Wachs [SW12, SW16].

A *decorated* unit-interval graph $\Gamma = (V, E, S)$ is a pair of a unit-interval graph $\Gamma = (V, E)$ and a subset $S \subseteq E$ of *strict edges*. To a Schröder path P in $\mathcal{S}_n$ we associate a decorated unit-interval graph Γ_P on vertex set $[n]$ as follows. For $x, y \in \mathbb{Z}$ with $0 < x < y$ there is an edge xy of Γ_P if there is a cell in column x and row y below the path P. For every diagonal step of P ending at the point (x, y) we have a corresponding strict edge xy in Γ_P. See Figure 4 for an example of this correspondence.

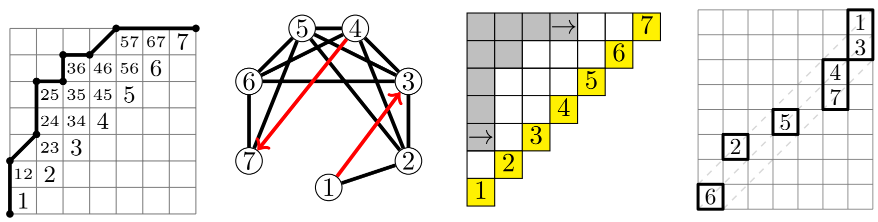
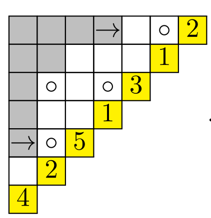


FIGURE 4. (a) The Schröder path $P = \texttt{nndnnenedeee}$. (b) The unit-interval graph Γ_P. (c) A sligthly simplified diagram (Dyck diagram) describing P. We have that $\text{area}(P) = 12$, and it is the number of white squares in this diagram. (d) The vertical strips (French notation!) corresponding to P. Thus, $\nu' = (2/1, 3/2, 2/1, 2, 3/1)$, and $\mathrm{G}_P = \mathrm{G}_\nu$, using the Bylund–Haiman convention for indexing LLT polynomials with tuples of skew shapes. The labeling of the boxes indicate the correspondence with the vertices in Γ_P.

2.7. **Colorings.** Let $P \in \mathcal{S}_n$ be a Schröder path. A *coloring* of Γ_P is a map $\kappa : [n] \to \mathbb{N}^+$ such that $\kappa(x) < \kappa(y)$ whenever the edge xy is strict. An *ascent* of a coloring is a a non-strict edge xy of Γ_P such that $x < y$ and $\kappa(x) < \kappa(y)$. We let $\text{asc}(\kappa)$ denote the number of ascents of κ.

We remark that our terminology stems from the world of chromatic quasisymmetric functions. Colorings are also closely related to parking functions, and the ascent statistic is called area′ in that setting, see [Hag07, Ch. 5]. We define the *area* of P, denoted $\text{area}(P) := |E \setminus S|$, as the number of non-strict edges in Γ_P.

Example 2.4. *Recall the path $P = \texttt{nndnnenedeee}$ from Figure 4. The following diagram illustrates a coloring of P where we have labeled i in Γ_P with $\kappa(i)$:*



The four edges contributing to asc *have been marked with* $\circ$, *and this coloring contributes with* $q^4 x_1^2 x_2^2 x_3 x_4 x_5$ *to the sum in* (5).

2.8. **A characterization of vertical-strip LLT polynomials.** For $P \in \mathcal{S}_n$ the *vertical-strip LLT polynomial* $\mathrm{G}_P(\mathbf{x};q)$ is defined as

$$\mathrm{G}_P(\mathbf{x};q) \coloneqq \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}, \tag{5}$$

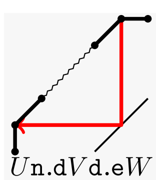
where the sum ranges over all colorings of Γ_P. One can show that $\mathrm{G}_P(\mathbf{x};q)$ is a symmetric function and a straightforward proof of this fact is given in [AP18], which is an adaptation of the proof given in the appendix of [HHL05a]. Alternatively this also follows from the results below. Moreover, every LLT polynomial indexed by a k-tuple of vertical-strip skew shapes in the Bylund–Haiman model can be realized as some $\mathrm{G}_P(\mathbf{x};q)$, see [CM17, AP18]. The family of vertical-strip LLT polynomials indexed by Dyck paths are referred to as *unicellular LLT polynomials*.

The vertical-strip LLT polynomials satisfy the recursion in Theorem 2.1.

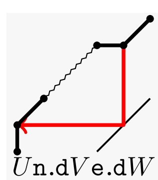
Theorem 2.5. *Let $F : \mathcal{S} \to \Lambda$ be defined by $F_P = \mathrm{G}_P$. Then F satisfies the conditions in Theorem 2.1* (i)–(iii)*. Moreover let $P \in \mathcal{S}$ be a Schröder path, and let $(x,z) \in \mathbb{Z}^2$ be a point on P with $x+1<z$ such that the bounce decomposition of P at (x,z) is given by $P = Us.tV\,\mathtt{d.e}W$ for some $st \in \{\mathtt{nn}, \mathtt{dn}, \mathtt{nd}\}$. Then*

$$F_P = \begin{cases} qF_{U\mathtt{nn}V\mathtt{ed}W} & \text{if } st = \mathtt{nn}, \\ F_{U\mathtt{nd}V\mathtt{ed}W} & \text{if } st = \mathtt{dn}, \\ (q-1)F_{U\mathtt{nd}V\mathtt{ed}W} + qF_{U\mathtt{dn}V\mathtt{ed}W} & \text{if } st = \mathtt{nd}. \end{cases}$$

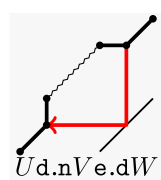
A similar result was shown by M. D'Adderio in [D'A19] in the language of the Dyck path algebra. D'Adderio in turn built on the results that E. Carlsson and A. Mellit derived in their proof of the Shuffle Theorem [CM17]. In Section 4 we give an alternative proof of the fact that the vertical-strip LLT polynomials satisfy the conditions of Theorem 2.1. This proof makes use of the combinatorics of colorings of unit-interval graphs and bijective arguments rather than relations between operators in the Dyck path algebra. Note that Theorem 2.5 contains more general relations than Theorem 2.1. In particular it contains a third type of bounce relation which is illustrated by



$= \quad (q-1) \times$



$+ \quad q \times$



(6)

A proof of Theorem 2.5 is achieved by combining the results of Section 4 with Proposition 3.1, which deduces the third bounce relation from the first two, and with Theorem 3.3, which yields bounce relations with arbitrarily many bounce points.

Theorem 2.5 clearly implies the existence statement in Theorem 2.1. Moreover Theorems 2.1 and 2.5 provide a new characterization of vertical-strip LLT polynomials.

Corollary 2.6. *The vertical-strip LLT-polynomials G_P are uniquely determined by the initial condition and relations in Theorem 2.1.*

In this paper we make use of this characterization to prove an explicit positive expansion of $\mathrm{G}_P(\mathbf{x}, q+1)$ in terms of elementary symmetric functions.

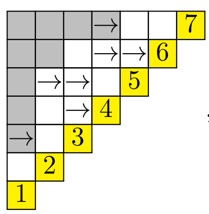
2.9. **Orientations.** Let $\Gamma = (V, E)$ be a graph. An *orientation* of Γ is a function $\theta : E \to V^2$ that assigns to each edge $uv \in E$ a directed edge $\overrightarrow{uv}$ or $\overrightarrow{vu}$. Alternatively, we may view $\theta = \{\theta(e) : e \in E\} \subseteq V^2$ as a set that contains a unique directed edge for each edge in E. We adopt the convention of viewing orientations as sets, which is more convenient if we want to add or remove edges. Let $\Gamma' = (V, E')$ be a subgraph of Γ, that is, $E' \subseteq E$. An orientation θ of Γ yields an orientation θ' of Γ' defined by

$$\theta' = \{\overrightarrow{xy} \in \theta : xy \in E'\}.$$

We call θ' the *restriction* of θ to E'. If $V = [n]$ then Γ is equipped with the *natural orientation* that assigns to each edge uv the directed edge $\overrightarrow{uv}$ where $u < v$.

Let $P \in \mathcal{S}_n$ and let $\Gamma_P = (V, E, S)$ be the corresponding decorated unit-interval graph. We let $\mathcal{O}(P)$ denote the set of orientations θ of Γ_P such that the restriction of θ to S is the natural orientation. Given $\theta \in \mathcal{O}(P)$, an edge $\overrightarrow{uv}$ is an *ascending edge* in θ if $u < v$ and $uv \notin S$. We let $\mathrm{asc}(\theta)$ denote the number of ascending edges in θ.

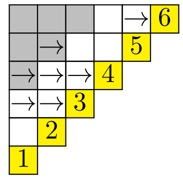
Example 2.7. *We use the diagram notation as in Figure 4, to illustrate an orientation θ. The ascending edges in θ are marked with $\to$ in the diagram, and* $\mathrm{asc}(\theta) = 5$.



$$, \quad \theta = \{\overrightarrow{13}, \overrightarrow{25}, \overrightarrow{34}, \overrightarrow{35}, \overrightarrow{46}, \overrightarrow{47}, \overrightarrow{56}\}$$

2.10. **The highest reachable vertex.** For a vertex u of Γ_P, the the *highest reachable vertex*, denoted $\mathrm{hrv}(\theta, u)$, is defined as the maximal v such that there is a directed path from u to v in θ *using only strict and ascending edges*. By definition, $\mathrm{hrv}(\theta, u) \geq u$ for all u. The orientation θ defines a set partition $\pi(\theta)$ of the vertices of Γ_P, where two vertices are in the same block if and only if they have the same highest reachable vertex. Finally, let $\lambda(\theta)$ denote the integer partition given by the sizes of the blocks in $\pi(\theta)$.

Example 2.8 (Taken from [Ale20])**.** *Below, we illustrate an orientation $\theta \in O(P)$, where $P =$ `nnnddeneee`. Strict edges and edges contributing to $\mathrm{asc}(\theta)$ are marked with $\to$.*



We have that $\mathrm{hrv}(\theta, 2) = \mathrm{hrv}(\theta, 5) = \mathrm{hrv}(\theta, 6) = 6$ *and* $\mathrm{hrv}(\theta, 1) = \mathrm{hrv}(\theta, 3) = \mathrm{hrv}(\theta, 4) = 4$. *Thus* $\pi(\theta) = \{652, 431\}$ *and the orientation θ contributes with* $q^5 \mathrm{e}_{33}(\mathbf{x})$ *in* (7).

2.11. **An explicit e-expansion.** Given a Schröder path $P \in \mathcal{S}_n$ define the symmetric function $\hat{\mathrm{G}}_P(\mathbf{x};q)$ by

$$\hat{\mathrm{G}}_P(\mathbf{x};q+1) \coloneqq \sum_{\theta \in \mathcal{O}(P)} q^{\mathrm{asc}(\theta)} \mathrm{e}_{\lambda(\theta)}(\mathbf{x}). \tag{7}$$

The second main result of this paper states that the symmetric functions defined in (7) satisfies the initial condition and relations in Theorem 2.1.

Theorem 2.9. *Let $F : \mathcal{S} \to \Lambda$ be defined by $F_P = \hat{\mathrm{G}}_P$. Then F satisfies the conditions in Theorem 2.1.*

Combining Theorems 2.1, 2.5 and 2.9 we obtain the first explicit expansion of the vertical-strip LLT polynomials in terms of elementary symmetric functions.

Corollary 2.10. *For all $P \in \mathcal{S}$ we have $\mathrm{G}_P = \hat{\mathrm{G}}_P$. In particular $\mathrm{G}_P(\mathbf{x};q+1)$ expands positively into elementary symmetric functions.*

The positivity result in Corollary 2.10 was obtained by M. D'Adderio in [D'A19]. The formula in (7) was conjectured in [GHQR19] and independently by the first named author in [Ale20]. Combinatorial positive expansions into elementary symmetric functions were obtained in a few special cases in [AP18, Ale20], albeit with a different statistic in place of $\lambda(\theta)$.

It is worth noting that the right-hand side of (7) is manifestly symmetric. In contrast, previously known formulas for LLT polynomials in terms of power-sum symmetric functions or Schur functions (with signs) rely on the expansion into fundamental quasisymmetric functions. To derive these formulas one needs to use the fact that LLT polynomials are symmetric in the first place, and not just quasisymmetric.

3. Bounce relations

In this section we prove the uniqueness statement in Theorem 2.1. In a first step we show that the bounce relations in Theorem 2.1 (iv) are equivalent to, and hence imply, other types of bounce relations. This is the content of Proposition 3.1. We then prove the presence of bounce relations with arbitrarily many bounce points in Theorem 3.3. With these more general relation at our disposal the proof of the uniqueness statement can be carried out. We conclude the section with two corollaries including a characterization of unicellular LLT polynomials that uses bounce relations for Dyck paths in Theorem 3.5.

Recall that we use dots in the bounce decomposition to highlight the starting point and endpoint of the bounce path in consideration. Note that the location of these dots does not change the word encoding a path. Moreover, slightly different choices of starting points may result (essentially) in the same bounce path. For example,

$$U\mathtt{ne.n}V\mathtt{en.e}W \text{ and } U\mathtt{ne.n}V\mathtt{e.ne}W$$

describe the same Schröder path, and the indicated bounce paths have all bounce points in common.

Proposition 3.1. *Let $F : \mathcal{S} \to \Lambda$, $P \mapsto F_P$ be a function that satisfies the unicellular relation, that is,*

(iii) For all $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$ we have $F_{U\mathtt{ne}V} - F_{U\mathtt{en}V} = (q-1)F_{U\mathtt{d}V}$.

Then the following additional sets of relations are equivalent:

(iv) ***Schröder Relations A.*** *Let $P \in \mathcal{S}$ be a Schröder path, and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x + 1 < z$ such that the bounce path of P at (x, z) has only one bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = Us.tV\mathtt{d.e}W$ for some $st \in \{\mathtt{nn}, \mathtt{dn}\}$. Then*

$$F_P = \begin{cases} qF_{U\mathtt{n.n}V\mathtt{e.d}W} & \text{if } st\text{=}\mathtt{nn}, \quad (8) \\ F_{U\mathtt{n.d}V\mathtt{e.d}W} & \text{if } st\text{=}\mathtt{dn}. \quad (9) \end{cases}$$

(v) ***Schröder Relations B.*** *Let $P \in \mathcal{S}$ be a Schröder path, and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x + 1 < z$ such that the bounce path of P at (x, z) has only one bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = Us.tV\mathtt{d.e}W$ for some $st \in \{\mathtt{nn}, \mathtt{nd}\}$. Then*

$$F_P = \begin{cases} qF_{U\mathtt{n.n}V\mathtt{e.d}W} & \text{if } st\text{=}\mathtt{nn}, \quad (10) \\ (q-1)F_{U\mathtt{n.d}V\mathtt{e.d}W} + qF_{U\mathtt{d.n}V\mathtt{e.d}W} & \text{if } st\text{=}\mathtt{nd}. \quad (11) \end{cases}$$

(vi) ***Dyck Relations.*** *Let $P \in \mathcal{S}$ be a Schröder path, and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x+1 < z$ such that the bounce path of P at (x, z) has only one bounce point. If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{n.n}V\mathtt{ne.e}W$ then*

$$F_{U\mathtt{n.n}V\mathtt{ne.e}W} = (q+1)F_{U\mathtt{n.n}V\mathtt{e.ne}W} - qF_{U\mathtt{n.n}V\mathtt{e.en}W}. \tag{12}$$

If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{ne.n}V\mathtt{ne.e}W$ then

$$\begin{aligned} &F_{U\mathtt{en.n}V\mathtt{en.e}W} + F_{U\mathtt{ne.n}V\mathtt{ne.e}W} + F_{U\mathtt{n.ne}V\mathtt{e.en}W} = \\ &F_{U\mathtt{en.n}V\mathtt{ne.e}W} + F_{U\mathtt{ne.n}V\mathtt{e.en}W} + F_{U\mathtt{n.ne}V\mathtt{e.ne}W}. \end{aligned} \tag{13}$$

Remark 3.2. *Curiously* (13) *is reminiscent of a cross-product and can be expressed as the following vanishing determinant, where multiplication is now replaced by concatenation:*

$$\begin{vmatrix} U & U & U \\ \mathtt{enn}V & \mathtt{nen}V & \mathtt{nne}V \\ \mathtt{een}W & \mathtt{ene}W & \mathtt{nee}W \end{vmatrix} = 0.$$

Expanding this determinant according to Sarrus' rule gives exactly the terms in (13) *with correct sign.*

Proof of Proposition 3.1. We first show that (8) and (10) are equivalent to (12). We start with

$$F_{U\mathrm{n.n}V\mathrm{ne.e}W} = (q+1)F_{U\mathrm{n.n}V\mathrm{e.ne}W} - qF_{U\mathrm{n.n}V\mathrm{e.en}W}.$$

Bringing one term to the other side we obtain

$$F_{U\mathrm{n.n}V\mathrm{ne.e}W} - F_{U\mathrm{n.n}V\mathrm{en.e}W} = q(F_{U\mathrm{n.n}V\mathrm{en.e}W} - F_{U\mathrm{n.n}V\mathrm{e.en}W}).$$

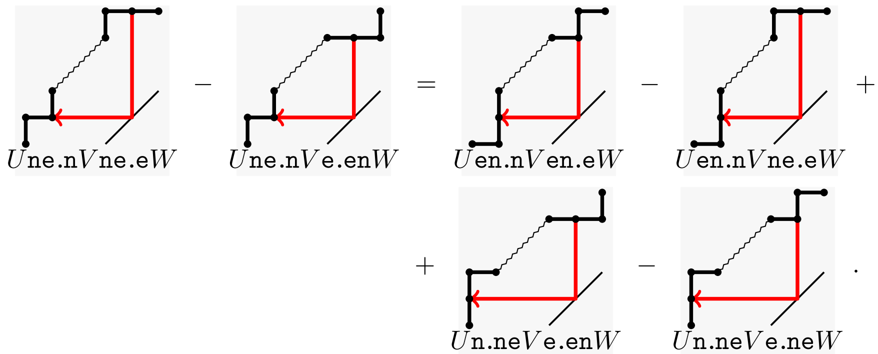
Applying the unicellular relation (iii) on both sides we obtain

$$(q-1)F_{U\mathrm{n.n}V\mathrm{d.e}W} = q(q-1)F_{U\mathrm{n.n}V\mathrm{e.d}W},$$

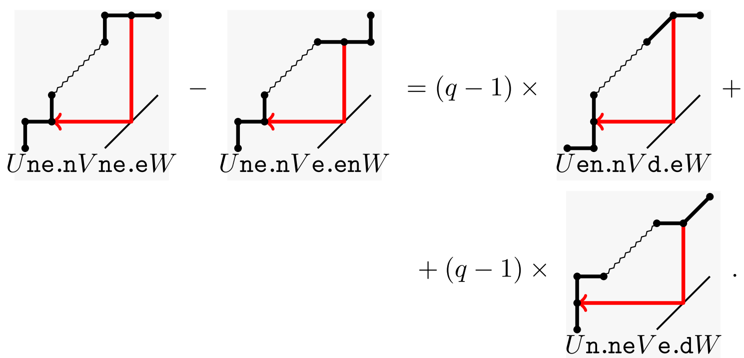
which after a cancellation becomes

$$F_{U\mathrm{n.n}V\mathrm{d.e}W} = qF_{U\mathrm{n.n}V\mathrm{e.d}W}.$$

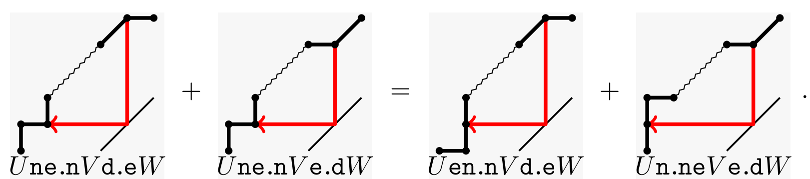
Next we show that (9) is equivalent to (13). We start with (13), which is rearranged as



Applying the unicellular relation twice on the right-hand side we get

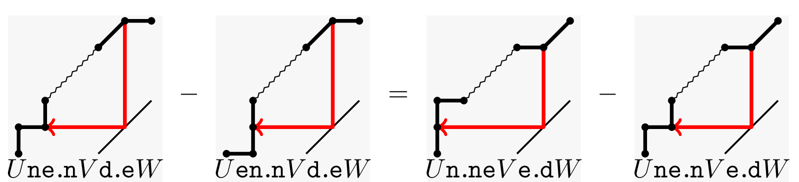


After additional applications of the unicellular relation on the left-hand side we get (after dividing by $q-1$)



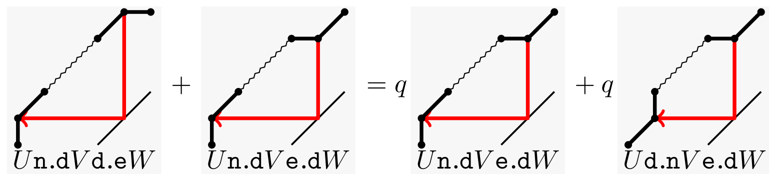
(14)

Rearranging the terms again we arrive at



which after a final application of the unicellular relation on both sides gives (9).

To complete the proof we now show that (11) is equivalent to (13) assuming that (10) and (12) are valid. We start with (11) which becomes



after moving one term to the left-hand side. We now use the unicellular relation to eliminate one of the diagonals. In the left-hand side, we keep the first diagonal step and eliminate the second one. In the right-hand side we keep the second diagonal step. After cancelling a common factor of $(q-1)$, the left-hand side simplifies to

$$(F_{U\mathtt{n.d}V\mathtt{ne.e}W} - F_{U\mathtt{n.d}V\mathtt{e.ne}W}) + (F_{U\mathtt{n.d}V\mathtt{e.ne}W} - F_{U\mathtt{n.d}V\mathtt{e.en}W}) \\ = F_{U\mathtt{n.d}V\mathtt{ne.e}W} - F_{U\mathtt{n.d}V\mathtt{e.en}W}$$

Similarly the right-hand side becomes

$$q(F_{U\mathtt{n.ne}V\mathtt{e.d}W} - F_{U\mathtt{ne.n}V\mathtt{e.d}W}) + q(F_{U\mathtt{ne.n}V\mathtt{e.d}W} - F_{U\mathtt{en.n}V\mathtt{e.d}W}) \\ = q(F_{U\mathtt{n.ne}V\mathtt{e.d}W} - F_{U\mathtt{en.n}V\mathtt{e.d}W}).$$

Now we use (10) on the both terms in the righ hand side, that is, $qF_{U\mathtt{n.ne}V\mathtt{e.d}W} = F_{U\mathtt{n.ne}V\mathtt{d.e}W}$ and $qF_{U\mathtt{en.n}V\mathtt{e.d}W} = F_{U\mathtt{en.n}V\mathtt{d.e}W}$. Moving all terms to the left-hand side, we obtain the relation

$$F_{U\mathtt{en.n}V\mathtt{d.e}W} - F_{U\mathtt{n.ne}V\mathtt{d.e}W} + F_{U\mathtt{n.d}V\mathtt{ne.e}W} - F_{U\mathtt{n.d}V\mathtt{e.en}W} = 0.$$

Finally, we eliminate the remaining diagonal steps using the unicellular relation. After cancellation of a common factor this gives the identity

$$0 = (F_{U\mathtt{en.n}V\mathtt{ne.e}W} - F_{U\mathtt{en.n}V\mathtt{e.ne}W}) - \left(\underline{F_{U\mathtt{n.ne}V\mathtt{ne.e}W}} - F_{U\mathtt{n.ne}V\mathtt{e.ne}W}\right) \\ + \left(\underline{F_{U\mathtt{n.ne}V\mathtt{ne.e}W}} - F_{U\mathtt{ne.n}V\mathtt{ne.e}W}\right) - (F_{U\mathtt{n.ne}V\mathtt{e.en}W} - F_{U\mathtt{ne.n}V\mathtt{e.en}W}).$$

After cancelling the underlined terms we arrive at (13). Since all steps in this deduction are invertible we have the desired equivalence. ■

We next show that the relations in Theorem 2.1 are strong enough to impose relations involving bounce paths with *arbitrarily many* bounce points.

Theorem 3.3. *Let $F : \mathcal{S} \to \Lambda$, $P \mapsto F_P$ be a function that satisfies Conditions* (iii) *and* (iv) *in Theorem 2.1. Let $P \in \mathcal{S}$ be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x + 1$ such that the bounce decomposition of P at (x, z) is given by $P = Us.tV\,\mathtt{d.e}W$ for some $V \in \{\mathtt{n}, \mathtt{d}\}^*$ and some $st \in \{\mathtt{nn}, \mathtt{nd}, \mathtt{dn}\}$. Then*

$$F_P = \begin{cases} qF_{U\mathtt{nn}V\mathtt{ed}W} & \text{if } st = \mathtt{nn}, \\ F_{U\mathtt{nd}V\mathtt{ed}W} & \text{if } st = \mathtt{dn}, \\ (q-1)F_{U\mathtt{nd}V\mathtt{ed}W} + qF_{U\mathtt{dn}V\mathtt{ed}W} & \text{if } st = \mathtt{nd}. \end{cases}$$

Proof. The claim is shown by induction on the number of bounce points of the bounce path of P at (x, z). First assume that the bounce path of P at (x, z) has only one bounce point. If $st \in \{\mathtt{nn}, \mathtt{dn}\}$ then there is nothing to show. If $st = \mathtt{nd}$ then the claim follows from Proposition 3.1. To see this note that the proof of the equivalence of Proposition 3.1 (iv) and (v) goes through in the same way if we impose the restriction $V \in \{\mathtt{n}, \mathtt{d}\}^*$ in both cases. Thus the base case of the induction is taken care of.

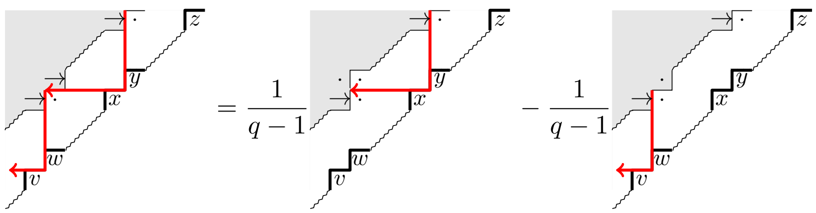
Otherwise the bounce partition of P at (x, z) has length $r > 3$ and is given by $(z, x, v, \dots)$. Set $y = x + 1$ and $w = v + 1$. In this case there are at least two bounce points and we may write

$$P = Us.tV'\mathtt{dd}V''\mathtt{d.e}W,$$

where $V', V'' \in \{\mathtt{n}, \mathtt{d}\}^*$ such that $\mathtt{d}V''\mathtt{d}$ defines a path from (v, x) to (x, z). Note that here we use the fact that $z > y$, which also implies $x > w$. By Theorem 2.1 (iii) we have

$$F_P = \frac{1}{q-1}\Big(F_{UstV'\mathtt{d.ne}V''\mathtt{d.e}W} - F_{Us.tV'\mathtt{d.en}V''\mathtt{de}W}\Big), \tag{15}$$

which is illustrated via diagrams as

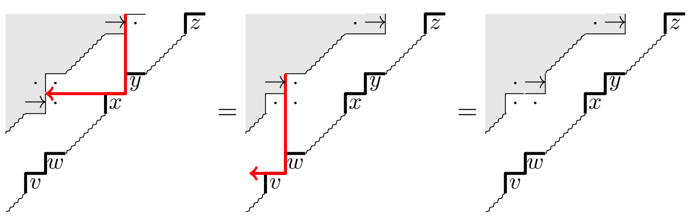


(16)

To simplify notation suppose for the moment that $st = \mathtt{dn}$. By applying first Theorem 2.1 (iv) at the point (x, z), and then the induction hypothesis at the point (v, y) we obtain

$$F_{U\mathtt{dn}V'\mathtt{d.ne}V''\mathtt{d.e}W} = F_{U\mathtt{d.n}V'\mathtt{n.de}V''\mathtt{e.d}W} = F_{U\mathtt{nd}V'\mathtt{ned}V''\mathtt{ed}W}, \tag{17}$$

which corresponds to



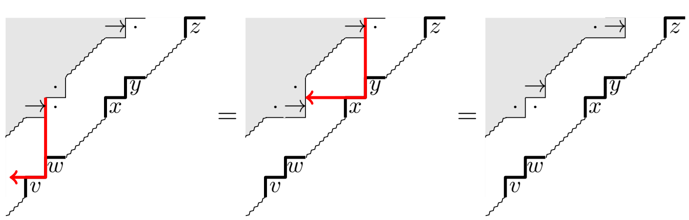
(18)

Note that the bounce path of $Us.tV'\mathtt{nde}V''\mathtt{e.d}W$ at (v, y) has the same bounce points as the bounce path of P at (x, z) except for (x, x). Thus the induction hypothesis can be applied.

Similarly, by applying first induction hypothesis at the point (v, x), and then Theorem 2.1 (iv) at the point (x, z) the we obtain

$$F_{U\mathtt{d.n}V'\mathtt{d.en}V''\mathtt{de}W} = F_{U\mathtt{nd}V'\mathtt{ed.n}V''\mathtt{d.e}W} = F_{U\mathtt{nd}V'\mathtt{end}V''\mathtt{ed}W}, \tag{19}$$

which corresponds to



(20)

Note that the bounce path of $Us.tV'\mathtt{den}V''\mathtt{e.d}W$ at (v,x) has the same bounce points as the bounce path of P at (x,z) except for (x,x). Thus the induction hypothesis can be applied.

Combining (15), (17) and (19), and using Theorem 2.1 (iii) one more time, we obtain

$$\begin{aligned} F_P &= \frac{1}{q-1}\Big(F_{U\mathtt{nd}V'\mathtt{ned}V''\mathtt{ed}W} - F_{U\mathtt{nd}V'\mathtt{end}V''\mathtt{ed}W}\Big) \\ &= F_{U\mathtt{n.d}V'\mathtt{dd}V''\mathtt{e.d}W} \\ &= F_{U\mathtt{n.d}V\mathtt{e.d}W}. \end{aligned}$$

This proves the claim in the case $st = \mathtt{dn}$. The other two cases $st \in \{\mathtt{nn}, \mathtt{nd}\}$ are proved in the exact same manner. ■

The uniqueness in Theorem 2.1 can now be shown using an intricate induction argument devised by M. D'Adderio in the proof of [D'A19, Thm. 5.4]. Given a Schröder path P our strategy is to show that F_P can be expressed in terms of elementary symmetric functions as well as functions F_Q where Q has size less than P, or Q has fewer east steps than P, or Q is obtained from P by moving an east step of P to the left.

Proof of uniqueness in Theorem 2.1. Let P be a Schröder path of size n with r east steps, and assume that $P = X\mathtt{e}W$ where $X \in \{\mathtt{n},\mathtt{d}\}^*$ defines a path from $(0,0)$ to (x,z), and $W \in \{\mathtt{n},\mathtt{d},\mathtt{e}\}$. That is, X is the initial segment of P that leads up to the first east step of P.

We first use induction on the size n. By Theorem 2.1 (ii) the empty path is sent to 1 by F. Thus we may assume that $n > 0$ and that F_Q is uniquely determined for all $Q \in \mathcal{S}_m$ for all $m < n$.

Secondly we use induction on r. If P has only one east step, then P is of the form $P = \mathtt{nd}^{n-1}\mathtt{e}$ and $F_P = \mathrm{e}_n(\mathbf{x})$ is determined by the initial condition in Theorem 2.1 (i). Thus the base case of the second induction is taken care of. We may assume that P has at least two east steps, and that F_Q is determined uniquely for all Schröder paths $Q \in \mathcal{S}_n$ with fewer than r east steps.

Thirdly we use induction on $x+z$. The base case for the third induction is $z = x+1$. If $z = x+1$ then $X\mathtt{e}$ and W are Schröder paths of size less than n. Note that here we use the fact that P contains two or more east steps. By Theorem 2.1 (ii) and by induction on n it follows that $F_P = F_{X\mathtt{e}}F_W$ is determined uniquely. Thus we may assume that $z > x+1$ and that F_Q is determined uniquely for all $Q \in \mathcal{S}_n$ with r east steps that are of the form $Q = X'\mathtt{e}W'$ where $W' \in \{\mathtt{n},\mathtt{d}\}^*$ defines a path from $(0,0)$ to (x',z') with $x'+z' < x+z$. Let $A \subseteq \mathcal{S}_n$ denote the set of all such Schröder paths.

Now if $X = Y\mathtt{n}$, that is $P = Y\mathtt{ne}W$, then $Y\mathtt{d}W$ is a Schröder path of size n with fewer east steps than P. Note that here we use the fact that $z > x+1$. Similarly $Y\mathtt{en}W$ is a Schröder path in A. By Theorem 2.1 (iii) and by induction on r and $x+z$ it follows that $F_P = (q-1)F_{Y\mathtt{d}W} + F_{Y\mathtt{en}W}$ is determined uniquely.

If on the other hand $X = Y\mathtt{d}$, that is $P = Y\mathtt{de}W$, then let $P = Us.tV\mathtt{d.e}W$ be the bounce decomposition of P at (x,z). Clearly $V \in \{\mathtt{n},\mathtt{d}\}^*$ and $st \in \{\mathtt{nn},\mathtt{nd},\mathtt{dn}\}$. Thus by Theorem 3.3 we may write F_P as a linear combination of functions F_Q for certain paths $Q \in A$. Therefore F_P is determined uniquely by induction on $x+z$. This completes the proof. ■

Using Proposition 3.1 we obtain an analog of Theorem 2.1 using a different bounce relation.

Corollary 3.4. *There exists a unique function $F : \mathcal{S} \to \Lambda$, $P \mapsto F_P$ that satisfies the following conditions:*

(i) For all $k \in \mathbb{N}$ the initial condition $F_{\mathtt{n}\mathtt{d}^k\mathtt{e}} = \mathrm{e}_{k+1}$ is satisfied.
(ii) The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{S}$.
(iii) For all $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$ we have $F_{U\mathtt{ne}V} - F_{U\mathtt{en}V} = (q-1)F_{U\mathtt{d}V}$.
(v) Let $P \in \mathcal{S}$ be a Schröder path, and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x+1 < z$ such that the bounce path of P at (x, z) has only one bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = Us.tV\mathtt{d.e}W$ for some $V \in \{\mathtt{n}, \mathtt{d}\}^*$ and some $st \in \{\mathtt{nn}, \mathtt{nd}\}$. Then

$$F_P = \begin{cases} qF_{U\mathtt{nn}V\mathtt{ed}W} & \text{if } st = \mathtt{nn}, \\ (q-1)F_{U\mathtt{nd}V\mathtt{ed}W} + qF_{U\mathtt{dn}V\mathtt{ed}W} & \text{if } st = \mathtt{nd}. \end{cases}$$

Moreover we can prove a Dyck path analog of Theorem 2.1. Let $\mathcal{D}$ denote the set of all Dyck paths, that is, the set of Schröder paths that contain no diagonal steps. Note that using our conventions, if P is a Dyck path and (x, z) is a point on P then the bounce path of P at (x, z) must have a single bounce point.

Theorem 3.5. *The function that assigns to each Dyck path the corresponding unicellular LLT polynomial is the unique function $F : \mathcal{D} \to \Lambda$, $P \mapsto F_P$ that satisfies the following conditions:*

(i) For all $k \in \mathbb{N}$ there holds the initial condition

$$F_{\mathtt{n}(\mathtt{ne})^k\mathtt{e}}(\mathbf{x}; q) = \sum_{\alpha \vDash k+1} (q-1)^{k-\ell(\alpha)+1} \mathrm{e}_\alpha(\mathbf{x}).$$

(ii) The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{D}$.
(vi) Let $P \in \mathcal{D}$ be a Dyck path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x + 1 < z$. If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{n.n}V\mathtt{ne.e}W$ then

$$F_{U\mathtt{n.n}V\mathtt{ne.e}W} = (q+1)F_{U\mathtt{n.n}V\mathtt{e.ne}W} - qF_{U\mathtt{n.n}V\mathtt{e.en}W}.$$

If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{ne.n}V\mathtt{ne.e}W$ then

$$\begin{aligned} &F_{U\mathtt{en.n}V\mathtt{en.e}W} + F_{U\mathtt{ne.n}V\mathtt{ne.e}W} + F_{U\mathtt{n.ne}V\mathtt{e.en}W} = \\ &F_{U\mathtt{en.n}V\mathtt{ne.e}W} + F_{U\mathtt{ne.n}V\mathtt{e.en}W} + F_{U\mathtt{n.ne}V\mathtt{e.ne}W}. \end{aligned}$$

Proof. Suppose that $F : \mathcal{D} \to \Lambda$ satisfies the conditions in theorem. First note that F can be extended to a function $\bar{F} : \mathcal{S} \to \Lambda$ on Schröder paths in a unique way such that the unicellular relation Proposition 3.1 (iii) holds. Moreover $\bar{F}$ is multiplicative on Schröder paths.

We next show that a function $\bar{F}$ satisfies the relations Proposition 3.1 (vi). To see this let P be a Schröder path and $(x, z) \in \mathbb{Z}^2$ be a point on P with $x + 1 < z$, such that the bounce path of P at (x, z) has only one bounce point, and such that the bounce decomposition of P at (x, z) is of the form $P = U\mathtt{n.n}V\mathtt{ne.e}W$ for some $U, V, W \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$. Using the unicellular relation (many times) we can write $\bar{F}_P$ as a linear combination of symmetric functions of $\bar{F}_Q = F_Q$, where Q is a Dyck path of the form $Q = U'\mathtt{n.n}V'\mathtt{ne.e}W'$. In particular the bounce paths of P and Q at (x, z) agree. We may now apply the linear relation from the assumption to the symmetric functions F_Q and use the unicellular relation to reintroduce the diagonal steps. In this way we have show that $\bar{F}$ satisfies (12) and the same can be done for (13).

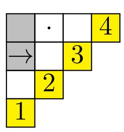
It now follows from Proposition 3.1 and Theorem 2.1 that the function $\bar{F}$, and in particular the function F must be unique. Moreover, by Theorem 2.5 and Proposition 3.1 the function that assigns to each Dyck path the corresponding unicellular LLT polynomial satisfies Conditions (ii) and (vi) above. The fact that this function also satisfy the initial condition in (i) is an easy consequence of the e-expansion in Corollary 2.10[1]. This completes the proof. ■

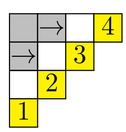
4. Bijections on colorings

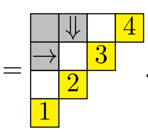
In this section we prove Theorem 2.5. We first verify Theorem 2.1 (i)–(iii) in Proposition 4.2. We then proceed to demonstrate the two bounce relations of Theorem 2.1 (iv) in Theorems 4.4 and 4.5. The method of proof is purely bijective and similar to that in [Ale20, Prop. 18]. Indeed, (iii) as well as the $st = \texttt{nn}$ case of (iv) were already proved therein. To obtain the full claim of Theorem 2.5 one can then use the results of Section 3, namely Proposition 3.1 and Theorem 3.3. We remark that it is also possible to give a bijective proof for the third type of bounce relations in Theorem 2.5, that is, the case $st = \texttt{nd}$, if there is only one bounce point. However, Theorem 3.3 is currently our only way to prove bounce relations with many bounce points.

In the proof of Theorem 3.3, we used diagrams to get a good overview of the recursions. In this section we use such diagrams to denote weighted sums over vertex colorings of decorated unit-interval graphs. To simplify the notation, we shall only show the vertices and edges of the diagrams which matter. The following example introduces the necessary conventions.

Example 4.1. *The LLT polynomials* $\mathrm{G}_P(\mathbf{x}; q)$ *and* $\mathrm{G}_Q(\mathbf{x}; q)$ *for* $P = \texttt{nndneee}$ *and* $Q = \texttt{nnddee}$ *are given as sums over vertex colorings. We let* $A(\mathbf{x}; q)$ *be the sum over colorings of* Γ_P*, with the extra condition that* $\kappa(2) \geq \kappa(4)$*. Expressed using diagrams, we have*

$\mathrm{G}_P(\mathbf{x}; q) =$ 

$\mathrm{G}_Q(\mathbf{x}; q) =$ 

$A(\mathbf{x}; q) =$ 

.

Note that we use ⇓ *to impose a* weak *inequality. That is, if an edge* $\overrightarrow{xy}$ *is marked with* ⇓*, then we require that* $\kappa(x) \geq \kappa(y)$*. In the same manner,* ↓ *is used to indicate a strict inequality which is a non-ascent. We utilize the convention that edges in gray boxes do not contribute to the ascent statistic (this is consistent with marking all strict edges gray). Moreover, edges of particular importance are marked with a center-dot; we shall possibly refine our argument depending on if the colorings under consideration have this edge as ascending or not.*

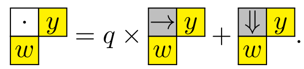
Now, a simple bijective argument shows that

$$\mathrm{G}_P(\mathbf{x}; q) = q \times \mathrm{G}_Q(\mathbf{x}; q) + A(\mathbf{x}; q),$$

[1]The e-expansion of the unicellular LLT polynomial of the path graph was also obtained in [AP18, Prop. 5.18]

since a coloring on the left-hand side appears as a coloring of exactly one of the diagrams in the righ hand side. Note that it is only the edge from 2 to 4 that really matters.

Stated in full generality, for any edge $\overrightarrow{wy}$ in any diagram, we have the identity



(21)

This simply expresses the fact that a set of colorings κ can be partitioned into two sets, the first set where $\kappa(w) > \kappa(y)$ and the second set where $\kappa(w) \leq \kappa(y)$.

Note that most diagrams we use do not represent vertical-strip LLT polynomials, but merely a sum $\sum_\kappa q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$, where the sum ranges over colorings compatible with the restrictions imposed by the diagram.

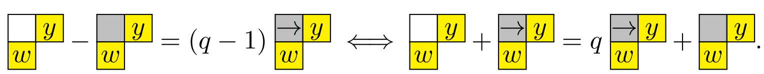
Proposition 4.2. *Let $F : \mathcal{S} \to \Lambda$ be the function that assigns to each Schröder path P the symmetric function $F_P = \mathrm{G}_P(\mathbf{x}; q)$ defined in (5).*

(i) Then F satisfies the initial condition $F_{\mathtt{n}\mathtt{d}^k\mathtt{e}} = \mathrm{e}_{k+1}$ for all $k \in \mathbb{N}$.
(ii) The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{S}$.
(iii) For all $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$ we have $F_{U\mathtt{ne}V} - F_{U\mathtt{en}V} = (q-1)F_{U\mathtt{d}V}$.

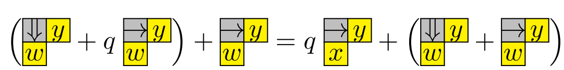
Proof. For (i), we have by definition that $F_{\mathtt{nd}^k\mathtt{e}}$ is a sum over colorings $\kappa : [k+1] \to \mathbb{N}^+$, such that $\kappa(1) < \kappa(2) < \cdots < \kappa(k+1)$. Now it is clear that we indeed obtain e_{k+1}.

Moreover, (ii) is immediate from the definition, as the graph Γ_{PQ} is (up to a relabeling of the vertices) the disjoint union of the graphs Γ_P and Γ_Q. Colorings can thus be done on each component independently and the statement follows.

It remains to prove (iii). Using diagrams as in (21), this identity can be expressed as follows:



The first and last diagram are now split into subcases, depending on whether $\kappa(w) < \kappa(y)$ or $\kappa(w) \geq \kappa(y)$ as in (21),



and now it is evident that the identity holds. ■

In the remainder of this section, we shall assume that $x + 1 = y$, so that they are adjacent entries on the diagram diagonal.

4.1. **Bounce relations for colorings.** We shall describe a simple bijection between certain colorings which is referred to as the *swap map*. The swap map sends a coloring κ to the coloring κ' defined as

$$\kappa'(j) := \begin{cases} \kappa(j) & \text{if } j \notin \{x, y\} \\ \kappa(y) & \text{if } j = x \\ \kappa(x) & \text{if } j = y. \end{cases} \tag{22}$$

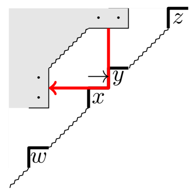
This map is used in all further proofs in this section.

Lemma 4.3 (Swap Map)**.** *Let $P \in \mathcal{S}$ and suppose that the bounce decomposition at $(x, z-1)$ has one bounce point and is given by $P = U\mathtt{n.n}V\,\mathtt{e.e}W$ and set $y = x+1$. Then*

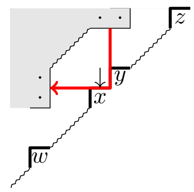
$$\sum_{\kappa:\kappa(x)<\kappa(y)} q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)} = q \times \sum_{\kappa':\kappa'(x)>\kappa'(y)} q^{\operatorname{asc}(\kappa')} x_{\kappa'(1)} \cdots x_{\kappa'(n)},$$

where we sum over all colorings of P on both sides, with the indicated added restriction on the colors of x and y.

Proof. Expressing the statement as an identity on diagrams, we have



$= q \times$



(23)

In order to prove (23), we must show that the swap preserves the number of ascents, not counting the ascent given by the $\overrightarrow{xy}$ edge. But note that for all j with $w < j < x$ or $y < j < z$ we have that

$$\kappa(y) < \kappa(j) \iff \kappa'(x) < \kappa'(j).$$

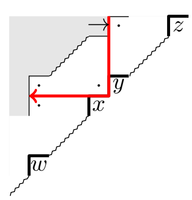
This implies that for any coloring κ appearing in the left-hand side of (23), we have that $\operatorname{asc}(\kappa) = 1 + \operatorname{asc}(\kappa')$. ■

We now proceed with showing that the LLT polynomials satisfy the two bounce relations in Theorem 2.1.

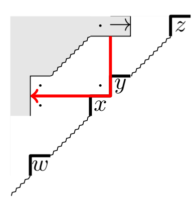
Theorem 4.4 (The first bounce relation for LLT polynomials)**.** *Let $P \in \mathcal{S}$ be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x+1$ such that the bounce path of P at (x, z) has a single bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = U\mathtt{n.n}V\,\mathtt{d.e}W$. Then*

$$\mathrm{G}_P(\mathbf{x}; q) = q\mathrm{G}_{U\mathtt{n.n}V\,\mathtt{e.d}W}(\mathbf{x}; q).$$

Proof. We want to prove the following identity expressed as diagrams:

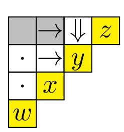


$= q \times$

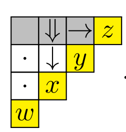


. (24)

If $\overrightarrow{xz}$ and $\overrightarrow{yz}$ are ascending edges on both sides, we use the identity map on the set of colorings. By using the transitivity of the forced inequalities, it then suffices to prove the identity



$= q \times$



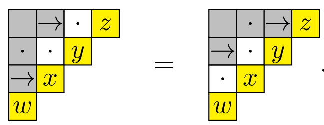
.

But this now follows from Lemma 4.3. ■

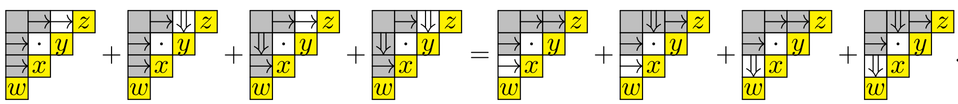
Theorem 4.5 (The second bounce relation for LLT polynomials)**.** *Let $P \in \mathcal{S}$ be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x+1$ such that the bounce path of P at (x, z) has a single bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = U\,\mathtt{d.n}V\,\mathtt{d.e}W$. Then*

$$\mathrm{G}_P(\mathbf{x}; q) = \mathrm{G}_{U\mathtt{n.d}V\mathtt{e.d}W}(\mathbf{x}; q).$$

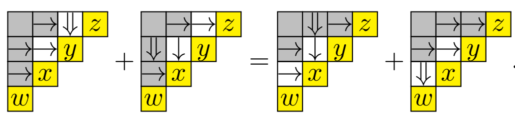
Proof. We want to prove the identity expressed as diagrams in (4). For convenience, it is presented here as follows:



We split both sides into cases according to (21):



The first term on both sides cancel, and there are no colorings compatible with the inequalities in the last term on both sides. Moreover, transitivity forces the last inequality in the remaining cases:



All four remaining diagrams have q-weight 1 (recall the convention for shaded boxes) and we can now see that the swap map sends the terms on the left-hand side to the terms on the right-hand side. This concludes the proof. ■

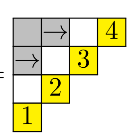
5. Bijections on orientations

In this section we prove Theorem 2.9. To be precise we first treat the easier claims in Proposition 5.2. We then show that the symmetric functions $\hat{\mathrm{G}}_P(\mathbf{x}; q)$ satisfy the bounce relations of Theorem 2.1 (iv) in Theorems 5.4 and 5.17. Since we want to work with orientations we actually show that the symmetric functions $\hat{\mathrm{G}}_P(\mathbf{x}; q+1)$ satisfy a shifted version of these relations. Our proofs are purely bijective and the involved bijections are very simple. However, the proofs of the bounce relations are a bit tedious in the sense that they require the distinction of quite a few different cases. We remark that the third type of bounce relations in Theorem 2.5 ending in a pattern $st = \mathtt{nd}$ can be given a bijective proof in similar style, assuming that there is only one bounce point.

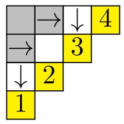
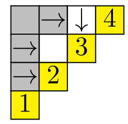
As with the colorings, we use diagrams to illustrate the different cases. However, in this section, a diagram now represents a *sum over orientations* θ weighted by $q^{\mathrm{asc}(\theta)}\mathrm{e}_{\lambda(\theta)}$.

Example 5.1. *The symmetric function $\hat{\mathrm{G}}_Q(\mathbf{x}; q+1)$ where $Q = \mathtt{nnddee}$ is described by the diagram*

$$\hat{\mathrm{G}}_Q(\mathbf{x}; q+1) =$$



which represents a sum over eight orientations (all possible ways to orient the edges 12*,* 23 *and* 34*). By specifying additional edges in the diagram, we obtain a sum over a smaller set of orientations. For example, the two following diagrams represent the indicated sums.*


$= (1+q)\mathrm{e}_{22}$ *and*

$= (1+q)\mathrm{e}_{31}$

Notice that in the second diagram, we again use the convention that edges marked gray do not contribute to ascents.

We start out by proving that the symmetric function $P \mapsto \hat{\mathrm{G}}_P(\mathbf{x}; q+1)$ satisfies the correct initial conditions, is multiplicative, and obeys the shifted version of the unicellular relation.

Proposition 5.2. *Let $F : \mathcal{S} \to \Lambda$ be the function that assigns to each Schröder path P the symmetric function $F_P = \hat{\mathrm{G}}_P(\mathbf{x}; q+1)$ defined in* (7)*.*

(i) Then F satisfies the initial condition $F_{\mathtt{nd}^k\mathtt{e}} = \mathrm{e}_{k+1}$ for all $k \in \mathbb{N}$.
(ii) The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{S}$.
(iii) For all $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$ we have $F_{U\mathtt{ne}V} - F_{U\mathtt{en}V} = qF_{U\mathtt{d}V}$.

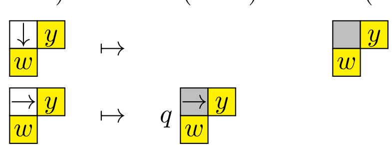
Proof. To see (i) let $P = \mathtt{nd}^k\mathtt{e} \in \mathcal{S}_{k+1}$. Note that the decorated unit-interval graph Γ_P has k strict edges and no other edges. Thus $\mathcal{O}(P)$ contains a unique orientation θ which has no ascents. Its highest reachable vertex partition is given by $\lambda(\theta) = (k+1)$ since the vertex $k+1$ can be reached from every other vertex using the strict edges. Hence $F_P = \mathrm{e}_{k+1}$ as claimed.

To see (ii) let $P, Q \in \mathcal{S}$. Note that Γ_{PQ} is (up to a relabeling of the vertices) the disjoint union of the graphs Γ_P and Γ_Q. Thus every orientation of Γ_{PQ} is (again up to a relabeling) the disjoint union of an orientation of Γ_P and Γ_Q. It follows that

$$F_{PQ} = \sum_{\theta \in \mathcal{O}(P)} \sum_{\theta' \in \mathcal{O}(Q)} q^{\mathrm{asc}(\theta)+\mathrm{asc}(\theta')} \mathrm{e}_{\lambda(\theta)} \mathrm{e}_{\lambda(\theta')} = F_P F_Q.$$

To see (iii) let $U, V \in \{\mathtt{n}, \mathtt{d}, \mathtt{e}\}^*$ such that $U\mathtt{d}V \in \mathcal{S}$. Let wy denote the strict edge in $\Gamma_{U\mathtt{d}V}$ corresponding to the diagonal step between U and V. Given $\theta \in \mathcal{O}(U\mathtt{ne}V)$ let $\phi(\theta) = \theta \in \mathcal{O}(U\mathtt{d}V)$ if $\overrightarrow{wy} \in \theta$, and $\phi(\theta) = \theta \setminus \{\overrightarrow{yw}\} \in \mathcal{O}(U\mathtt{en}V)$ if $\overrightarrow{yw} \in \theta$. This yields a bijection

$$\phi : \mathcal{O}(U\mathtt{ne}V) \quad \to \quad \mathcal{O}(U\mathtt{d}V) \quad \sqcup \quad \mathcal{O}(U\mathtt{en}V)$$



that preserves highest reachable vertices. Moreover $\mathrm{asc}(\phi(\theta)) = \mathrm{asc}(\theta) - 1$ if $\overrightarrow{wy} \in \theta$, and $\mathrm{asc}(\phi(\theta)) = \mathrm{asc}(\theta)$ if $\overrightarrow{yw} \in \theta$. The claim follows. ■

We now turn our attention to the bounce relations. To begin with we need some additional definitions. Given a finite totally ordered set $X = \{u_1 < u_2 < \cdots < u_r\}$ and a function $\sigma : X \to \mathbb{N}^+$, we identify σ with the word $\sigma_1\sigma_2\ldots\sigma_r$ where $\sigma_i = \sigma(u_i)$ for all $i \in [r]$. Note that this word contains all the information on σ if the domain

X is known. The *weak standardisation* of σ is defined as the unique surjection $\mathrm{wst}(\sigma) : X \to [m]$ that satisfies

$$\mathrm{wst}(\sigma)(u) \leq \mathrm{wst}(\sigma)(v) \quad \Leftrightarrow \quad \sigma(u) \leq \sigma(v)$$

for all $u, v \in X$, where m is (necessarily) the cardinality of the image of σ.

Let $\mathcal{O}(n)$ denote the set of orientations of unit-interval graphs on n vertices. Given a Schröder path $P \in \mathcal{S}_n$ and a set of directed edges $D \subseteq [n]^2$, define

$$\mathcal{O}(P, D)$$

as the set of all orientations θ of P such that $D \subseteq \theta$. Moreover, given $I \subseteq [n]$, $m \in [n]$, and a *surjection* $\sigma : I \to [m]$, define

$$\mathcal{O}(P, \sigma, D) \subset \mathcal{O}(P, D)$$

as the set of all orientations θ of P such that $D \subseteq \theta$, and σ is the weak standardization of $\mathrm{hrv}(\theta, .\,)$ on I. That is, for all $i, j \in I$ we have

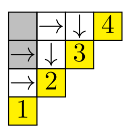
$$\sigma_i \leq \sigma_j \iff \mathrm{hrv}(\theta, i) \leq \mathrm{hrv}(\theta, j).$$

If $D = \{\overrightarrow{vw}, \overrightarrow{xy}, \dots\}$ has small cardinality we write

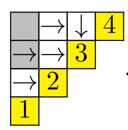
$$\mathcal{O}(P; \overrightarrow{vw}, \overrightarrow{xy}, \dots\,) \quad \text{and} \quad \mathcal{O}(P, \sigma; \overrightarrow{vw}, \overrightarrow{xy}, \dots\,)$$

instead of $\mathcal{O}(P, D)$ and $\mathcal{O}(P, \sigma, D)$ in order to make notation less heavy.

Example 5.3. *Let $P = \texttt{nndneee}$. Then the set $\mathcal{O}(P, 2212; \overrightarrow{12})$ consists of the two orientations*



and



.

Throughout the remainder of this section let P be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x + 1$ such that the bounce partition of P at (x, z) is given by (z, x, v), and the bounce decomposition of P at (x, z) is given by $P = Us.tV\texttt{d}.\texttt{e}W$ for some $st \in \{\texttt{nn}, \texttt{dn}\}$ and $U, V, W \in \{\texttt{n}, \texttt{d}, \texttt{e}\}$. In particular, the bounce path of P at (x, z) has only one bounce point (x, x). Set $w := v + 1$ and $y := x + 1$, and

$$Q := \begin{cases} U\texttt{nn}V\texttt{ed}W & \text{if } st = \texttt{nn}, \\ U\texttt{nd}V\texttt{ed}W & \text{if } st = \texttt{dn}. \end{cases}$$

Let $\tau := (x, y) \in \mathfrak{S}_n$ be a transposition. Let $A \subseteq \mathcal{O}(n)$ and $f : A \to \mathcal{O}(n)$ be a function. Then f is called *π-preserving on A* if $\pi(f(\theta)) = \pi(\theta)$ for all $\theta \in A$. The function f is called *π-switching on A* if $\pi(f(\theta)) = \tau(\pi(\theta))$. Both of these properties imply that f is *λ-preserving on A*, that is $\lambda(f(\theta)) = \lambda(\theta)$ for all $\theta \in A$. To see this note that permuting the entries of a set partition does not change the block structure.

Define a function

$$\rho_v : \mathcal{O}(n) \to [n]$$

by letting $\rho_v(\theta)$ be the maximal vertex in $[n]$ that can be reached from v using only ascending and strict edges in θ without using any of the edges $\vec{vx}$ or $\vec{vy}$. Similarly define

$$\rho_x : \mathcal{O}(n) \to [n]$$

by letting $\rho_x(\theta)$ be the maximal vertex in $[n]$ that can be reached from x using only ascending and strict edges in θ without using any of the edges $\vec{xy}$ or $\vec{xz}$. Finally define

$$\rho_y : \mathcal{O}(n) \to [n]$$

by letting $\rho_y(\theta)$ be the maximal vertex in $[n]$ that can be reached from y using only ascending and strict edges in θ without using the edge $\vec{yz}$. Note that $\rho_w(\theta) \leq \mathrm{hrv}(\theta, w)$, $\rho_x(\theta) \leq \mathrm{hrv}(\theta, x)$ and $\rho_y(\theta) \leq \mathrm{hrv}(\theta, y)$.

5.1. **The first bounce relation for orientations.** We now treat the case $st = \mathtt{nn}$. Our goal is to prove the following theorem.

Theorem 5.4. *Let $P \in \mathcal{S}$ be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x + 1$ such that the bounce path of P at (x, z) has a single bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = U\mathtt{n.n}V\mathtt{d.e}W$. Set $Q := U\mathtt{n.n}V\mathtt{e.d}W$. Then*

$$\hat{\mathrm{G}}_P(\mathbf{x}; q + 1) = (q + 1)\hat{\mathrm{G}}_Q(\mathbf{x}; q + 1).$$

A proof of Theorem 5.4 is given at the end of this subsection. Our strategy is to find a λ-preserving two-to-one function $\Phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ such that the fibres of Φ satisfy

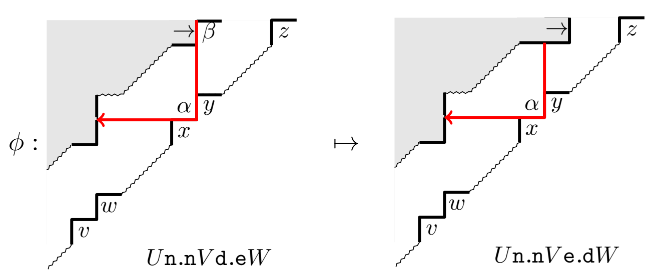
$$\sum_{\theta \in \Phi^{-1}(\theta')} q^{\mathrm{asc}(\theta)} = (q + 1)q^{\mathrm{asc}(\theta')}$$

for all $\theta' \in \mathcal{O}(Q)$. We build such a function Φ from three building blocks ϕ, ϕ', and ψ defined further down.

Let $\phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ be the function defined by

$$\phi(\theta) := \theta \setminus \{\vec{xz}, \vec{zy}\} \cup \{\vec{yz}\}.$$

Example 5.5. *The map ϕ is a bijection for all choices $\alpha, \beta \in \{\rightarrow, \downarrow\}$.*



The following result is immediate from this definition.

Lemma 5.6. *Let $a \in \{\vec{xy}, \vec{yx}\}$, $b \in \{\vec{yz}, \vec{zy}\}$. Then*

$$\phi : \mathcal{O}(P; a, b) \to \mathcal{O}(Q; a)$$

is a bijection. Moreover $\operatorname{asc}(\phi(\theta)) = \operatorname{asc}(\theta) - 1$ *for all* $\theta \in \mathcal{O}(P; \overrightarrow{yz})$*, and* $\operatorname{asc}(\phi(\theta)) = \operatorname{asc}(\theta)$ *for all* $\theta \in \mathcal{O}(P; \overrightarrow{zy})$*.* ■

If we impose restrictions via σ we can say more.

Lemma 5.7. *Let* $\sigma : \{x, y, z\} \to [m]$ *be a surjection such that* $\sigma_3 < \sigma_1$ *and* $\sigma_3 < \sigma_2$*, let* $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$*, and let* $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$*. Then*

$$\phi : \mathcal{O}(P, \sigma; a, b) \to \mathcal{O}(Q, \sigma; a)$$

is a π*-preserving bijection.*

Proof. Set $A = \mathcal{O}(P, \sigma; a, b)$ and $B = \mathcal{O}(Q, \sigma; a)$, and let $\theta \in A$. Note that $\operatorname{hrv}(\theta, y) > \operatorname{hrv}(\theta, z)$ implies

$$\operatorname{hrv}(\theta, y) = \rho_y(\theta) = \rho_y(\theta') = \operatorname{hrv}(\theta', y).$$

Similarly $\operatorname{hrv}(\theta, x) > \operatorname{hrv}(\theta, z)$ implies that the edge $\overrightarrow{xz}$ is not needed to reach $\operatorname{hrv}(\theta, x)$ from x using only strict and ascending edges in θ. Hence $\operatorname{hrv}(\theta', x) = \operatorname{hrv}(\theta, x)$. Consequently $\phi : A \to B$ is well-defined and π-preserving on A.

To see that ϕ is also surjective let $\theta' \in B$. By Lemma 5.6 there exists $\theta \in \mathcal{O}(P; a, b)$ with $\phi(\theta) = \theta'$. Since $\operatorname{hrv}(\theta', y) > \operatorname{hrv}(\theta', z)$ we have

$$\operatorname{hrv}(\theta', y) = \rho_y(\theta') = \rho_y(\theta) = \operatorname{hrv}(\theta, y).$$

Similarly $\operatorname{hrv}(\theta', x) > \operatorname{hrv}(\theta', z)$ implies that $\operatorname{hrv}(\theta, x) = \operatorname{hrv}(\theta', x)$. Thus $\theta \in A$ and the proof is complete. ■

We now collect the orientations on which we plan to use the map ϕ in the proof of Theorem 5.4.

Lemma 5.8. *The function* ϕ *is a* λ*-preserving bijection between the sets*

$$\begin{aligned} \mathcal{O}(P, 221; a, b) &\to \mathcal{O}(Q, 221; a) \\ \mathcal{O}(P, 231; \overrightarrow{yx}, b) &\to \mathcal{O}(Q, 231; \overrightarrow{yx}) \\ \mathcal{O}(P, 321; a, b) &\to \mathcal{O}(Q, 321; a) \end{aligned} \tag{25}$$

for all $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$ *and* $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$*, where all surjections have domain* $\{x, y, z\}$*.*

Let A *be the union of all domains in* (25) *taken over all possible choices of* a *and* b*. Similarly let* B *be the union of all codomains in* (25) *taken over all possible choices of* a*. Then* $\phi : A \to B$ *is a two-to-one function, and the fibres of* ϕ *satisfy*

$$\sum_{\theta \in \phi^{-1}(\theta')} q^{\operatorname{asc}(\theta)} = (q+1) q^{\operatorname{asc}(\theta')}.$$

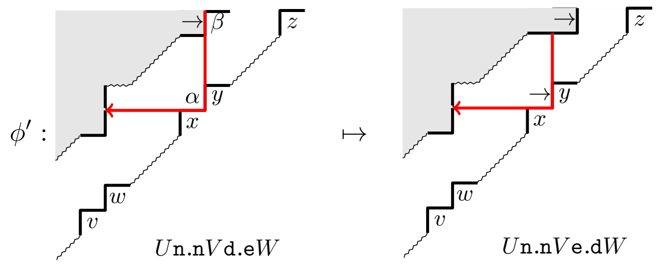
for all $\theta' \in B$*.*

Proof. All cases satisfy the conditions $\sigma_1 > \sigma_3$ and $\sigma_2 > \sigma_3$. Hence Lemma 5.7 applies. Each codomain is in bijection with two distinct domains. Depending on the choice of b the q-weight of θ and $\phi(\theta)$ is either equal or differs by one. ■

Next let $\phi' : \mathcal{O}(P) \to \mathcal{O}(Q)$ be the function defined by

$$\phi'(\theta) \coloneqq \phi(\theta) \setminus \{\overrightarrow{yx}\} \cup \{\overrightarrow{xy}\}.$$

Example 5.9. *The map ϕ' is a bijection for all choices $\alpha, \beta \in \{\rightarrow, \downarrow\}$.*



Our analysis of ϕ' follows the same pattern as for the map ϕ.

Lemma 5.10. *Let $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$ and $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$. Then*

$$\phi' : \mathcal{O}(P; a, b) \to \mathcal{O}(Q; \overrightarrow{xy})$$

is a bijection. Moreover $\operatorname{asc}(\phi(\theta)) = \operatorname{asc}(\theta) - 1$ *for all $\theta \in \mathcal{O}(P; \overrightarrow{xy}, \overrightarrow{yz})$, and* $\operatorname{asc}(\phi(\theta)) = \operatorname{asc}(\theta)$ *for all $\theta \in \mathcal{O}(P; \overrightarrow{xy}, \overrightarrow{zy})$ and all $\theta \in \mathcal{O}(P; \overrightarrow{yx}, \overrightarrow{yz})$.* ■

If we impose restrictions on σ we can say more.

Lemma 5.11. *Let $\sigma : \{x, y, z\} \to [m]$ be a surjection with $\sigma_2 \leq \sigma_1$ and $\sigma_2 = \sigma_3$, and let $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$. Then*

$$\phi' : \mathcal{O}(P, \sigma; a, \overrightarrow{yz}) \to \mathcal{O}(Q, \sigma; \overrightarrow{xy})$$

is a π-preserving bijection on $\mathcal{O}(P, \sigma; a, \overrightarrow{yz})$.

Proof. Set $A = \mathcal{O}(P, \sigma; a, \overrightarrow{yz})$ and $B = \mathcal{O}(Q, \sigma; \overrightarrow{xy})$. Let $\theta \in A$ and set $\theta' = \phi'(\theta)$. Note that $\operatorname{hrv}(\theta, y) = \operatorname{hrv}(\theta, z)$ implies

$$\rho_y(\theta') = \rho_y(\theta) \leq \operatorname{hrv}(\theta, z) = \operatorname{hrv}(\theta', z).$$

Since $\overrightarrow{yz} \in \theta'$ is a strict edge it follows that

$$\operatorname{hrv}(\theta', y) = \operatorname{hrv}(\theta', z) = \operatorname{hrv}(\theta, y).$$

On the other hand $\overrightarrow{xy} \in \theta'$ implies that $\operatorname{hrv}(\theta', x) \geq \operatorname{hrv}(\theta', y)$. If $\operatorname{hrv}(\theta, x) > \operatorname{hrv}(\theta, z)$ then

$$\operatorname{hrv}(\theta, x) = \rho_x(\theta) = \rho_x(\theta') = \operatorname{hrv}(\theta', x).$$

Otherwise $\rho_x(\theta') \leq \operatorname{hrv}(\theta', y)$ and

$$\operatorname{hrv}(\theta', x) = \operatorname{hrv}(\theta', y) = \operatorname{hrv}(\theta, y) = \operatorname{hrv}(\theta, x).$$

Consequently $\phi' : A \to B$ is well-defined and π-preserving on A.

To see that ϕ' is also surjective let $\theta' \in B$. By Lemma 5.10 there exists a unique $\theta \in \mathcal{O}(P; a, \overrightarrow{yz})$ with $\phi'(\theta) = \theta'$. From $\operatorname{hrv}(\theta', y) = \operatorname{hrv}(\theta', z)$ it follows that

$$\rho_y(\theta) = \rho_y(\theta') \leq \operatorname{hrv}(\theta', z) = \operatorname{hrv}(\theta, z).$$

Since $\overrightarrow{yz} \in \theta$ by assumption we obtain $\operatorname{hrv}(\theta, y) = \operatorname{hrv}(\theta, z)$. If $\operatorname{hrv}(\theta', x) > \operatorname{hrv}(\theta', z)$ then

$$\operatorname{hrv}(\theta', x) = \rho_x(\theta') = \rho_x(\theta) = \operatorname{hrv}(\theta, x).$$

Otherwise $\rho_x(\theta) \leq \mathrm{hrv}(\theta, z)$ and

$$\mathrm{hrv}(\theta, x) = \mathrm{hrv}(\theta, z) = \mathrm{hrv}(\theta', x)$$

since $\overrightarrow{xz}$ is a strict edge in θ. We conclude that $\theta \in A$ and the proof is complete. ■

We now collect the orientations on which we plan to use the map ϕ' in the proof of Theorem 5.4.

Lemma 5.12. *The function ϕ' is a λ-preserving bijection between the sets*

$$\begin{aligned} &\mathcal{O}(P, 111; a, \overrightarrow{yz}) \to \mathcal{O}(Q, 111; \overrightarrow{xy}) \\ &\mathcal{O}(P, 211; a, \overrightarrow{yz}) \to \mathcal{O}(Q, 211; \overrightarrow{xy}) \end{aligned} \tag{26}$$

fro all $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$, where all surjections σ have domain $\{x, y, z\}$.

Let A be the union of all domains in (26) taken over all possible choices of a. Similarly let B be the union of all codomains in (26). Then $\phi' : A \to B$ is a two-to-one function, and the fibres of ϕ' satisfy

$$\sum_{\theta \in (\phi')^{-1}(\theta')} q^{\mathrm{asc}(\theta)} = (q+1) q^{\mathrm{asc}(\theta')}$$

for all $\theta' \in B$.

Proof. In each case σ satisfies $\sigma_1 \geq \sigma_2 = \sigma_3$. Hence Lemma 5.11 applies. Each codomain is in bijection with two distinct domains depending on a choice of a. The claim on the q-weights follows from Lemma 5.10. ■

Next let $\psi : \mathcal{O}(P) \to \mathcal{O}(Q)$ be the function defined by

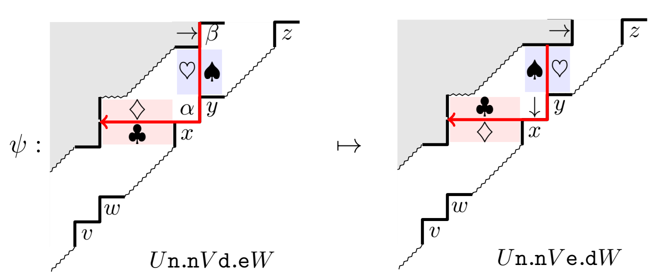
$$\begin{aligned} \overrightarrow{xi} \in \psi(\theta) &\iff \overrightarrow{yi} \in \phi(\theta) \\ \overrightarrow{yi} \in \psi(\theta) &\iff \overrightarrow{xi} \in \phi(\theta) \end{aligned}$$

for all $i \in \{1, 2, \dots, x-1\} \cup \{y+1, \dots, z-1\}$, and

$$\overrightarrow{ij} \in \psi(\theta) \iff \overrightarrow{ij} \in \phi(\theta) \setminus \{\overrightarrow{xy}\} \cup \{\overrightarrow{yx}\}$$

for all other edges. In words, $\psi(\theta)$ is obtained from $\phi(\theta)$ by exchanging the rows x and y, and the columns x and y (excluding the edge xy and the strict edge yz), and by forcing the inclusion of $\overrightarrow{yx}$.

Example 5.13. *As seen in the figure here, the map ψ swaps the marked columns above x and y, and the marked rows to the left of x and y. It is a bijection for all choices $\alpha, \beta \in \{\rightarrow, \downarrow\}$, and it may decrease the number of ascents depending on orientations of α and β.*



Our analysis of the map ψ follows the same pattern as for the functions ϕ and ϕ' above.

Lemma 5.14. *Let $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$ and $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$. Then*

$$\psi : \mathcal{O}(P; a, b) \to \mathcal{O}(Q; \overrightarrow{yx})$$

is a bijection. Moreover $\operatorname{asc}(\psi(\theta)) = \operatorname{asc}(\theta) - 1$ *for all* $\theta \in \mathcal{O}(P; \overrightarrow{yx}, \overrightarrow{yz})$ *and all* $\theta \in \mathcal{O}(P; \overrightarrow{xy}, \overrightarrow{zy})$*, and* $\operatorname{asc}(\psi(\theta)) = \operatorname{asc}(\theta)$ *for all* $\theta \in \mathcal{O}(P; \overrightarrow{yx}, \overrightarrow{zy})$. ■

If we impose restrictions on the data we can say more.

Lemma 5.15. *Let $\sigma : \{x, y, z\} \to [m]$ be a surjection, let $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$, and let $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$, such that the following two conditions are satisfied:*

(i) $a = \overrightarrow{yx}$ *or* $\sigma_1 > \sigma_2$ *or* $\sigma_2 = \sigma_3$.
(ii) $b = \overrightarrow{zy}$ *or* $\sigma_2 > \sigma_3$.

Then

$$\psi : \mathcal{O}(P, \sigma; a, b) \to \mathcal{O}(Q, \sigma_2\sigma_1\sigma_3; \overrightarrow{yx})$$

is a π-switching bijection.

Proof. Let $A = \mathcal{O}(P, \sigma; a, b)$ and $B = \mathcal{O}(Q, \sigma_2\sigma_1\sigma_3; \overrightarrow{yx})$. We first show that $\psi : A \to \mathcal{O}(Q)$ is π-switching. This implies that $\psi : A \to B$ is well-defined, that is, that ψ really maps A to B. The injectivity of ψ follows from Lemma 5.14. To conclude the proof we then show that $\psi : A \to B$ is also surjective.

Let $\theta \in A$ and set $\theta' = \psi(\theta)$. Clearly $\operatorname{hrv}(\theta', i) = \operatorname{hrv}(\theta, i)$ for all $i > y$. Since $\overrightarrow{yz}$ is a strict edge in θ' we have

$$\operatorname{hrv}(\theta', y) = \max(\rho_y(\theta'), \operatorname{hrv}(\theta', z)) = \max(\rho_x(\theta), \operatorname{hrv}(\theta, z)).$$

By assumption (i) this implies $\operatorname{hrv}(\theta', y) = \operatorname{hrv}(\theta, x)$. On the other hand $\operatorname{hrv}(\theta', x) = \rho_y(\theta)$. By assumption (ii) this implies $\operatorname{hrv}(\theta', x) = \operatorname{hrv}(\theta, y)$. For all $i \in [x-1]$ we have

$$\{\operatorname{hrv}(\theta', j) : \overrightarrow{ij} \in \theta'\} = \{\operatorname{hrv}(\theta, j) : \overrightarrow{ij} \in \theta\}$$

and therefore $\operatorname{hrv}(\theta', i) = \operatorname{hrv}(\theta, i)$. Thus ψ is π-switching on A as claimed.

To show that $\psi : A \to B$ is surjective let $\theta' \in B$. By Lemma 5.14 there exists a unique $\theta \in \mathcal{O}(P; a, b)$ such that $\psi(\theta) = \theta'$. Clearly $\operatorname{hrv}(\theta, z) = \operatorname{hrv}(\theta', z)$. First note that

$$\operatorname{hrv}(\theta, y) \geq \rho_y(\theta) = \rho_x(\theta') = \operatorname{hrv}(\theta', x). \tag{27}$$

Condition (ii) implies equality in (27). Since $\overrightarrow{xz} \in \theta$ and $\overrightarrow{yz} \in \theta'$ are strict edges we obtain

$$\operatorname{hrv}(\theta, x) \geq \max(\rho_x(\theta), \operatorname{hrv}(\theta, z)) = \max(\rho_y(\theta'), \operatorname{hrv}(\theta', z)) = \operatorname{hrv}(\theta', y). \tag{28}$$

Condition (i) implies equality (28), and the proof is complete. ■

We now collect the orientations on which we plan to use the map ψ in the proof of Theorem 5.4.

Lemma 5.16. *The function ψ is a λ-preserving bijection between sets*

$$\begin{aligned}\mathcal{O}(P,111;a,\overrightarrow{zy}) &\to \mathcal{O}(Q,111;\overrightarrow{yx})\\ \mathcal{O}(P,211;a,\overrightarrow{zy}) &\to \mathcal{O}(Q,121;\overrightarrow{yx})\\ \mathcal{O}(P,212;a,\overrightarrow{zy}) &\to \mathcal{O}(Q,122;\overrightarrow{yx})\\ \mathcal{O}(P,121;\overrightarrow{yx},b) &\to \mathcal{O}(Q,211;\overrightarrow{yx})\\ \mathcal{O}(P,312;a,\overrightarrow{zy}) &\to \mathcal{O}(Q,132;\overrightarrow{yx})\end{aligned} \tag{29}$$

for all $a \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$ and $b \in \{\overrightarrow{yz}, \overrightarrow{zy}\}$, where all surjections have domain $\{x, y, z\}$.

Let A be the union of all domains in (29) *taken over all possible choices of a and b. Similarly let B be the union of all codomains in* (29)*. Then $\psi : A \to B$ is a two-to-one function, and the fibres of ψ satisfy*

$$\sum_{\theta \in (\psi)^{-1}(\theta')} q^{\operatorname{asc}(\theta)} = (q+1)q^{\operatorname{asc}(\theta')}$$

for all $\theta' \in B$.

Proof. The data σ, a, b in each of the domains satisfies Conditions Lemma 5.15 (i) and (ii). Hence the lemma applies. Each codomain is in bijection with two distinct domains depending on a choice of a or b. The claim on the q-weights follows from Lemma 5.14. ∎

The results of this section combine to a proof of the desired bounce relation among the symmetric functions $\hat{\mathrm{G}}_P(\mathbf{x}; q+1)$.

Proof of Theorem 5.4. Define a function $\Phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ by letting

$$\Phi(\theta) = \begin{cases} \phi(\theta) \\ \phi'(\theta) \\ \psi(\theta) \end{cases}$$

according to Table 2. The reader should verify that the table covers all orientations in $\mathcal{O}(P)$ and $\mathcal{O}(Q)$. Lemmas 5.8, 5.12 and 5.16 show that Φ is a λ-preserving two-to-one function such that the fibres of Φ satisfy

$$\sum_{\theta \in \phi^{-1}(\theta')} q^{\operatorname{asc}(\theta)} = (q+1)q^{\operatorname{asc}(\theta')}$$

for all $\theta' \in \mathcal{O}(Q)$. The claim follows. ∎

5.2. **The second bounce relation for orientations.** We now treat the case $st = \mathtt{dn}$. Our goal is to prove the following theorem.

Theorem 5.17. *Let $P \in \mathcal{S}$ be a Schröder path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $z > x + 1$ such that the bounce path of P at (x, z) has a single bounce point, and such that the bounce decomposition of P at (x, z) is given by $P = U\,\mathtt{d.n}V\,\mathtt{d.e}W$. Set $Q := U\,\mathtt{n.d}V\,\mathtt{e.d}W$. Then*

$$\hat{\mathrm{G}}_P(\mathbf{x}; q+1) = \hat{\mathrm{G}}_Q(\mathbf{x}; q+1).$$

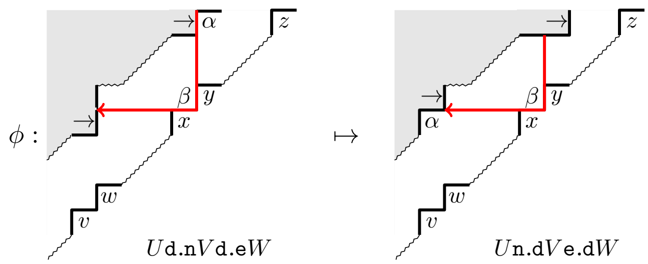
A proof of Theorem 5.17 is given at the end of this subsection. Our strategy is to find a λ-preserving bijection $\Phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ such that the $\mathrm{asc}(\Phi(\theta)) = \mathrm{asc}(\theta)$ for all $\theta \in \mathcal{O}(P)$. The bijection Φ is made up of two building blocks ϕ and ψ.

Let $\phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ be the function defined by

$$\phi(\theta) := \begin{cases} \theta \setminus \{\overrightarrow{xz}\} \cup \{\overrightarrow{vy}\} & \text{if } \overrightarrow{yz} \in \theta, \\ \theta \setminus \{\overrightarrow{vx}, \overrightarrow{xz}, \overrightarrow{zy}\} \cup \{\overrightarrow{xv}, \overrightarrow{vy}, \overrightarrow{yz}\} & \text{if } \overrightarrow{zy} \in \theta. \end{cases}$$

In words, $\phi(\theta)$ is obtained from θ by adding and removing some strict edges as needed, by demanding $\overrightarrow{xv} \in \phi(\theta)$ if and only if $\overrightarrow{yz} \in \theta$, and by leaving all other edges unchanged.

Example 5.18. *The function ϕ is a bijection for all choices $\alpha, \beta \in \{\to, \downarrow\}$.*



The following result is immediate from this definition.

Lemma 5.19. *Let $b \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$, and let $(a, c) \in \{(\overrightarrow{vx}, \overrightarrow{yz}), (\overrightarrow{xv}, \overrightarrow{zy})\}$. Then*

$$\phi : \mathcal{O}(P; b, c) \to \mathcal{O}(Q; a, b)$$

is a bijection. Moreover $\mathrm{asc}(\phi(\theta)) = \mathrm{asc}(\theta)$ *for all* $\theta \in \mathcal{O}(P)$. ■

If we impose restrictions on σ we can say more.

Lemma 5.20. *Let $\sigma : \{x, y, z\} \to [m]$ be a surjection, let $b \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$, and let $(a, c) \in \{(\overrightarrow{vx}, \overrightarrow{yz}), (\overrightarrow{xv}, \overrightarrow{zy})\}$, such that the following conditions are satisfied:*

(i) Either $\sigma_1 = \sigma_2$, or $\sigma_1 > \sigma_2$ and $a = \overrightarrow{vx}$.
(ii) $\sigma_1 > \sigma_3$.
(iii) $\sigma_2 > \sigma_3$ or $c = \overrightarrow{yz}$.

Then

$$\phi : \mathcal{O}(P, \sigma; b, c) \to \mathcal{O}(Q, \sigma; a, b)$$

is a bijection, and $\pi(\phi(\theta)) = \pi(\theta)$ for all $\theta \in \mathcal{O}(P, \sigma; b, c)$.

Proof. Set $A = \mathcal{O}(P, \sigma; b, c)$ and $B = \mathcal{O}(Q, \sigma; a, b)$.

Let $\theta \in A$, and set $\theta' = \phi(\theta)$. By Condition (iii) we have

$$\mathrm{hrv}(\theta, y) = \max(\rho_y(\theta), \mathrm{hrv}(\theta, z)) = \max(\rho_y(\theta'), \mathrm{hrv}(\theta', z)) = \mathrm{hrv}(\theta', y). \tag{30}$$

Similarly Condition (ii) implies that the edge $\overrightarrow{xz}$ is not needed to reach $\mathrm{hrv}(\theta, x)$ from x using strict and ascending edges in θ. Hence $\mathrm{hrv}(\theta', x) = \mathrm{hrv}(\theta, x)$. Consequently Condition (i) yields

$$\mathrm{hrv}(\theta, v) = \max(\rho_v(\theta), \mathrm{hrv}(\theta, x)) = \max(\rho_v(\theta'), \mathrm{hrv}(\theta', x)) = \mathrm{hrv}(\theta', v).$$

It follows that $\phi : A \to B$ is well-defined and π-preserving on A.

The function ϕ is injective by Lemma 5.19. To see that $\phi : A \to B$ is also surjective let $\theta' \in B$. By Lemma 5.19 there exists a unique $\theta \in \mathcal{O}(P; b, c)$ with $\phi(\theta) = \theta'$. Condition (iii) implies $\operatorname{hrv}(\theta, y) = \operatorname{hrv}(\theta', y)$ as in (30). Similarly $\operatorname{hrv}(\theta', x) > \operatorname{hrv}(\theta', z)$ implies that $\operatorname{hrv}(\theta, x) = \operatorname{hrv}(\theta', x)$. Thus $\theta \in A$ and the proof is complete. ■

We now collect the orientations on which we plan to use the map ϕ in the proof of Theorem 5.17.

Lemma 5.21. *The function ϕ is a λ-preserving bijection between the sets*

$$\begin{aligned} \mathcal{O}(P, 211; b, \overrightarrow{yz}) &\to \mathcal{O}(Q, 211; \overrightarrow{vx}, b) \\ \mathcal{O}(P, 221; b, c) &\to \mathcal{O}(Q, 221; a, b) \\ \mathcal{O}(P, 321; b, \overrightarrow{yz}) &\to \mathcal{O}(Q, 321; \overrightarrow{vx}, b), \end{aligned} \tag{31}$$

for all $b \in \{\overrightarrow{xy}, \overrightarrow{yx}\}$, and all $(a, c) \in \{(\overrightarrow{vx}, \overrightarrow{yz}), (\overrightarrow{xv}, \overrightarrow{zy})\}$, where all surjections have domain $\{x, y, z\}$. Moreover ϕ preserves the number of ascents.

Proof. All cases satisfy the conditions of Lemma 5.20. The claim on the ascents follows from Lemma 5.19. ■

Next we define a second map that covers the remaining cases. Let $\psi : \mathcal{O}(P) \to \mathcal{O}(Q)$ be the function defined by

$$\begin{aligned} \overrightarrow{xi} \in \psi(\theta) &\Leftrightarrow \overrightarrow{yi} \in \theta \\ \overrightarrow{yi} \in \psi(\theta) &\Leftrightarrow \overrightarrow{xi} \in \theta \end{aligned}$$

for all $i \in \{v + 1, \ldots, x - 1\} \cup \{y + 1, \ldots, z - 1\}$,

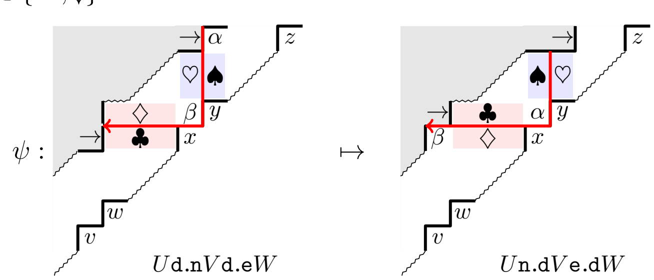
$$\begin{aligned} \overrightarrow{vx} \in \psi(\theta) &\Leftrightarrow \overrightarrow{xy} \in \theta, \\ \overrightarrow{xy} \in \psi(\theta) &\Leftrightarrow \overrightarrow{yz} \in \theta, \end{aligned}$$

and

$$\overrightarrow{ij} \in \psi(\theta) \Leftrightarrow \overrightarrow{ij} \in \theta \setminus \{\overrightarrow{xz}, \overrightarrow{zy}\} \cup \{\overrightarrow{vy}, \overrightarrow{yz}\}$$

for all other edges. In words, $\psi(\theta)$ is obtained from θ by exchanging the rows x and y, and the columns x and y, except that the edges vx, xy and yz (as well as some strict edges) have to be treated separately.

Example 5.22. *As seen in the figure here, the map ψ swaps the marked columns above x and y, and the marked rows to the left of x and y. It is a bijection for all choices $\alpha, \beta \in \{\to, \downarrow\}$.*



It is immediate from the definition that this map is invertible and preserves the number of ascents.

Lemma 5.23. *Let* $(a,b) \in \{(\overrightarrow{vx}, \overrightarrow{xy}), (\overrightarrow{xv}, \overrightarrow{yx})\}$ *and* $(b',c) \in \{(\overrightarrow{xy}, \overrightarrow{yz}), (\overrightarrow{yx}, \overrightarrow{zy})\}$. *Then*

$$\psi : \mathcal{O}(P; b, c) \to \mathcal{O}(Q; a, b')$$

is a bijection. Moreover $\operatorname{asc}(\psi(\theta)) = \operatorname{asc}(\theta)$ *for all* $\theta \in \mathcal{O}(P)$. ■

If we impose restrictions on σ we can say more.

Lemma 5.24. *Let* $\sigma : \{x,y,z\} \to [m]$ *be a surjection, let* $(a,b) \in \{(\overrightarrow{vx}, \overrightarrow{xy}), (\overrightarrow{xv}, \overrightarrow{yx})\}$, *and* $(b',c) \in \{(\overrightarrow{xy}, \overrightarrow{yz}), (\overrightarrow{yx}, \overrightarrow{zy})\}$, *such that the following conditions are satisfied:*

(i) $b = \overrightarrow{yx}$ *or* $\sigma_1 > \sigma_2$ *or* $\sigma_2 = \sigma_3$.
(ii) $c = \overrightarrow{zy}$ *or* $\sigma_2 > \max(\sigma_1, \sigma_3)$ *or* $\sigma_1 = \sigma_3$.

Then

$$\psi : \mathcal{O}(P, \sigma; b, c) \to \mathcal{O}(Q, \sigma_2\sigma_1\sigma_3; a, b')$$

is a π*-switching bijection.*

Proof. Let $A = \mathcal{O}(P, \sigma; b, c)$ and $B = \mathcal{O}(Q, \sigma_2\sigma_1\sigma_3; a, b')$. We first show that $\psi : A \to \mathcal{O}(Q)$ is π-switching. By Lemma 5.23 this implies that $\psi : A \to B$ is injective and well-defined, that is, that ψ really maps A to B. To conclude the proof we then show that $\psi : A \to B$ is also surjective.

Let $\theta \in A$ and set $\theta' = \psi(\theta)$. Clearly $\operatorname{hrv}(\theta', i) = \operatorname{hrv}(\theta, i)$ for all $i > y$. Since $\overrightarrow{yz}$ is a strict edge in θ' we have

$$\operatorname{hrv}(\theta', y) = \max(\rho_y(\theta'), \operatorname{hrv}(\theta', z)) = \max(\rho_x(\theta), \operatorname{hrv}(\theta, z)).$$

Condition (i) implies that the edge xy is not needed to reach $\operatorname{hrv}(\theta, x)$ from x using only strict and ascending edges in θ. Therefore we obtain $\operatorname{hrv}(\theta', y) = \operatorname{hrv}(\theta, x)$. If $(b', c) = (\overrightarrow{yx}, \overrightarrow{zy})$ then

$$\operatorname{hrv}(\theta', x) = \rho_x(\theta') = \rho_y(\theta) = \operatorname{hrv}(\theta, y). \tag{32}$$

On the other hand, if $(b', c) = (\overrightarrow{xy}, \overrightarrow{yz})$ then Condition (ii) implies

$$\begin{aligned} \operatorname{hrv}(\theta', x) &= \max(\rho_x(\theta'), \operatorname{hrv}(\theta', y)) \\ &= \max(\rho_y(\theta), \operatorname{hrv}(\theta, x)) \\ &= \max(\rho_y(\theta), \operatorname{hrv}(\theta, z)) \\ &= \operatorname{hrv}(\theta, y). \end{aligned} \tag{33}$$

For all $i \in \{v+1, \dots, x-1\}$ we have

$$\{\operatorname{hrv}(\theta', j) : \overrightarrow{ij} \in \theta'\} = \{\operatorname{hrv}(\theta, j) : \overrightarrow{ij} \in \theta\}$$

and therefore $\operatorname{hrv}(\theta', i) = \operatorname{hrv}(\theta, i)$. If $a = \overrightarrow{xv}$ then

$$\{\operatorname{hrv}(\theta', j) : \overrightarrow{vj} \in \theta'\} = \{\operatorname{hrv}(\theta, j) : \overrightarrow{vj} \in \theta\}.$$

On the other hand, if $(a, b) = (\overrightarrow{vx}, \overrightarrow{xy})$ then

$$\{\operatorname{hrv}(\theta', j) : \overrightarrow{vj} \in \theta'\} = \{\operatorname{hrv}(\theta, j) : \overrightarrow{vj} \in \theta\} \cup \{\operatorname{hrv}(\theta, y)\}.$$

Condition (i) implies $\mathrm{hrv}(\theta, x) \geq \mathrm{hrv}(\theta, y)$ and consequently $\mathrm{hrv}(\theta', v) = \mathrm{hrv}(\theta, v)$. It now follows that $\mathrm{hrv}(\theta', i) = \mathrm{hrv}(\theta, i)$ also for all $i \in [v]$, and thus ψ is π-switching on A as claimed.

To show that $\psi : A \to B$ is surjective let $\theta' \in B$. By Lemma 5.23 there exists a unique $\theta \in \mathcal{O}(P; b, c)$ such that $\psi(\theta) = \theta'$. Clearly $\mathrm{hrv}(\theta, z) = \mathrm{hrv}(\theta', z)$. If $(b', c) = (\overrightarrow{yx}, \overrightarrow{zy})$ then $\mathrm{hrv}(\theta, y) = \mathrm{hrv}(\theta', x)$ as in (32). If $(b', c) = (\overrightarrow{xy}, \overrightarrow{yz})$ then Condition (ii) yields

$$\begin{aligned} \mathrm{hrv}(\theta, y) &= \max(\rho_y(\theta), \mathrm{hrv}(\theta, z)) \\ &= \max(\rho_x(\theta'), \mathrm{hrv}(\theta', z)) \\ &= \max(\rho_x(\theta'), \mathrm{hrv}(\theta', y)) \\ &= \mathrm{hrv}(\theta', x). \end{aligned}$$

If $b = \overrightarrow{yx}$ then

$$\mathrm{hrv}(\theta, x) = \max(\rho_x(\theta), \mathrm{hrv}(\theta, z)) = \max(\rho_y(\theta'), \mathrm{hrv}(\theta', z)) = \mathrm{hrv}(\theta', y).$$

Otherwise $b = \overrightarrow{xy}$ and Condition (i) implies

$$\mathrm{hrv}(\theta, x) = \max(\rho_y(\theta'), \mathrm{hrv}(\theta', x), \mathrm{hrv}(\theta', z)) = \mathrm{hrv}(\theta', y).$$

This completes the proof. ∎

We now collect the orientations on which we plan to use the map ψ in the proof of Theorem 5.17.

Lemma 5.25. *The function ψ is a λ-preserving bijection between the sets*

$$\begin{aligned} &\mathcal{O}(P, 111; b, c) \to \mathcal{O}(Q, 111; a, b') \\ &\mathcal{O}(P, 211; b, \overrightarrow{zy}) \to \mathcal{O}(Q, 121; a, \overrightarrow{yx}) \\ &\mathcal{O}(P, 212; b, \overrightarrow{zy}) \to \mathcal{O}(Q, 122; a, \overrightarrow{yx}) \\ &\mathcal{O}(P, 121; \overrightarrow{yx}, c) \to \mathcal{O}(Q, 211; \overrightarrow{xv}, b') \\ &\mathcal{O}(P, 312; b, \overrightarrow{zy}) \to \mathcal{O}(Q, 132; a, \overrightarrow{yx}) \\ &\mathcal{O}(P, 231; \overrightarrow{yx}, c) \to \mathcal{O}(Q, 321; \overrightarrow{xv}, b') \\ &\mathcal{O}(P, 321; b, \overrightarrow{zy}) \to \mathcal{O}(Q, 231; a, \overrightarrow{yx}) \end{aligned} \tag{34}$$

for all $(a, b) \in \{(\overrightarrow{vx}, \overrightarrow{xy}), (\overrightarrow{xv}, \overrightarrow{yx})\}$, and all $(b', c) \in \{(\overrightarrow{xy}, \overrightarrow{yz}), (\overrightarrow{yx}, \overrightarrow{zy})\}$, where all surjections have domain $\{x, y, z\}$. Moreover the map ψ preserves the number of ascents in each case.

Proof. The data σ, b, c in each of the domains satisfies the conditions of Lemma 5.24. The claim on the ascents follows from Lemma 5.23. ∎

The results of this section combine to a proof of the desired relation among the symmetric functions $\hat{\mathrm{G}}_P(\mathbf{x}; q+1)$.

Proof of Theorem 5.4. Define a function $\Phi : \mathcal{O}(P) \to \mathcal{O}(Q)$ by letting

$$\Phi(\theta) = \begin{cases} \phi(\theta) \\ \psi(\theta) \end{cases}$$

according to Table 3. The reader should verify that the table covers all orientations in $\mathcal{O}(P)$ and $\mathcal{O}(Q)$. Lemmas 5.21 and 5.25 provide that Φ is a λ-preserving bijection that preserves the number of ascents. The claim follows. ■

6. Applications and further directions

6.1. Schur expansions. It is a major open problem to find an explicit positive formula for the Schur expansion of $\mathrm{G}_P(\mathbf{x};q)$. In this subsection we compare a new signed Schur expansion with the previously known signed Schur expansion. The following description is a straightforward application of techniques in [HHL05a], and appears in [GHQR19, Eq. 3.16]. Given a permutation $\sigma \in \mathfrak{S}_n$, let the *inverse ascent set* $\mathrm{ASC}(\sigma)$ be defined as

$$\mathrm{ASC}(\sigma) \coloneqq \{i \in [n-1] : \sigma^{-1}(i) > \sigma^{-1}(i+1)\}.$$

With $\mathrm{ASC}(\sigma) = \{d_1, d_2, \dots, d_\ell\}$ set

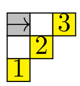
$$\alpha(\sigma) \coloneqq (d_1, d_2 - d_1, d_3 - d_2 \dots, d_\ell - d_{\ell-1}, n - d_\ell).$$

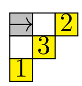
Let P be a Schröder path of size n and let $\mathfrak{S}_n(P)$ denote the set of colorings σ of P such that $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation in $\mathfrak{S}_n$. By using the Egge–Loehr–Warrington [ELW10] theorem, we have the Schur expansion

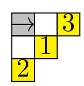
$$\mathrm{G}_P(\mathbf{x};q) = \sum_{\sigma \in \mathfrak{S}_n(P)} q^{\mathrm{asc}(\sigma)} \mathrm{s}_{\alpha(\sigma)}(\mathbf{x}). \tag{35}$$

Note that $\alpha(\sigma)$ is a composition in general, and $\mathrm{s}_{\alpha(\sigma)}$ should be computed via the Jacobi–Trudi identity. This simplifies either to 0, or a Schur function with a sign.

Example 6.1. *For $P = \texttt{nndee}$ we have three colorings in $\mathfrak{S}_n(P)$ with the inverse ascent sets $\{1,2\}$, $\{2\}$ and $\{1\}$:*



$\{1,2\}$


$\{2\}$


$\{1\}$.

The corresponding compositions are $(1,1,1)$, $(2,1)$ and $(1,2)$. The function $\mathrm{s}_{12}(\mathbf{x})$ is identically 0, so

$$\mathrm{G}_P(\mathbf{x};q) = q^2 \mathrm{s}_{111}(\mathbf{x}) + q\mathrm{s}_{21}(\mathbf{x}).$$

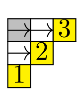
The results in this paper yield a new Schur expansion of vertical-strip LLT polynomials.

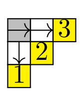
Corollary 6.2. *The Schur expansion of $\mathrm{G}_P(\mathbf{x};q)$ is given by*

$$\mathrm{G}_P(\mathbf{x};q) = \sum_{\mu} \sum_{\theta \in \mathcal{O}(P)} (q-1)^{\mathrm{asc}(\theta)} K_{\mu',\lambda(\theta)} \mathrm{s}_\mu(\mathbf{x}), \tag{36}$$

where $K_{\mu\lambda}$ is a Kostka coefficient.

Example 6.3. *For $P = \texttt{nndee}$ we have four orientations in $\mathcal{O}(P)$ with $\lambda(\theta)$ given by*



3


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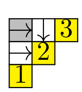
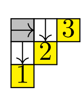


21


21.

Since $K_{\mu\lambda} = 0$ unless μ dominates λ, we have

$$\begin{aligned}\mathrm{G}_P(\mathbf{x};q) = &\big((q-1)^2K_{3,3} + (q-1)^1K_{3,3} + (q-1)^1K_{3,21} + (q-1)^0K_{3,21}\big)\,\mathrm{s}_{111}\\ &+ \big((q-1)^1K_{21,21} + (q-1)^0K_{21,21}\big)\,\mathrm{s}_{21}.\end{aligned}$$

Since $K_{3,3} = K_{3,21} = K_{21,21} = 1$, we see that this expression matches the previous example.

There is one advantage of (36) over the formula in (35). We have a simpler formula for specific Schur coefficients. Moreover, it might be easier to come up with a sign-reversing involution which on (36) which solves the Schur-positivity problem. Finally we note that the Schur coefficients of hook shapes in the unicellular LLT polynomials are determined in [HNY20].

6.2. **Dual bounce relations.** Given $P = s_1 s_2 \dots s_\ell \in \mathcal{S}$ define the *reverse of P*, denoted by $\mathrm{rev}(P)$, be the Schröder path $\hat{s}_\ell \hat{s}_{\ell-1} \dots \hat{s}_1$ where $\hat{\mathtt{e}} = \mathtt{n}$, $\hat{\mathtt{n}} = \mathtt{e}$ and $\hat{\mathtt{d}} = \mathtt{d}$. A quite surprising fact about vertical-strip LLT polynomials is that they are invariant under reversal of the Schröder path.

Lemma 6.4. *Let $P \in \mathcal{S}_n$. Then*

$$\mathrm{G}_P(\mathbf{x};q) = \mathrm{G}_{\mathrm{rev}(P)}(\mathbf{x};q). \tag{37}$$

Similar observations have been made earlier, see for example [CM17, Prop. 3.3].

Proof. Since the LLT polynomials G_P and $\mathrm{G}_{\mathrm{rev}(P)}$ are homogeneous symmetric functions of degree n, it is enough to show that

$$\mathrm{G}_P(x_1, x_2, \dots, x_n; q) = \mathrm{G}_{\mathrm{rev}(P)}(x_n, \dots, x_2, x_1; q). \tag{38}$$

Given a coloring κ of P, define a coloring κ' of $\mathrm{rev}(P)$ via $\kappa'(j) := n+1-\kappa(n+1-j)$ for all $j \in [n]$. Then $\mathrm{asc}(\kappa) = \mathrm{asc}(\kappa')$. By definition of vertical-strip LLT polynomials this implies (38). ∎

As a corollary, vertical-strip LLT polynomials satisfy the *dual bounce relations*. These are the relations obtained from the bounce relations in Theorem 2.5 by reversing all paths.

Corollary 6.5. *The function $F : \mathcal{S} \to \Lambda$ defined by $P \mapsto \mathrm{G}_P$ satisfies the relations obtained from the bounce relations in Theorem 2.5 by reversing all paths. Moreover this function is uniquely determined by the conditions in Theorem 2.1* (i)–(iii) *together with the dual bounce relations obtained by reversing all paths in Theorem 2.1* (iv) *(or, alternatively, in Proposition 3.1* (v)*).*

Corollary 6.6. *The function $F : \mathcal{D} \to \Lambda$ defined by $P \mapsto \mathrm{G}_P$ is uniquely determined by the conditions in Theorem 3.5* (i) *and* (ii) *together with the dual bounce relations obtained by reversing all paths in Theorem 3.5* (vi).

Note that some sets of relations we have encountered in this paper are *self-dual*, that is, invariant under reversal of all underlying paths. Examples are the unicellular relation and the second type of bounce relations in Theorem 2.5 — the case $st = \mathtt{dn}$. Moreover the initial conditions in Theorem 2.1 (i) and Theorem 3.5 (i) are self-dual. However Lemma 6.4 is not manifestly evident, neither from the characterization of vertical-strip LLT poylnomials in Theorem 2.1, nor from the e-expansion in Corollary 2.10.

Open Problem 6.7. *Give a combinatorial explanation of the identity*

$$\sum_{\theta \in \mathcal{O}(P)} q^{\operatorname{asc}(\theta)} \mathrm{e}_{\lambda(\theta)}(\mathbf{x}) = \sum_{\theta' \in \mathcal{O}(\operatorname{rev}(P))} q^{\operatorname{asc}(\theta')} \mathrm{e}_{\lambda(\theta')}(\mathbf{x}).$$

Open Problem 6.8. *Deduce the dual bounce relations directly from the bounce relations (in the style of the proof of Proposition 3.1).*

Moreover it is an interesting question if there are other sets of relations that uniquely determine the vertical-strip LLT polynomials.

Open Problem 6.9. *Combining bounce relations and dual bounce relations, can we find other results in the style of Theorem 2.1.*

6.3. **Hall–Littlewood polynomials.** For a Dyck path P it is noted in [AP18] that $\mathrm{G}_P(\mathbf{x}; q) = \mathrm{G}_{\operatorname{rev}(P)}(\mathbf{x}; q)$ and that

$$\omega \mathrm{G}_P(\mathbf{x}; q) = q^{\operatorname{area}(P)} \mathrm{G}_P(\mathbf{x}; q^{-1}), \tag{39}$$

and a similar relation holds for general Schröder paths, where now vertical-strip LLT polynomials are mapped to horizontal-strip LLT polynomials. Hence, there is a formula analogous to (7) for the horizontal-strip LLT polynomials. As a special case, we can say something about the transformed Hall–Littlewood polynomials. The *transformed Hall–Littlewood polynomial* $\mathrm{H}_\mu(\mathbf{x}; q)$ may be defined as follows (see [TZ03, Eq. (2)+(11)] for a reference). For $\lambda \vdash n$ we let

$$\mathrm{H}_\lambda(\mathbf{x}; q) \coloneqq \prod_{1 \leq i < j \leq n} \frac{1 - R_{ij}}{1 - q R_{ij}} \mathrm{h}_\lambda(\mathbf{x}) \tag{40}$$

where the R_{ij} are *raising operators* acting on the partitions (or compositions) indexing the complete homogeneous symmetric functions as

$$R_{ij} \mathrm{h}_{(\lambda_1, \dots, \lambda_n)}(\mathbf{x}) \coloneqq \mathrm{h}_{(\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots, \lambda_n)}(\mathbf{x}).$$

The transformed Hall–Littlewood polynomials can be expressed (see for example [Ale20, Prop. 38]) via a vertical-strip LLT polynomial as

$$\mathrm{H}_{\mu'}(\mathbf{x}; q) = q^{-\sum_{i \geq 2} \binom{\mu_i}{2}} \omega \mathrm{G}_{P_\mu}(\mathbf{x}; q)$$

where

$$P_\mu \coloneqq (\mathtt{n}^{\mu_1})(\mathtt{e}^{\mu_1 - \mu_2})(\mathtt{d}^{\mu_2})(\mathtt{e}^{\mu_2 - \mu_3})(\mathtt{d}^{\mu_3}) \cdots (\mathtt{e}^{\mu_{\ell-1} - \mu_\ell})(\mathtt{d}^{\mu_\ell})(\mathtt{e}^{\mu_\ell}).$$

Hence, our main result implies that

$$\mathrm{H}_{\mu'}(\mathbf{x}; q+1) = (q+1)^{-\sum_{i \geq 2} \binom{\mu_i}{2}} \sum_{\theta \in \mathcal{O}(P_\mu)} q^{\operatorname{asc}(\theta)} \mathrm{h}_{\lambda(\theta)}(\mathbf{x}). \tag{41}$$

It would be interesting to see if there is a direct combinatorial proof of this identity, connecting Kostka–Foulkes polynomials and ascents. Moreover, perhaps it is possible to interpret the raising operators used to define the transformed Hall–Littlewood polynomials on the level of orientations.

6.4. **Diagonal harmonics.** We shall briefly sketch some consequences in diagonal harmonics. Let us first recall the definition of the ∇ and Δ'_f operators. They acts on the space $\Lambda(q,t)$, and the modified Macdonald polynomials are eigenvectors. First set $B(\mu) \coloneqq \sum_{(i,j)\in\mu}(q^{i-1}t^{j-1})$, where the sum ranges over all (row,column) coordinates of the boxes in μ. We then define the operators ∇ and Δ'_f via the relations

$$\begin{aligned} \nabla \tilde{\mathrm{H}}_\lambda(\mathbf{x};q,t) &\coloneqq \mathrm{e}_n[B(\mu)] \cdot \tilde{\mathrm{H}}_\lambda(\mathbf{x};q,t), \\ \Delta'_f \tilde{\mathrm{H}}_\lambda(\mathbf{x};q,t) &\coloneqq f[B(\mu)-1] \cdot \tilde{\mathrm{H}}_\lambda(\mathbf{x};q,t), \end{aligned} \tag{42}$$

where we use plethystic substitution. For example,

$$\mathrm{e}_2[B(2,1)] = \mathrm{e}_2[1+q+t] = \mathrm{e}_2(1,q,t) = q+t+qt.$$

Let $\mathcal{F}_n(\mathbf{x};q,t) \coloneqq \nabla \mathrm{e}_n$. M. Haiman showed [Hai94] that $\mathcal{F}_n(\mathbf{x};q,t)$ is the bigraded Frobenius series of the space of diagonal coinvariants in $2n$ variables where $\mathfrak{S}_n$ act diagonally. A great deal of research has been devoted to give combinatorial interpretations of ∇f, for various choices of symmetric function f.

In [HHL$^+$05b], the authors presented a conjectured combinatorial formula for $\mathcal{F}_n(\mathbf{x};q,t)$, the *Shuffle Conjecture*. This conjecture was later refined in [HMZ12], where the *Compositional Shuffle Conjecture* was stated. This refined conjecture was proved by E. Carlsson and A. Mellit [CM17, Prop. 2.4].

Theorem 6.10 (The Shuffle Theorem)**.** *We have that*

$$\mathcal{F}_n(\mathbf{x};q,t) = \sum_{w\in \mathrm{WPF}(n)} q^{\mathrm{area}(w)} t^{\mathrm{dinv}(w)} \mathbf{x}_w, \tag{43}$$

where the sum is over all word parking functions *of size n.*

By applying the ζ-map of J. Haglund [Hag07] and interpreting the result, we obtain the equivalent statement now using vertical-strip LLT polynomials as they are defined in this paper. We have that

$$\mathcal{F}_n(\mathbf{x};q,t) = \sum_{P\in\mathcal{D}_n} t^{\mathrm{bounce}(P)} \mathrm{G}_{P^*}(\mathbf{x};q), \tag{44}$$

where P^* is the Schröder path obtained from the Dyck path P by making every corner (`en`) into a diagonal step (`d`). By applying Corollary 2.10, we now have a new combinatorial formula for $\mathcal{F}_n(\mathbf{x};q+1,t)$.

Corollary 6.11. *We have that*

$$\mathcal{F}_n(\mathbf{x};q+1,t) = \sum_{P\in\mathcal{D}_n} t^{\mathrm{bounce}(P)} \sum_{\theta\in\mathcal{O}(P^*)} q^{\mathrm{asc}(\theta)} \mathrm{e}_{\lambda(\theta)}(\mathbf{x}).$$

Similar observations can be used to show that the conjectured expressions for $\Delta'_{\mathrm{e}_k}\mathrm{e}_n$ appearing in the *Delta Conjecture* by J. Haglund, J. Remmel and A.Wilson [HRW18], are also e-positive after the $q \mapsto q+1$ substitution.

Another interesting result is the combinatorial interpretation of $(-1)^{n-1}\nabla \mathrm{p}_n$. The *square path conjecture* was formulated by N. Loehr and G. Warrington [LW07]. Later, E. Sergel [Ser17] gave a proof of the square path conjecture, by using the Compositional Shuffle Theorem.

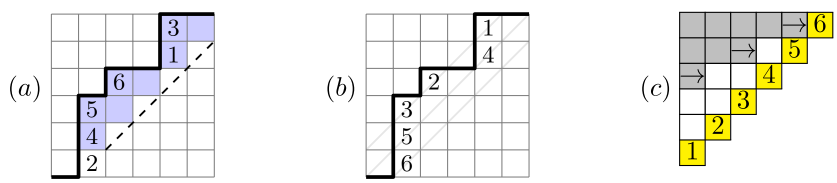
Theorem 6.12 (E. Sergel, (2017))**.** *We have the expansion in the fundamental quasisymmetric functions*

$$(-1)^{n-1}\nabla \mathrm{p}_n = \sum_{w \in \mathrm{Pref}(n)} t^{\mathrm{area}(w)} q^{\mathrm{dinv}(w)} \mathrm{F}_{\mathrm{IDES}(\sigma(w))}(\mathbf{x}),$$

where the sum is taken over all preference functions *of size n, and $\sigma(w)$ is the reading word of w, and* dinv *is a statistic closely related to the* asc*-statistic.*

There is again a close connection with vertical-strip LLT polynomials. We shall give an example explaining the connection.

Example 6.13. *A preference function is simply a map $f : [n] \to [n]$. We can describe such an f by writing $f^{-1}(i)$ in column i in an $n \times n$-diagram, sorted increasingly from the bottom. Moreover, we demand that the entries $f^{-1}(i)$ appear in rows lower than $f^{-1}(j)$ for all $i < j$. This uniquely determines the diagram. The numbers in the digram are called* cars[2]*. There is a unique North-East path from $(0,0)$ to (n,n) where each North step is immediately to the left of a car. We set $\alpha_i := |f^{-1}(i)|$. For example, the preference function $f = (5,2,5,2,2,3)$ give rise to the diagram (a) here:*



(45)

The path P is given by `ennneneeennee`*. Moreover, $\alpha = (0,3,1,0,2,0)$, and note that P is uniquely determined by α. There are a few other statistics on preference functions that only depend on α. We let* area(α) *be the number of squares below P and strictly above the lowest diagonal of the form $y = x + k$, containing a car. In our diagram above,* $\mathrm{area}(\alpha) = 8$*. Moreover, we let* below(α) *be the number of cars strictly below the $y = x$ line. In this case,* $\mathrm{below}(\alpha) = 1$*, as the car labeled 2 is below the main diagonal.*

Finally, given α, we associate a Schröder path as follows. We read the cars diagonal by diagonal as in (b), this gives the reading order *of the cars. Then $\overrightarrow{xy}$ is a diagonal steps in the Schröder path if and only if label x is immediately below y in the reading word order (b). We note that the map that takes P to the Schröder path is also a type of ζ-map, see* [AHJ14, Sect. 5.2].

We can now state the expansion of $(-1)^{n-1}\nabla \mathrm{p}_n$ in terms of vertical-strip LLT polynomials. It is straightforward to deduce this from Theorem 6.12, by unraveling the definitions.

Theorem 6.14. *We have the expansion*

$$(-1)^{n-1}\nabla \mathrm{p}_n = \sum_{\alpha} t^{\mathrm{area}(\alpha)} q^{\mathrm{below}(\alpha)} \mathrm{G}_{\boldsymbol{\nu}(\alpha)}(\mathbf{x}; q),$$

where the sum is taken over all weak compositions α with n parts, of size n.

Again by applying Corollary 2.10, we obtain a new combinatorial expression for $(-1)^{n-1}\nabla \mathrm{p}_n$ and note that it is e-positive if we replace q with $q+1$. We also mention [Ber13] and [Ber17, Sec. 4], where related e-positivity conjectures are made.

[2]This is due to the close connection with parking functions.

n	$(-1)^{n-1}\nabla \mathrm{p}_n$
1	s_1
2	$\mathrm{s}_2 + (q + t + qt)\mathrm{s}_{11}$
3	$\mathrm{s}_3 + (q + q^2 + t + qt + q^2t + t^2 + qt^2 + q^2t^2)\mathrm{s}_{21} +$
	$+(q^3 + qt + q^2t + q^3t + qt^2 + q^2t^2 + q^3t^2 + t^3 + qt^3 + q^2t^3)\mathrm{s}_{111}$

TABLE 1. The Schur expansion of $(-1)^{n-1}\nabla \mathrm{p}_n$ for the first few n.

6.5. **Chromatic quasisymmetric functions.** Carlsson and Mellit observed a plethystic relationship between chromatic quasisymmetric functions and unicellular LLT polynomials. This was implicitly done already in [HHL05a, Sec. 5.1]. It is also hidden in [GP16, Eq. (178)]. The relationship is as follows:

Lemma 6.15 ([CM17, Prop. 3.5])**.** *Let P be a Dyck path of size n. Then*

$$(q-1)^{-n}\mathrm{G}_P[\mathbf{x}(q-1); q] = \mathrm{X}_P(\mathbf{x}; q), \tag{46}$$

where the bracket denotes a substitution using plethysm.

Using Lemma 6.15 and Theorem 3.5 we obtain a new characterization of chromatic quasisymmetric functions of unit-interval graphs.

Corollary 6.16. *The function that assigns to each Dyck path P the chromatic quasisymmetric function of the unit-interval graph Γ_P is the unique function $F : \mathcal{D} \to \Lambda$, $P \mapsto F_P$ that satisfies the following conditions:*

(i) For all $k \in \mathbb{N}$ the initial condition $F_{\mathtt{n}(\mathtt{ne})^k\mathtt{e}} = \mathrm{X}_{\mathtt{n}(\mathtt{ne})^k\mathtt{e}}$ holds, where $\mathrm{X}_{\mathtt{n}(\mathtt{ne})^k\mathtt{e}}$ denotes the chromatic quasisymmetric function of the path on $k+1$ vertices[3].

(ii) The function F is multiplicative, that is, $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{D}$.

(vi) Let $P \in \mathcal{D}$ be a Dyck path and let $(x, z) \in \mathbb{Z}^2$ be a point on P with $x + 1 < z$. If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{n.n}V\mathtt{ne.e}W$ then

$$F_{U\mathtt{n.n}V\mathtt{ne.e}W} = (q+1)F_{U\mathtt{n.n}V\mathtt{e.ne}W} - qF_{U\mathtt{n.n}V\mathtt{e.en}W}.$$

If the bounce decomposition of P at (x, z) is given by $P = U\mathtt{ne.n}V\mathtt{ne.e}W$ then

$$F_{U\mathtt{en.n}V\mathtt{en.e}W} + F_{U\mathtt{ne.n}V\mathtt{ne.e}W} + F_{U\mathtt{n.ne}V\mathtt{e.en}W} =$$
$$F_{U\mathtt{en.n}V\mathtt{ne.e}W} + F_{U\mathtt{ne.n}V\mathtt{e.en}W} + F_{U\mathtt{n.ne}V\mathtt{e.ne}W}.$$

We remark that it is an open problem to prove that the chromatic quasisymmetric functions $\mathrm{X}_P(\mathbf{x}; q)$ expand positively in the e-basis. This is known as the Shareshian–Wachs conjecture [SW12, Conj. 4.9], and for $q = 1$, it reduces to the Stanley–Stembridge conjecture [SS93, Sta95]. Some special cases have been solved, for example in [CH19] and [HP19], using different methods. Theorem 2.9 together with Lemma 6.15 implies that the functions $\mathrm{X}_P(\mathbf{x}; q)$ can be expressed as

$$\mathrm{X}_P(\mathbf{x}; q) = \sum_{\theta \in \mathcal{O}(P)} (q-1)^{\mathrm{asc}(\theta)-n} \mathrm{e}_{\lambda(\theta)}[\mathbf{x}(q-1)]. \tag{47}$$

Why the right-hand side in (47) is e-positive remains very much unclear.

[3]See [SW16] or [Ath15] for explicit formulas in terms of elementary symmetric functions, respectively power-sum symmetric functions.

6.6. **Hessenberg varieties.** Hessenberg varieties have been studied since [MPS92]. Recently unicellular LLT polynomials and chromatic quasisymmetric functions of unit-interval graphs have been connected to the graded Frobenius series of certain actions of the symmetric group on the equivariant cohomology rings of regular semisimple Hessenberg varieties [BC18, GP16]. Here we use more or less the notation of [GP16].

A *flag* in $\mathbb{C}^n$ is a sequence $F_\bullet \coloneqq (F_1, F_2, \dots, F_n)$ of subspaces

$$\{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$$

such that $\dim F_i = i$. The *full flag variety* $\mathrm{Flag}(\mathbb{C}^n)$ is defined as the set of all flags in $\mathbb{C}^n$. Each Dyck path $P \in \mathcal{D}_n$ is readily identified with a *Hessenberg function*, that is, a function $h : [n] \to [n]$ such that $h(i) \geq i$ for all $i \in [n]$, and $h(i+1) \geq h(i)$ for all $i \in [n-1]$. Let $M \in \mathrm{GL}(n, \mathbb{C})$ be a diagonal matrix with n distinct eigenvalues. The *regular semisimple Hessenberg variety of type A* indexed by M and P is the subvariety of $\mathrm{Flag}(\mathbb{C}^n)$ given by

$$\mathrm{Hess}(M, P) \coloneqq \{F_\bullet \in \mathrm{Flag}(\mathbb{C}^n) : MF_i \subseteq F_{h(i)} \text{ for all } i \in [n]\}.$$

Let $T \cong (\mathbb{C}^*)^n$ be the group of invertible diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$. Since the elements of T commute with M we have an action of T on $\mathrm{Hess}(M, P)$ and can consider the *equivariant cohomology ring* $H_T^*(M, P) \coloneqq H_T^*(\mathrm{Hess}(M, P))$. The reader is referred to [Tym05, Tym08] for excellent accessible expositions of this topic. The ring $H_T^*(M, P)$ is isomorphic to the subring of the direct product

$$A \coloneqq \prod_{w \in \mathfrak{S}_n} \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_n] = \{(f_w)_{w \in \mathfrak{S}_n} : f_w \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_n] \text{ for all } w \in \mathfrak{S}_n\}$$

whose elements satisfy divisibility conditions imposed by the edges of the unit-interval graph $\Gamma_P = (V, E)$:

$$H_T^*(M, P) \cong \{(f_w)_{w \in \mathfrak{S}_n} \in A : (x_i - x_j) | (f_w - f_{(i,j)w}) \text{ for all } ij \in E\}.$$

The symmetric group $\mathfrak{S}_n$ acts on $H_T^*(M, P)$ by

$$u \cdot (f_w)_{w \in \mathfrak{S}_n} \coloneqq (u \cdot f_{u^{-1}w})_{w \in \mathfrak{S}_n}.$$

This action respects the grading. There are two natural ways to embed $\mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ into $H_T^*(M, P)$, namely

$$L \coloneqq \{(f)_{w \in \mathfrak{S}_n} : f \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_n]\} \quad \text{and}$$
$$R \coloneqq \{(w \cdot f)_{w \in \mathfrak{S}_n} : f \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_n]\}.$$

One can show that $H_T^*(M, P)$ is a free module over L and over R. This reflects the fact that $H_T^*(M, P)$ is a free module over $H_T^*(\mathrm{pt})$, the equivariant cohomology ring of a point. Note that the action of $\mathfrak{S}_n$ fixes L as a set, and fixes R pointwise. Define the *graded Frobenius series* of $H_T^*(M, P)$ over L as

$$\mathrm{Frob}(H_T^*(M, P), L; \mathbf{x}, q) \coloneqq \sum_{w \in \mathfrak{S}_n} \mathrm{trace}(w, H_T^*(M, P), L; q)\, \mathrm{p}_{\lambda(w)}(\mathbf{x}),$$

where $\lambda(w)$ denotes the cycle type of w, and

$$\mathrm{trace}(w, H_T^*(M, P), L, q) \coloneqq \sum_{b \in B} q^{\deg(b)} (\text{coefficient of } b \text{ in } w \cdot b),$$

where B is a homogeneous basis of $H_T^*(M,P)$ over L. Then the chromatic quasisymmetric function satisfies

$$\mathrm{X}_P(\mathbf{x};q) = \omega\,\mathrm{Frob}(H_T^*(M,P), L; \mathbf{x}, q). \tag{48}$$

This was conjectured in [SW12] and proven in [BC18]. A independent proof was obtained by M. Guay-Paquet. Moreover it is shown in [GP16, Lem. 158] that

$$\mathrm{G}_P(\mathbf{x};q) = \mathrm{Frob}(H_T^*(M,P), R; \mathbf{x}, q). \tag{49}$$

The results obtained in this paper therefore imply relations among $\mathfrak{S}_n$-modules built from the modules $H_T^*(M,P)$ for various Dyck paths P. For example consider the six-term relation in Theorem 3.5 (vi). It follows that there exists an isomorphism of graded $\mathfrak{S}_n$-modules

$$H_T^*(M, U\mathtt{enn}V\mathtt{ene}W) \oplus H_T^*(M, U\mathtt{nen}V\mathtt{nee}W) \oplus H_T^*(M, U\mathtt{nne}V\mathtt{een}W) \cong$$
$$H_T^*(M, U\mathtt{enn}V\mathtt{nee}W) \oplus H_T^*(M, U\mathtt{nen}V\mathtt{een}W) \oplus H_T^*(M, U\mathtt{nne}V\mathtt{ene}W).$$

It would be interesting to know whether such relations can be understood directly or provide some insight in the geometry of these representations. Note that the relations considered in [BC18, HP19] are not linear in the sense that the relations involve chromatic quasisymmetric functions indexed by Dyck paths of different sizes.

6.7. **Circular unit-interval digraphs.** The class of Dyck paths and unit-interval graphs can be extended to so called *circular Dyck paths* and *circular unit-interval graphs*, see [AP18, Ell17a, Ell17b]. There are corresponding chromatic quasisymmetric functions and analogs of vertical-strip LLT polynomials in this setting. Modifying the notion of highest reachable vertex slightly, there is an analog of (7) in this extended setting. It remains to prove e-positivity in this extended setting. The positivity in the power-sum basis[4] has been proved for the circular unit-interval graphs, both for chromatic quasisymmetric functions and the vertical-strip LLT polynomials, via a uniform method in [AS19]. Note that the plethystic relationship in Lemma 6.15 no longer holds in the circular setting.

6.8. **Other open problems.**

Open Problem 6.17. *Consider the coefficients $a_\mu(q) \in \mathbb{N}[q]$, defined via*

$$\mathrm{G}(\mathbf{x}; q+1) = \sum_{\mu} a_\mu(q) \mathrm{e}_\mu(q).$$

It seems that $a_\mu(q)$ is unimodal with mode $\lfloor \mu_1/2 \rfloor$. Moreover, if $a_\mu(q) = b_0 + b_1 q + \dots + b_\ell q^\ell$, then for all $j \in \{1, 2, \dots, \ell-1\}$, we seem to have $b_{j-1}b_{j+1} \leq b_j^2$. In other words, the coefficients of $a_\mu(q)$ seem to be log-concave.

Open Problem 6.18. *In* [HW17, TWZ19]*, a non-symmetric extension of the chromatic quasisymmetric functions is considered. It is then natural to ask if there the recurrences we have extend to this setting.*

Open Problem 6.19. *There is a notion of non-commutative unicellular LLT polynomials introduced in* [NT19]*. Can one extend the* e*-positivity result to the non-commutative setting?*

[4]After applying ω of course.

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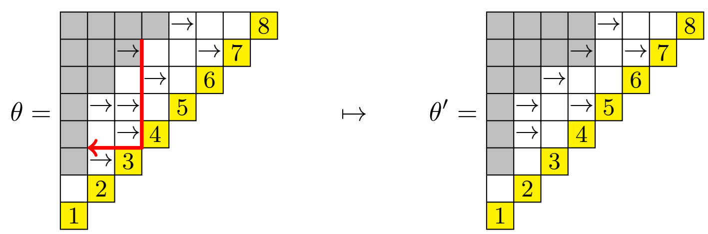
Appendix: Tables

This appendix contains two tables that list all possible types of orientations that appear in the proofs of Theorem 5.4 and Theorem 5.17, and show how they are matched. The following example illustrates how to read these tables.

Example 6.20. *Consider the following orientation of the Schröder path* $P =$ $\texttt{nnennnendeneeee}$ *and let* $(x, z) = (3, 7)$*.*



The bounce decomposition of P *at* (x, z) *is* $P = U\texttt{n.n}V\texttt{d.e}W$ *where* $U = \texttt{nne}$*,* $V = \texttt{nen}$ *and* $W = \texttt{neeee}$*. Therefore we need to consult Table 2 below. We have* $v = 1$*,* $w = 2$*,* $y = 4$*,* $Q = U\texttt{n.n}V\texttt{e.d}W$*, and importantly* $\mathrm{hrv}(\theta, x) = 8$*,* $\mathrm{hrv}(\theta, y) = 7$*, and* $\mathrm{hrv}(\theta, z) = 7$*. Thus we need to consider the weak standardisation of* 877 *which is* 211*. Table 2 shows that there are four subcases in the row indexed by* $\sigma = 211$*. That is, all four orientations of the edges* xy *and* yz *could lead to this word. In our case* $\overrightarrow{xy}, \overrightarrow{zy} \in \theta$*. Thus we are in the second subcase, which belongs to* ⟨5⟩*. This means we should apply the map* ψ *of Lemma 5.14 and end up in case* ⟨5⟩ *in the column corresponding to* Q*. In accordance with what the table predicts, the resulting orientation* θ' *contains* $\overrightarrow{yx}$*, and the weak standardization of* $\mathrm{hrv}(\theta', x), \mathrm{hrv}(\theta', y), \mathrm{hrv}(\theta', z)$ *is* 121*. Since* θ *has one more descent than* θ' *we should regard* θ' *as the second subcase of* ⟨5⟩ *(in column* Q*), which is multiplied by* q*. The reader can check that if* ψ *is applied to an orientation from the first subcase of* ⟨5⟩ *(in column* P*), then the number of ascents is preserved.*

	P			Q		
σ	xy	yz	Map	q^*	xy	Map
111	↓	↓	⬡1	1	↓	⬡1
	→	↓		q	↓	
	↓	→	◇2	1	→	◇2
	→	→		q	→	
122				1	↓	⬡3
				q	↓	
212	↓	↓	⬡3			
	↓	→				
211	↓	↓	⬡5	1	↓	⬡6
	→	↓		q	↓	
	↓	→	◇4	1	→	◇4
	→	→		q	→	
121	↓	↓	⬡6	1	↓	⬡5
	→	↓		q	↓	
221	↓	↓	⑦	1	↓	⑦
	→	↓		q	↓	
	↓	→		1	→	
	→	→		q	→	
132				1	↓	⬡8
				q	↓	
312	↓	↓	⬡8			
	↓	→				
231	↓	↓	⑨	1	↓	⑨
	→	↓		q	↓	
321	↓	↓	⓪	1	↓	⓪
	→	↓		q	↓	
	↓	→		1	→	
	→	→		q	→	

TABLE 2. Subcases for orientations, first bounce relation. The missing $\sigma \in \{112, 123, 213\}$ are not possible, since the edge $\overrightarrow{xz}$ is present in P and $\overrightarrow{yz}$ is present in Q. The maps marked with ⬡i are covered by Lemma 5.16, while ◇i and ⓘ are covered by Lemma 5.12 and Lemma 5.8, respectively.

	P			Q		
σ	xy	yz	Map	xy	wx	Map
111	$\downarrow$ $\downarrow$ $\rightarrow$ $\rightarrow$	$\downarrow$ $\rightarrow$ $\downarrow$ $\rightarrow$	⟨1⟩	$\downarrow$ $\downarrow$ $\rightarrow$ $\rightarrow$	$\downarrow$ $\rightarrow$ $\downarrow$ $\rightarrow$	⟨1⟩
122				$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨2⟩
212	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨2⟩			
211	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨3⟩	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨5⟩
	$\downarrow$ $\rightarrow$	$\rightarrow$ $\rightarrow$	④	$\downarrow$ $\rightarrow$	$\rightarrow$ $\rightarrow$	④
121	$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨5⟩	$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨3⟩
221	$\downarrow$ $\rightarrow$ $\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$ $\rightarrow$ $\rightarrow$	⑥	$\downarrow$ $\rightarrow$ $\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$ $\rightarrow$ $\rightarrow$	⑥
132				$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨7⟩
312	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨7⟩			
231	$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨8⟩	$\downarrow$ $\downarrow$	$\downarrow$ $\rightarrow$	⟨9⟩
321	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨9⟩	$\downarrow$ $\rightarrow$	$\downarrow$ $\downarrow$	⟨8⟩
	$\downarrow$ $\rightarrow$	$\rightarrow$ $\rightarrow$	⑩	$\downarrow$ $\rightarrow$	$\rightarrow$ $\rightarrow$	⑩

TABLE 3. Subcases for orientations, second bounce relation. The maps marked with ⓘ correspond to Lemma 5.21 and the maps marked with ⟨i⟩ are covered by Lemma 5.25. The missing $\sigma \in \{112, 123, 213\}$ are not possible, due to the edges $\overrightarrow{xz}$ and $\overrightarrow{yz}$ being present in P and Q, respectively.

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