A Complete Solution of the Partitions of a Number into Arithmetic Progressions

F. Javier de Vega
King Juan Carlos University
Paseo de los Artilleros, 38, 28032 Madrid
Spain
javier.devega@urjc.es

Abstract

The paper solves the enumeration of the set $\mathrm{AP}(n)$ of partitions of a positive integer n in which the nondecreasing sequence of parts form an arithmetic progression. In particular, we prove a formula for the number of arithmetic progressions of positive, integers, nondecreasing with sum n. We also present an explicit method to calculate all the partitions of $\mathrm{AP}(n)$.

1 Introduction

A partition of a positive integer n is a nondecreasing sequence of positive integers whose sum is n. The summands are called parts of the partition. We consider the problem of enumerating the set $\mathrm{AP}(n)$ of partitions of n in which the nondecreasing sequence of parts form an arithmetic progression (AP), that is, the arithmetic progressions of positive, integers, nondecreasing with sum n. See A049988.
Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n, $|\mathrm{AP}(n)|$ denote the cardinality of the set $\mathrm{AP}(n)$, $\mathrm{D_E}(n)$ denote the set of divisors of n and $\mathrm{D_O}(n)$ denote the set of divisors of $2n$ except the even divisors of n. We will prove the following result:

$$|\mathrm{AP}(n)| = \tau(n) + \sum_{\substack{d\,\in\,\mathrm{D_E}(n)\\ 1\,<\,d\,\leq\,\sqrt{n}}} \left\lfloor \frac{1}{2}\left(\left\lceil \frac{2n}{d(d-1)} \right\rceil - 1\right) \right\rfloor + \sum_{\substack{d\,\in\,\mathrm{D_O}(n)\\ 1\,<\,d\,<\,\sqrt{2n}}} \left\lfloor \frac{1}{2} \left\lceil \frac{2n}{d(d-1)} \right\rceil \right\rfloor \tag{1}$$

We also present an explicit method to calculate all the partitions of $\mathrm{AP}(n)$.

These results are a consequence of **Lemma 13** and **Example 14** of [6]. In the following section, we present a brief introduction to the methods used in [6].

2 Arithmetic progressions and the usual arithmetic

The usual product $m{\cdot}n$ on $\mathbb{Z}$ can be viewed as the sum of n terms of an arithmetic progression, (a_n), whose first term is $a_1 = m - n + 1$ and whose difference is $d = 2$.

Example 1.

$$5 \cdot 3 = (5 - 3 + \underbrace{1) +}_{+2} \underbrace{5 +}_{+2} 7$$

$$3 \cdot 5 = (3 - 5 + \underbrace{1) +}_{+2} \underbrace{1 +}_{+2} \underbrace{3 +}_{+2} \underbrace{5 +}_{+2} 7$$

The previous example motivates the following definition.

Definition 2 (*k-arithmetic product* $\odot_k$)**.** Given $m, k \in \mathbb{Z}$, for all positive integers n, we define the following expression

$$m \odot_k n = (m - n + 1) + (m - n + 1 + k) + \ldots + (m - n + 1 + k + \overset{(n-1)}{\cdots} + k)$$

as the *k-arithmetic* product.

This arithmetic progression can be added to obtain the following formula:

$$m \odot_k n = (m - n + 1) \cdot n + \frac{n \cdot (n-1) \cdot k}{2}. \tag{2}$$

We take (2) as Definition (2) and consider $n \in \mathbb{Z}$.
In connection with the above result, for each $k \in \mathbb{Z}$, the expression "*given a k-arithmetic*" refers to the fact that we are going to work with integers, the sum, the new product and the usual order. This means that we are going to work on

$$\mathcal{Z}_k = \{\mathbb{Z}, +, \odot_k, <\}.$$

Clearly, $\mathcal{Z}_2$, the $2-$*arithmetic*, will be the usual arithmetic.

Definition 3 (*k-arithmetic divisor*)**.** Given a *k-arithmetic*, an integer $d > 0$ is called a divisor of a (*arith k*) if there exists some integer b such that $a = b \odot_k d$. We can write:

$$d \mid a \ (artih\ k) \ \Leftrightarrow \ \exists b \in \mathbb{Z} \ such\ that \ b \odot_k d = a$$

In other words, d is the number of terms of the summation that represents the *k-arithmetic* product (see the following example).

Example 4. Consider the following expression:

$$\underbrace{6}_{where\ to\ begin} \odot_3 \underbrace{5}_{number\ of\ terms} = 2 + 5 + 8 + 11 + 14 = 40$$

The number of terms is 5; hence, we can say that 5 is a divisor of 40 in $3-$*arithmetic*, that is, 5 is a divisor of 40 (*arith* 3).
Notably, a divisor is always a positive number, and the number 6 indicates where we should start the summation. However, we cannot be sure that 6 is a divisor of 40 (*arith* 3).

To characterize the set of divisors, we define the *k-arithmetic* quotient:

Definition 5 (*k-arithmetic quotient* $\oslash_k$)**.** Given a *k-arithmetic*, an integer c is called a quotient of a divided by b ($arith\ k$) if and only if $c \odot_k b = a$. We write:

$$a \oslash_k b = c \Leftrightarrow c \odot_k b = a$$

By means of the following proposition, we can use the usual quotient to study the new one.

Proposition 6. *Given a k-arithmetic and $a, b, k \in \mathbb{Z}$ ($b \neq 0$),*

$$a \oslash_k b = \tfrac{a}{b} + (b-1) \cdot (1 - \tfrac{k}{2}).$$

Proof. $a \oslash_k b = c \Leftrightarrow c \odot_k b = a \Leftrightarrow (c - b + 1)b + \frac{1}{2}b(b-1)k = a \Leftrightarrow c = \frac{a}{b} + (b-1) \cdot (1 - \frac{k}{2})$ □

We must consider $\oslash_k$ in the following manner. If we want to write a as the sum of b terms of an arithmetic progression, then the quotient will give us the place to start the summation (see the following example).

Example 7. Express 81 as the sum of 6 terms of an arithmetic progression whose difference is 3.

We can then obtain $81 \oslash_3 6 = \frac{81}{6} + 5 \cdot (1 - \frac{3}{2}) = 11$. Hence, $81 = 11 \odot_3 6$. The first term is $11 - 6 + 1 = 6$, and the solution is $6 + 9 + 12 + 15 + 18 + 21 = 81$.
Clearly, 6 is a divisor of 81 ($arith$ 3).

Corollary 8. *$a \oslash_k b$ is an integer $\Leftrightarrow$ b is a divisor of a ($arith\ k$).*

Proof.
$$a \oslash_k b = c \in \mathbb{Z} \overset{Def.(5)}{\Leftrightarrow} c \odot_k b = a \overset{Def.(3)}{\Leftrightarrow} b \mid a \ (arith\ k)$$ □

Consider the Example (4): $40 = 6 \odot_3 5$ but 6 is not a divisor of 40 ($arith$ 3) because $40 \oslash_3 6 = \frac{40}{6} + 5 \cdot (1 - \frac{3}{2}) = \frac{25}{6} \notin \mathbb{Z}$.

For the upcoming Lemma and the rest of this paper, we use the following notation for even and odd numbers.

Notation 9. We write the set of even and odd numbers as follows:

- $E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$.
- $O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$.

We now come to the lemma and the example mentioned in the introduction. (After the references of this pre-print, we paste the proof of this lemma appear in [6]).

Lemma 10. *Given a k-arithmetic and $a \in \mathbb{Z}$, the divisors of a ($arith\ k$) are:*

1. *The usual divisors of a if $k \in E$.*
2. *The usual divisors of $2a$ except the even usual divisors of a if $k \in O$.*

Example 11. Express the number 12 in all possible ways as a sum of an arithmetic progression whose difference is 3.

The divisors of 12 (*arith* 3) are the usual divisors of 24 except the even usual divisors of 12: $\{1, \cancel{2}, 3, \cancel{4}, \cancel{6}, 8, \cancel{12}, 24\}$. We repeat Example (7) in each case.

- $d = 1 \Rightarrow a = \frac{12}{1} + (1-1)(1-\frac{3}{2}) = 12 \Rightarrow 12 = 12 \odot_3 1 \Rightarrow 12 = 12.$
- $d = 3 \Rightarrow a = \frac{12}{3} + (3-1)(1-\frac{3}{2}) = 3 \Rightarrow 12 = 3 \odot_3 3 \Rightarrow 12 = 1+4+7.$
- $d = 8 \Rightarrow a = \frac{12}{8} + (8-1)(1-\frac{3}{2}) = -2 \Rightarrow 12 = -2 \odot_3 8 \Rightarrow 12 = -9-6-3+0+3+6+9+12.$
- $d = 24 \Rightarrow a = \frac{12}{24} + (24-1)(1-\frac{3}{2}) = -11 \Rightarrow 12 = -11 \odot_3 24 \Rightarrow 12 = -34-31-28-\ldots+29+32+35.$

Definition 12. Given a positive integer n and $k \in \mathbb{Z}$, let $\mathrm{D}_k(n)$ denote the set of divisors of n (*arith* k).

By Lemma (10) we have two options:

- If $k \in E$, $\mathrm{D}_k(n)$ is the set of the usual divisors of n. $\mathrm{D}_E(n)$ denote this case.
- If $k \in O$, $\mathrm{D}_k(n)$ is the set of the usual divisors of $2n$ except the even usual divisors of n. $\mathrm{D}_O(n)$ denote this case.

We are now ready to prove the main results of this paper.

3 Main result

If we think about the previous example, we have a constructive method to solve the main problem of this paper. For instance, if we want to calculate AP(6), we need to repeat Example (11) varying k from 0 to 4.

Example 13. Calculate AP(6).

- $k = 0 \Rightarrow \mathrm{D}_0(6) = \left\{ \begin{array}{cc} 1 & 2 \\ 6 & 3 \end{array} \right\}$. We calculate a such that $a \odot_k d = n$. Hence, $a = \frac{n}{d} + (d-1)(1-\frac{k}{2})$. Now, if $d \in \mathrm{D}_0(6)$, we have the following possibilities:

⋆ $d = 1 \Rightarrow a = \frac{6}{1} + (1-1)(1-\frac{0}{2}) = 6 \Rightarrow 6 \odot_0 1 = 6 \Rightarrow 6 = 6.$

⋆ $d = 2 \Rightarrow a = 4 \Rightarrow 4 \odot_0 2 = 6 \Rightarrow 3+3 = 6.$

⋆ $d = 3 \Rightarrow a = 4 \Rightarrow 4 \odot_0 3 = 6 \Rightarrow 2+2+2 = 6.$

⋆ $d = 6 \Rightarrow a = 6 \Rightarrow 6 \odot_0 6 = 6 \Rightarrow 1+1+1+1+1+1 = 6.$

- $k = 1 \Rightarrow \mathrm{D}_1(6) = \left\{ \begin{array}{ccc} 1 & \cancel{2} & 3 \\ 12 & \cancel{6} & 4 \end{array} \right\}$.

$\star$ $d = 1 \Rightarrow a = \frac{6}{1} + (1-1)(1-\frac{1}{2}) = 6 \Rightarrow 6 \odot_1 1 = 6 \Rightarrow 6 = 6$.

$\star$ $d = 3 \Rightarrow a = 3 \Rightarrow 3 \odot_1 3 = 6 \Rightarrow 1 + 2 + 3 = 6$.

$\star$ $d = 4 \Rightarrow a = 3 \Rightarrow 3 \odot_1 4 = 6 \Rightarrow 0 + 1 + 2 + 3 = 6$.

$\star$ $d = 12 \Rightarrow a = 6 \Rightarrow 6 \odot_1 12 = 6 \Rightarrow -5 - 4 - 3 - 2 - 1 - 0 + 1 + 2 + 3 + 4 + 5 + 6 = 6$.

- $k = 2 \Rightarrow \mathrm{D}_2(6) = \left\{ \begin{array}{cc} 1 & 2 \\ 6 & 3 \end{array} \right\}$.

$\star$ $d = 1 \Rightarrow a = \frac{6}{1} + (1-1)(1-\frac{2}{2}) = 6 \Rightarrow 6 \odot_2 1 = 6 \Rightarrow 6 = 6$.

$\star$ $d = 2 \Rightarrow a = 3 \Rightarrow 3 \odot_2 2 = 6 \Rightarrow 2 + 4 = 6$.

$\star$ $d = 3 \Rightarrow a = 2 \Rightarrow 2 \odot_2 3 = 6 \Rightarrow 0 + 2 + 4 = 6$.

$\star$ $d = 6 \Rightarrow a = 1 \Rightarrow 1 \odot_2 6 = 6 \Rightarrow -4 + -2 - 0 + 2 + 4 + 6 = 6$.

- $k = 3 \Rightarrow \mathrm{D}_3(6) = \left\{ \begin{array}{cc} 1 & 3 \\ 12 & 4 \end{array} \right\}$.

$\star$ $d = 1 \Rightarrow a = \frac{6}{1} + (1-1)(1-\frac{3}{2}) = 6 \Rightarrow 6 \odot_3 1 = 6 \Rightarrow 6 = 6$.

$\star$ $d = 3 \Rightarrow a = 1 \Rightarrow 1 \odot_3 3 = 6 \Rightarrow -1 + 2 + 5 = 6$.

$\star$ $d = 4 \Rightarrow a = 0 \Rightarrow 0 \odot_3 4 = 6 \Rightarrow -3 + 0 + 3 + 6 = 6$.

$\star$ $d = 12 \Rightarrow a = -5 \Rightarrow -5 \odot_3 12 = 6 \Rightarrow -16 - 13 - 10 - 7 - 4 - 1 + 2 + 5 + 8 + 11 + 14 + 17 = 6$.

- $k = 4 \Rightarrow \mathrm{D}_4(6) = \left\{ \begin{array}{cc} 1 & 2 \\ 6 & 3 \end{array} \right\}$.

$\star$ $d = 1 \Rightarrow a = \frac{6}{1} + (1-1)(1-\frac{4}{2}) = 6 \Rightarrow 6 \odot_4 1 = 6 \Rightarrow 6 = 6$.

$\star$ $d = 2 \Rightarrow a = 2 \Rightarrow 2 \odot_4 2 = 6 \Rightarrow 1 + 5 = 6$.

$\star$ $d = 3 \Rightarrow a = 0 \Rightarrow 0 \odot_4 3 = 6 \Rightarrow -2 + 2 + 6 = 6$.

$\star$ $d = 6 \Rightarrow a = -4 \Rightarrow -4 \odot_4 6 = 6 \Rightarrow -9 - 5 - 1 + 3 + 7 + 11 = 6$.

We have now to observe the following remarks:

1. The case $k=0$, produces the trivial partitions. Now we can understand the beginning of Formula (1).

2. k varies from 0 to 4. In the case $k=4$, we have only the partition $1+5=6$.

3. We are interested in the partitions whose first term is greater than 0. If $d \in \mathrm{D}_k(n)$, the first term of the partition is given by the following expression:

$$\begin{aligned}&\frac{n}{d}+(d-1)(1-\frac{k}{2})-d+1. \text{ Hence,}\\ &\frac{n}{d}+(d-1)(1-\frac{k}{2})-d+1>0 \Leftrightarrow k<\frac{2n}{d(d-1)}.\end{aligned} \tag{3}$$

4. The divisor $d=1$, always produce the partition $6=6$.

With the above, we are prepare to count the partitions of AP(6).

We consider k=1, 2, 3, 4. In the case $k=0$, we have the 4 trivial partitions.

- $k \in E \;\Rightarrow\; \mathrm{D}_E(6)=\left\{\begin{array}{cc}1 & 2\\ 6 & 3\end{array}\right\}$.

$\star$ $d=2 \Rightarrow k_2=\frac{2\cdot 6}{2(2-1)}=6$. The divisor $d=2$ produces partitions of AP(6) in cases such that $k \in E$, $k \in \{1,2,3,4\}$, $k<6$. Hence $d=2$ produces 2 partitions. ($k=2,\ k=4$).

$\star$ $d=3 \Rightarrow k_3=\frac{2\cdot 6}{3(3-1)}=2$. The divisor $d=3$ produces partitions of AP(6) in cases such that $k \in E$, $k \in \{1,2,3,4\}$, $k<2$. Hence $d=3$ produces 0 partitions.

$\star$ $d=6 \Rightarrow k_6=\frac{2\cdot 6}{6(6-1)}<1$. Hence $d=6$ produces 0 partitions. As we will see later, this case will not have to be made.

- $k \in O \;\Rightarrow\; \mathrm{D}_O(6)=\left\{\begin{array}{cc}1 & 3\\ 12 & 4\end{array}\right\}$.

$\star$ $d=3 \Rightarrow k_3=\frac{2\cdot 6}{3(3-1)}=2$. The divisor $d=3$ produces partitions of AP(6) in cases such that $k \in O$, $k \in \{1,2,3,4\}$, $k<2$. Hence $d=3$ produces 1 partition. ($k=1$).

$\star$ $d=4 \Rightarrow k_4=\frac{2\cdot 6}{4(4-1)}=1$. The divisor $d=4$ produces partitions of AP(6) in cases such that $k \in O$, $k \in \{1,2,3,4\}$, $k<1$. Hence $d=4$ produces 0 partitions. As we will see later, this case will not have to be made.

$\star$ $d=12 \Rightarrow k_{12}=\frac{2\cdot 6}{12(12-1)}<1$. Hence $d=12$ produces 0 partitions. As we will see later, this case will not have to be made.

Hence $\mathrm{AP}(6)=4+2+1=7$.

Remark 14. If we want to calculate $|\mathrm{AP}(n)|$, k varies from 0 to $(n-2)$.

Remark 15. If $k \in E$, $k > 0$, we have only to study the divisors d such that $1 < d \leq \sqrt{n}$.

Proof. By (3), if $k \in E$, and $\frac{2n}{d(d-1)} < 2$, then there will be no partition with a positive first term. Hence if $d > \sqrt{n}$, there will be no partition with a positive first term. □

Remark 16. If $k \in O$, $k > 0$, we have only to study the divisors d such that $1 < d < \sqrt{2n}$.

Proof. By (3), if $k \in O$, and $\frac{2n}{d(d-1)} < 1$, then there will be no partition with a positive first term. Hence if $d > \sqrt{2n}$, there will be no partition with a positive first term.
If $d = \sqrt{2n}$, then $\sqrt{2n}$ is even and $\sqrt{2n} \mid n$. Thus, by Lemma (10), we don't have to consider this case. □

Now we have a clear method to calculate $\mathrm{AP}(n)$.

Example 17. Calculate $|\mathrm{AP}(100)|$.

k varies from 0 to 98.
The case $k = 0$, produces the trivial partitions. We have $\tau(100) = 9$ partitions in this case.

- $k \in E$.

$$\mathrm{D}_E(100) = \left\{ \begin{matrix} 1 & 2 & 4 & 5 & 10 \\ 100 & 50 & 25 & 20 & \end{matrix} \right\}.$$

$\star$ $d = 2 \Rightarrow k_2 = \frac{2 \cdot 100}{2 \cdot 1} = 100$. The divisor $d = 2$ produces partitions of AP(100) in cases such that $k \in E$, $k \in \{1, 2, ..., 98\}$, $k < 100$. Hence $d = 2$ produces 49 partitions. ($k = 2$, $k = 4$, $\ldots$, $k = 98$).

$\star$ $d = 4 \Rightarrow k_4 = \frac{2 \cdot 100}{4 \cdot 3} = 16.66\ldots$. The divisor $d = 4$ produces partitions of AP(100) in cases such that $k \in E$, $k \in \{1, 2, ..., 98\}$, $k < 16.66\ldots$. Hence $d = 4$ produces 8 partitions. ($k = 2$, $k = 4$, $\ldots$, $k = 16$).

$\star$ $d = 5 \Rightarrow k_5 = \frac{2 \cdot 100}{5 \cdot 4} = 10$. The divisor $d = 5$ produces partitions of AP(100) in cases such that $k \in E$, $k \in \{1, 2, ..., 98\}$, $k < 10$. Hence $d = 5$ produces 4 partitions. ($k = 2$, $k = 4$, $k = 6$, $k = 8$).

$\star$ $d = 10 \Rightarrow k_{10} = \frac{2 \cdot 100}{10 \cdot 9} = 2.22\ldots$. The divisor $d = 10$ produces partitions of AP(100) in cases such that $k \in E$, $k \in \{1, 2, ..., 98\}$, $k < 2.22\ldots$. Hence $d = 10$ produces 1 partition. ($k = 2$).

- $k \in O$.

$$\mathrm{D}_O(100) = \left\{ \begin{matrix} 1 & \cancel{2} & \cancel{4} & 5 & 8 & \cancel{10} \\ 200 & \cancel{100} & \cancel{50} & 40 & 25 & \cancel{20} \end{matrix} \right\}.$$

$\star$ $d = 5 \Rightarrow k_5 = \frac{2\cdot 100}{5\cdot 4} = 10$. The divisor $d = 5$ produces partitions of AP(100) in cases such that $k \in O$, $k \in \{1, 2, ..., 98\}$, $k < 10$. Hence $d = 5$ produces 5 partitions. ($k = 1$, $k = 3$, $k = 5$, $k = 7$, $k = 9$).

$\star$ $d = 8 \Rightarrow k_8 = \frac{2\cdot 100}{8\cdot 7} = 3.57\ldots$. The divisor $d = 4$ produces partitions of AP(100) in cases such that $k \in O$, $k \in \{1, 2, ..., 98\}$, $k < 3.57\ldots$. Hence $d = 8$ produces 2 partitions. ($k = 1$, $k = 3$).

Hence $|\text{AP}(100)| = 9 + 49 + 8 + 4 + 1 + 5 + 2 = 78$.

Remark 18. We can use "floor and ceil functions" to count the even and odd numbers in each case.

With all of the above, we have proven the following theorem.

Theorem 19. *Given a positive integer n,*

$$|AP(n)| = \tau(n) + \sum_{\substack{d \,\in\, D_E(n) \\ 1 \,<\, d \,\leq\, \sqrt{n}}} \left\lfloor \frac{1}{2}(\left\lceil \frac{2n}{d(d-1)} \right\rceil - 1) \right\rfloor + \sum_{\substack{d \,\in\, D_O(n) \\ 1 \,<\, d \,<\, \sqrt{2n}}} \left\lfloor \frac{1}{2} \left\lceil \frac{2n}{d(d-1)} \right\rceil \right\rfloor .$$

4 Conclusion

The methods used in [6] have been fundamental to the development of this paper. Hence, more work related to the topic of [6] is necessary.

References

[1] R. Cook and D. Sharp, Sums of arithmetic progressions, *Fib. Quart.* **33** (1995), 218-221.

[2] A. O. Munagi, T. Shonhiwa, On the Partitions of a Number into Arithmetic Progressions, *Journal of Integer Sequences* **11** (2008), 08.5.4.

[3] A. O. Munagi, Combinatorics of Integer Partitions in Arithmetic Progression, *Integers* **10** (2010), 73-82.

[4] M. A. Nyblom and C. Evans, On the enumeration of partitions with summands in arithmetic progression, *Australas. J. Combin* **28** (2003), 149-159.

[5] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org.

[6] F. J. de Vega, An Extension of Furstenberg's Theorem of the Infinitude of Primes, https://arxiv.org/abs/2003.13378 [math.NT], 2020.

2010 *Mathematics Subject Classification*: Primary 11P81.
Keywords: partition, arithmetic progression, polygonal numbers, arithmetic generated by a sequence.

(Concerned with sequence A049988)

Notes for helping the referee.

With this program, you can check the main result of the paper. Please, copy and paste in a Maple worksheet or download the maple document "A.P.partitions.mw" from the Google site https://sites.google.com/site/extensionfurstenbergprimes/

```
> restart:

> with(numtheory):

> APpartitionformula := proc (n)
> local i, j, counter, A:
> global total:
> total:=0:
> counter:=tau(n):
> A:=divisors(n):
> for i to counter do
> if A[i]>1 and A[i]<=evalf(sqrt(n)) then
> total:=total + floor(((ceil(2*n/(A[i]*(A[i]-1))))-1)/2);
> fi:
> od:
> counter:=tau(2*n):
> A:=divisors(2*n):
> for j to counter do
> if A[j]>1 and A[j]<evalf(sqrt(2*n)) then
> if (A[j] mod 2) <>0 then
> total:=total + floor((ceil(2*n/(A[j]*(A[j]-1))))/2);
> else if (n mod A[j])<>0 then
> total:=total + floor((ceil(2*n/(A[j]*(A[j]-1))))/2);
> fi:
> fi:
> fi:
> od:
> total:=tau(n)+total:
> end:
```

```
> APpartitionformula(100):  total;
```

$$78$$

```
> A:=array(1..20):
> for i to 20 do
> APpartitionformula(i):
> A[i]:=total:
> od:
> print(A);
```

$$[\,1\ \ 2\ \ 3\ \ 4\ \ 4\ \ 7\ \ 5\ \ 7\ \ 9\ \ 9\ \ 7\ \ 14\ \ 8\ \ 11\ \ 16\ \ 13\ \ 10\ \ 20\ \ 11\ \ 17\,]$$

Number of nondecreasing arithmetic progressions of positive integers with sum n. See A049988.

Lemma 10. Given a *k-arithmetic* and $a \in \mathbb{Z}$, the divisors of a $(arith\ k)$ are:

1. The usual divisors of a if $k \in E$.
2. The usual divisors of $2a$ except the even usual divisors of a if $k \in O$.

Proof. We use Proposition (6) and Corollary (8) in the following cases:

1) $k \in E$. $d \mid a \Leftrightarrow d \mid a\ (arith\ k)$.

2a) $k \in O$. If d is odd: $d \mid 2a \Leftrightarrow d \mid a\ (arith\ k)$.

2b) $k \in O$. If d is even: $d \mid 2a$ and $d \nmid a \Leftrightarrow d \mid a\ (arith\ k)$.

1. $k \in E$. *Suppose d is a usual divisor of a:*

$$a \oslash_k d = \underbrace{\frac{a}{d}}_{\in\mathbb{Z}} + (d-1)(1 - \underbrace{\frac{k}{2}}_{\in\mathbb{Z}}) \in \mathbb{Z} \Rightarrow d \mid a\ (arith\ k).$$

Hence, d is a divisor of a $(arith\ k)$.

$k \in E$. *Suppose d is a divisor of a $(arith\ k)$:*

$$d \mid a\ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z} \text{ such that } b \odot_k d = a \Leftrightarrow (b-d+1) + \underbrace{\frac{(d-1)k}{2}}_{\in\mathbb{Z}} = \frac{a}{d} \in \mathbb{Z}.$$

Hence, $\frac{a}{d} \in \mathbb{Z}$, and d is a usual divisor of a.

2. We consider two cases:
a) $k \in O$. *Suppose d is an odd usual divisor of $2a$:*

If $d \mid 2a \Rightarrow d \mid a$ because d is an odd number. Then, $a \oslash_k d = \underbrace{\frac{a}{d}}_{\in\mathbb{Z}} + \underbrace{(d-1)}_{even}(1 - \frac{k}{2}) \in \mathbb{Z}$.

Hence, a$\oslash_k d \in \mathbb{Z}$, and d is a divisor of a $(arith\ k)$.

$k \in O$. *Suppose d is an odd divisor of a $(arith\ k)$:*

$d \mid a\ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z}$ such that $b \odot_k d = a \Leftrightarrow (b - d + 1) + \underbrace{(d-1)}_{even}\dfrac{k}{2} = \dfrac{a}{d} \in \mathbb{Z}.$

Then, $\frac{a}{d} \in \mathbb{Z}$, and d is a usual divisor of a. Hence, d is a usual divisor of $2a$.

b) $k \in O$. *Suppose d is an even usual divisor of $2a$ but d is not a divisor of a:*

$d \mid 2a \Rightarrow \exists h \in \mathbb{Z}$ such that $2a = dh \Rightarrow \frac{a}{d} = \frac{h}{2}$. By hypothesis $d \nmid a$; hence, $\frac{h}{2} \notin \mathbb{Z}$ and h is odd.

Then, $a \oslash_k d = \dfrac{a}{d} + (d-1)(1 - \dfrac{k}{2}) = \dfrac{h}{2} + (d-1)(1 - \dfrac{k}{2}) = \dfrac{1}{2}(\underbrace{\underbrace{h}_{odd} - \underbrace{(d-1)k}_{odd}}_{even}) + d - 1 \in \mathbb{Z}.$

Hence, d is a divisor of a $(arith\ k)$.

$k \in O$. *Suppose d is an even number and d is a divisor of a $(arith\ k)$:*

$$d \mid a\ (arith\ k) \Leftrightarrow \exists b \in \mathbb{Z} \text{ such that } b \odot_k d = a \Leftrightarrow \begin{cases} (b - d + 1) + \underbrace{\dfrac{(d-1)k}{2}}_{\notin\mathbb{Z}} = \frac{a}{d} \notin \mathbb{Z}. \\ \\ 2(b - d + 1) + (d-1)k = \frac{2a}{d} \in \mathbb{Z}. \end{cases}$$

Hence, $d \nmid a$ and $d \mid 2a$ in the usual sense. $\square$