CONICAL TESSELLATIONS ASSOCIATED WITH WEYL CHAMBERS

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Abstract. We consider d-dimensional random vectors $Y_1, \dots, Y_n$ that satisfy a mild general position assumption a.s. The hyperplanes

$$(Y_i + Y_j)^\perp \;\; (1 \le i < j \le n), \quad (Y_i - Y_j)^\perp \;\; (1 \le i < j \le n), \quad Y_i^\perp \;\; (1 \le i \le n)$$

generate a conical tessellation of the Euclidean d-space, which is closely related to the Weyl chambers of type B_n. We determine the number of cones in this tessellation and show that it is a.s. constant. For a random cone chosen uniformly at random from this random tessellation, we compute expectations for a general series of geometric functionals. These include the face numbers, as well as the conical intrinsic volumes and the conical quermassintegrals. Under the additional assumption of symmetric exchangeability on $Y_1, \dots, Y_n$, the same is done for the dual random cones which have the same distribution as the positive hull of $Y_1 - Y_2, \dots, Y_{n-1} - Y_n, Y_n$ given that this positive hull is not equal to $\mathbb{R}^d$. All these expectations turn out to be distribution-free.
Similarly, we consider the conical tessellation induced by the hyperplanes

$$(Y_i - Y_j)^\perp \;\; (1 \le i < j \le n).$$

This tessellation is closely related to the Weyl chambers of type A_{n-1}. We compute the number of cones in this tessellation and the expectations of the same geometric functionals for the random cones obtained from this random tessellation.
The main ingredient in the proofs is a connection between the number of faces of the tessellation and the number of faces of the Weyl chambers of the corresponding type that are intersected by a certain linear subspace in general position.

1. Introduction

Let $\{H_1, \dots, H_n\}$ be a set of distinct hyperplanes in $\mathbb{R}^d$ passing through the origin. These hyperplanes dissect $\mathbb{R}^d$ into finitely many polyhedral cones forming a *conical tessellation* of $\mathbb{R}^d$. More precisely, the set $\mathbb{R}^d \setminus \bigcup_{i=1}^n H_i$ consists of open connected components whose closures define the polyhedral cones of the tessellation. Under the condition that the hyperplanes satisfy some minor assumption, which is referred to as *general position*, Schlfli [14] derived the well-known formula for the number $C(n,d)$ of cones induced by these hyperplanes:

$$C(n,d) = 2 \sum_{i=0}^{d-1} \binom{n-1}{i}. \tag{1.1}$$

For a simple inductive proof of this formula, see [16, Lemma 8.2.1].

If the hyperplanes $H_1, \dots, H_n$ are chosen at random, for example independently and uniformly on the space of all linear hyperplanes, we obtain a *random* conical tessellation. By intersecting the cones of a conical tessellation with the unit sphere $\mathbb{S}^{d-1}$ we obtain a tessellation of the unit sphere by spherical polytopes; see Figure 1 for a sample realization in dimension $d = 3$. This tessellation

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has been studied by Cover and Efron [4] and Hug and Schneider [5]. For further results on this and other types of random tessellations of the sphere we refer to [13, 2, 3, 15, 9, 6, 7, 8].

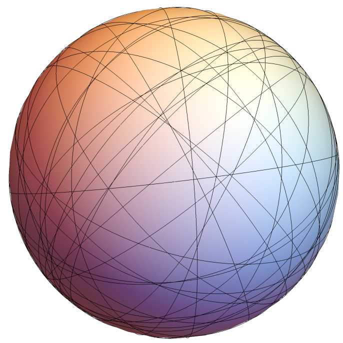

FIGURE 1. Tessellation of the unit sphere in $\mathbb{R}^3$ induced by $n = 36$ uniform and independent hyperplanes.

In this paper we want to introduce two new classes of conical tessellations that are related to reflection groups of types A_{n-1} and B_n. Let us start with tessellations of type B_n. Take some vectors $y_1, \dots, y_n \in \mathbb{R}^d$, where $n \geq d$. By definition, the hyperplane arrangement $\mathcal{A}^B(y_1, \dots, y_n)$ consists of the hyperplanes in $\mathbb{R}^d$ given by

$$\begin{aligned} (y_i + y_j)^\perp, &\quad 1 \leq i < j \leq n, \\ (y_i - y_j)^\perp, &\quad 1 \leq i < j \leq n, \\ y_i^\perp, &\quad 1 \leq i \leq n, \end{aligned}$$

where $x^\perp = \{y \in \mathbb{R}^d : \langle x, y \rangle = 0\}$ denotes the orthogonal complement of a vector $x \in \mathbb{R}^d \setminus \{0\}$ and $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. For these hyperplane to be well-defined, we assume that $y_i \neq \pm y_j$ and $y_i \neq 0$ for all $1 \leq i < j \leq n$. Then the *Weyl tessellation of type* B_n, denoted by $\mathcal{W}^B(y_1, \dots, y_n)$, is defined as the conical tessellation generated by the hyperplanes from $\mathcal{A}^B(y_1, \dots, y_n)$. Now, the natural question arises if we can evaluate the number of cones in the Weyl tessellation of type B_n, which we denote by

$$D^B(n, d) := \#\mathcal{W}^B(y_1, \dots, y_n).$$

Note that $D^B(n, d)$ initially depends on the choice of vectors $y_1, \dots, y_n$. We will not indicate this fact in the notation, since it turns out that $D^B(n, d)$ is constant, under certain mild conditions on $y_1, \dots, y_n$ which we will state in Theorem 1.1.

Denote the group of permutations of the set $\{1, \dots, n\}$ by $\mathcal{S}_n$. Our first result in analogy to (1.1) is as follows.

Theorem 1.1. *Let $y_1, \dots, y_n \in \mathbb{R}^d$, where $n \geq d$, satisfy the following assumption:*

(B1) For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ and every permutation $\sigma \in \mathcal{S}_n$ let any d or fewer of the vectors $\varepsilon_1 y_{\sigma(1)} - \varepsilon_2 y_{\sigma(2)}, \varepsilon_2 y_{\sigma(2)} - \varepsilon_3 y_{\sigma(3)}, \dots, \varepsilon_{n-1} y_{\sigma(n-1)} - \varepsilon_n y_{\sigma(n)}, \varepsilon_n y_{\sigma(n)}$ be linearly independent.

Then the number of cones in the Weyl tessellation $\mathcal{W}^B(y_1, \dots, y_n)$ is given by

$$D^B(n, d) = 2\big(B(n, n - d + 1) + B(n, n - d + 3) + \dots\big),$$

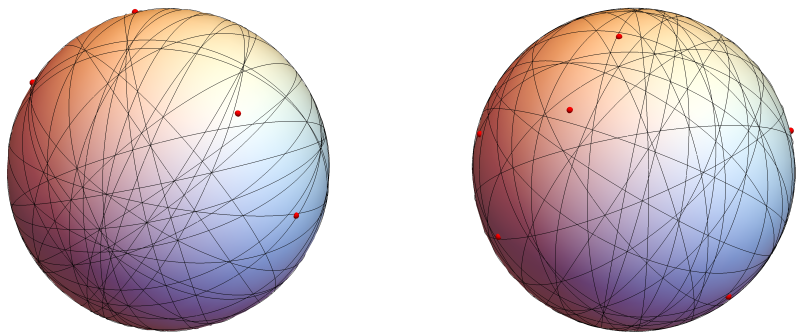

FIGURE 2. Left: Weyl tessellation of type B_n of the unit sphere in $\mathbb{R}^3$ with $n = 6$. Right: Weyl tessellation of type A_{n-1} of the unit sphere in $\mathbb{R}^3$ with $n = 9$. Both tessellations are generated by 36 hyperplanes. The vectors $Y_1, \dots, Y_n$ (red points) were sampled independently and uniformly on the unit sphere.

where $B(n,k)$ are the coefficients of the polynomial

$$(t+1)(t+3)\cdot \ldots \cdot (t+2n-1) = \sum_{k=0}^{n} B(n,k)t^k \tag{1.2}$$

and, by convention, $B(n,k) = 0$ for $k \notin \{0, \dots n\}$.

The assumption on $y_1, \dots, y_n$ in Theorem 1.1 may seem very specific and unnatural, but in the course of this paper we will show that, in certain random settings, it is satisfied with probability 1; see Lemma 5.2. Moreover, in Theorem 3.2 we will state an equivalent assumption, called (B2), which allows to view (B1) from the larger perspective of *general position*.

We may also define a conical tessellation of type A_{n-1}. Take some pairwise distinct vectors $y_1, \dots, y_n \in \mathbb{R}^d$, where $n \geq d+1$. By definition, the hyperplane arrangement $\mathcal{A}^A(y_1, \dots, y_n)$ consists of the hyperplanes given by

$$(y_i - y_j)^\perp, \quad 1 \leq i < j \leq n.$$

Then the *Weyl tessellation of type* A_{n-1}, denoted by $\mathcal{W}^A(y_1, \dots, y_n)$, is defined as the conical tessellation generated by the hyperplane arrangement $\mathcal{A}^A(y_1, \dots, y_n)$. We denote the number of cones in the Weyl tessellation of type A_{n-1} by

$$D^A(n,d) = \#\mathcal{W}^A(y_1, \dots, y_n).$$

The next result is an analogue of Theorem 1.1.

Theorem 1.2. *Let $y_1, \dots, y_n \in \mathbb{R}^d$, where $n \geq d+1$, satisfy the following assumption:*

(A1) For every permutation $\sigma \in \mathcal{S}_n$ let any d or fewer of the vectors $y_{\sigma(1)} - y_{\sigma(2)}, \dots, y_{\sigma(n-1)} - y_{\sigma(n)}$ be linearly independent.

Then the number of cones in the Weyl tessellation $\mathcal{W}^A(y_1, \dots, y_n)$ of type A_{n-1} is given by

$$D^A(n,d) = 2\left(\begin{bmatrix} n \\ n-d+1 \end{bmatrix} + \begin{bmatrix} n \\ n-d+3 \end{bmatrix} + \dots \right),$$

where $\left[{n \atop k}\right]$ *are the Stirling numbers of first kind defined by the formula*

$$t(t+1)\cdot\ldots\cdot(t+n-1)=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}t^k \tag{1.3}$$

and, by convention, $\left[{n \atop k}\right]=0$ *for* $k\notin\{1,\ldots n\}$.

As a byproduct of the proofs of these and some more general theorems, we shall also compute the total number of j-dimensional faces in the tessellations $\mathcal{W}^B(y_1,\ldots,y_n)$ and $\mathcal{W}^A(y_1,\ldots,y_n)$; see Theorems 3.10 and 4.7.

The Schlfli formula (1.1) has several probabilistic consequences [20, 4, 5]. For example, Hug and Schneider [5] (who continued the work of Cover and Efron [4]) defined the *random Schlfli cone* S_n as the random cone obtained by picking uniformly at random one of the cones induced by random, independent linear hyperplanes $H_1,\ldots,H_n$ having a distribution satisfying some minor condition (for example, the uniform distribution on the set of all hyperplanes). These authors evaluated the expectations of a few geometric functionals like the expected number of j-faces of S_n, which is given by

$$\mathbb{E}\,f_j(S_n)=\frac{2^{d-j}\binom{n}{d-j}C(n-d+j,j)}{C(n,d)}, \tag{1.4}$$

for $j=1,\ldots,d$. Hug and Schneider [5] generalized these results by introducing a series of general geometric functionals $Y_{k,j}$, called the *size functionals*. In order to define them, we need to introduce the *conical quermassintegrals*. For a cone C, which is not a linear subspace, the j-th conical quermassintegral $U_j(C)$ is defined as $1/2$ times the probability that that the intersection of C with a uniform random $(d-j)$-dimensional linear hyperplane is different from $\{0\}$. Then the functional $Y_{k,j}(C)$ is defined as the sum of $U_j(F)$ over all k-faces F of C. In [5, Theorem 4.1], Hug and Schneider derived a formula for the expected size functionals of S_n, namely

$$\mathbb{E}\,Y_{d-k+j,\,d-k}(S_n)=\frac{2^{k-j}\binom{n}{k-j}C(n-k+j,j)}{2C(n,d)} \tag{1.5}$$

for $1\leq j\leq k\leq d$ and $n>k-j$. The quantities $Y_{k,j}$ are significant, since they comprise a lot of important geometric functionals, such as the number of k-faces of C and the conical quermassintegrals $U_j(C)$ mentioned above, as special cases. Furthermore, the j-th conical intrinsic volume $\upsilon_j(C)$, which is essentially defined as the probability that the projection of a standard Gaussian vector in $\mathbb{R}^d$ onto C lies in the relative interior of a j-face of C, can be expressed through the quermassintegrals.

Again, the natural question arises whether similar calculations are possible for a random cone chosen from the Weyl tessellations. At first, we consider the type B_n. Let $Y_1,\ldots,Y_n$ be (possibly dependent) random vectors in $\mathbb{R}^d$ with $n\geq d$ satisfying assumption (B1) a.s. For example, (B1) is satisfied a.s. if $(Y_1,\ldots,Y_n)$ has a joint density function on $(\mathbb{R}^d)^n$ with respect to μ^n, where μ is a Lebesgue measure or, more generally, any σ-finite measure on $\mathbb{R}^d$ that assigns measure zero to each affine hyperplane; see Lemma 5.2. Then the *random Weyl cone* $\mathcal{D}_n^B$ of type B_n is defined as follows: Among the cones of the random Weyl tessellation $\mathcal{W}^B(Y_1,\ldots,Y_n)$ choose one uniformly at random. For a realization of the random tessellation $\mathcal{W}^B(Y_1,\ldots,Y_n)$, see the left panel of Figure 2. One of our main results is the following formula for the expected size functionals of $\mathcal{D}_n^B$.

Theorem 1.3. *Let $\mathcal{D}_n^B$ be a random Weyl cone of type B_n in $\mathbb{R}^d$ defined as above. Then*

$$\mathbb{E}\, Y_{d-k+j,\, d-k}(\mathcal{D}_n^B) = \frac{2^{k-j}\binom{n}{k-j} D^B(n-k+j, j)}{2D^B(n,d)} \frac{n!}{(n-k+j)!}$$

holds for all $1 \le j \le k \le d$.

For type A_{n-1}, we can make similar calculations. Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$, where $n \ge d+1$, which satisfy assumption (A1) a.s. Then the *random Weyl cone* $\mathcal{D}_n^A$ of type A_{n-1} can be constructed as follows: Among the cones of the random Weyl tessellation $\mathcal{W}^A(Y_1, \dots, Y_n)$ choose one uniformly at random. For a realization of the random tessellation $\mathcal{W}^A(Y_1, \dots, Y_n)$, see the right panel of Figure 2. We can now state an analogue of Theorem 1.3.

Theorem 1.4. *Let $\mathcal{D}_n^A$ be a random Weyl cone of type A_{n-1}. Then*

$$\mathbb{E}\, Y_{d-k+j,\, d-k}(\mathcal{D}_n^A) = \frac{\binom{n-1}{k-j} D^A(n-k+j, j)}{2D^A(n,d)} \frac{n!}{(n-k+j)!}$$

holds for all $1 \le j \le k \le d$.

The similarities to the analogous result (1.5) for Schlfli cones are obvious. From Theorems 1.3 and 1.4 we will derive the values of several interesting expected geometrical functionals as special cases. In the following corollaries we assume that $n \ge d$ and (B1) holds a.s. (in the B_n case) or that $n \ge d+1$ and (A1) holds a.s. (in the A_{n-1} case).

Corollary 1.5. *For $j = 1, \dots, d$ the expected numbers of j-faces of the random Weyl cones $\mathcal{D}_n^B$ and $\mathcal{D}_n^A$ are given by*

$$\mathbb{E}\, f_j(\mathcal{D}_n^B) = \frac{2^{d-j}\binom{n}{d-j} D^B(n-d+j, j)}{D^B(n,d)} \frac{n!}{(n-d+j)!},$$

$$\mathbb{E}\, f_j(\mathcal{D}_n^A) = \frac{\binom{n-1}{d-j} D^A(n-d+j, j)}{D^A(n,d)} \frac{n!}{(n-d+j)!}.$$

Corollary 1.6. *For $j = 0, \dots, d-1$ the expected conical quermassintegrals of the random Weyl cones $\mathcal{D}_n^B$ and $\mathcal{D}_n^A$ are given by*

$$\mathbb{E}\, U_j(\mathcal{D}_n^B) = \frac{D^B(n, d-j)}{2D^B(n,d)}, \qquad \mathbb{E}\, U_j(\mathcal{D}_n^A) = \frac{D^A(n, d-j)}{2D^A(n,d)}.$$

Corollary 1.7. *For $j = 1, \dots, d$ the expected conical intrinsic volumes of the random Weyl cones $\mathcal{D}_n^B$ and $\mathcal{D}_n^A$ are given by*

$$\mathbb{E}\, \upsilon_j(\mathcal{D}_n^B) = \frac{B(n, n-d+j)}{D^B(n,d)}, \qquad \mathbb{E}\, \upsilon_j(\mathcal{D}_n^A) = \begin{bmatrix} n \\ n-d+j \end{bmatrix} \frac{1}{D^A(n,d)}.$$

The papers [11] and [10] studied convex hulls of the d-dimensional random walks (and bridges) of the form $Y_1, Y_1 + Y_2, \dots, Y_1 + \ldots + Y_n$, where $Y_1, \dots, Y_n$ are random vectors satisfying certain exchangeability conditions; see also [19]. The main results of these works are formulas for the probability that such convex hull contains the origin, as well as for the expected number of j-faces of the convex hull. These formulas (which are distribution-free) also involve the numbers $D^B(n,d)$ and $D^A(n,d)$. In the following, we shall describe the dual cones of $\mathcal{D}_n^B$, respectively $\mathcal{D}_n^A$. Under natural exchangeability assumptions on $Y_1, \dots, Y_n$, these turn out to be the positive hulls of $Y_1 - Y_2, \dots, Y_{n-1} - Y_n, Y_n$, respectively $Y_1 - Y_2, \dots, Y_{n-1} - Y_n$; see Section 5.1. For these positive hulls, we are able to compute the expected values of the size functionals $Y_{k,j}$, thus showing that the differences of exchangeable random variables also exhibit a distribution-free behavior.

Let us finally mention that it is possible to extend the results of the present paper to Weyl tessellations corresponding to the reflection groups of the product type $B_{n_1} \times \ldots \times B_{n_r} \times A_{k_1-1} \times \ldots \times A_{k_l-1}$. These arrangements of product type are just unions of the arrangements corresponding to the individual factors. In particular, Weyl tessellations of type B_1^n coincide with the tessellations studied by Cover and Efron [4] and Hug and Schneider [5]. Thus, their results become special cases of this more general setting. We refrain from stating the results in the product type setting since they require introducing heavy notation.

The rest of the paper is mostly devoted to the proofs.

2. Preliminaries

In this section we collect some notation and facts on polyhedral cones and integral geometry. Let σ_d, $d \in \mathbb{N}$, be the $(d-1)$-dimensional spherical Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$. The spherical content of $\mathbb{S}^{d-1}$ is given by

$$\omega_d := \sigma_{d-1}(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

2.1. Polyhedral cones.

A *polyhedral cone* (or, for simplicity, just a *cone*) $C \subseteq \mathbb{R}^d$ is a finite intersection of closed half-spaces whose boundaries pass through the origin. We denote the set of polyhedral cones in $\mathbb{R}^d$ by $\mathcal{PC}^d$. A supporting hyperplane for a cone C is a linear hyperplane H, such that C lies entirely in one of the closed half-spaces H^+ and H^- induced by H. If not explicitly stated otherwise, all hyperplanes are assumed to be linear.

A *face* of C is a set of the form $F = C \cap H$, for a supporting hyperplane H, or the cone C itself. We denote by $\mathcal{F}(C)$ the set of all faces of C and by $\mathcal{F}_k(C)$ the set of all k-dimensional faces of C. Note that the dimension of a cone C is defined as the dimension of its linear hull, i.e. $\dim C = \dim \operatorname{lin}(C)$. Let $f_k(C) = \#\mathcal{F}_k(C)$ be the number of k-faces of C. Equivalently, the faces of C are obtained by replacing some of the half-spaces, whose intersection defines the polyhedral cone, by their boundaries and taking the intersection.

Furthermore, let $\operatorname{linsp}(C) = C \cap (-C)$ denote the *lineality space* of C, which is the linear subspace contained in C and having the maximal possible dimension. Additionally, $\operatorname{linsp}(C)$ is contained in every face of C. A cone C is *pointed* if it does not contain a non-trivial linear subspace, i.e. if $\{0\}$ is a 0-dimensional face, or equivalently, if $\operatorname{linsp}(C) = \{0\}$.

2.2. Duality.

We will introduce the dual of a cone and state some useful results referring to [1, Section 2.1] for the proofs.

The *dual cone* of a cone $C \subseteq \mathbb{R}^d$ is defined as

$$C^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \le 0 \ \forall y \in C\}.$$

If $C = L$ is a linear subspace, then $C^\circ = L^\perp$ is its orthogonal complement. There is a one-to-one correspondence between the k-faces $\mathcal{F}_k(C)$ and the $(d-k)$-faces $\mathcal{F}_{d-k}(C)$. The following theorem is a conical version of the Hahn-Banach theorem.

Theorem 2.1 (Seperating hyperplane for cones)**.** *Let C, D be cones in $\mathbb{R}^d$. Then* $\operatorname{relint}(C) \cap \operatorname{relint}(D) = \emptyset$ *if and only if there exists a linear hyperplane H, not containing $C \cup D$, such that $C \subseteq H^+$ and $D \subseteq H^-$.*

Note that $\operatorname{relint} X$ denotes the interior of a set X relative to its linear hull and is called the *relative interior* of X. The theorem implies that $C^{\circ\circ} := (C^\circ)^\circ = C$, which was shown in Proposition 2.3 of [1]. The dual cone of the intersection of two cones C and D is given by

$$(C \cap D)^\circ = C^\circ + D^\circ, \tag{2.1}$$

where $C + D := \{x + y : x \in C,\, y \in D\}$ denotes the Minkowski sum. The following result is a variation of Farkas' Lemma for cones.

Lemma 2.2. *Let C, D be cones in $\mathbb{R}^d$. Then*

$$\operatorname{relint}(C) \cap D = \emptyset \Leftrightarrow C^\circ \cap -D^\circ \neq \{0\}.$$

In particular, if $D = L$ is a linear subspace, then

$$\operatorname{relint}(C) \cap L = \emptyset \Leftrightarrow C^\circ \cap L^\perp \neq \{0\}.$$

We are able to derive a related result, which will be of use in Section 3.5.

Lemma 2.3. *Let C be a cone and L be a subspace in $\mathbb{R}^d$. Then*

$$C \cap L \nsubseteq \operatorname{linsp}(C) \Leftrightarrow \operatorname{relint}(C^\circ) \cap L^\perp = \emptyset.$$

Proof. Suppose $\operatorname{relint}(C^\circ) \cap L^\perp \neq \emptyset$. Then, by Farkas' Lemma 2.2, $C \cap L = \{0\}$, which implies $C \cap L \subseteq \operatorname{linsp}(C)$.

To prove the other direction, assume $\operatorname{relint}(C^\circ) \cap L^\perp = \emptyset$. Note that $(\operatorname{linsp}(C))^\circ = \operatorname{lin}(C^\circ)$, since $\operatorname{linsp}(C)$ is the subspace contained in C of maximal dimension, and thus, $(\operatorname{linsp}(C))^\circ$ is the subspace of smallest dimension containing C°. Then, we also have

$$\operatorname{relint}(C^\circ) \cap (L^\perp \cap \operatorname{lin}(C^\circ)) = \emptyset.$$

Applying Theorem 2.1 in the ambient linear subspace $\operatorname{lin}(C^\circ)$, we can find a separating hyperplane H in $\operatorname{lin}(C^\circ)$, such that $C^\circ \subseteq H^-$ and $L^\perp \cap \operatorname{lin}(C^\circ) \subseteq H^+$, where H^-, H^+ denote the closed half-spaces in $\operatorname{lin}(C^\circ)$ defined by H. This implies that $L^\perp \cap \operatorname{lin}(C^\circ) \subseteq H \subseteq H^-$, since $L^\perp \cap \operatorname{lin}(C^\circ)$ is a linear subspace. It follows that

$$C^\circ + (L^\perp \cap \operatorname{lin}(C^\circ)) \subseteq H^-,$$

which implies $C^\circ + (L^\perp \cap \operatorname{lin}(C^\circ)) \nsupseteq \operatorname{lin}(C^\circ)$, and thus, $C^\circ + L^\perp \nsupseteq \operatorname{lin}(C^\circ)$. This is equivalent to $C \cap L \nsubseteq (\operatorname{lin}(C^\circ))^\circ = \operatorname{linsp}(C)$, due to (2.1). □

2.3. **Geometric functionals of convex cones.** We will introduce the geometric functionals for convex cones which we want to evaluate in Section 5. For general information regarding spherical integral geometry we refer to [16, Section 6.5]. At first, we will define the conical quermassintegrals and state some important properties. They are taken from [5, Section 2]. For $k \in \{0, \ldots, d\}$ denote by $G(d, k)$ the Grassmannian of k-dimensional linear subspaces in $\mathbb{R}^d$, and let ν_k be its normalized Haar measure, meaning the unique rotation invariant Borel probability measure on $G(d, k)$. Rotation invariance will always refer to the invariance with respect to the action of the special orthogonal group SO_d, which is the group of linear mappings $\vartheta : \mathbb{R}^d \to \mathbb{R}^d$ that preserve scalar product and orientation.

Definition 2.4. For a cone $C \subseteq \mathbb{R}^d$ that is not a linear subspace the *conical quermassintegrals* are defined by

$$U_j(C) = \frac{1}{2} \int_{G(d,d-j)} \mathbb{1}_{\{C \cap L \neq \{0\}\}}\, \nu_{d-j}(\mathrm{d}L), \quad j = 0, \ldots, d. \tag{2.2}$$

Hence, $2U_j(C)$ is the probability that a subspace, which is uniformly distributed on $G(d,k)$, intersects with C in a non-trivial way. By definition, if $L_k \subseteq \mathbb{R}^d$ is a k-dimensional linear subspace, then

$$U_j(L_k) = \begin{cases} 1 & \text{if } k-j>0 \text{ and odd,} \\ 0 & \text{if } k-j \le 0 \text{ or even.} \end{cases}$$

Note that the conical quermassintegrals may be defined in a unified way via the Euler characteristic, see [5, (1)]. For a d-dimensional cone $C \subset \mathbb{R}^d$, we have

$$U_{d-1}(C) = \frac{\sigma_{d-1}(C \cap \mathbb{S}^{d-1})}{\omega_d}.$$

If the cone C is not a linear subspace, then we have the duality relation

$$U_j(C) + U_{d-j}(C^\circ) = \frac{1}{2}, \qquad j = 0, \dots, d. \tag{2.3}$$

Now, we define the conical intrinsic volumes. The definition and further properties are taken from [1, Section 2.2] and [5, Section 2].

Definition 2.5. Let C be a polyhedral cone, and g be a d-dimensional standard Gaussian random vector. Then

$$\upsilon_k(C) := \sum_{F \in \mathcal{F}_k(C)} \upsilon_F(C)$$

defines the k-th *conical intrinsic volume* (or, for simplicity, just *intrinsic volume*) of C, where for a face $F \in \mathcal{F}(C)$, we put

$$\upsilon_F(C) := \mathbb{P}(\Pi_C(g) \in \operatorname{relint}(F)).$$

Here, Π_C denotes the orthogonal projection on C, that is $\Pi_C(x)$ is the vector in C minimizing the Euclidean distance to $x \in \mathbb{R}^d$.

Again, for a d-dimensional cone $C \subset \mathbb{R}^d$ we have

$$\upsilon_d(C) = \frac{\sigma_{d-1}(C \cap \mathbb{S}^{d-1})}{\omega_d}.$$

In this case, $\upsilon_d(C)$ is also called the *solid angle* of C, denoted by $\alpha(C)$.

The conical intrinsic volumes and quermassintegrals are essentially different functionals, yet there is a linear relation, which follows from a spherical integral-geometry formula of Crofton type (see [16, (6.63)]):

$$U_j(C) = \sum_{k=0}^{\lfloor \frac{d-1-j}{2} \rfloor} \upsilon_{j+2k+1}(C),$$

for a cone C and $j = 0, \dots, d-1$. This implies the relations

$$\begin{cases} \upsilon_j = U_{j-1} - U_{j+1} & \text{for } j = 1, \dots d-2, \\ \upsilon_{d-1} = U_{d-2}, \\ \upsilon_d = U_{d-1}. \end{cases} \tag{2.4}$$

Using (2.4) and the duality relation (2.3) for the quermassintegrals, we have

$$\upsilon_j(C) = \upsilon_{d-j}(C^\circ), \quad j = 0, \dots, d. \tag{2.5}$$

Following Hug and Schneider [5], we will use the conical quermassintegrals to define a more general series of functionals which comprises some interesting geometric functionals as special cases.

Definition 2.6. For a cone C, $k = 1, \ldots, d$ and $j = 0, \ldots, k-1$ define the *size functionals* $Y_{k,j}$ by

$$Y_{k,j}(C) := \sum_{F \in \mathcal{F}_k(C)} U_j(F).$$

If we set $k = \dim C$, we get the conical quermassintegrals

$$Y_{\dim C, j} = \sum_{F \in \mathcal{F}_{\dim C}(C)} U_j(F) = U_j(C), \quad j < \dim C.$$

Thus, using the relation (2.4), we obtain the conical intrinsic volumes as a suitable linear transformation of the size functionals. On the other hand, for $j = 0$ and $k = 1, \ldots, d$, the size functional yields the number of k-faces for a cone C whose k-faces are not linear subspaces, since

$$Y_{k,0}(C) = \sum_{F \in \mathcal{F}_k(C)} U_0(F) = \frac{1}{2} \sum_{F \in \mathcal{F}_k(C)} \mathbb{1}_{\{F \neq \{0\}\}} = \frac{1}{2} f_k(C). \tag{2.6}$$

2.4. **Results on general position.** Before proceeding to the Weyl tessellations, we need to introduce the definitions of *general position* in various contexts and state some results that will be used throughout this paper. For a vector $x \in \mathbb{R}^d \setminus \{0\}$, let

$$x^\perp = \{y \in \mathbb{R}^d : \langle y, x \rangle = 0\}, \quad x^- = \{y \in \mathbb{R}^d : \langle y, x \rangle \leq 0\}.$$

We will make use of the duality relation

$$(\operatorname{pos}\{x_1, \ldots, x_n\})^\circ = \bigcap_{i=1}^n x_i^-, \tag{2.7}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^d$, where $\operatorname{pos} X$ denotes the positive hull of a set X.

Definition 2.7. A set of vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ is said to be in *general position* if for any $k \leq d$ and $1 \leq i_1 < \ldots < i_k \leq n$ the set of vectors $x_{i_1}, \ldots, x_{i_k}$ is linearly independent. A set of hyperplanes $H_1, \ldots, H_n \in G(d, d-1)$ is said to be in *general position* if

$$\dim(H_{i_1} \cap \ldots \cap H_{i_k}) = d - k$$

for any $k \leq d$ and $1 \leq i_1 < \ldots < i_k \leq n$.

It is easy to see that $x_1, \ldots, x_n$ are in general position if and only if the hyperplanes $x_1^\perp, \ldots, x_n^\perp$ are in general position. If this is the case, then

$$\operatorname{pos}\{x_1, \ldots, x_n\} \neq \mathbb{R}^d \Leftrightarrow \bigcap_{i=1}^n x_i^- \neq \{0\} \Leftrightarrow \dim \bigcap_{i=1}^n x_i^- = d. \tag{2.8}$$

We refer to [5, (14)] for the proof of the equivalences.

A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^d$ is a finite set of distinct linear hyperplanes. The *rank* of the arrangement $\mathcal{A}$ is defined as

$$\operatorname{rank}(\mathcal{A}) = d - \dim\Big(\bigcap_{H \in \mathcal{A}} H \Big),$$

where $\operatorname{rank}(\emptyset) := 0$.

We already saw in the introduction that a hyperplane arrangement $\mathcal{A}$ induces a set of cones. Denote by $\mathcal{R}(\mathcal{A})$ the set of open connected components ("regions" or "chambers") of the complement $\mathbb{R}^d \setminus \bigcup_{H\in\mathcal{A}} H$ of the hyperplanes. Then, $\overline{\mathcal{R}}(\mathcal{A}) := \{\bar{R} : R \in \mathcal{R}(\mathcal{A})\}$ is the *conical tessellation* or the *conical mosaic* induced by $\mathcal{A}$, where $\bar{R}$ denotes the closure of R. The set of faces $\mathcal{F}(\overline{\mathcal{R}}(\mathcal{A}))$ of $\overline{\mathcal{R}}(\mathcal{A})$ is defined as the union of the sets of faces of the polyhedral cones $C \in \overline{\mathcal{R}}(\mathcal{A})$.

Thus, the conical mosaic $\overline{\mathcal{R}}(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$ consists precisely of the cones of the form

$$\bigcap_{H\in\mathcal{A}} \varepsilon_H H^-, \quad \varepsilon_H = \pm 1,$$

which have non-empty interior. Here, H^- denotes one of the closed half-spaces induced by H.

For a lot of results, e.g. results on the faces of a conical tessellation, we need the concept of general position of a linear subspace with respect to a hyperplane arrangement.

Definition 2.8. Let $L \in G(d,k)$ be a linear subspace of dimension $k \in \{0,\dots,d-1\}$. Then L is said to be in *general position* with respect to $\mathcal{A}$ if for all finite subsets $\mathcal{B} \subseteq \mathcal{A}$

$$\dim\left(\bigcap_{H\in\mathcal{B}} (H\cap L)\right) = \max\{0, k - \operatorname{rank}(\mathcal{B})\}. \tag{2.9}$$

If $L \in G(d,k)$ is in general with respect to $\mathcal{A} = \{H_1,\dots,H_n\}$, then it is easy to see that the induced hyperplane arrangement $\mathcal{A}|L := \{H\cap L : H \in \mathcal{A}\}$ also consists of distinct hyperplanes in L and thus induces a conical mosaic in L. And more importantly, the general position assumption provides that the subspaces $H\cap L$ are hyperplanes in L, i.e. they have dimension $k-1$. If $y_1,\dots,y_n$ are the normal vectors to the hyperplanes $H_1,\dots,H_n$, respectively, then their orthogonal projections on L, denoted by $\Pi_L(y_1),\dots,\Pi_L(y_n)$, are the normal vectors of $H_1\cap L,\dots,H_n\cap L$ inside L, respectively, since

$$\langle v, y_i\rangle = \langle v, y_i - \Pi_L(y_i)\rangle + \langle v, \Pi_L(y_i)\rangle = \langle v, \Pi_L(y_i)\rangle$$

holds for all $v \in L$.

Remark 2.9. It is important to note the following result, which follows from the definition above. If a hyperplane arrangement $\mathcal{A}$ consists of hyperplanes $H_1 = y_1^\perp,\dots,H_n = y_n^\perp$ in general position, then the fact that a linear subspace L is in general position to $\mathcal{A}$ implies that the induced hyperplanes $H_1\cap L,\dots,H_n\cap L$ are in general position in L, and thus, their respective normal vectors $\Pi_L(y_1),\dots,\Pi_L(y_n)$ are also in general position in L.

Lemma 2.10. *For a linear subspace $L \subseteq \mathbb{R}^d$ in general position with respect to a hyperplane arrangement $\mathcal{A}$, the closed chambers generated by the induced arrangement $\mathcal{A}|L$ are obtained by intersecting the closed chambers of $\overline{\mathcal{R}}(\mathcal{A})$ by L. Thus, we have*

$$\{C\cap L : C \in \overline{\mathcal{R}}(\mathcal{A}), C\cap L \neq \{0\}\} = \overline{\mathcal{R}}(\mathcal{A}|L).$$

Proof. Let $\mathcal{A} = \{H_1,\dots,H_n\}$. Denote by $y_1,\dots,y_n$ the normal vectors to hyperplanes $H_1,\dots,H_n$, respectively. Then for all $\varepsilon_1,\dots,\varepsilon_n \in \{\pm1\}^n$

$$\bigcap_{i=1}^n \varepsilon_i y_i^- \cap L = \bigcap_{i=1}^n \varepsilon_i \Pi_L(y_i)^-,$$

where $\Pi_L(y_i)^- = \{v \in L : \langle v, \Pi_L(y_i)\rangle \le 0\}$. Thus, if $C \in \overline{\mathcal{R}}(\mathcal{A})$ satisfies $C\cap L \neq \{0\}$, there are $\varepsilon_1,\dots,\varepsilon_n \in \{\pm1\}^n$, such that $C = \bigcap_{i=1}^n \varepsilon_1 y_i^-$. This means that $\{0\} \neq C\cap L = \bigcap_{i=1}^n \varepsilon_i \Pi_L(y_i)^- \in$

$\overline{\mathcal{R}}(\mathcal{A}|L)$. If otherwise $D \in \overline{\mathcal{R}}(\mathcal{A}|L)$, then $D = \bigcap_{i=1}^n \varepsilon_i \Pi_L(y_i)^-$ for some $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}^n$. It follows that $\{0\} \neq D = \bigcap_{i=1}^n \varepsilon_i y_i^- \cap L$, where $\bigcap_{i=1}^n \varepsilon_i y_i^-$ is obviously different from $\{0\}$ and therefore a cone in $\overline{\mathcal{R}}(\mathcal{A})$. □

Lemma 2.11. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$ and let L be a linear subspace in $\mathbb{R}^d$. If L is in general position with respect to $\mathcal{A}$, then for all $R \in \mathcal{R}(\mathcal{A})$*

$$\bar{R} \cap L \neq \{0\} \Leftrightarrow R \cap L \neq \emptyset,$$

or equivalently

$$C \cap L \neq \{0\} \Leftrightarrow \operatorname{relint}(C) \cap L \neq \emptyset$$

holds true for all $C \in \overline{\mathcal{R}}(\mathcal{A})$.

For the proof of this lemma, we refer to [12, Section 6.3]. A similar result can be proven for the faces in a conical tessellation.

Lemma 2.12. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^d$ and let L be a linear subspace in $\mathbb{R}^d$. If L is in general position with respect to $\mathcal{A}$, then for all faces $F \in \mathcal{F}(\overline{\mathcal{R}}(\mathcal{A}))$*

$$F \cap L \neq \{0\} \Leftrightarrow \operatorname{relint}(F) \cap L \neq \emptyset.$$

Proof. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$. We define $G(i) := H_{i_1} \cap \ldots \cap H_{i_k}$ for all $k \leq d$ and $1 \leq i_1 < \ldots < i_k \leq n$. Then, the linear subspace $L \cap G(i)$ is in general position with respect to the hyperplane arrangement $\{H_j \cap G(i) : j \notin \{i_1, \ldots, i_k\}\}$ in $G(i)$. This is a direct consequence of the definition of general position in (2.9). Since every face $F \in \mathcal{F}(\overline{\mathcal{R}}(\mathcal{A}))$ is contained in such a subspace $G(i)$ for a suitable collection of indices and and is furthermore a cone of the induced tessellation $\overline{\mathcal{R}}(\mathcal{A}|G(i))$, Lemma 2.11 applied to the linear subspace $G(i)$ yields

$$F \cap L \neq \{0\} \Leftrightarrow F \cap (L \cap G(i)) \neq \{0\} \Leftrightarrow \operatorname{relint}(F) \cap (L \cap G(i)) \neq \emptyset \Leftrightarrow \operatorname{relint}(F) \cap L \neq \emptyset,$$

which completes the proof. □

Similarly, we can define the notion of general position for two arbitrary linear subspaces.

Definition 2.13. Linear subspaces L, L' of $\mathbb{R}^d$ are in *general position* if

$$\dim(L \cap L') = \max\{0, \dim L + \dim L' - d\}.$$

This definition implies that a linear subspace L is in general position to a hyperplane arrangement $\mathcal{A}$ if and only if L is in general position to each subspace K that can be represented as an intersection of the hyperplanes from $\mathcal{A}$.

Lemma 2.14. *Let $k \in \{0, \ldots, d\}$ and let $M \subseteq \mathbb{R}^d$ be a linear subspace. Define B as the set of all $L \in G(d,k)$, for which L and M are not in general position. Then $\nu_k(B) = 0$.*

Proof. The result follows from [16, Lemma 13.2.1], since $\nu_k = \nu \circ \beta_k^{-1}$ is defined as the image measure of ν under the mapping $\beta_k : SO_d \to G(d,k)$, $\vartheta \mapsto \vartheta L_k$ for a fixed linear subspace $L_k \in G(d,k)$. Note that ν denotes the unique rotation invariant probability measure on the rotation group SO_d. Thus

$$\nu_k(B) = \nu\big(\{\vartheta \in SO_d : M \text{ is not in general position to } \vartheta L_k\big) = 0.$$

□

Remark 2.15. Let L be a random k-dimensional subspace with distribution ν_k and let $\mathcal{A}$ be a hyperplane arrangement. Then L is a.s. in general position with respect to $\mathcal{A}$. This is a direct consequence of Lemma 2.14 and (2.9).

3. Conical tessellations and Weyl chambers of type B_n

In this section, we introduce the Weyl chambers in $\mathbb{R}^n$ of type B_n and a conical tessellation of $\mathbb{R}^d$, which is closely related to it. Our main result is a formula on the expected k-face number of a cone chosen uniformly at random from this tessellation. We will state it in Theorem 3.3 and Corollary 3.5. In the present section, we always assume $n \geq d$.

3.1. **The reflection arrangement and Weyl chambers of type B_n.**

At first, we consider Weyl chambers of type B_n and introduce the necessary notation, taken from [12, Section 2.1]. We call $\mathcal{G}(B_n)$ the *reflection group* of type B_n which acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way and by multiplying any number of coordinates by -1. This means that the $2^n n!$ elements of $\mathcal{G}(B_n)$ are the linear mappings

$$g_{\varepsilon,\sigma} : \mathbb{R}^n \to \mathbb{R}^n, \quad (\beta_1, \ldots, \beta_n) \mapsto (\varepsilon_1\beta_{\sigma(1)}, \ldots, \varepsilon_n\beta_{\sigma(n)}),$$

where $\sigma \in \mathcal{S}_n$ is a permutation of the set $\{1, \ldots, n\}$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$.

The *closed Weyl chambers of type B_n* are the cones of the conical tessellation induced by the hyperplane arrangement $\mathcal{A}(B_n)$, which consists of the hyperplanes given by

$$\begin{aligned} &\{\beta \in \mathbb{R}^n : \beta_i = \beta_j\} \quad (1 \leq i < j \leq n),\\ &\{\beta \in \mathbb{R}^n : \beta_i = -\beta_j\} \quad (1 \leq i < j \leq n),\\ &\{\beta \in \mathbb{R}^n : \beta_i = 0\} \quad (1 \leq i \leq n). \end{aligned} \tag{3.1}$$

It is called the *reflection arrangement of type B_n*. The name is due to the fact that reflections with respect to the hyperplanes of this arrangement generate the group $\mathcal{G}(B_n)$. Thus, it is easy to see that the closed Weyl chambers of type B_n are given by

$$C^B_{\varepsilon,\sigma} := \{(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \varepsilon_1\beta_{\sigma(1)} \leq \ldots \leq \varepsilon_n\beta_{\sigma(n)} \leq 0\}, \quad \sigma \in \mathcal{S}_n,\, \varepsilon \in \{\pm 1\}^n.$$

The superscript B indicates the type of the Weyl chamber. Equivalently, the Weyl chambers of type B_n are defined as the reflections $g\,\mathcal{C}(B_n)$, $g \in \mathcal{G}(B_n)$, of the *fundamental Weyl chamber of type B_n* given by

$$\mathcal{C}(B_n) = \{(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : 0 \leq \beta_1 \leq \ldots \leq \beta_n\}.$$

The k-dimensional faces of the Weyl chamber $C^B_{\varepsilon,\sigma}$ are determined by the collection of indices $1 \leq l_1 < \ldots < l_k \leq n$ and have the form

$$\begin{aligned} C^B_{\varepsilon,\sigma}(l_1, \ldots, l_k) := \{\beta \in \mathbb{R}^n : \ &\varepsilon_1\beta_{\sigma(1)} = \ldots = \varepsilon_{l_1}\beta_{\sigma(l_1)} \leq \varepsilon_{l_1+1}\beta_{\sigma(l_1+1)} = \ldots = \varepsilon_{l_2}\beta_{\sigma(l_2)}\\ &\leq \ldots \leq \varepsilon_{l_{k-1}+1}\beta_{\sigma(l_{k-1}+1)} = \ldots = \varepsilon_{l_k}\beta_{\sigma(l_k)}\\ &\leq \beta_{\sigma(l_k+1)} = \ldots = \beta_{\sigma(n)} = 0\}. \end{aligned} \tag{3.2}$$

In the case $i_k = n$, no β_i's are required to be 0. Thus, $\#\mathcal{F}_k(C^B_{\varepsilon,\sigma}) = \binom{n}{k}$.

3.2. **Weyl tessellation of type B_n.**

As mentioned in the introduction, we can define a conical tessellation, which is closely related to the Weyl chambers of type B_n.

Definition 3.1. (Weyl tessellation of type B_n) Let $y_1, \ldots, y_n \in \mathbb{R}^d$ and let the hyperplane arrangement $\mathcal{A}^B(y_1, \ldots, y_n)$ consists of the hyperplanes given by

$$\begin{aligned} (y_i + y_j)^\perp, &\quad 1 \leq i < j \leq n,\\ (y_i - y_j)^\perp, &\quad 1 \leq i < j \leq n,\\ y_i^\perp, &\quad 1 \leq i \leq n. \end{aligned} \tag{3.3}$$

Then the *Weyl tessellation of type* B_n or *Weyl mosaic of type* B_n is defined as the conical tessellation induced by $\mathcal{A}^B(y_1,\dots,y_n)$ and is denoted by $\mathcal{W}^B(y_1,\dots,y_n)$. We denote the number of cones in $\mathcal{W}^B(y_1,\dots,y_n)$ by

$$D^B(n,d) := \#\mathcal{W}^B(y_1,\dots,y_n).$$

The set of k-faces of $\mathcal{W}^B(y_1,\dots,y_n)$ is denoted by

$$\mathcal{F}_k^B(y_1,\dots,y_n) = \bigcup_{C\in\mathcal{W}^B(y_1,\dots,y_n)} \mathcal{F}_k(C), \quad \mathcal{F}_d^B(y_1,\dots,y_n) = \mathcal{W}^B(y_1,\dots,y_n).$$

As mentioned in the introduction, we did not bother to indicate that $D^B(n,d)$ depends on the choice of $y_1,\dots,y_n$, because we will show in Corollary 3.4 that it is constant under certain conditions on $y_1,\dots,y_n$. It is easy to see that the cones in $\mathcal{W}^B(y_1,\dots,y_n)$ are the cones different from $\{0\}$ of the form

$$D^B_{\epsilon,\sigma} := \{v\in\mathbb{R}^d : \langle v,\varepsilon_1 y_{\sigma(1)}\rangle \le \dots \le \langle v,\varepsilon_n y_{\sigma(n)}\rangle \le 0\}, \quad \varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n, \sigma\in\mathcal{S}_n.$$

We will refer to these cones as *Weyl cones of type* B_n or just *Weyl cones* when the type we are referring to is obvious from the context.

The initial problem we encounter is that the hyperplanes of $\mathcal{A}^B(y_1,\dots,y_n)$ are not in general position even if $y_1,\dots,y_n$ are in general position. Thus, without further conditions on $y_1,\dots,y_n$, we cannot say with certainty that $\mathcal{A}^B(y_1,\dots,y_n)$ is even a hyperplane arrangement, i.e. the hyperplanes are distinct. Yet, we may formulate two equivalent conditions, under which we are able derive general results on the faces and the number of cones in the Weyl tessellation of type B_n.

Theorem 3.2. *For arbitrary $y_1,\dots,y_n\in\mathbb{R}^d$ the following conditions (B1) and (B2) are equivalent:*

(B1) For every $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$ the vectors $\varepsilon_1 y_{\sigma(1)}-\varepsilon_2 y_{\sigma(2)}, \varepsilon_2 y_{\sigma(2)}-\varepsilon_3 y_{\sigma(3)},\dots,\varepsilon_{n-1}y_{\sigma(n-1)}-\varepsilon_n y_{\sigma(n)}, \varepsilon_n y_{\sigma(n)}$ are in general position.

(B2) The linear subspace $L^\perp$ has dimension d and is in general position with respect to the hyperplane arrangement $\mathcal{A}(B_n)$, where $L:=\{\beta\in\mathbb{R}^n : \beta_1y_1+\dots+\beta_ny_n=0\}$.

We will often refer to these conditions as the *general position assumptions* (B1) and (B2). It is not obvious that the conditions (B1) and (B2) are equivalent and the proof will be postponed to Section 6. In some cases it is more natural to use condition (B1) and sometimes it will be more convenient to use (B2).

Now, we state the main result of this section.

Theorem 3.3. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (B1) or (B2). For $1\le k\le d$, we have*

$$\sum_{F\in\mathcal{F}_k^B(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}_d^B(y_1,\dots,y_n)} \mathbb{1}_{\{F\subseteq C\}}$$
$$= 2^{d-k+1}\binom{n}{d-k}\frac{n!}{(n-d+k)!}(B(n-d+k,n-d+1)+B(n-d+k,n-d+3)+\dots).$$

Recall that the $B(n,k)$'s are the coefficients of the polynomial

$$(t+1)(t+3)\cdot\ldots\cdot(t+2n-1) = \sum_{k=0}^n B(n,k)t^k \tag{3.4}$$

and, by convention, $B(n,k)=0$ for $k\notin\{0,\dots n\}$.

We will postpone the proof of Theorem 3.3 to Section 3.5 because we need to establish some results on the faces of the Weyl tessellation first. As a special case of this theorem, we are able to derive the number of cones in the Weyl tessellation $\mathcal{W}^B(y_1,\dots,y_n)$ of type B_n under one of the equivalent conditions (B1) or (B2). We introduced this result as Theorem 1.1 in the introduction and will now restate it here as a corollary of Theorem 3.3. Note that the condition on $y_1,\dots,y_n$ stated in Theorem 1.1 coincides with the general position assumptions (B1).

Corollary 3.4. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (B1) or (B2). Then the number of cones in the Weyl mosaic $\mathcal{W}^B(y_1,\dots,y_n)$ of type B_n is given by*

$$D^B(n,d)=2\big(B(n,n-d+1)+B(n,n-d+3)+\dots\big).$$

Proof. This follows from Theorem 3.3 in the special case $k=d$. □

Note that under the assumptions of Theorem 3.3, we may also write

$$\sum_{F\in\mathcal{F}_k^B(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}_d^B(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}=2^{d-k}\binom{n}{d-k}\frac{n!}{(n-d+k)!}D^B(n-d+k,k).$$

It is evident that Theorem 3.3 also carries the following probabilistic meaning.

Corollary 3.5. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (B1) or (B2). Let Q^B be sampled randomly and uniformly among the $D^B(n,d)$ cones of $\mathcal{W}^B(y_1,\dots,y_n)$. Then the expected number of k-dimensional faces of Q^B is given by*

$$\begin{aligned}\mathbb{E}\,f_k(Q^B)&=\frac{1}{D^B(n,d)}\sum_{C\in\mathcal{F}_d^B(y_1,\dots,y_n)}\ \sum_{F\in\mathcal{F}_k^B(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}\\&=2^{d-k}\binom{n}{d-k}\frac{n!}{(n-d+k)!}\frac{D^B(n-d+k,k)}{D^B(n,d)}.\end{aligned}$$

3.3. **Characterizing the faces of the Weyl tessellation of type B_n.** Before we are able to prove Theorem 3.3, it is necessary to consider the faces of the Weyl tessellation more closely. At first, we introduce helpful notation. For $y_1,\dots,y_n\in\mathbb{R}^d$, a collection of indices $1\le l_1<\dots<l_{n-d+k}\le n$, a vector of signs $\varepsilon\in\{\pm1\}^n$ and a permutation $\sigma\in\mathcal{S}_n$, we define

$$\begin{aligned}F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k}):=\{v\in\mathbb{R}^d:{}&\varepsilon_1f_{\sigma(1)}=\dots=\varepsilon_{l_1}f_{\sigma(l_1)}\le\varepsilon_{l_1+1}f_{\sigma(l_1+1)}=\dots=\varepsilon_{l_2}f_{\sigma(l_2)}\\&\le\dots\le\varepsilon_{l_{n-d+k-1}+1}f_{\sigma(l_{n-d+k-1}+1)}=\dots=\varepsilon_{l_{n-d+k}}f_{\sigma(l_{n-d+k})}\\&\le f_{\sigma(l_{n-d+k}+1)}=\dots=f_{\sigma(n)}=0\},\end{aligned}\tag{3.5}$$

where the functionals f_i are defined by $f_i=f_i(v):=\langle v,y_i\rangle$, $i=1,\dots,n$. If $l_{n-d+k}=n$, no f_i's are required to be 0. These cones will represent the k-faces of the Weyl tessellation, according to the following proposition.

Proposition 3.6. *Let $1\le k\le d$ and let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy the general position assumption (B1). Then it holds:*

(i) For every $F\in\mathcal{F}_k^B(y_1,\dots,y_n)$ there exist a collection of indices $1\le l_1<\dots<l_{n-d+k}\le n$, a vector of signs $\varepsilon\in\{\pm1\}^n$ and a permutation $\sigma\in\mathcal{S}_n$, such that $F=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})$.

(ii) Let $1\le l_1<\dots<l_{n-d+k}\le n$ and $\varepsilon\in\{\pm1\}^n$, $\sigma\in\mathcal{S}_n$. If $F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})\neq\{0\}$, then $F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})\in\mathcal{F}_k^B(y_1,\dots,y_n)$.

Proof. We start by proving (i). Let $F \in \mathcal{F}_k^B(y_1, \dots, y_n)$. Then there exists a Weyl cone $D \in \mathcal{F}_d^B(y_1, \dots, y_n)$, such that $D = D_{\varepsilon,\sigma}^B$ and $F \in \mathcal{F}_k^B(D)$ for suitable $\varepsilon \in \{\pm 1\}^n$ and $\sigma \in \mathcal{S}_n$. Every face of a polyhedral cone is obtained by replacing some of the half-spaces, whose intersection defines the cone, by their boundaries, or in this case equivalently, replacing some of the inequalities in the defining condition of $D_{\varepsilon,\sigma}^B$ by equalities. Thus there exist a number $1 \le m \le n$ and a collection of indices $1 \le l_1 < \dots < l_m \le n$, such that $F = F_{\varepsilon,\sigma}^B(l_1, \dots, l_m)$. It is left to show that $m = n - d + k$. We have

$$F \subseteq L_m := \{v \in \mathbb{R}^d : \varepsilon_1 f_{\sigma(1)} = \dots = \varepsilon_{l_1} f_{\sigma(l_1)}, \dots, \varepsilon_{l_{m-1}+1} f_{\sigma(l_{m-1}+1)} = \dots = \varepsilon_{l_m} f_{\sigma(l_m)},$$
$$f_{\sigma(l_m+1)} = \dots = f_{\sigma(n)} = 0\}.$$

Note that the condition which defines L_m effectively consists of $n - m$ equations. Thus, the general position assumption (B1) implies $\dim(L_m) = \max\{0, d - n + m\}$. Since $\dim F = k \ge 1$, it is obvious that $\dim L_m = d - n + m$. Now, we want to show that the dimensions of F and L_m are equal. Due to (B1), L_m is in general position to the arrangement

$$\mathcal{A}_m := \left\{ \left(\varepsilon_{l_1} y_{\sigma(l_1)} - \varepsilon_{l_1+1} y_{\sigma(l_1+1)}\right)^\perp, \dots, \left(\varepsilon_{l_m} y_{\sigma(l_m)} - \varepsilon_{l_m+1} y_{\sigma(l_m+1)}\right)^\perp \right\},$$

where $\varepsilon_{l_m+1} = 0$ for $l_m = n$, and additionally, the hyperplanes of $\mathcal{A}_m$ are itself in general position. Thus, the hyperplanes in the induced arrangement $\mathcal{A}_m | L_m = \{H \cap L_m : H \in \mathcal{A}_m\}$ are in general position in L_m and generate a mosaic of $(d - n + m)$-dimensional cones in L_m, following (2.8). Since the cones in the induced mosaic are obtained by intersecting the cones of $\overline{\mathcal{R}}(\mathcal{A}_m)$ with L_m and $F = D \cap L_m$ is such a cone, $F \ne \{0\}$ implies that $\dim F = d - n + m$. On the other hand, F is a k-dimensional face, thus $k = d - n + m$ holds true.

The proof of (ii) is similar. Obviously $F_{\varepsilon,\sigma}^B(l_1, \dots, l_{n-d+k})$ is a face of the Weyl tessellation $\mathcal{W}^B(y_1, \dots, y_n)$ for all $1 \le l_1 < \dots < l_{n-d+k} \le n$ and $\varepsilon \in \{\pm 1\}^n$, $\sigma \in \mathcal{S}_n$. If not $\{0\}$, then it is already a k-dimensional face, due to the general position arguments we stated above. $\square$

To conclude this section, we want to evaluate the number of Weyl cones $C \in \mathcal{W}^B(y_1, \dots, y_n)$ that contain a k-face $F = F_{\varepsilon,\sigma}^B(l_1, \dots, l_{n-d+k}) \in \mathcal{F}_k^B(y_1, \dots, y_n)$, which will be necessary for the proof of Theorem 3.3. In order to avoid heavy notation, we consider an example from which the general case should become evident.

Example 3.7. Consider the case $n = 7$, $d = 6$, $k = 2$ and the face of the Weyl tessellation $\mathcal{W}^B(y_1, \dots, y_7)$ in $\mathbb{R}^6$ given by

$$F = \{v \in \mathbb{R}^6 : \underbrace{-f_5 = f_2}_{\text{group 1}} \le \underbrace{-f_3}_{\text{group 2}} \le \underbrace{f_4 = f_1}_{\text{group 3}} \le \underbrace{f_6 = f_7 = 0}_{\text{group 4}}\}.$$

Assume that $F \ne \{0\}$. Under assumption (B1), the cone F is a 2-dimensional face of the Weyl tessellation of type B_7, due to Proposition 3.6. In particular, it is a 2-face of the Weyl cone

$$\{v \in \mathbb{R}^6 : -f_5 \le f_2 \le -f_3 \le f_4 \le f_1 \le f_6 \le f_7 \le 0\}. \tag{3.6}$$

However, it is also a 2-face of

$$\{v \in \mathbb{R}^6 : f_2 \le -f_5 \le -f_3 \le f_4 \le f_1 \le -f_7 \le f_6 \le 0\}$$

and, more generally any number of cones obtained from (3.6) by permuting the f's inside the groups $(-f_5, f_2), (-f_3), (f_4, f_1), (f_6, f_7)$, and by changing any number of signs in the *last* group. The total number of cones obtained in this way is $2!1!2!2!2^2$.

The question arises, if there are any other Weyl cones that contain F. We will show that the answer is "no". Indeed, if we change the sign of any f_i that is not in the last group (for example, f_1), we see that F is not contained in this cone any longer, e.g.

$$F \nsubseteq \{v \in \mathbb{R}^6 : -f_5 \leq f_2 \leq -f_3 \leq f_4 \leq -f_1 \leq f_6 \leq f_7 \leq 0\}.$$

Otherwise, F would be contained in the hyperplanes $\{f_1 \leq 0\}$ and $\{-f_1 \leq 0\}$, and thus, $F \subseteq \{f_1 = 0\}$. This implies

$$F = \{v \in \mathbb{R}^d : -f_5 = f_2 \leq -f_3 \leq f_4 = f_1 = f_6 = f_7 = 0\}.$$

The number of groups in this representation is strictly smaller than in the original one because f_1 was not in the last group. In fact, the cone on the right-hand side is a 1-face of the Weyl tessellation, due to Proposition 3.6 under the general position assumption (B1). This means that any cone where we altered a sign of any f_i, which is not in the last group, does not contain F.

From now on, we can consider only the Weyl cones in whose representations the signs of all f_i's are the same as in the original representation of F, except for the f_i's in the last group. Take such a cone and assume that in its representation we have an inequality $\pm f_j \leq \pm f_i$, while in the representation of F we have the converse inequality $\pm f_i \leq \pm f_j$. For example take the cone

$$\{v \in \mathbb{R}^d : -f_5 = f_2 \leq -f_3 \leq f_4 = f_6 \leq f_1 = f_7 = 0\}$$

which satisfies $f_6 \leq f_1$, while in the representation of F we have $f_1 \leq f_6$. We claim that F is not contained in this cone. Indeed, otherwise, F would be contained in the hyperplane $\{f_1 = f_6\}$, which implies that

$$F = \{v \in \mathbb{R}^6 : -f_5 = f_2 \leq -f_3 \leq f_4 = f_1 = f_6 = f_7 = 0\}.$$

Again, the number of groups in this representation is strictly smaller than in the original representation of F. In fact, the cone on the right-hand side is a 1-face, similar to the previous case. That is a contradiction, since F is 2-dimensional. This means that there are no other Weyl cones containing F. Generalizing this argument, yields the following proposition.

Proposition 3.8. *Let $1 \leq k \leq d$ and let $y_1, \dots, y_n \in \mathbb{R}^d$ satisfy the general position assumption (B1). Then, each k-face $F^B_{\varepsilon,\sigma}(l_1, \dots, l_{n-d+k}) \in \mathcal{F}^B_k(y_1, \dots, y_n)$ belongs to exactly*

$$l_1!(l_2 - l_1)! \cdot \ldots \cdot (n - l_{n-d+k})! 2^{n - l_{n-d+k}}$$

cones $C \in \mathcal{W}^B(y_1, \dots, y_n)$.

The same argument proves also the following proposition stating that there is a one-to-one correspondence between the k-faces of the Weyl mosaic and those combinatorial representations leading to a non-trivial face.

Proposition 3.9. *Let $y_1, \dots, y_n \in \mathbb{R}^d$ satisfy the general position assumption (B1). Let furthermore $F = F^B_{\varepsilon,\sigma}(l_1, \dots, l_{n-d+k})$ and $G = F^B_{\delta,\pi}(i_1, \dots, i_{n-d+m})$ be such that $F \neq \{0\}$, with some $\varepsilon, \delta \in \{\pm 1\}^n$, $\sigma, \pi \in \mathcal{S}_n$ and $1 \leq k \leq d$, $1 \leq m \leq d$ and $1 \leq l_1 < \ldots < l_{n-d+k} \leq n$, $1 \leq i_1 < \ldots < i_{n-d+m} \leq n$. If $F = G$, then $\varepsilon = \delta$, $\sigma = \pi$, $k = m$, and $l_j = i_j$ for all admissible j.*

3.4. **Counting the faces of Weyl tessellations of type B_n.** Now, the question arises if we can evaluate the total number of k-faces in the Weyl tessellation $\mathcal{W}^B(y_1, \dots, y_n)$.

Theorem 3.10. *Let $y_1, \dots, y_n \in \mathbb{R}^d$ satisfy one of the equivalent general position assumptions (B1) or (B2). Then the number of k-faces in the Weyl mosaic of type B_n is given by*

$$\#\mathcal{F}^B_k(y_1, \dots, y_n) = T(n, n-d+k) D^B(n-d+k, k)$$

for $k \in \{1, \dots, d\}$, where the $T(n,k)$'s are given by

$$T(n,k) = \sum_{r=0}^{n-k} \binom{n}{r} \left\{ {n-r \atop k} \right\} 2^{n-r-k}.$$

Here, $\left\{ {n \atop k} \right\}$ is the Stirling number of second kind, that is the number of partitions of an n-element set into k non-empty subsets.

Remark 3.11. The numbers $T(n,k)$ are known as the B-analogons of Stirling numbers of the second kind [18] and appear as Entry A039755 in [17]. The A-analogons of the Stirling numbers of the second kind are the usual Stirling numbers $\left\{ {n \atop k} \right\}$. The generating functions of both sequences are given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} T(n,k) \frac{x^n}{n!} y^k = e^x \exp\Big(\frac{y}{2}\big(e^{2x}-1\big)\Big), \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{x^n}{n!} y^k = \exp\Big(y\big(e^x-1\big)\Big).$$

The numbers $T(n,k)$ and $\left\{ {n \atop k} \right\}$ appear as the coefficients in the formulas

$$t^n = \sum_{k=0}^{n} (-1)^{n-k} T(n,k)(t+1)(t+3)\dots(t+2k-1), \quad t^n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} t(t+1)(t+2)\dots(t+k-1);$$

see Entry A039755 in [17]. This should be compared to the following formulas for the Stirling numbers of the first kind $\left[{n \atop k} \right]$ and their B-analogues $B(n,k)$:

$$(t+1)(t+3)\cdot \ldots \cdot (t+2n-1) = \sum_{k=0}^{n} B(n,k) t^k, \qquad t(t+1)\cdot \ldots \cdot (t+n-1) = \sum_{k=1}^{n} \left[{n \atop k} \right] t^k.$$

Proof of Theorem 3.10. Due to Proposition 3.6(i), each k-face $F \in \mathcal{F}_k^B(y_1, \dots, y_n)$ is contained in a k-dimensional linear subspace of the form

$$\begin{aligned} L(l, \varepsilon, \sigma) := \{v \in \mathbb{R}^d : \;& \varepsilon_1 f_{\sigma(1)} = \ldots = \varepsilon_{l_1} f_{\sigma(l_1)}, \dots, \\ & \varepsilon_{l_{n-d+k-1}+1} f_{\sigma(l_{n-d+k-1}+1)} = \ldots = \varepsilon_{l_{n-d+k}} f_{\sigma(l_{n-d+k})}, \\ & f_{\sigma(l_{n-d+k}+1)} = \ldots = f_{\sigma(n)} = 0\} \end{aligned}$$

for $1 \le l_1 < \ldots < l_{n-d+k} \le n$, $l = (l_1, \dots, l_{n-d+k})$, $\varepsilon \in \{\pm 1\}^n$ and $\sigma \in \mathcal{S}_n$. At first, we want to evaluate the number of distinct subspaces of this form. For each fixed number of elements $r \in \{0, \dots, d-k\}$ in the last group of equations there are $\binom{n}{r}$ possibilities to choose its elements among $\{f_1, \dots, f_n\}$. Then, we are left with a set of $n-r$ elements, which we want to partition in $n-d+k$ non-empty sets. Thus, there are $\left\{ {n-r \atop n-d+k} \right\}$ possibilities to choose the partition. Furthermore, we can choose the signs of the f_i's in the first $n-d+k$ groups arbitrarily, for which there are 2^{n-r} possibilities. But since we obtain the same subspace if we multiply any group of equations by -1, we have to divide the 2^{n-r} possibilities by 2^{n-d+k}. This yields a total of

$$\sum_{r=0}^{d-k} \binom{n}{r} \left\{ {n-r \atop n-d+k} \right\} \frac{2^{n-r}}{2^{n-d+k}} = \sum_{r=0}^{d-k} \binom{n}{r} \left\{ {n-r \atop n-d+k} \right\} 2^{d-k-r}$$

possible subspaces of the form $L(l, \varepsilon, \sigma)$. All these subspaces are pairwise different, which can be shown in the same way as in Example 3.7 and relies on the general position assumption (B1).

Now, we want to show that the k-faces of $\mathcal{W}^B(y_1,\dots,y_n)$ contained in $L(l,\varepsilon,\sigma)$ form a Weyl mosaic in $L(l,\varepsilon,\sigma)$ and that the number of these k-faces is $D^B(n-d+k,k)$. To simplify the notation, we consider the special case $\varepsilon_i=1$ and $\sigma(i)=i$, for all $i=1,\dots,n$, and define

$$L:=\{v\in\mathbb{R}^d: f_1=\dots=f_{l_1},\dots,f_{l_{n-d+k-1}+1}=\ldots=f_{l_{n-d+k}}, f_{l_{n-d+k}+1}=\ldots=f_n=0\}.$$

Now, we show that the projections $\Pi_L(y_{l_1}),\dots,\Pi_L(y_{l_{n-d+k}})$ satisfy the general position assumption (B1) in L, and thus, induce a mosaic of type B_n in L. Corollary 3.4 would then imply that the number of these k-dimensional cones is $D^B(n-d+k,k)$. Take some $\delta\in\{\pm1\}^{n-d+k}$ and $\pi\in\mathcal{S}_{n-d+k}$. Due to (B1), the hyperplanes

$$\begin{aligned}&\big\{(y_1-y_2)^\perp,\dots,(y_{l_1-1}-y_{l_1})^\perp,\dots,(y_{l_{n-d+k-1}+1}-y_{l_{n-d+k-1}+2})^\perp,\dots,(y_{l_{n-d+k}-1}-y_{l_{n-d+k}})^\perp,\\&(y_{l_{n-d+k}+1}-y_{l_{n-d+k}+2})^\perp,\dots,(y_{n-1}-y_n)^\perp,y_n^\perp,\\&(\delta_1y_{l_{\pi(1)}}-\delta_2y_{l_{\pi(2)}})^\perp,\dots,(\delta_{n-d+k}y_{l_{\pi(n-d+k)}}-\delta_{n-d+k+1}y_{l_{\pi(n-d+k+1)}})^\perp\big\}\end{aligned}$$

are in general position. Therefore, L (being the intersection of the hyperplanes in the first two lines) is in general position with the hyperplanes constituting the third line. Thus, by Remark 2.9, the hyperplanes in L given by

$$L\cap(\delta_1y_{l_{\pi(1)}}-\delta_2y_{l_{\pi(2)}})^\perp,\dots,L\cap(\delta_{n-d+k}y_{l_{\pi(n-d+k)}}-\delta_{n-d+k+1}y_{l_{\pi(n-d+k+1)}})^\perp$$

are in general position in L. It follows from the definition of L that the representation of the last hyperplane in this list can be simplified as follows

$$L\cap(\delta_{n-d+k}y_{l_{\pi(n-d+k)}}-\delta_{n-d+k+1}y_{l_{\pi(n-d+k+1)}})^\perp=L\cap(\delta_{n-d+k}y_{l_{\pi(n-d+k)}})^\perp.$$

It follows that the normal vectors of the hyperplanes from the above list (taken inside L) are in general positions. So,

$$\delta_1\Pi_L(y_{l_{\pi(1)}})-\delta_2\Pi_L(y_{l_{\pi(2)}}),\dots,\delta_{n-d+k-1}\Pi_L(y_{l_{\pi(n-d+k-1)}})-\delta_{n-d+k}\Pi_L(y_{l_{\pi(n-d+k)}}),\\ \delta_{n-d+k}\Pi_L(y_{l_{\pi(n-d+k)}})$$

are in general position in L, for all $\delta\in\{\pm1\}^{n-d+k}$ and $\pi\in\mathcal{S}_{n-d+k}$, which proves (B1) for the projected vectors $\Pi_L(y_{l_1}),\dots,\Pi_L(y_{l_{n-d+k}})$.

Thus, the orthogonal complements of the vectors $\Pi_L(y_{l_1}),\dots,\Pi_L(y_{l_{n-d+k}})$ induce in the k-dimensional linear space L a Weyl mosaic consisting of $D^B(n-d+k,k)$ k-dimensional cones. These are the cones different from $\{0\}$ of the form

$$\begin{aligned}&\{v\in L:\delta_1\langle v,\Pi_L(y_{l_{\pi(1)}})\rangle\le\ldots\le\delta_{n-d+k}\langle v,\Pi_L(y_{l_{\pi(n-d+k)}})\rangle\le0\}\\&\quad=\{v\in L:\delta_1f_{l_{\pi(1)}}\le\ldots\le\delta_{n-d+k}f_{l_{\pi(n-d+k)}}\le0\}\\&\quad=\{v\in\mathbb{R}^d:\delta_1f_{l_{\pi(1)-1}+1}=\ldots=\delta_1f_{l_{\pi(1)}}\le\delta_2f_{l_{\pi(2)-1}+1}=\ldots=\delta_2f_{l_{\pi(2)}}\\&\qquad\qquad\le\ldots\le\delta_{n-d+k}f_{l_{\pi(n-d+k)-1}+1}=\ldots=\delta_{n-d+k}f_{l_{\pi(n-d+k)}}\le f_{l_{n-d+k}+1}=\ldots=f_n=0\},\end{aligned}$$

where in the first equality we used that $\langle v,\Pi_L(y_{l_{\pi(i)}})\rangle=\langle v,y_{l_{\pi(i)}}\rangle$, for all $v\in L$ and $i=1,\dots,n-d+k$. The second equality follows from the definition of L. The last representation basically says that we keep the groups of equations from L (except for the last one), permute them according to π and change the signs the groups according to δ. Since we are interested only in cones different from $\{0\}$, the above representations define the k-faces of $\mathcal{W}^B(y_1,\dots,y_n)$ due to Proposition 3.6(ii). Since the cones different from $\{0\}$ cover L, these are already all of the k-faces contained L. The same arguments, with y_i replaced by $\varepsilon_iy_{\sigma(i)}$, are valid for the general case of a subspace $L(l,\varepsilon,\sigma)$.

In summary, we know that every k-face of $\mathcal{W}^B(y_1,\dots,y_n)$ is contained in a unique subspace of the form $L(l,\varepsilon,\sigma)$ and every such subspace contains $D^B(n-d+k,k)$ faces of dimension k. This yields a total of

$$\sum_{r=0}^{d-k}\binom{n}{r}\left\{{n-r \atop n-d+k}\right\}2^{d-k-r}D^B(n-d+k,k)=T(n,n-d+k)D^B(n-d+k,k)$$

k-faces of $\mathcal{W}^B(y_1,\dots,y_n)$. □

3.5. **Proof of Theorem 3.3.** In this section, we are finally going to prove Theorem 3.3. We do this in a separate section, since the proof requires a lot of results we need to establish first. These results will explain the connection between the Weyl tessellation and the Weyl chambers of type B_n. But first, we need to prove a lemma stating a useful result on vectors in general position, which we will need for the proof.

Lemma 3.12. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ be in general position. Then $\operatorname{pos}\{y_1,\dots,y_n\}=\mathbb{R}^d$ holds if and only if there exists an $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n_{\geq 0}\setminus\{0\}$, such that $\alpha_1y_1+\dots+\alpha_ny_n=0$.*

Proof. Suppose $\operatorname{pos}\{y_1,\dots,y_n\}=\mathbb{R}^d$. Let $x\in\mathbb{R}^d\setminus\{0\}$, thus also $\pm x\in\operatorname{pos}\{y_1,\dots,y_n\}\setminus\{0\}$. It follows that we can find a $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{R}^n_{\geq 0}\setminus\{0\}$, such that

$$x=\beta_1y_1+\dots+\beta_ny_n.$$

Moreover, there exists a $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{R}^n_{\geq 0}\setminus\{0\}$, such that

$$-x=\gamma_1y_1+\ldots+\gamma_ny_n.$$

Then

$$0=x-x=(\beta_1+\gamma_1)y_1+\ldots+(\beta_n+\gamma_n)y_n.$$

We define $\alpha_i=\beta_i+\gamma_i\geq 0$ for $i=1,\dots,n$ and observe that $\alpha_i>0$ has to hold true for at least one $i\in\{1,\dots,n\}$.

To prove the other direction, let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n_{\geq 0}\setminus\{0\}$ satisfying $\alpha_1y_1+\ldots+\alpha_ny_n=0$. Since $y_1,\dots,y_n$ are in general position, $\alpha_i>0$ holds true for at least $d+1$ indices $i\in\{1,\dots,n\}$. That is easily seen by considering the contraposition. If there is a $k\in\{1,\dots,d\}$ and indices $1\leq i_1<\ldots<i_k\leq n$, such that $\alpha_{i_1},\dots,\alpha_{i_k}>0$ and $\alpha_j=0$ for all $j\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}$, then

$$\alpha_{i_1}y_{i_1}+\ldots+\alpha_{i_k}y_{i_k}=0.$$

But since $y_{i_1},\dots,y_{i_k}$ are linearly independent by the general position assumption, $\alpha_{i_1},\dots,\alpha_{i_k}$ are also required to be 0, which is a contradiction.

Additionally, suppose $\operatorname{pos}\{y_1,\dots,y_n\}\neq\mathbb{R}^d$, then $\bigcap_{i=1}^n y_i^-=\operatorname{pos}\{y_1,\dots,y_n\}^\circ\neq\{0\}$. That is, there exists a $w\in\mathbb{R}^d\setminus\{0\}$, such that

$$\langle w,y_i\rangle\leq 0,\quad i=1,\dots,n.$$

We claim that $\langle w,y_i\rangle=0$ holds true for at most $d-1$ indices $i\in\{1,\dots,n\}$. To see this, observe that there is a $k\geq d$ and indices $1\leq i_1<\ldots<i_k\leq n$ satisfying $\langle w,y_{i_r}\rangle=0$ for all $r=1,\dots,k$, then $y_{i_1},\dots,y_{i_k}\in w^\perp\in G(d,d-1)$. But this is a contradiction to the general position of $y_1,\dots,y_n$.

Taking these results into consideration, we have

$$0=\langle w,0\rangle=\langle w,\alpha_1y_1+\dots+\alpha_ny_n\rangle=\alpha_1\langle w,y_1\rangle+\dots+\alpha_n\langle w,y_n\rangle<0,$$

since all summands are non-positive, $\alpha_i > 0$ holds at least $d+1$ times and $\langle w, y_i\rangle = 0$ at most $d-1$ times, thus at least two summands $\alpha_i\langle w, y_i\rangle$ are negative. That is a contradiction, yielding $\operatorname{pos}\{y_1,\dots,y_n\} = \mathbb{R}^d$. $\square$

The main ingredient in the proof of Theorem 3.3 is the following result on the number of k-faces of Weyl chambers of type B_n intersecting a linear subspace only in a trivial way. It was proven in [12, Theorem 2.1] in a probabilistic version.

Theorem 3.13. *Let $L_d \in G(n,d)$ be a deterministic d-dimensional subspace of $\mathbb{R}^n$ in general position with respect to the reflection arrangement $\mathcal{A}(B_n)$. Then*

$$\sum_{\varepsilon\in\{\pm1\}^n}\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C^B_{\varepsilon,\sigma})}\mathbb{1}_{\{F\cap L_d=\{0\}\}} = 2^{n-k+1}\binom{n}{k}\frac{n!}{k!}(B(k,n-d-1)+B(k,n-d-3)+\dots),$$

where the $B(k,j)$'s are defined in Theorem 3.3.

Recall that the cones $C^B_{\varepsilon,\sigma} := \{(\beta_1,\dots,\beta_n)\in\mathbb{R}^n : \varepsilon_1\beta_{\sigma(1)}\le\dots\le\varepsilon_n\beta_{\sigma(n)}\le 0\}$ denote the Weyl chambers of type B_n as introduced in Section 3.1.

Remark 3.14. We can easily derive the analogous result for the faces that do intersect the subspace L_d in a non-trivial way. By taking $t=\pm1$ in (3.4) we get

$$B(k,1)+B(k,3)+\dots = B(k,0)+B(k,2)+\dots = 2^{k-1}k!.$$

Under the assumptions of Theorem 3.13, it follows that

$$\begin{aligned}\sum_{\varepsilon\in\{\pm1\}^n}\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C^B_{\varepsilon,\sigma})}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}} &= 2^n n!\binom{n}{k} - \sum_{\varepsilon\in\{\pm1\}^n}\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C^B_{\varepsilon,\sigma})}\mathbb{1}_{\{F\cap L_d=\{0\}\}}\\ &= 2^{n-k+1}\binom{n}{k}\frac{n!}{k!}\big(B(k,n-d+1)+B(k,n-d+3)+\dots\big)\\ &= 2^{n-k}\binom{n}{k}\frac{n!}{k!}D^B(k,d-n+k),\end{aligned}$$

where we used in the first step that the number of Weyl chambers is $2^n n!$ and each chamber has $\binom{n}{k}$ faces of dimension k.

To make use of Theorem 3.13, we will derive a connection between the faces of the Weyl tessellation $\mathcal{W}^B(y_1,\dots,y_n)$ and the faces of the Weyl chambers in $\mathbb{R}^n$ of type B_n. Recall the notation $F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})$ for the k-faces of the Weyl mosaic of type B_n from (3.5) and the notation $C^B_{\varepsilon,\sigma}(l_1,\dots,l_k)$ for the k-faces of the Weyl chambers of type B_n in $\mathbb{R}^n$ from (3.2).

Lemma 3.15. *Let $1\le k\le d$ and let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (B1) or (B2). For $L=\{\beta\in\mathbb{R}^n : \beta_1y_1+\dots+\beta_ny_n=0\}$ the equivalence*

$$F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k}) = \{0\} \Leftrightarrow C^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})\cap L^\perp = \{0\} \tag{3.7}$$

holds true for all $1\le l_1<\dots<l_{n-d+k}\le n$, $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$.

Before we prove this lemma, we want to state separately the special case $\varepsilon_i=+1$, $\sigma(i)=i$ and $k=d$. In this case the lemma states that

$$\{v\in\mathbb{R}^d : \langle v,y_1\rangle\le\dots\le\langle v,y_n\rangle\le 0\} = \{0\} \Leftrightarrow \{\beta\in\mathbb{R}^n : \beta_1\le\dots\le\beta_n\le 0\}\cap L^\perp = \{0\}.$$

This means that the Weyl cone on the left hand side is degenerate, i.e. equal to $\{0\}$, if and only if the corresponding Weyl chamber, having the same arrangement of inequalities, intersects $L^\perp$ only in a trivial way. Thus, using this lemma, we can count the number of faces of the Weyl chambers intersected by $L^\perp$ in a non-trivial way, which is already done in Theorem 3.13, instead of counting the faces of the Weyl mosaic. That is the basic idea behind the proof of Theorem 3.3.

Proof of Lemma 3.15. For the sake of simplicity, we first derive a result for the special case $\sigma(i) = i$ and $\varepsilon_i = 1$, for all $i = 1, \ldots, n$. For $1 \leq l_1 < \ldots < l_{n-d+k} \leq n$ we define

$$F := \{v \in \mathbb{R}^d : f_1 = \ldots = f_{l_1} \leq f_{l_1+1} = \ldots = f_{l_2} \leq \ldots \leq f_{l_{n-d+k}+1} = \ldots = f_n = 0\},$$

which is just a shorthand notation for $F^B_{\varepsilon,\sigma}(l_1, \ldots, l_{n-d+k})$ in the special case $\varepsilon_i = 1$ and $\sigma(i) = i$. Note that we the linear functionals f_i are defined by $f_i = f_i(v) := \langle v, y_i \rangle$ as above. We already saw in Proposition 3.6 that if F is not $\{0\}$, it is a k-face of the Weyl mosaic $\mathcal{W}^B(y_1, \ldots, y_n)$, due to the general position assumption (B1). Then, the linear span of F is given by

$$\operatorname{lin} F = \{v \in \mathbb{R}^d : f_1 = \ldots = f_{l_1}, f_{l_1+1} = \ldots = f_{l_2}, \ldots, f_{l_{n-d+k}+1} = \ldots = f_n = 0\},$$

since the condition of $\operatorname{lin} F$ consists of $d-k$ equations, and therefore, the general position assumption (B1) implies that $\dim(\operatorname{lin} F) = d - (d - k) = k$ holds true. Using the duality properties (2.1) and (2.7), we get

$$\begin{aligned} F^\circ &= \left(\operatorname{lin} F \cap \bigcap_{i=1}^{n-d+k} (y_{l_i} - y_{l_i+1})^-\right)^\circ = (\operatorname{lin} F)^\circ + \left(\bigcap_{i=1}^{n-d+k} (y_{l_i} - y_{l_i+1})^-\right)^\circ \\ &= (\operatorname{lin} F)^\perp + \operatorname{pos}\{y_{l_1} - y_{l_1+1}, \ldots, y_{l_{n-d+k}} - y_{l_{n-d+k}+1}\}, \end{aligned}$$

where we set $y_{n+1} = 0$. Thus, we get

$$F^\circ \cap \operatorname{lin} F = \operatorname{pos}\{\Pi(y_{l_1} - y_{l_1+1}), \ldots, \Pi(y_{l_{n-d+k}} - y_{l_{n-d+k}+1})\},$$

where $\Pi : \mathbb{R}^d \to \operatorname{lin}(F)$ is the orthogonal projection onto $\operatorname{lin}(F)$. In order to prove this equation, we can represent the vectors $y_{l_i} - y_{l_i+1}$ as (z_i, x_i), where z_i is the projection on $\operatorname{lin} F$, and x_i is the projection on $(\operatorname{lin} F)^\perp$. Thus, the vectors in $F^\circ = (\operatorname{lin} F)^\perp + \operatorname{pos}\{y_{l_1} - y_{l_1+1}, \ldots, y_{l_{n-d+k}} - y_{l_{n-d+k}+1}\}$ take the form

$$\begin{pmatrix} \alpha_1 z_1 + \ldots + \alpha_{n-d+k} z_{n-d+k} \\ v + \alpha_1 x_1 + \ldots + \alpha_{n-d+k} x_{n-d+k} \end{pmatrix}$$

for $\alpha_1, \ldots, \alpha_{n-d+k} \geq 0$ and $v \in (\operatorname{lin} F)^\perp$. Here, the first entry denotes the component in $\operatorname{lin} F$ and the second entry denotes the component in $(\operatorname{lin} F)^\perp$. Such a vector is contained in $\operatorname{lin} F$ if and only if the second component is 0, that is, $-v = \alpha_1 x_1 + \ldots + \alpha_{n-d+k} x_{n-d+k}$. Therefore, $F^\circ \cap \operatorname{lin} F = \operatorname{pos}(z_1, \ldots, z_{n-d+k})$.

Taking all of that into consideration, we obtain

$$\begin{aligned} F = \{0\} &\Leftrightarrow F^\circ = \mathbb{R}^d \\ &\Leftrightarrow F^\circ \cap \operatorname{lin} F = \operatorname{lin} F \\ &\Leftrightarrow \operatorname{pos}\{\Pi(y_{l_1} - y_{l_1+1}), \ldots, \Pi(y_{l_{n-d+k}} - y_{l_{n-d+k}+1})\} = \operatorname{lin} F. \end{aligned}$$

Note that we used the decomposition $F^\circ = (\operatorname{lin} F)^\perp + (F^\circ \cap \operatorname{lin} F)$ for the second equivalence. Applying Lemma 3.12 to the k-dimensional linear subspace $\operatorname{lin} F$, we have that $\operatorname{lin} F = \operatorname{pos}\{\Pi(y_{l_1} -$

$y_{l_1+1}),\dots,\Pi(y_{l_{n-d+k}}-y_{l_{n-d+k}+1})\}$ if and only if there exist $\alpha_{l_1},\dots\alpha_{l_{n-d+k}}\geq 0$ that do not vanish simultaneously and such that

$$\begin{aligned}0&=\alpha_{l_1}\Pi(y_{l_1}-y_{l_1+1})+\ldots+\alpha_{l_{n-d+k}}\Pi(y_{l_{n-d+k}}-y_{l_{n-d+k}+1})\\&=\Pi\big(\alpha_{l_1}(y_{l_1}-y_{l_1+1})+\ldots+\alpha_{l_{n-d+k}}(y_{l_{n-d+k}}-y_{l_{n-d+k}+1})\big).\end{aligned}$$

This holds if and only if there exists an $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$ with $\alpha_{l_1},\dots,\alpha_{l_{n-d+k}}\geq 0$ not all being 0, such that

$$0=\alpha_1(y_1-y_2)+\ldots+\alpha_{n-1}(y_{n-1}-y_n)+\alpha_n y_n$$

After regrouping the terms, the condition is of the form

$$0=\alpha_1y_1+(\alpha_2-\alpha_1)y_2+\ldots+(\alpha_n-\alpha_{n-1})y_n,$$

since $(\mathrm{lin}(F))^\perp=\mathrm{lin}\{y_i-y_{i+1}:i\in\{1,\dots,n\}\backslash\{l_1,\dots,l_{n-d+k}\}\}$, $y_{n+1}:=0$. By defining $\beta_1=\alpha_1$, $\beta_i=\alpha_i-\alpha_{i-1}$ for $i=2,\dots,n$, we see that this is equivalent to the existence of a vector $\beta\in\mathbb{R}^n$ with $\beta_1+\ldots+\beta_{l_1}\geq 0,\beta_1+\ldots+\beta_{l_2}\geq 0,\dots,\beta_1+\dots+\beta_{l_{n-d+k}}\geq 0$, where at least one inequality is strict, such that

$$0=\beta_1y_1+\dots+\beta_{l_n}y_{l_n}.$$

By defining $M:=\{\beta\in\mathbb{R}^n:\beta_1+\ldots+\beta_{l_1}\geq 0,\dots,\beta_1+\dots+\beta_{l_{n-d+k}}\geq 0\}$, recalling $L=\{\beta\in\mathbb{R}^n:\beta_1y_1+\ldots+\beta_ny_n=0\}$ and taking the previous results into account, we get

$$F=\{0\}\Leftrightarrow M\cap L\not\subseteq\mathrm{linsp}(M),$$

since $\mathrm{linsp}\,M:=M\cap(-M)=\{\beta\in\mathbb{R}^n:\beta_1+\ldots+\beta_{l_1}=0,\dots,\beta_1+\ldots+\beta_{l_{n-d+k}}=0\}$ is the lineality space of M. Using Lemma 2.3, we get

$$M\cap L\not\subseteq\mathrm{linsp}(M)\Leftrightarrow\mathrm{relint}(M^\circ)\cap L^\perp=\emptyset.$$

For the dual cone of M, the following holds:

$$\begin{aligned}M^\circ&=\big(\{\beta\in\mathbb{R}^n:\langle(\overbrace{1,\dots,1}^{l_1},0,\dots,0)^T,\beta\rangle\geq 0,\dots,\langle(\overbrace{1,\dots,1}^{l_{n-d+k}},0\dots,0)^T,\beta\rangle\geq 0\}\big)^\circ\\&=-\mathrm{pos}\left\{(1,\dots,1,0,\dots,0)^T,\dots,(1,\dots,1,0\dots,0)^T\right\}\\&=\{x\in\mathbb{R}^n:x_1=\ldots=x_{l_1}\leq x_{l_1+1}=\ldots=x_{l_2}\leq\ldots\leq x_{l_{n-d+k}+1}=\ldots=x_n=0\}\\&=:G,\end{aligned}$$

where G is just a shorthand notation for the $(n-d+k)$-dimensional face $C^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})$ of the Weyl chambers of type B_n in the special case $\varepsilon_i=1$ and $\sigma(i)=i$. Taking all equivalences into consideration, applying Lemma 2.12 and using (B2), we get

$$F=\{0\}\Leftrightarrow\mathrm{relint}(G)\cap L^\perp=\emptyset\Leftrightarrow G\cap L^\perp=\{0\},$$

which is the special case $\varepsilon_i=1$ and $\sigma(i)=i$ of the equivalence (3.7).

Now we return to the general case and apply this equivalence, replacing $y_1,\dots,y_n$ by the vectors $\varepsilon_1y_{\sigma(1)},\dots,\varepsilon_ny_{\sigma(n)}$ for $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$. It follows

$$F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})=\{0\}\Leftrightarrow G\cap(L_{\epsilon,\sigma})^\perp=\{0\},$$

where $L_{\varepsilon,\sigma}:=\{\beta\in\mathbb{R}^n:\beta_1\epsilon_1y_{\sigma(1)}+\ldots+\beta_{l_{n-d+k}}\varepsilon_{l_{n-d+k}}y_{\sigma(l_{n-d+k})}+\beta_{l_{n-d+k}+1}y_{\sigma(l_{n-d+k}+1)}+\ldots+\beta_ny_{\sigma(n)}=0\}$.

Now, it is left to show that

$$G\cap (L_{\epsilon,\sigma})^{\perp}=\{0\}\Leftrightarrow C^{B}_{\epsilon,\sigma}(l_1,\dots,l_{n-d+k})\cap L^{\perp}=\{0\}$$

holds for every $\varepsilon\in\{\pm1\}^n$, $\sigma\in\mathcal{S}_n$. At first, we see that for any rotation $\vartheta\in SO_d$ and linear subspace U

$$\begin{aligned}(\vartheta U)^{\perp}&=\{x\in\mathbb{R}^d:\langle\vartheta u,x\rangle=0\ \forall u\in U\}=\{x\in\mathbb{R}^d:\langle u,\vartheta^{-1}x\rangle=0\ \forall u\in U\}\\&=\{\vartheta x\in\mathbb{R}^d:\langle u,x\rangle=0\ \forall u\in U\}=\vartheta U^{\perp}\end{aligned}$$

holds true. Moreover, we can see that

$$\begin{aligned}L_{\varepsilon,\sigma}&=\big\{\beta\in\mathbb{R}^d:\beta_{\sigma^{-1}(1)}\varepsilon_{\sigma^{-1}(1)}y_1+\ldots+\beta_{\sigma^{-1}(l_{n-d+k})}\varepsilon_{\sigma^{-1}(l_{n-d+k})}y_{l_{n-d+k}}\\&\qquad+\beta_{\sigma^{-1}(l_{n-d+k}+1)}y_{l_{n-d+k}+1}+\ldots+\beta_{\sigma^{-1}(n)}y_n=0\big\}\\&=\{\beta\in\mathbb{R}^n:\gamma_1y_1+\ldots+\gamma_ny_n=0\}\end{aligned}$$

for $\gamma_i=\beta_{\sigma^{-1}(i)}\varepsilon_{\sigma^{-1}(i)}$, $i\in\{1,\dots,l_{n-d+k}\}$ and $\gamma_i=\beta_{\sigma^{-1}(i)}$, $i\in\{l_{n-d+k}+1,\dots,n\}$. Now we choose the unique $g\in\mathcal{G}(B_n)$ satisfying $\beta=g\gamma$, that is

$$g:(x_1,\dots,x_n)\mapsto\big(\varepsilon_1x_{\sigma(1)},\dots,\varepsilon_{l_{n-d+k}}x_{\sigma(l_{n-d+k})},x_{\sigma(l_{n-d+k}+1)},\dots,x_{\sigma(n)}\big),$$

and we get

$$L_{\varepsilon,\sigma}=\{g\gamma\in\mathbb{R}^n:\gamma_1y_1+\cdots+\gamma_ny_n=0\}=gL.$$

It follows that

$$\begin{aligned}G\cap L^{\perp}_{\epsilon,\sigma}=\{0\}&\Leftrightarrow G\cap gL^{\perp}=\{0\}\\&\Leftrightarrow g^{-1}G\cap L^{\perp}=\{0\}\\&\Leftrightarrow C^{B}_{\epsilon,\sigma}(l_1,\dots,l_{n-d+k})\cap L^{\perp}=\{0\},\end{aligned}$$

since g^{-1} is defined by

$$(x_1,\dots,x_n)\mapsto\big(\varepsilon_{\sigma^{-1}(1)}x_{\sigma^{-1}(1)},\dots,\varepsilon_{\sigma^{-1}(l_{n-d+k})}x_{\sigma^{-1}(l_{n-d+k})},x_{\sigma^{-1}(l_{n-d+k}+1)},\dots,x_{\sigma^{-1}(n)}\big)$$

and therefore it is easy to check that

$$g^{-1}G=C^{B}_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k}).$$

Taking all into consideration this yields

$$F^{B}_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})=\{0\}\Leftrightarrow C^{B}_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})\cap L^{\perp}=\{0\},$$

which completes the proof. □

Finally, we are able to prove Theorem 3.3.

Proof of Theorem 3.3. Let $1\le k\le d$ and let $y_1,\dots,y_n$ satisfy one of the equivalent general position assumptions (B1) or (B2). We want to evaluate the number of k-faces of $F\in\mathcal{F}^B_k(y_1,\dots,y_n)$, each face counted with the multiplicity equal to the number of d-dimensional cones $C\in\mathcal{F}^B_d(y_1,\dots,y_n)$ containing it. We define $\Omega_n(l):=\{\varepsilon\in\{\pm1\}^n:\varepsilon_{l_{n-d+k}+1}=\ldots=\varepsilon_n=1\}$ and use Proposition 3.6 and Proposition 3.8 to obtain

$$\begin{aligned}&\sum_{F\in\mathcal{F}^B_k(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}^B_d(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}\\&=\sum_{1\le l_1<\ldots<l_{n-d+k}\le n}l_1!(l_2-l_1)!\cdots(n-l_{n-d+k})!2^{n-l_{n-d+k}}\sum_{\varepsilon\in\Omega_n(l)}\sum_{\sigma\in\mathcal{S}_n}\mathbb{1}_{\{F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-d+k})\neq\{0\}\}}\end{aligned}$$

$$= \sum_{1\le l_1<\ldots<l_{n-d+k}\le n} l_1!(l_2-l_1)!\cdots(n-l_{n-d+k})!2^{n-l_{n-d+k}} \sum_{\varepsilon\in\Omega_n(l)}\sum_{\sigma\in\mathcal{S}_n} \mathbb{1}_{\{C^B_{\varepsilon,\sigma}(l_1,\ldots,l_{n-d+k})\cap L^\perp\neq\{0\}\}}.$$

Note that we applied the equivalence (3.7) from Lemma 3.15 in the last equation. Now, we use that each face $C^B_{\varepsilon,\sigma}(l_1,\ldots,l_{n-d+k})$ is contained in exactly $l_1!(l_2-l_1)!\cdots(n-l_{n-d+k})!2^{n-l_{n-d+k}}$ Weyl chambers $C^B_{\varepsilon,\sigma}$ of type B_n. For this fact, we refer to [12, Proof of Theorem 2.1]. Furthermore $L^\perp$ is a d-dimensional subspace and in general position with respect to the reflection arrangement $\mathcal{A}(B_n)$, due to (B2). Thus, we can apply Theorem 3.13, or rather Remark 3.14, replace k by $n-d+k$ and get

$$\begin{aligned}&\sum_{F\in\mathcal{F}^B_k(y_1,\ldots,y_n)}\sum_{C\in\mathcal{F}^B_d(y_1,\ldots,y_n)}\mathbb{1}_{\{F\subseteq C\}}\\&\quad=\sum_{\varepsilon\in\{\pm1\}^n}\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_{n-d+k}(C^B_{\varepsilon,\sigma})}\mathbb{1}_{\{F\cap L^\perp\neq\{0\}\}}\\&\quad=2^{d-k+1}\binom{n}{d-k}\frac{n!}{(n-d+k)!}(B(n-d+k,n-d+1)+B(n-d+k,n-d+3)+\ldots),\end{aligned}$$

which completes the proof. □

4. Conical tessellations and Weyl chambers of type A_{n-1}

4.1. **The reflection arrangement and Weyl chambers of type A_{n-1}.** Our calculations for the Weyl chambers and the Weyl tesselation of type B_n suggest that we should be able to prove similar results for the Weyl chambers of type A_{n-1}. We introduce the necessary notation, which is taken from [12, Section 2.5]. In this section, we always assume that $n\ge d+1$.

We call $\mathcal{G}(A_{n-1})$ the *reflection group* of type A_{n-1}, which acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way. This means that the $n!$ elements of $\mathcal{G}(B_n)$ are the linear mappings

$$g_\sigma:\mathbb{R}^n\to\mathbb{R}^n,\quad(\beta_1,\ldots,\beta_n)\mapsto(\beta_{\sigma(1)},\ldots,\beta_{\sigma(n)}),$$

where $\sigma\in\mathcal{S}_n$.

The *closed Weyl chambers of type* A_{n-1} are the cones of the conical tessellation induced by the hyperplane arrangement $\mathcal{A}(A_{n-1})$ consisting of the hyperplanes given by

$$\{\beta\in\mathbb{R}^n:\beta_i=\beta_j\}\quad(1\le i<j\le n).\tag{4.1}$$

It is called the *reflection arrangement of type* A_{n-1}. Thus, the closed Weyl chambers of type A_{n-1} are given by

$$C^A_\sigma:=\{(\beta_1,\ldots,\beta_n)\in\mathbb{R}^n:\beta_{\sigma(1)}\le\ldots\le\beta_{\sigma(n)}\},$$

where $\sigma\in\mathcal{S}_n$. The superscript A indicates the type of the Weyl chamber. Similar to the B_n-case, the the Weyl chambers of type A_{n-1} may equivalently be defined as the reflections $g\,\mathcal{C}(A_{n-1})$, $g\in\mathcal{G}(A_{n-1})$, of the *fundamental Weyl chamber of type* A_{n-1} given by

$$\mathcal{C}(A_{n-1})=\{(\beta_1,\ldots,\beta_n)\in\mathbb{R}^n:\beta_1\le\ldots\le\beta_n\}.$$

The k-dimensional faces of the Weyl chambers C^A_σ are determined by the collection of indices $1\le l_1<\ldots<l_{k-1}\le n-1$ and have the form

$$\begin{aligned}&C^A_\sigma(l_1,\ldots,l_{k-1})\\&\quad:=\{\beta\in\mathbb{R}^n:\beta_{\sigma(1)}=\ldots=\beta_{\sigma(l_1)}\le\beta_{\sigma(l_1+1)}=\ldots=\beta_{\sigma(l_2)}\le\ldots\le\beta_{\sigma(l_{k-1}+1)}=\ldots=\beta_{\sigma(n)}\}.\end{aligned}\tag{4.2}$$

For this fact, we refer to [12, Section 2.7]. Thus, the number of k-faces of C_σ^A is given by $\#\mathcal{F}_k(C_\sigma^A) = \binom{n-1}{k-1}$.

4.2. **Weyl tessellation of type A_{n-1}.** The definition of the Weyl tessellation of type A_{n-1} is somewhat simpler than for tessellations of type B_n.

Definition 4.1. (Weyl tessellation of type A_{n-1}) Let $y_1, \ldots, y_n \in \mathbb{R}^d$ and let the hyperplane arrangement $\mathcal{A}^A(y_1, \ldots, y_n)$ consist of the hyperplanes given by

$$(y_i - y_j)^\perp, \quad 1 \le i < j \le n. \tag{4.3}$$

Then the *Weyl tessellation of type* A_{n-1} or *Weyl mosaic of type* A_{n-1} is defined as the conical tessellation generated by $\mathcal{A}^A(y_1, \ldots, y_n)$ and is denoted by $\mathcal{W}^A(y_1, \ldots, y_n)$. We denote the number of cones in $\mathcal{W}^A(y_1, \ldots, y_n)$ by

$$D^A(n,d) := \#\mathcal{W}^A(y_1, \ldots, y_n).$$

Similarly, the set of k-faces of $\mathcal{W}^A(y_1, \ldots, y_n)$ is denoted by $\mathcal{F}_k^A(y_1, \ldots, y_n)$, where $\mathcal{F}_d^A(y_1, \ldots, y_n) = \mathcal{W}^A(y_1, \ldots, y_n)$.

We will show in Corollary 4.4 that $D^A(n,d)$ is constant under certain mild conditions on $y_1, \ldots, y_n$. The cones in $\mathcal{W}^A(y_1, \ldots, y_n)$ are the cones different from $\{0\}$ of the form

$$D_\sigma^A := \{v \in \mathbb{R}^d : \langle v, y_{\sigma(1)} \rangle \le \cdots \le \langle v, y_{\sigma(n)} \rangle\}, \quad \sigma \in \mathcal{S}_n.$$

Again, we will refer to these cones as *Weyl cones of type* A_n or just *Weyl cones.*

Since the definition of the Weyl chambers and Weyl cones of type A_{n-1} is similar and even somewhat simpler than that of type B_n, it suggests that we can prove similar results on the number of Weyl cones and Weyl faces of type A_{n-1}. We will state the results but will not give each proof in full detail. At first, we need assumptions on $y_1, \ldots, y_n$ similar to (B1) and (B2).

Theorem 4.2. *For arbitrary $y_1, \ldots, y_n \in \mathbb{R}^d$ the following conditions (A1) and (A2) are equivalent:*

- *(A1) For every $\sigma \in \mathcal{S}_n$ the vectors $y_{\sigma(1)} - y_{\sigma(2)}, y_{\sigma(2)} - y_{\sigma(3)}, \ldots, y_{\sigma(n-1)} - y_{\sigma(n)}$ are in general position.*
- *(A2) The linear subspace $L^\perp$ has dimension d and is in general position with respect to the hyperplane arrangement $\mathcal{A}(A_{n-1})$, where $L := \{\beta \in \mathbb{R}^n : \beta_1 y_1 + \cdots + \beta_n y_n = 0\}$.*

We will prove this theorem in Section 6. Note that the general position assumption (B1) implies (A1), and (B2) implies (A2), since $\mathcal{A}(A_{n-1}) \subseteq \mathcal{A}(B_n)$. The analogue to Theorem 3.3 is as follows.

Theorem 4.3. *Let $y_1, \ldots, y_n \in \mathbb{R}^d$ satisfy one of the equivalent general position assumptions (A1) or (A2). For $1 \le k \le d$, we have*

$$\sum_{F \in \mathcal{F}_k^A(y_1, \ldots, y_n)} \sum_{C \in \mathcal{F}_d^A(y_1, \ldots, y_n)} \mathbb{1}_{\{F \subseteq C\}} = 2\binom{n-1}{d-k} \frac{n!}{(n-d+k)!} \left(\begin{bmatrix} n-d+k \\ n-d+1 \end{bmatrix} + \begin{bmatrix} n-d+k \\ n-d+3 \end{bmatrix} + \ldots \right).$$

Recall that $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Stirling numbers of first kind defined by the formula

$$t(t+1) \cdot \ldots \cdot (t+n-1) = \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} t^k \tag{4.4}$$

and, by convention, $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $k \notin \{1, \ldots n\}$.

We postpone the proof Theorem 4.3 to Section 4.4. Again, we are able to compute the number of cones in the Weyl tessellation $\mathcal{W}^A(y_1,\dots,y_n)$ of type A_{n-1} under one of the conditions (A1) or (A2).

Corollary 4.4. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (A1) or (A2). Then the number of cones in the Weyl mosaic $\mathcal{W}^A(y_1,\dots,y_n)$ of type A_{n-1} is given by*

$$D^A(n,d)=2\left(\begin{bmatrix}n\\n-d+1\end{bmatrix}+\begin{bmatrix}n\\n-d+3\end{bmatrix}+\dots\right).$$

Proof. This follows from Theorem 4.3 in the special case $k=d$. $\square$

Note that we may rewrite the claim of Theorem 4.3 as follows:

$$\sum_{F\in\mathcal{F}_k^A(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}_d^A(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}=\binom{n-1}{d-k}\frac{n!}{(n-d+k)!}D^A(n-d+k,k).$$

Again, we can state the probabilistic version of Theorem 4.3.

Corollary 4.5. *Let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (A1) or (A2). Let Q^A be sampled randomly and uniformly among the $D^A(n,d)$ cones of $\mathcal{W}^A(y_1,\dots,y_n)$. Then the expected number of k-dimensional faces of Q^A is given by*

$$\begin{aligned}\mathbb{E}\,f_k(Q^A)&=\frac{1}{D^A(n,d)}\sum_{C\in\mathcal{F}_d^A(y_1,\dots,y_n)}\ \sum_{F\in\mathcal{F}_k^A(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}\\&=\binom{n-1}{d-k}\frac{n!}{(n-d+k)!}\frac{D^A(n-d+k,k)}{D^A(n,d)}.\end{aligned}$$

4.3. **Faces of the Weyl tessellation of type A_{n-1}.** For $y_1,\dots,y_n\in\mathbb{R}^d$, a collection of indices $1\le l_1<\dots<l_{n-d+k-1}\le n-1$ and $\sigma\in\mathcal{S}_n$, we define

$$\begin{aligned}&F_\sigma^A(l_1,\dots,l_{n-d+k-1})\\&\quad:=\{v\in\mathbb{R}^d: f_{\sigma(1)}=\dots=f_{\sigma(l_1)}\le f_{\sigma(l_1+1)}=\dots=f_{\sigma(l_2)}\le\dots\le f_{\sigma(l_{n-d+k-1}+1)}=\dots=f_{\sigma(n)}\},\end{aligned}\tag{4.5}$$

where the functionals f_i are defined by $f_i=f_i(v):=\langle v,y_i\rangle$, $i=1,\dots,n$. These cones will, similarly to the B_n-case, represent the k-faces of the Weyl tessellation of type A_{n-1}. The next result is an analogue to Propositions 3.6 and 3.8. Therefore, we omit the proof.

Proposition 4.6. *Let $1\le k\le d$ and let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy the general position assumption (A1). Then it holds:*

- *(i) For every $F\in\mathcal{F}_k^A(y_1,\dots,y_n)$ there are a collection of indices $1\le l_1<\dots<l_{n-d+k-1}\le n-1$ and a permutation $\sigma\in\mathcal{S}_n$, such that $F=F_\sigma^A(l_1,\dots,l_{n-d+k-1})$.*
- *(ii) Let $1\le l_1<\dots<l_{n-d+k-1}\le n-1$ and $\sigma\in\mathcal{S}_n$. If $F_\sigma^A(l_1,\dots,l_{n-d+k-1})\neq\{0\}$, then $F_\sigma^A(l_1,\dots,l_{n-d+k-1})\in\mathcal{F}_k^A(y_1,\dots,y_n)$.*
- *(iii) Every k-face $F_\sigma^A(l_1,\dots,l_{n-d+k-1})\in\mathcal{F}_k^A(y_1,\dots,y_n)$ is contained in exactly*

$$l_1!(l_2-l_1)!\cdot\ldots\cdot(n-l_{n-d+k-1})!$$

cones $C\in\mathcal{F}_d^A(y_1,\dots,y_n)$.

Theorem 4.7. *Let $y_1,\dots,y_n$ satisfy one of the equivalent general position assumptions (A1) or (A2). Then the number of k-faces in the Weyl mosaic of type A_{n-1} is given by*
$$\#\mathcal{F}_k^A(y_1,\dots,y_n)=\begin{Bmatrix} n\\ n-d+k\end{Bmatrix}D^A(n-d+k,k),$$
for all $k\in\{1,\dots,d\}$.

Proof. This is proven similar to Theorem 3.10. Due to Proposition 4.6(i), each k-face of $F\in\mathcal{F}_k^A(y_1,\dots,y_n)$ is contained in a subspace of the form
$$L(l,\sigma):=\{v\in\mathbb{R}^d: f_{\sigma(1)}=\ldots=f_{\sigma(l_1)},\dots,f_{\sigma(l_{n-d+k-1}+1)}=\ldots=f_{\sigma(n)}\}$$
for suitable $1\le l_1<\ldots<l_{n-d+k-1}\le n-1$ and $\sigma\in\mathcal{S}_n$. There are a total of $\left\{{n\atop n-d+k}\right\}$ distinct subspaces of the given form, since these are in one-to-one correspondence with partitions of the set $\{1,\dots,n\}$ into $n-d+k$ non-empty sets.

Now, it is left to prove that every subspace $L(l,\sigma)$ contains exactly $D^A(n-d+k,k)$ k-faces of $\mathcal{W}^A(y_1,\dots,y_n)$. Again, consider only the case $L:=L(l,\sigma)$ for $\sigma(i)=i$, $i=1,\dots,n$. For this, we need to show that $\Pi_L(y_{l_1}),\dots,\Pi_L(y_{l_{n-d+k-1}}),\Pi_L(y_n)$ satisfy the general position assumption (A1) in L, which is shown in the same way as in Theorem 3.10. This completes the proof. $\square$

4.4. **Proof of Theorem 4.3.** The proof of Theorem 4.3 is similar to that of Theorem 3.3. Again, the main ingredient is a result on the number of k-faces of Weyl chambers of type A_{n-1} intersecting a linear subspace in a trivial way. It was proven in [12, Theorem 2.8] in a probabilistic version, and we will state it as follows.

Theorem 4.8. *Let $L_d\in G(n,d)$ be a deterministic d-dimensional subspace of $\mathbb{R}^n$ in general position with respect to the reflection arrangement $\mathcal{A}(A_{n-1})$. Then*
$$\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C_\sigma^A)}\mathbb{1}_{\{F\cap L_d=\{0\}\}}=\frac{2n!}{k!}\binom{n-1}{k-1}\left(\begin{bmatrix}k\\ n-d-1\end{bmatrix}+\begin{bmatrix}k\\ n-d-3\end{bmatrix}+\ldots\right)$$
where the $\left[{n\atop k}\right]$'s are defined as in Theorem 4.3.

Recall that the cones $C_\sigma^A:=\{(\beta_1,\dots,\beta_n)\in\mathbb{R}^n:\beta_{\sigma(1)}\le\ldots\le\beta_{\sigma(n)}\}$ denote the Weyl chambers of type A_{n-1} as introduced in Section 4.1.

Remark 4.9. Similar to the case of B_n, we can easily derive the analogous result for the faces that intersect the subspace L_d in a non-trivial way. By taking $t=\pm1$ in (4.4) we get
$$\begin{bmatrix}n\\ 1\end{bmatrix}+\begin{bmatrix}n\\ 3\end{bmatrix}+\ldots=\begin{bmatrix}n\\ 2\end{bmatrix}+\begin{bmatrix}n\\ 4\end{bmatrix}+\ldots=\frac{n!}{2}.$$
It follows that under the assumptions of Theorem 4.8
$$\begin{aligned}\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C_\sigma^A)}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}&=n!\binom{n-1}{k-1}-\sum_{\sigma\in\mathcal{S}_n}\sum_{F\in\mathcal{F}_k(C_\sigma^A)}\mathbb{1}_{\{F\cap L_d=\{0\}\}}\\&=\frac{2n!}{k!}\binom{n-1}{k-1}\left(\begin{bmatrix}k\\ n-d+1\end{bmatrix}+\begin{bmatrix}k\\ n-d+3\end{bmatrix}+\ldots\right)\\&=\frac{n!}{k!}\binom{n-1}{k-1}D^A(k,n-d+k),\end{aligned}$$
where we used in the first step that the number of Weyl chambers is $n!$ and each chamber has $\binom{n-1}{k-1}$ k-faces.

The following lemma is an analogue to Lemma 3.15. Recall the notation $F_\sigma^A(l_1,\ldots,l_{n-d+k-1})$ for the k-faces of the Weyl mosaic of type A_{n-1} from (4.5), and the notation $C_\sigma^A(l_1,\ldots,l_{k-1})$ for the k-faces of the Weyl chambers of type A_{n-1} in $\mathbb{R}^n$ from (4.2).

Lemma 4.10. *Let $1\le k\le d$ and let $y_1,\ldots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (A1) or (A2). For $L=\{\beta\in\mathbb{R}^n:\beta_1y_1+\ldots+\beta_ny_n=0\}$ the equivalence*

$$F_\sigma^A(l_1,\ldots,l_{n-d+k-1})=\{0\}\Leftrightarrow C_\sigma^A(l_1,\ldots,l_{n-d+k-1})\cap L^\perp=\{0\}$$

holds true for all $1\le l_1<\ldots<l_{n-d+k-1}\le n-1$ and $\sigma\in\mathcal{S}_n$.

Proof. The proof is similar to that of Lemma 3.15 and we will not explain each argument in full detail. Let $1\le l_1<\ldots<l_{n-d+k-1}\le n-1$. We start with the special case $\sigma(i)=i$ for all $i\in\{1,\ldots,n\}$. Define

$$F=\{v\in\mathbb{R}^d:f_1=\ldots=f_{l_1}\le f_{l_1+1}=\ldots=f_{l_2}\le\ldots\le f_{l_{n-d+k-1}+1}=\ldots=f_n\},$$

where the f_i's are the functionals defined by $f_i=f_i(v):=\langle v,y_i\rangle$. Then, we have

$$\begin{aligned}F=\{0\}&\Leftrightarrow F^\circ\cap\operatorname{lin}F=\operatorname{lin}F\\&\Leftrightarrow\operatorname{pos}\{\Pi(y_{l_1}-y_{l_1+1}),\ldots,\Pi(y_{l_{n-d+k-1}}-y_{l_{n-d+k-1}+1})\}=\operatorname{lin}F,\end{aligned}$$

where Π denotes the orthogonal projection onto $\operatorname{lin}F$. Using Lemma 3.12 and the general position assumption (A1), this is equivalent to the fact that there are $\alpha_{l_1},\ldots\alpha_{l_{n-d+k-1}}\ge 0$ that do not vanish simultaneously and satisfy

$$0=\Pi\big(\alpha_{l_1}(y_{l_1}-y_{l_1+1})+\ldots+\alpha_{l_{n-d+k-1}}(y_{l_{n-d+k-1}}-y_{l_{n-d+k-1}+1})\big).$$

Since

$$\begin{aligned}(\operatorname{lin}F)^\perp=\operatorname{lin}\{&y_1-y_2,\ldots,y_{l_1-1}-y_{l_1},y_{l_1+1}-y_{l_1+2},\ldots,y_{l_2-1}-y_{l_2},\\&\ldots,y_{l_{n-d+k-1}+1}-y_{l_{n-d+k-1}+2},\ldots,y_{n-1}-y_n\},\end{aligned}$$

the above is equivalent to the existence of a vector $(\alpha_1,\ldots,\alpha_{n-1})\in\mathbb{R}^{n-1}$, where $\alpha_{l_1},\ldots,\alpha_{l_{n-d+k-1}}\ge 0$ do not vanish simultaneously, such that

$$0=\alpha_1(y_1-y_2)+\ldots+\alpha_{n-1}(y_{n-1}-y_n).$$

After regrouping the terms, the condition takes the form

$$0=\alpha_1y_1+(\alpha_2-\alpha_1)y_2+\ldots+(\alpha_{n-1}-\alpha_{n-2})y_{n-1}+(-\alpha_{n-1})y_n.$$

By setting $\beta_1:=\alpha_1,\beta_i:=\alpha_i-\alpha_{i-1}$, $2\le i\le n-1$ and $\beta_n:=-\alpha_{n-1}=-(\beta_1+\ldots+\beta_{n-1})$, this is equivalent to the fact that there exists a vector $(\beta_1,\ldots,\beta_n)\in\mathbb{R}^n$ satisfying $\beta_1+\ldots+\beta_n=0$ and

$$\beta_1+\ldots+\beta_{l_1}\ge 0,\beta_1+\ldots+\beta_{l_2}\ge 0,\ldots,\beta_1+\ldots+\beta_{l_{n-d+k-1}}\ge 0,$$

where at least one inequality is strict, such that $\beta_1y_1+\ldots+\beta_ny_n=0$. Similarly to the proof of Lemma 3.15, we define $M:=\{\beta\in\mathbb{R}^n:\beta_1+\ldots+\beta_{l_1}\ge 0,\ldots,\beta_1+\cdots+\beta_{l_{n-d+k-1}}\ge 0,\beta_1+\ldots+\beta_n=0\}$ and obtain

$$F=\{0\}\Leftrightarrow L\cap M\nsubseteq\operatorname{linsp}(M)\Leftrightarrow\operatorname{relint}(M^\circ)\cap L^\perp=\emptyset,$$

where

$$\begin{aligned}M^\circ=\big(\{\beta\in\mathbb{R}^n:\langle(\overbrace{1,\ldots,1}^{l_1},0,\ldots,0)^T,\beta\rangle\ge 0,\ldots,\langle(\overbrace{1,\ldots,1}^{l_{n-d+k-1}},0\ldots,0)^T,\beta\rangle\ge 0,\\\langle(1,\ldots,1)^T,\beta\rangle\ge 0,\langle(1,\ldots,1)^T,\beta\rangle\le 0\}\big)^\circ\end{aligned}$$

$$
\begin{aligned}
&= -\operatorname{pos}\left\{(1,\dots,1,0,\dots,0)^T,\dots,(1,\dots,1,0\dots,0)^T,(1,\dots,1)^T,(-1,\dots-1)^T\right\}\\
&= \{x\in\mathbb{R}^d : x_1=\ldots=x_{l_1}\le x_{l_1+1}=\ldots=x_{l_2}\le\ldots\le x_{l_{n-d+k-1}+1}=\ldots=x_n\}\\
&=: G.
\end{aligned}
$$

Note that G is just a shorthand notation for the k-face $D_\sigma^A(l_1,\dots,l_{n-d+k-1})$ of the Weyl tessellation of type A_{n-1}, in the special case $\sigma(i)=i$. Using Lemma 2.12 and (A2), we get

$$
F=\{0\}\Leftrightarrow \operatorname{relint}(G)\cap L^\perp=\emptyset\Leftrightarrow G\cap L^\perp=\{0\}, \tag{4.6}
$$

which is the special case $\sigma(i)=i$ of Lemma 4.10. The general case is obtained in the same way as in the proof of Lemma 3.15. We can replace $y_1,\dots,y_n$ by $y_{\sigma(1)},\dots,y_{\sigma(n)}$ in (4.6) for $\sigma\in\mathcal{S}_n$ and get

$$
F_\sigma^A(l_1,\dots,l_{n-d+k-1})=\{0\}\Leftrightarrow G\cap(L_\sigma)^\perp=\{0\},
$$

where $L_\sigma:=\{\beta\in\mathbb{R}^n:\beta_1y_{\sigma(1)}+\ldots+\beta_ny_{\sigma(n)}=0\}$. In the same way, we are able to derive that

$$
G\cap(L_\sigma)^\perp=\{0\}\Leftrightarrow g^{-1}\,G\cap L^\perp=\{0\}\Leftrightarrow C_\sigma^A(l_1,\dots,l_{n-d+k-1})\cap L^\perp=\{0\}
$$

for the transformation $g=g_\sigma\in\mathcal{G}(A_{n-1})$, and thus, $g^{-1}=g_{\sigma^{-1}}$. □

Proof of Theorem 4.3. Let $1\le k\le d$ let $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy one of the equivalent general position assumptions (A1) or (A2). The proof follows that of Theorem 3.3. Using Proposition 4.6 and Lemma 4.10, we get

$$
\begin{aligned}
&\sum_{F\in\mathcal{F}_k^A(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}_d^A(y_1,\dots,y_n)}\mathbb{1}_{\{F\subseteq C\}}\\
&=\sum_{1\le l_1<\ldots<l_{n-d+k-1}\le n-1} l_1!(l_2-l_1)!\cdots(n-l_{n-d+k-1})!\sum_{\sigma\in\mathcal{S}_n}\mathbb{1}_{\{F_\sigma^A(l_1,\dots,l_{n-d+k-1})\neq\{0\}\}}\\
&=\sum_{1\le l_1<\ldots<l_{n-d+k-1}\le n-1} l_1!(l_2-l_1)!\cdots(n-l_{n-d+k-1})!\sum_{\sigma\in\mathcal{S}_n}\mathbb{1}_{\{C_\sigma^A(l_1,\dots,l_{n-d+k-1})\cap L^\perp\neq\{0\}\}}\\
&=\sum_{\sigma\in\mathcal{S}_n}\ \sum_{F\in\mathcal{F}_{n-d+k}(C^B_{\varepsilon,\sigma})}\mathbb{1}_{\{F\cap L^\perp\neq\{0\}\}}\\
&=\binom{n-1}{d-k}\frac{n!}{(n-d+k)!}\left(\begin{bmatrix}n-d+k\\n-d+1\end{bmatrix}+\begin{bmatrix}n-d+k\\n-d+3\end{bmatrix}+\ldots\right)
\end{aligned}
$$

which completes the proof. Here, we used that each face $C_\sigma^A(l_1,\dots,l_{n-d+k-1})$ is contained in exactly $l_1!(l_2-l_1)!\cdots(n-l_{n-d+k-1})!$ Weyl chambers C_σ^A. For this fact, we refer to [12, Proof of Theorem 2.8]. Since (A2) is satisfied, we were able to apply Theorem 4.8. □

5. Expectations for random Weyl cones

In this section, we will formally define the random Weyl cones $\mathcal{D}_n^B$ and $\mathcal{D}_n^A$, whose definitions was already sketched in the introduction. Furthermore, we want to evaluate the expected size functionals $Y_{k,j}$ of $\mathcal{D}_n^B$ and $\mathcal{D}_n^A$, like Hug and Schneider did in [5, Theorem 4.1] for the Random Schlfli cone, and thus, derive results for the expected geometric functionals, which we introduced in Section 2.3.

5.1. **Random Weyl cones.** In this Section, we will define the random cones chosen from the Weyl tessellations we introduced in the Sections 3 and 4.

Type B_n. At first, we consider the B_n-case.

Definition 5.1 (Random Weyl cone of type B_n)**.** Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy one of the equivalent general position assumptions (B1) or (B2) a.s. Then the *random Weyl cone $\mathcal{D}_n^B$ of type B_n* is obtained as follows: Among the cones of the random Weyl tessellation $\mathcal{W}^B(Y_1, \dots, Y_n)$ we pick one uniformly at random.

Due to Corollary 3.4, the number of Weyl cones in the induced tessellation $\mathcal{W}^B(Y_1, \dots, Y_n)$ is a.s. constant and the distribution of $\mathcal{D}_n^B$ is therefore given by

$$\mathbb{P}(\mathcal{D}_n^B \in B) = \int_{(\mathbb{R}^d)^n} \frac{1}{D^B(n,d)} \sum_{C \in \mathcal{F}_d^B(y_1,\dots,y_n)} \mathbb{1}_B(C)\, \mathbb{P}_Y(\mathrm{d}(y_1, \dots, y_n)) \tag{5.1}$$

for $B \in \mathcal{B}(\mathcal{PC}^d)$, where $\mathbb{P}_Y$ here and from now on denotes the joint probability law of $(Y_1, \dots, Y_n)$ on $(\mathbb{R}^d)^n$. The following lemma states that under some mild assumptions on the distribution of $(Y_1, \dots, Y_n)$ the general position assumptions (B1) and (B2) are a.s. satisfied. We will postpone the proof to Section 6.2.

Lemma 5.2. *Let μ be a σ-finite Borel measure on $\mathbb{R}^d$ that assigns measure zero to each affine hyperplane, i.e. each $(d-1)$-dimensional affine subspace. Furthermore, let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$ having a joint μ^n-density on $(\mathbb{R}^d)^n$. Then $Y_1, \dots, Y_n$ satisfy the general position assumptions (B1) and (B2) almost surely.*

Remark 5.3. Special cases of the measure μ are the Lebesgue measure λ^d on $\mathbb{R}^d$ and the spherical Lebesgue measure σ_{d-1} on $\mathbb{S}^{d-1}$, since both assign measure zero to affine hyperplanes. Thus, in the interesting case where $Y_1, \dots, Y_n$ have a joint $(\lambda^d)^n$- or a joint $(\sigma_{d-1})^n$-density, the general position assumptions (B1) and (B2) are a.s. satisfied. This includes the case where $Y_1, \dots, Y_n$ are independent and uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$.

If we additionally assume $Y_1, \dots, Y_n$ to be *symmetrically exchangeable*, that is

$$(Y_1, \dots, Y_n) \stackrel{d}{=} (\varepsilon_1 Y_{\sigma(1)}, \dots, \varepsilon_n Y_{\sigma(n)})$$

for every $\varepsilon \in \{\pm 1\}^n$ and $\sigma \in \mathcal{S}_n$, we can find an equivalent Definition of $\mathcal{D}_n^B$. At first, we need the following proposition, which is an analogue to Theorem 8.2.1 in [16], going back to the results of Wendel in [20].

Proposition 5.4. *Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$, which are symmetrically exchangeable and satisfy one of the equivalent general position assumptions (B1) or (B2) a.s. Then*

$$q_n^{(d)} := \mathbb{P}(\{v \in \mathbb{R}^d : \langle v, Y_1 \rangle \le \ldots \le \langle v, Y_n \rangle \le 0\} \neq \{0\}) = \frac{D^B(n,d)}{2^n n!}.$$

Proof. Since the random vector $(Y_1, \dots, Y_n)$ is symmetrically exchangeable, we get

$$\begin{aligned} q_n^{(d)} &= \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\{\{v \in \mathbb{R}^d : \langle v, y_1 \rangle \le \ldots \le \langle v, y_n \rangle \le 0\} \neq \{0\}\}}\, \mathbb{P}_Y(\mathrm{d}(y_1, \dots, y_n)) \\ &= \int_{(\mathbb{R}^d)^n} \frac{1}{2^n n!} \sum_{(\varepsilon,\sigma) \in \{\pm 1\}^n \times \mathcal{S}_n} \mathbb{1}_{\{D^B_{\varepsilon,\sigma} \neq \{0\}\}}\, \mathbb{P}_Y(\mathrm{d}(y_1, \dots, y_n)) \\ &= \frac{D^B(n,d)}{2^n n!}. \end{aligned}$$

We used Corollary 3.4 in last equation, which holds for $\mathbb{P}_Y$-a.e. $(y_1, \dots, y_n)$. ∎

Proposition 5.5. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$, which are symmetrically exchangeable satisfy one of the equivalent general position assumptions (B1) or (B2) a.s. Let $\mathcal{G}$ be defined as the random cone whose distribution is that of $\{v\in\mathbb{R}^d:\langle v,Y_1\rangle\le\dots\le\langle v,Y_n\rangle\le 0\}$ conditioned on the event that $\{v\in\mathbb{R}^d:\langle v,Y_1\rangle\le\dots\le\langle v,Y_n\rangle\le 0\}$ is different from $\{0\}$. Then*

$$\mathcal{G}\overset{d}{=}\mathcal{D}_n^B.$$

Proof. Thus, $\mathcal{G}$ is a random cone with distribution given by $\mathbb{P}(\mathcal{G}=\{0\})=0$ and for $B\in\mathcal{B}(\mathcal{PC}^d\setminus\{\{0\}\})$ by

$$\begin{aligned}
\mathbb{P}(\mathcal{G}\in B)&=\frac{1}{q_n^{(d)}}\mathbb{P}(\{v\in\mathbb{R}^d:\langle v,Y_1\rangle\le\dots\le\langle v,Y_n\rangle\le 0\}\in B)\\
&=\frac{2^n n!}{D^B(n,d)}\int_{(\mathbb{R}^d)^n}\mathbb{1}_B(\{v\in\mathbb{R}^d:\langle v,y_1\rangle\le\ldots\le\langle v,y_n\rangle\le 0\})\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n)))\\
&=\frac{2^n n!}{D^B(n,d)}\int_{(\mathbb{R}^d)^n}\frac{1}{2^n n!}\sum_{(\varepsilon,\sigma)\in\{\pm1\}^n\times\mathcal{S}_n}\mathbb{1}_B(D^B_{\varepsilon,\sigma})\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))\\
&=\int_{(\mathbb{R}^d)^n}\frac{1}{D^B(n,d)}\sum_{C\in\mathcal{F}_d^B(y_1,\dots,y_n)}\mathbb{1}_B(C)\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))\\
&=\mathbb{P}(\mathcal{D}_n^B\in B).
\end{aligned}$$

Note that we used the symmetrical exchangeability of $(Y_1,\dots,Y_n)$. ◻

Similar to the Cover-Efron cone in [5], we may define the cone, which is dual to $\mathcal{D}_n^B$ in distribution.

Definition 5.6. Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$, which are symmetrically exchangeable and satisfy one of the equivalent general position assumptions (B1) or (B2). Then the random cone $\mathcal{C}_n^B$ is defined as the cone whose distribution is that of $\operatorname{pos}\{Y_1-Y_2,\dots,Y_{n-1}-Y_n,Y_n\}$ conditioned on the event that $\operatorname{pos}\{Y_1-Y_2,\dots,Y_{n-1}-Y_n,Y_n\}$ is not equal to $\mathbb{R}^d$.

Proposition 5.7. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$, which are symmetrically exchangeable and satisfy one of the equivalent general position assumptions (B1) or (B2) a.s. Then*

$$\mathcal{C}_n^B\overset{d}{=}\big(\mathcal{D}_n^B\big)^\circ.$$

Proof. The distribution of $\mathcal{C}_n^B$ satisfies $\mathbb{P}(\mathcal{C}_n^B=\mathbb{R}^d)=0=\mathbb{P}(\mathcal{D}_n^B=\{0\})=\mathbb{P}\big(\big(\mathcal{D}_n^B\big)^\circ=\mathbb{R}^d\big)$. Moreover, for $B\in\mathcal{B}(\mathcal{PC}^d\setminus\{\{0\}\})$, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{C}_n^B\in B)&=\frac{\mathbb{P}(\operatorname{pos}\{Y_1-Y_2,\dots,Y_{n-1}-Y_n,Y_n\}\in B)}{\mathbb{P}(\operatorname{pos}\{Y_1-Y_2,\dots,Y_{n-1}-Y_n,Y_n\}\ne\mathbb{R}^d)}\\
&=\frac{1}{q_n^{(d)}}\int_{(\mathbb{R}^d)^n}\mathbb{1}_B(\operatorname{pos}\{y_1,-y_2,\dots,y_{n-1}-y_n,y_n\})\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))\\
&=\frac{2^n n!}{D^B(n,d)}\int_{(\mathbb{R}^d)^n}\frac{1}{2^n n!}\sum_{(\varepsilon,\sigma)\in\mathcal{S}_n\times\{\pm1\}^n}\mathbb{1}_B\big(\operatorname{pos}\{\varepsilon_1y_{\sigma(1)}-\varepsilon_2y_{\sigma(2)},\dots,\varepsilon_ny_{\sigma(n)}\}\big)\\
&\quad\times\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))\\
&=\int_{(\mathbb{R}^d)^n}\frac{1}{D^B(n,d)}\sum_{(\varepsilon,\sigma)\in\mathcal{S}_n\times\{\pm1\}^n}\mathbb{1}_B\big(\big(D^B_{\varepsilon,\sigma}\big)^\circ\big)\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))
\end{aligned}$$

$$= \int_{(\mathbb{R}^d)^n} \frac{1}{D^B(n,d)} \sum_{C \in \mathcal{F}_d^B(y_1,\dots,y_n)} \mathbb{1}_B\big(C^\circ\big)\, \mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))$$
$$= \mathbb{P}\big(\big(\mathcal{D}_n^B\big)^\circ \in B\big),$$

where we used (2.7) and the symmetric exchangeability. The last equation follows from (5.1). □

Type A_{n-1}. Now, we introduce the analogous random cones chosen from the Weyl tessellation of type A_{n-1}, which we defined in Section 4.2.

Definition 5.8 (Random Weyl cone of type A_{n-1})**.** Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy one of the equivalent general position assumptions (A1) or (A2) a.s. Then the *random Weyl cone $\mathcal{D}_n^A$ of type A_{n-1}* is obtained as follows: Among the cones of the random Weyl tessellation $\mathcal{W}^A(Y_1, \dots, Y_n)$ we pick one uniformly at random.

Due to Corollary 4.4, the number of Weyl cones in the induced tessellation $\mathcal{W}^A(Y_1, \dots, Y_n)$ is a.s. constant. Then, the distribution of $\mathcal{D}_n^A$ is given by

$$\mathbb{P}(\mathcal{D}_n^A \in B) = \int_{(\mathbb{R}^d)^n} \frac{1}{D^A(n,d)} \sum_{C \in \mathcal{F}_d^A(y_1,\dots,y_n)} \mathbb{1}_B(C)\, \mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n)) \tag{5.2}$$

for $B \in \mathcal{B}(\mathcal{PC}^d)$. The following lemma states that the same mild conditions on the distribution of $(Y_1, \dots, Y_n)$ imply that (A1) and (A2) are a.s. satisfied.

Lemma 5.9. *Let μ be a σ-finite Borel measure on $\mathbb{R}^d$ that assigns measure zero to each affine hyperplane. Furthermore, let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$ having a joint μ^n-density on $(\mathbb{R}^d)^n$. Then $Y_1, \dots, Y_n$ satisfy the general position assumptions (A1) and (A2) almost surely.*

Proof. This follows from Lemma 5.2, since (B1) implies (A1) and (B2) implies (A2). □

If we additionally assume $Y_1, \dots, Y_n$ to be *exchangeable*, that is

$$(Y_1, \dots, Y_n) \stackrel{d}{=} (Y_{\sigma(1)}, \dots, Y_{\sigma(n)})$$

for every $\sigma \in \mathcal{S}_n$, we can provide the following equivalent construction of $\mathcal{D}_n^A$.

Proposition 5.10. *Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$, which are exchangeable and satisfy one of the equivalent general position assumptions (A1) or (A2) a.s. Let $\mathcal{G}$ be defined as the random cone whose distribution is that of $\{v \in \mathbb{R}^d : \langle v, Y_1\rangle \le \dots \le \langle v, Y_n\rangle\}$ conditioned on the event that $\{v \in \mathbb{R}^d : \langle v, Y_1\rangle \le \dots \le \langle v, Y_n\rangle\}$ is different from $\{0\}$. Then,*

$$\mathcal{G} \stackrel{d}{=} \mathcal{D}_n^A.$$

The equivalence is proven in the same way as Proposition 5.5. Note that under the assumptions of Proposition 5.10 we have

$$\mathbb{P}(\{v \in \mathbb{R}^d : \langle v, Y_1\rangle \le \ldots \le \langle v, Y_n\rangle\} \ne \{0\}) = \frac{D^A(n,d)}{n!}.$$

Again, we may define the cone, which is dual to $\mathcal{D}_n^A$ in distribution.

Definition 5.11. Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$, which are exchangeable and satisfy one of the equivalent general position assumptions (A1) or (A2) a.s. Then the random cone $\mathcal{C}_n^A$ is defined as the cone whose distribution is that of $\operatorname{pos}\{Y_1 - Y_2, \dots, Y_{n-1} - Y_n\}$ conditioned on the event that $\operatorname{pos}\{Y_1 - Y_2, \dots, Y_{n-1} - Y_n\}$ it is not equal to $\mathbb{R}^d$.

Proposition 5.12. *Let $Y_1, \dots, Y_n$ be random vectors in $\mathbb{R}^d$, which are exchangeable and satisfy one of the equivalent general position assumptions (A1) or (A2) a.s. Then*

$$\mathcal{C}_n^A \stackrel{d}{=} (\mathcal{D}_n^A)^\circ.$$

Proof. The proof is similar to the proof of Proposition 5.7 and is left to the reader. □

5.2. **Expected size functionals of random Weyl cones.** In this section, we want to prove our main results on the expected size functionals of the random Weyl cones, which we already stated in the introduction as Theorem 1.3 and Theorem 1.4. Again, we divide this section into the results for type B_n and the results for type A_{n-1}.

Type B_n. At first, we need to state a result on the faces of the Weyl mosaic, induced in a linear subspace. For this, we introduce the following notation. Let $U \subseteq \mathbb{R}^d$ be a k-dimensional linear subspace in $\mathbb{R}^d$. Recall from Section 2.4 that the hyperplane arrangement induced by $\mathcal{A}^B(y_1, \dots, y_n)$ in U is defined as $\mathcal{A}^B|_U(y_1, \dots, y_n) = \{H \cap U : H \in \mathcal{A}^B(y_1, \dots, y_n)\}$. The induced arrangement $\mathcal{A}^B|_U(y_1, \dots, y_n)$ is explicitly given by the following hyperplanes in U:

$$\begin{aligned} \big(\Pi_U(y_i) + \Pi_U(y_j)\big)^\perp \cap U, &\quad 1 \le i < j \le n, \\ \big(\Pi_U(y_i) - \Pi_U(y_j)\big)^\perp \cap U, &\quad 1 \le i < j \le n, \\ \Pi_U(y_i)^\perp \cap U, &\quad 1 \le i \le n. \end{aligned}$$

By definition, the induced Weyl tessellation in U, which we will denote by $\mathcal{W}^B|_U(y_1, \dots, y_n)$, consists of the cones of the conical tessellation in U generated by the hyperplane arrangement $\mathcal{A}^B|_U(y_1, \dots, y_n)$. We denote the set of j-faces of $\mathcal{W}^B|_U(y_1, \dots, y_n)$ by $\mathcal{F}_j^B|_U(y_1, \dots, y_n)$. To state an explicit representation of these faces, we define the cones

$$F^B_{\varepsilon,\sigma}|_U(l_1, \dots, l_{n-k+j}) := F^B_{\varepsilon,\sigma}(l_1, \dots, l_{n-k+j}) \cap U,$$

where $\varepsilon \in \{\pm 1\}^n$, $\sigma \in \mathcal{S}_n$, $1 \le j \le k \le d$, and $1 \le l_1 < \ldots < l_{n-k+j} \le n$. Let f_i' be the linear functionals on U given by $f_i' = f_i'(v) := \langle v, \Pi_U(y_i)\rangle$ for $i = 1, \dots, n$. Since

$$f_i(v) = \langle v, y_i\rangle = \langle v, \Pi_U(y_i)\rangle + \langle v, y_i - \Pi_U(y_i)\rangle = \langle v, \Pi_U(y_i)\rangle = f_i'(v)$$

holds for all $v \in U$, we have the explicit representation

$$\begin{aligned} F^B_{\varepsilon,\sigma}|_U(l_1, \dots, l_{n-k+j}) = \big\{v \in U : \varepsilon_1 f'_{\sigma(1)} = \ldots = \varepsilon_{l_1} f'_{\sigma(l_1)} \le \varepsilon_{l_1+1} f'_{\sigma(l_1+1)} = \ldots = \varepsilon_{l_2} f'_{\sigma(l_2)} \\ \le \ldots \le \varepsilon_{l_{n-k+j-1}+1} f'_{\sigma(l_{n-k+j-1}+1)} = \ldots = \varepsilon_{l_{n-k+j}} f'_{\sigma(l_{n-k+j})} \\ \le f'_{\sigma(l_{n-k+j}+1)} = \ldots = f'_{\sigma(n)} = 0\big\}, \end{aligned} \tag{5.3}$$

We shall see below that if not $\{0\}$, the cones $F^B_{\varepsilon,\sigma}|_U(l_1, \dots, l_{n-k+j})$ are the j-faces of the induced Weyl mosaic $\mathcal{W}^B|_U(y_1, \dots, y_n)$.

Lemma 5.13. *Let $1 \le j \le k \le d$ and let $y_1, \dots, y_n \in \mathbb{R}^d$ satisfy the general position assumption (B1). Furthermore, let $U \in G(d, k)$ be in general position with respect to the hyperplane arrangement $\mathcal{A}^B(y_1, \dots, y_n)$. Then the following holds:*

(i) For every j-face $F_j \in \mathcal{F}_j^B|_U(y_1, \dots, y_n)$ of the tessellation $\mathcal{W}^B|_U(y_1, \dots, y_n)$ there is a unique $(d-k+j)$-face $F \in \mathcal{F}^B_{d-k+j}(y_1, \dots, y_n)$ containing F_j and satisfying $F_j = F \cap U$.

(ii) If $F \in \mathcal{F}^B_{d-k+j}(y_1, \dots, y_n)$ and $F \cap U \neq \{0\}$, then $F \cap U \in \mathcal{F}_j^B|_U(y_1, \dots, y_n)$.

Proof. At first, we show that the projections $\Pi_U(y_1),\dots,\Pi_U(y_n)$ satisfy the general position assumption (B1). Take some $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$. Condition (B1) implies that

$$(\varepsilon_1 y_{\sigma(1)}-\varepsilon_2 y_{\sigma(2)})^\perp,\dots,(\varepsilon_{n-1}y_{\sigma(n-1)}-\varepsilon_n y_{\sigma(n)})^\perp,(\varepsilon_n y_{\sigma(n)})^\perp$$

are in general position. Since U is in general position with respect to the arrangement $\mathcal{A}^B(y_1,\dots,y_n)$ containing these hyperplanes, we have that the following hyperplanes in U

$$U\cap(\varepsilon_1 y_{\sigma(1)}-\varepsilon_2 y_{\sigma(2)})^\perp,\dots,U\cap(\varepsilon_{n-1}y_{\sigma(n-1)}-\varepsilon_n y_{\sigma(n)})^\perp,U\cap(\varepsilon_n y_{\sigma(n)})^\perp$$

are in general position in U. Since $U\cap(z^\perp)=(\Pi_U(z))^\perp\cap U$ for every $z\in\mathbb{R}^d$, it follows that

$$\big(\varepsilon_1\Pi_U(y_{\sigma(1)})-\varepsilon_2\Pi_U(y_{\sigma(2)})\big)^\perp\cap U,\dots,\big(\varepsilon_{n-1}\Pi_U(y_{\sigma(n-1)})-\varepsilon_n\Pi_U(y_{\sigma(n)})\big)^\perp\cap U,\big(\varepsilon_n\Pi_U(y_{\sigma(n)})\big)^\perp\cap U$$

are in general position in U. Equivalently, the vectors

$$\varepsilon_1\Pi_U(y_{\sigma(1)})-\varepsilon_2\Pi_U(y_{\sigma(2)}),\dots,\varepsilon_{n-1}\Pi_U(y_{\sigma(n-1)})-\varepsilon_n\Pi_U(y_{\sigma(n)}),\varepsilon_n\Pi_U(y_{\sigma(n)})$$

are in general position in U. This means that, under the given assumptions, (B1) is satisfied for $\Pi_U(y_1),\dots,\Pi_U(y_n)$.

Now, we prove part (i). Let $F_j\in\mathcal{F}^B_j|_U(y_1,\dots,y_n)$. We can apply Proposition 3.6(i) in the ambient linear subspace U to the projections $\Pi_U(y_1),\dots,\Pi_U(y_n)$. It follows from this proposition and the representation (5.3) that there are $1\le l_1<\ldots\le l_{n-k+j}\le n$ and $\varepsilon\in\{\pm1\}^n$, $\sigma\in\mathcal{S}_n$, such that

$$F_j=F^B_{\varepsilon,\sigma}|_U(l_1,\dots,l_{n-k+j})=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})\cap U.$$

Now, we define $F:=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})$. Note that $F\neq\{0\}$ because $F_j\neq\{0\}$. Since (B1) is satisfied for $y_1,\dots,y_n$, Proposition 3.6(ii) yields that $F\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)$. It follows from the construction that $F\cap U=F_j$.

The uniqueness of $F\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)$ such that $F\cap U=F_j$ follows from our general position assumptions or rather from the fact that the projections $\Pi_U(y_1),\dots,\Pi_U(y_n)$ satisfy the assumption (B1) in U. We will sketch the idea of the proof. Suppose there is another face $G\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)$ with $G\cap U=F_j$. By Proposition 3.6(i) this means that there are $1\le i_1<\ldots<i_{n-k+j}\le n$ and $\delta\in\{\pm1\}^n$, $\pi\in\mathcal{S}_n$, such that $G=F^B_{\delta,\pi}(i_1,\dots,i_{n-k+j})$. It follows that

$$F^B_{\delta,\pi}(i_1,\dots,i_{n-k+j})\cap U=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})\cap U=F_j.$$

Consequently,

$$F^B_{\delta,\pi}|_U(i_1,\dots,i_{n-k+j})=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})|_U=F_j\neq\{0\}.$$

Applying Proposition 3.9 in the ambient space U to the projected vectors $\Pi_U(y_1),\dots,\Pi_U(y_n)$, we get $\varepsilon=\delta$, $\sigma=\pi$, and $l_j=i_j$ for all admissible j. But this implies that $F=G$. This proves (i).

Now we will prove part (ii). Suppose $F\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)$ satisfying $F\cap U\neq\{0\}$. Proposition 3.6(i) implies that there are $1\le l_1<\ldots<l_{n-k+j}\le n$ and $\varepsilon\in\{\pm1\}^n$, $\sigma\in\mathcal{S}_n$, such that $F=F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})$. As we have seen in (5.3) above, it follows

$$\{0\}\neq F\cap U=F^B_{\varepsilon,\sigma}|_U(l_1,\dots,l_{n-k+j}).$$

Since $\Pi_U(y_1),\dots,\Pi_U(y_n)$ satisfy the condition (B1) in U, we can apply Proposition 3.6(ii) in the linear subspace U, which yields that $F\cap U$ is j-face of the induced Weyl mosaic $\mathcal{W}^B|_U(y_1,\dots,y_n)$. $\square$

The next theorem is our main result and gives a formula for the expected size functionals $Y_{k,j}$ of a random Weyl cone $\mathcal{D}_n^B$. We stated it as Theorem 1.3 in the introduction and will restate it here.

Theorem 5.14. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy one of the equivalent general position assumptions (B1) or (B2) a.s. Let $\mathcal{D}_n^B$ be a random Weyl cone of type B_n in $\mathbb{R}^d$ as defined in Section 5.1. Then*

$$\mathbb{E}\, Y_{d-k+j,\,d-k}(\mathcal{D}_n^B) = \frac{2^{k-j}\binom{n}{k-j} D^B(n-k+j,j)}{2D^B(n,d)}\frac{n!}{(n-k+j)!} \tag{5.4}$$

holds for all $1 \le j \le k \le d$.

Proof. Suppose $1 \le j \le k \le d$. Let $\mathbb{P}_Y$ be the joint probability law of $(Y_1,\dots,Y_n)$ on $(\mathbb{R}^d)^n$. Using the definition of the size functional and (5.1), we get

$$\begin{aligned}\mathbb{E}\, Y_{d-k+j,\,k-j}(\mathcal{D}_n^B) &= \mathbb{E}\sum_{F\in\mathcal{F}_{d-k+j}(\mathcal{D}_n^B)} U_{d-k}(F)\\ &= \int_{(\mathbb{R}^d)^n} \frac{1}{D^B(n,d)} \sum_{C\in\mathcal{F}_d^B(y_1,\dots,y_n)}\ \sum_{F\in\mathcal{F}^B_{d-k+j}(C)} U_{d-k}(F)\,\mathbb{P}_Y(\mathrm{d}(y_1,\dots,y_n))\end{aligned}$$

In order to apply the definition (2.2) of the quermassintegral U_{d-k} we need to verify that the $(d-k+j)$-faces $F\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)$ are a.s. no linear subspaces. Proposition 3.6(i) yields that every such $(d-k+j)$-face can be represented in the form $F = F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j})$ for suitable $\varepsilon\in\{\pm1\}^n$, $\sigma\in\mathcal{S}_n$ and $1\le l_1<\ldots<l_{n-k+j}\le n$ (see (3.5)). As we have seen several times before, assumption (B1) implies that the linear hull of F is a $(d-k+j)$-dimensional subspace and is given by

$$L_{d-k+j} := \{v\in\mathbb{R}^d : \varepsilon_1 f_{\sigma(1)} = \ldots = \varepsilon_{l_1} f_{\sigma(l_1)},\dots,\varepsilon_{l_{n-k+j}+1} f_{\sigma(l_{n-k+j}+1)} = \ldots = \varepsilon_n f_{\sigma(n)} = 0\}.$$

Note that the assumptions $n\ge d$ and $1\le j\le k\le d$ imply that $n>k-j$, and thus, $n-k+j\ge 1$. Therefore, there are at least two groups of equations in the defining condition of $\operatorname{lin} F$ (In the special case $l_1 = n$, the last group of equations is empty, but the following argument still holds). Suppose that F is a linear subspace, then we have $F^B_{\varepsilon,\sigma}(l_1,\dots,l_{n-k+j}) = L_{d-k+j}$. Due to the form of the representation of F, this implies

$$L_{d-k+j} \subseteq \big(\varepsilon_{l_1} y_{\sigma(l_1)} - \varepsilon_{l_1+1} y_{\sigma(l_1+1)}\big)^-$$

and therefore even

$$L_{d-k+j} \subseteq \big(\varepsilon_{l_1} y_{\sigma(l_1)} - \varepsilon_{l_1+1} y_{\sigma(l_1+1)}\big)^\perp.$$

Then, we have

$$L_{d-k+j} \cap \big(\varepsilon_{l_1} y_{\sigma(l_1)} - \varepsilon_{l_1+1} y_{\sigma(l_1+1)}\big)^\perp = L_{d-k+j},$$

but assumption (B1) implies that the left-hand side is a subspace of dimension $d-k+j-1$, which is a contradiction. Note that in the case $l_1 = n$, we may replace $\varepsilon_{l_1} y_{\sigma(l_1)} - \varepsilon_{l_1+1} y_{\sigma(l_1+1)}$ by $\varepsilon_{l_1} y_{\sigma(l_1)}$ in the argument above, which leads to the same result.

Now, we can apply (2.2) and then interchange the integral and the sums. This yields

$$\mathbb{E}\, Y_{d-k+j,\,k-j}(\mathcal{D}_n^B) = \frac{1}{2D^B(n,d)} \int_{(\mathbb{R}^d)^n} \sum_{F\in\mathcal{F}^B_{d-k+j}(y_1,\dots,y_n)}\ \sum_{C\in\mathcal{F}^B_d(y_1,\dots,y_n)} \mathbb{1}_{\{F\subseteq C\}}$$

$$
\begin{aligned}
&\times \int_{G(d,k)} \mathbb{1}_{\{F\cap U\neq\{0\}\}}\, \nu_k(\mathrm{d}U)\, \mathbb{P}_Y(\mathrm{d}(y_1,\ldots,y_n))\\
&= \frac{1}{2D^B(n,d)} \int_{(\mathbb{R}^d)^n} \int_{G(d,k)} \sum_{F\in\mathcal{F}^B_{d-k+j}(y_1,\ldots,y_n)} \mathbb{1}_{\{F\cap U\neq\{0\}\}} \sum_{C\in\mathcal{F}^B_d(y_1,\ldots,y_n)} \mathbb{1}_{\{F\subseteq C\}}\\
&\times \nu_k(\mathrm{d}U)\, \mathbb{P}_Y(\mathrm{d}(y_1,\ldots,y_n)). \qquad (5.5)
\end{aligned}
$$

Our goal is to show that the sums inside the integrals are constant for ν_k-almost every $U\in G(d,k)$ and $\mathbb{P}_Y$-almost every $(y_1,\ldots,y_n)\in(\mathbb{R}^d)^n$. Using Lemma 5.13, we obtain

$$
\sum_{F\in\mathcal{F}^B_{d-k+j}(y_1,\ldots,y_n)} \mathbb{1}_{\{F\cap U\neq\{0\}\}} \sum_{C\in\mathcal{F}^B_d(y_1,\ldots,y_n)} \mathbb{1}_{\{F\subseteq C\}} = \sum_{F_j\in\mathcal{F}^B_j|_U(y_1,\ldots,y_n)} \sum_{D\in\mathcal{F}^B_k|_U(y_1,\ldots,y_n)} \mathbb{1}_{\{F_j\subseteq D\}} \tag{5.6}
$$

for almost every $U\in G(d,k)$ and $\mathbb{P}_Y$-almost every $(y_1,\ldots,y_n)$. Indeed, by Lemma 5.13 there is a one-to-one correspondence between the pairs $F\subseteq C$ such that $F\cap U\neq\{0\}$ and the pairs $F_j\subseteq D$ as above. Note that Lemma 5.13 was applicable, since ν_k-almost every $U\in G(d,k)$ is in general position with respect to the arrangement $\mathcal{A}^B(y_1,\ldots,y_n)$, due to Remark 2.15, and almost every set of vectors $(y_1,\ldots,y_n)$ satisfies the general position assumption (B1).

Applying Theorem 3.3 to the ambient linear subspace U instead of $\mathbb{R}^d$ and the projections $\Pi_U(y_1),\ldots,\Pi_U(y_n)$ instead of $y_1,\ldots,y_n$, we obtain

$$
\sum_{F_j\in\mathcal{F}^B_j|_U(y_1,\ldots,y_n)} \sum_{D\in\mathcal{F}^B_k|_U(y_1,\ldots,y_n)} \mathbb{1}_{\{F_j\subseteq D\}} = 2^{k-j}\binom{n}{k-j}\frac{n!}{(n-k+j)!}D^B(n-k+j,j). \tag{5.7}
$$

To see that Theorem 3.3 is applicable, note that ν_k-a.e. U is in general position with respect to the arrangement $\mathcal{A}^B(y_1,\ldots,y_n)$ and hence the projections $\Pi_U(y_1),\ldots,\Pi_U(y_n)$ satisfy assumption (B1) as we have shown in the proof of Lemma 5.13.

Inserting (5.7) and (5.6) into (5.5), we arrive at

$$
\mathbb{E}\, Y_{d-k+j,\,k-j}(\mathcal{D}^B_n) = \frac{1}{2D^B(n,d)}\cdot 2^{k-j}\binom{n}{k-j}\frac{n!}{(n-k+j)!}D^B(n-k+j,j),
$$

which completes the proof. □

This theorem yields the expected number of faces, the expected quermassintegral and the expected intrinsic volumes of a random Weyl cone $\mathcal{D}^B_n$, as mentioned in the introduction. We will restate the results here. Under the additional assumption that $Y_1,\ldots,Y_n$ are symmetrically exchangeable, we obtain the same properties for the cone $\mathcal{C}^B_n$ defined in Definition 5.6, which is dual to $\mathcal{D}^B_n$ in distribution.

Corollary 5.15. *Let $Y_1,\ldots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (B1) or (B2) a.s. For $j=1,\ldots,d$, the expected number of j-faces of the random Weyl cone $\mathcal{D}^B_n$ of type B_n is given by*

$$
\mathbb{E}\, f_j(\mathcal{D}^B_n) = \frac{2^{d-j}\binom{n}{d-j}D^B(n-d+j,j)}{D^B(n,d)}\frac{n!}{(n-d+j)!}. \tag{5.8}
$$

If, additionally, we assume $Y_1,\ldots,Y_n$ to be symmetrically exchangeable, the expected number of j-faces of $\mathcal{C}^B_n$ for $j=0,\ldots,d-1$ is given by

$$
\mathbb{E}\, f_j(\mathcal{C}^B_n) = \frac{2^j\binom{n}{j}D^B(n-j,d-j)}{D^B(n,d)}\frac{n!}{(n-j)!}.
$$

Note that (5.8) coincides with the formula derived in Corollary 3.5, which is not surprising.

Proof. Every j-face $F \in \mathcal{F}_j^B(Y_1, \ldots, Y_n)$ is not a linear subspace a.s. Thus, the j-faces of $\mathcal{D}_n^B$ are a.s. not linear subspaces. Then we can use (2.6) and get

$$\mathbb{E}\, f_j(\mathcal{D}_n^B) = 2\,\mathbb{E}\, Y_{j,0}(\mathcal{D}_n^B).$$

Using (5.4) with $k = d$, yields the desired formula.

The second property follows from $\mathbb{E}\, f_j(\mathcal{C}_n^B) = \mathbb{E}\, f_j((\mathcal{D}_n^B)^\circ)$ and the 1:1-correspondence between the j-faces of a cone and the $(d-j)$-faces of its dual cone. □

Corollary 5.16. *Let $Y_1, \ldots, Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (B1) or (B2) a.s. Then, the expected conical quermassintegrals of the random Weyl cone $\mathcal{D}_n^B$ are given by*

$$\mathbb{E}\, U_j(\mathcal{D}_n^B) = \frac{D^B(n, d-j)}{2D^B(n,d)}.$$

for $j = 0, \ldots, d-1$. For $\mathcal{C}_n^B$ and $j = 1, \ldots, d$, it is given by

$$\mathbb{E}\, U_j(\mathcal{C}_n^B) = \frac{D^B(n,d) - D^B(n,j)}{2D^B(n,d)}$$

if we additionally assume that $Y_1, \ldots, Y_n$ are symmetrically exchangeable.

Proof. Replacing k and j in (5.4) both by $d-j$, we obtain

$$\mathbb{E}\, U_j(\mathcal{D}_n^B) = \mathbb{E}\, Y_{d,j}(\mathcal{D}_n^B) = \mathbb{E}\, Y_{d,\,d-(d-j)}(\mathcal{D}_n^B) = \frac{D^B(n, d-j)}{2D^B(n,d)}.$$

Using (2.3) and the fact that $\mathcal{C}_n^B$ is almost surely pointed, we get

$$\mathbb{E}\, U_j(\mathcal{C}_n^B) = \frac{1}{2} - \mathbb{E}\, U_{d-j}(\mathcal{D}_n^B) = \mathbb{E}\, U_j(\mathcal{C}_n^B) = \frac{D^B(n,d) - D^B(n,j)}{2D^B(n,d)}.$$

□

Corollary 5.17. *Let $Y_1, \ldots, Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (B1) or (B2) a.s. For $j = 1, \ldots, d$, the expected conical intrinsic volumes of the random Weyl cone $\mathcal{D}_n^B$ are given by*

$$\mathbb{E}\, \upsilon_j(\mathcal{D}_n^B) = \frac{B(n, n-d+j)}{D^B(n,d)}.$$

If we additionally assume $Y_1, \ldots, Y_n$ to be symmetrically exchangeable, then for $j = 0, \ldots, d-1$ the expected intrinsic volumes for $\mathcal{C}_n^B$ are given by

$$\mathbb{E}\, \upsilon_j(\mathcal{C}_n^B) = \frac{B(n, n-j)}{D^B(n,d)}$$

and it holds that

$$\mathbb{E}\, \upsilon_0(\mathcal{D}_n^B) = \mathbb{E}\, \upsilon_d(\mathcal{C}_n^B) = \frac{D^B(n,d) - D^B(n, d-1)}{2D^B(n,d)}.$$

Proof. We use the linear relation between the conical quermassintegrals and conical intrinsic volumes given in (2.4). For $j \in \{1, \ldots, d-2\}$, we obtain

$$\mathbb{E}\, \upsilon_j(\mathcal{D}_n^B) = \mathbb{E}\, U_{j-1}(\mathcal{D}_n^B) - \mathbb{E}\, U_{j+1}(\mathcal{D}_n^B) = \frac{D^B(n, d-j+1) - D^B(n, d-j-1)}{2D^B(n,d)} = \frac{B(n, n-d+j)}{D^B(n,d)}.$$

For $j=d-1$ and $j=d$, we get

$$\mathbb{E}\, \upsilon_{d-1}(\mathcal{D}_n^B) = \mathbb{E}\, U_{d-2}(\mathcal{D}_n^B) = \frac{D^B(n,2)}{2D^B(n,d)} = \frac{B(n,n-1)}{D^B(n,d)}$$

and

$$\mathbb{E}\, \upsilon_{d}(\mathcal{D}_n^B) = \mathbb{E}\, U_{d-1}(\mathcal{D}_n^B) = \frac{D^B(n,1)}{2D^B(n,d)} = \frac{1}{D^B(n,d)} = \frac{B(n,n)}{D^B(n,d)}.$$

In the case $j=0$, we get

$$\mathbb{E}\, \upsilon_{0}(\mathcal{D}_n^B) = \mathbb{E}\, \upsilon_{d}(\mathcal{C}_n^B) = \mathbb{E}\, U_{d-1}(\mathcal{C}_n^B) = \frac{D^B(n,d)-D^B(n,d-1)}{2D^B(n,d)}$$

The expected intrinsic volumes for $\mathcal{C}_n^B$ then follow from (2.5). □

In each of these corollaries we see a great similarity to the respective results on the expected geometric functionals of a random Schlfli cone and the random Cover-Efron cone, which Hug and Schneider stated in [5, Section 4]. Note that $\upsilon_d(C) = \sigma_{d-1}(C)/\omega_d$, thus the expected solid angle α of a random Weyl cone $\mathcal{D}_n^B$ is the special case $j=d$ of Corollary 5.17 and is given by

$$\mathbb{E}\, \alpha(\mathcal{D}_n^B) = \mathbb{E}\, \upsilon_d(\mathcal{D}_n^B) = \frac{1}{D^B(n,d)}.$$

Type A_{n-1}. Similarly, we obtain the expected size functionals of random Weyl cone of type A_{n-1}. This is an analogue to Theorem 1.3.

Theorem 5.18. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy one of the equivalent general postition assumptions (A1) or (A2) a.s. Let $\mathcal{D}_n^A$ be a random Weyl cone of type A_{n-1} as defined in Section 5.1. Then*

$$\mathbb{E}\, Y_{d-k+j,\,d-k}(\mathcal{D}_n^A) = \frac{\binom{n-1}{k-j} D^A(n-k+j,j)}{2D^A(n,d)} \frac{n!}{(n-k+j)!} \tag{5.9}$$

holds for all $1\le j\le k\le d$.

Proof. This is proven in the same way as Theorem 5.14 and uses the corresponding results for the Weyl tessellation of type A_{n-1}. □

Thus, we can formulate the same corollaries for the random Weyl cone of type A_{n-1}. We omit the proofs, since they are analogous to the B_n-case. Recall that the random cone $\mathcal{C}_n^A$ is defined in Definition 5.11 and is dual in distribution to $\mathcal{D}_n^A$ if we additionally assume $Y_1,\dots,Y_n$ to be exchangeable.

Corollary 5.19. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (A1) or (A2) a.s. For $j=1,\dots,d$, the expected number of j-faces of the random Weyl cone $\mathcal{D}_n^A$ of type A_{n-1} is given by*

$$\mathbb{E}\, f_j(\mathcal{D}_n^A) = \frac{\binom{n-1}{d-j} D^A(n-d+j,j)}{2D^A(n,d)} \frac{n!}{(n-d+j)!}.$$

If we additionally assume $Y_1,\dots,Y_n$ to be exchangeable, the expected number of j-faces of $\mathcal{C}_n^A$ for $j=0,\dots,d-1$ is given by

$$\mathbb{E}\, f_j(\mathcal{C}_n^A) = \frac{\binom{n-1}{j} D^A(n-j,d-j)}{2D^A(n,d)} \frac{n!}{(n-j)!}.$$

Corollary 5.20. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (A1) or (A2) a.s. The expected conical quermassintegrals of the random Weyl cone $\mathcal{D}_n^A$ are given by*

$$\mathbb{E}\, U_j(\mathcal{D}_n^B) = \frac{D^A(n,d-j)}{2D^A(n,d)}$$

for $j = 0\dots,d-1$. For $\mathcal{C}_n^A$ and $j=1,\dots,d$, it is given by

$$\mathbb{E}\, U_j(\mathcal{C}_n^A) = \frac{D^A(n,d)-D^A(n,j)}{2D^A(n,d)}$$

if we additionally assume that $Y_1,\dots,Y_n$ are exchangeable.

Corollary 5.21. *Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ that satisfy (A1) or (A2) a.s. For $j=1,\dots,d$, the expected conical intrinsic volumes of the random Weyl cone $\mathcal{D}_n^A$ are given by*

$$\mathbb{E}\, \upsilon_j(\mathcal{D}_n^A) = \begin{bmatrix} n \\ n-d+j \end{bmatrix} \frac{1}{D^A(n,d)}.$$

If we additionally assume $Y_1,\dots,Y_n$ to be exchangeable, then for $j=0,\dots,d-1$ the expected intrinsic volumes for $\mathcal{C}_n^A$ are given by

$$\mathbb{E}\, \upsilon_j(\mathcal{C}_n^A) = \begin{bmatrix} n \\ n-j \end{bmatrix} \frac{1}{D^A(n,d)}$$

and it holds that

$$\mathbb{E}\, \upsilon_0(\mathcal{D}_n^A) = \mathbb{E}\, \upsilon_d(\mathcal{C}_n^A) = \frac{D^A(n,d)-D^A(n,d-1)}{2D^A(n,d)}.$$

6. General Positon: Proofs of Theorems 3.2, 4.2 and Lemma 5.2

6.1. **Equivalences of (B1) and (B2), (A1) and (A2).** We will prove the equivalence of the general position assumptions. Recall that $y_1,\dots,y_n \in \mathbb{R}^d$ satisfy (B1) and (B2) if the following holds true.

- (B1) For every $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$ the vectors $\varepsilon_1 y_{\sigma(1)} - \varepsilon_2 y_{\sigma(2)}, \varepsilon_2 y_{\sigma(2)} - \varepsilon_3 y_{\sigma(3)},\dots,\varepsilon_{n-1}y_{\sigma(n-1)} - \varepsilon_n y_{\sigma(n)}, \varepsilon_n y_{\sigma(n)}$ are in general position.
- (B2) The linear subspace $L^\perp$ has dimension d and is in general position with respect to the hyperplane arrangement $\mathcal{A}(B_n)$, where $L := \{\beta\in\mathbb{R}^n : \beta_1 y_1 + \dots + \beta_n y_n = 0\}$.

Proof of Theorem 3.2. At first, we prove that (B2) implies (B1). Let (B2) hold true for $y_1,\dots,y_n \in \mathbb{R}^d$, but suppose (B1) is not satisfied. Then, there exist $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$, such that

$$\varepsilon_1 y_{\sigma(1)} - \varepsilon_2 y_{\sigma(2)},\dots,\varepsilon_{n-1}y_{\sigma(n-1)} - \varepsilon_n y_{\sigma(n)}, \varepsilon_n y_{\sigma(n)}$$

are not in general position. For sake of simplicity, we first assume that $\varepsilon_i = 1$ and $\sigma(i)=i$. Thus, $y_1-y_2,\dots,y_{n-1}-y_n,y_n$ are not in general position. This means that there is a subset of d or fewer linearly dependent vectors. In general, this set is of the form

$$\underbrace{y_1-y_2,\dots,y_{i_1-1}-y_{i_1}}_{\text{group 1}},\underbrace{y_{i_1+1}-y_{i_1+2},\dots,y_{i_2-1}-y_{i_2}}_{\text{group 2}},\dots,\underbrace{y_{i_k+1}-y_{i_k+2},\dots,y_{n-1}-y_n,y_n}_{\text{group } k+1}$$

for a $k\ge n-d$ and suitable indices $1\le i_1<i_2<\ldots<i_k\le n$. Note that each of these groups may be empty and the set consists of $n-k\le d$ vectors. This set is linearly dependent if and only if

there exist numbers λ_i with $i \in \{1,\dots,n\}\backslash\{i_1,\dots,i_k\}$ that do not vanish simultaneously and such that

$$\begin{aligned}0 &= \lambda_1(y_1-y_2)+\ldots+\lambda_{i_1-1}(y_{i_1-1}-y_{i_1})+\lambda_{i_1+1}(y_{i_1+1}-y_{i_1+2})+\ldots+\lambda_{i_2-1}(y_{i_2-1}-y_{i_2})\\ &\quad+\dots+\lambda_{i_k+1}(y_{i_k+1}-y_{i_k+2})+\ldots+\lambda_{n-1}(y_{n-1}-y_n)+\lambda_n y_n\end{aligned}$$

After regrouping the terms, the condition takes the form

$$\begin{aligned}0 &= \lambda_1 y_1+(\lambda_2-\lambda_1)y_2+\ldots+(\lambda_{i_1-1}-\lambda_{i_1-2})y_{i_1-1}+(-\lambda_{i_1-1})y_{i_1}\\ &\quad+\lambda_{i_1+1}y_{i_1+1}+(\lambda_{i_1+2}-\lambda_{i_1+1})y_{i_1+2}+\ldots+(\lambda_{i_2-1}-\lambda_{i_2-2})y_{i_2-1}+(-\lambda_{i_2-1})y_{i_2}\\ &\quad+\dots+\lambda_{i_k+1}y_{i_k+1}+(\lambda_{i_k+2}-\lambda_{i_k+1})y_{i_k+2}+\ldots+(\lambda_n-\lambda_{n-1})y_n,\end{aligned}\tag{6.1}$$

or equivalently,

$$\begin{aligned}\big(&\lambda_1,\lambda_2-\lambda_1,\dots,\lambda_{i_1-1}-\lambda_{i_1-2},-\lambda_{i_1-1},\lambda_{i_1+1},\lambda_{i_1+2}-\lambda_{i_1+1},\dots,\lambda_{i_2-1}-\lambda_{i_2-2},-\lambda_{i_2-1},\\ &\dots,\lambda_{i_k+1},\lambda_{i_k+2}-\lambda_{i_k+1},\dots,\lambda_n-\lambda_{n-1}\big)\in L,\end{aligned}$$

where $L=\{\beta\in\mathbb{R}^n:\beta_1y_1+\ldots+\beta_ny_n=0\}$. If we denote by $e_1,\dots,e_n$ the standard Euclidean basis in $\mathbb{R}^n$, this holds if and only if there exist numbers λ_i with $i\in\{1,\dots,n\}\backslash\{i_1,\dots,i_k\}$ that do not vanish simultaneously and such that the vector

$$\begin{aligned}&\lambda_1(e_1-e_2)+\ldots+\lambda_{i_1-1}(e_{i_1-1}-e_{i_1})+\lambda_{i_1+1}(e_{i_1+1}-e_{i_1+2})+\ldots+\lambda_{i_2-1}(e_{i_2-1}-e_{i_2})+\\ &\quad\dots+\lambda_{i_k+1}(e_{i_k+1}-e_{i_k+2})+\ldots+\lambda_{n-1}(e_{n-1}-e_n)+\lambda_ne_n\end{aligned}$$

lies in L. This is equivalent to

$$\begin{aligned}\operatorname{lin}\big\{&e_1-e_2,\dots,e_{i_1-1}-e_{i_1},e_{i_1+1}-e_{i_1+2},\dots,e_{i_2-1}-e_{i_2},\\ &\dots,e_{i_k+1}-e_{i_k+2},\dots,e_{n-1}-e_n,e_n\big\}\cap L\neq\{0\}.\end{aligned}$$

This holds if and only if $K^\perp\cap L\neq\{0\}$, for the k-dimensional subspace

$$\begin{aligned}K&=\{\beta\in\mathbb{R}^n:\beta_1=\ldots=\beta_{i_1},\dots,\beta_{i_{k-1}+1}=\ldots=\beta_{i_k},\beta_{i_k+1}=\ldots=\beta_n=0\}\\ &=\bigcap_{i\notin\{i_1,\dots,i_k\}}(e_i-e_{i+1})^\perp,\end{aligned}$$

where $e_{n+1}:=0$. We observe that K is the intersection of hyperplanes from the reflection arrangement $\mathcal{A}(B_n)$. Then, $K^\perp\cap L\neq\{0\}$ is equivalent to

$$\dim(L^\perp\cap K)=n-\dim(L+K^\perp)=d-n+k-\dim(L\cap K^\perp)\neq d-n+k,$$

since $\dim(L)=n-d$. This means that $L^\perp$ is not in general position to $\mathcal{A}(B_n)$, which is a contradiction to (B2).

Now, we only need show that the general case follows from the previous results. If there are $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$, such that $\varepsilon_1y_{\sigma(1)}-\varepsilon_2y_{\sigma(2)},\dots,\varepsilon_{n-1}y_{\sigma(n-1)}-\varepsilon_ny_{\sigma(n)},\varepsilon_ny_{\sigma(n)}$ are not in general position, we can apply the above reasoning to $\varepsilon_1y_{\sigma(1)},\dots,\varepsilon_ny_{\sigma(n)}$ instead of $y_1,\dots,y_n$. Thus, it follows that

$$\dim\big((L_{\varepsilon,\sigma})^\perp\cap K\big)\neq d-n+k,$$

where $L_{\varepsilon,\sigma}=\{\beta\in\mathbb{R}^n:\varepsilon_1y_{\sigma(1)}\beta_1+\ldots+\varepsilon_ny_{\sigma(n)}\beta_n=0\}$. It is easy to see that $L_{\varepsilon,\sigma}=g_{\varepsilon,\sigma}L$ for the reflection $g_{\varepsilon,\sigma}\in\mathcal{G}(B_n)$ given by

$$g_{\varepsilon,\sigma}:(x_1,\dots,x_n)\mapsto(\varepsilon_1x_{\sigma(1)},\dots,\varepsilon_nx_{\sigma(n)}).$$

Then, we have

$$\dim\left(L^\perp \cap g_{\varepsilon,\sigma}^{-1}K\right) = \dim\left(g_{\varepsilon,\sigma}L^\perp \cap K\right) \neq d-n+k.$$

Since $g_{\varepsilon,\sigma}^{-1}K$ is also an intersection of hyperplanes from $\mathcal{A}(B_n)$, we obtain that $L^\perp$ is not in general position to $\mathcal{A}(B_n)$, which is a contradiction to (B2).

It is left to prove that (B1) implies (B2). Let (B1) hold true for $y_1,\dots,y_n \in \mathbb{R}^d$. This implies that $\dim L^\perp = d$. In order to prove this, it is enough to show that, for example, the set of d vectors $y_{n-d+1},\dots,y_n$ is linearly independent. Suppose $\lambda_{n-d+1}y_{n-d+1}+\ldots+\lambda_n y_n = 0$ holds for some $\lambda_{n-d+1},\dots,\lambda_n \in \mathbb{R}$. Representing the individual y_j's as telescope sums, this implies

$$\begin{aligned}0 &= \lambda_{n-d+1}((y_{n-d+1}-y_{n-d+2})+\ldots+(y_{n-1}-y_n)+y_n)+\ldots+\lambda_{n-1}((y_{n-1}-y_n)+y_n)+\lambda_n y_n\\ &= \lambda_{n-d+1}(y_{n-d+1}-y_{n-d+2})+\ldots+(\lambda_{n-d+1}+\ldots+\lambda_n)y_n\end{aligned}$$

Since $y_{n-d+1}-y_{n-d+2},\dots,y_{n-1}-y_n,y_n$ are linearly independent, due to (B1), it follows that $\lambda_{n-d+1}=\ldots=\lambda_n=0$, which proves the linear independence of $y_{n-d+1},\dots,y_n$. Then we obtain

$$\dim L^\perp = n-\dim L = \operatorname{rank}(y_1,\dots,y_n) = d.$$

Now, suppose $L^\perp$ is not in general position to $\mathcal{A}(B_n)$. Therefore, it exists a k-dimensional subspace K' that can be represented as the intersections of hyperplanes from $\mathcal{A}(B_n)$, such that

$$\dim(K'\cap L^\perp) \neq \begin{cases} d-n+k & , k\geq n-d\\ 0 & , k<n-d\end{cases}.$$

The linear subspace K' is given by a set of equations of the following form. The coordinates $\beta_1,\dots,\beta_n$ are decomposed into $k+1$ distinguishable groups. These groups are required to be non-empty except the last one. All coordinates in the last group must be 0. For the remaining variables there is a unique choice of signs, which multiplies each variable by $+1$ or -1, such that the sign-changed variables are equal inside every group, except the last one. Then, there is a suitable transformation $g_{\varepsilon,\sigma} \in \mathcal{G}(B_n)$, such that $g_{\varepsilon,\sigma}K'$ is given by

$$\{\beta\in\mathbb{R}^n : \beta_1=\ldots=\beta_{i_1},\dots,\beta_{i_{k-1}+1}=\ldots=\beta_{i_k},\beta_{i_k+1}=\ldots=\beta_n=0\} \tag{6.2}$$

for some $1\leq i_1<\ldots<i_k\leq n$, and thus, $g_{\varepsilon,\sigma}K'$ coincides with the subspace K mentioned in the above argument.

At first, suppose $k\geq n-d$. Then $\dim(K'\cap L^\perp)\neq d-n+k$ implies that also

$$\dim\left(g_{\varepsilon,\sigma}K'\cap(L_{\varepsilon,\sigma})^\perp\right) = \dim\left(g_{\varepsilon,\sigma}K'\cap g_{\varepsilon,\sigma}L^\perp\right)\neq d-n+k$$

holds true. Now we can use the same arguments like in the first part of the proof, since all the steps in the argument are equivalent. This implies that $\varepsilon_1 y_{\sigma(1)}-\varepsilon_2 y_{\sigma(2)},\dots,\varepsilon_{n-1}y_{\sigma(n-1)}-\varepsilon_n y_{\sigma(n)},\varepsilon_n y_{\sigma(n)}$ are not in general position, and thus, (B1) is not satisfied. Note that the assumption $k\geq n-d$ was crucial for the arguments in the first part, since this implies that the resulting set of vectors consists of $n-k\leq d$ elements.

In the case $k\leq n-d$, we know that $\dim(K'\cap L^\perp)\neq d-n+k$. Due to

$$\dim(K'\cap L^\perp) = \dim(K')+\dim(L^\perp)-\dim(K'+L^\perp)\geq k+d-n,$$

this implies even $\dim(K'\cap L^\perp)>n-d+k$. Thus, there is a linear subspace $K'\subseteq K''$ that can also be represented as the intersection of hyperplanes from $\mathcal{A}(B_n)$, such that $\dim(K'')=n-d$ and $\dim(K''\cap L^\perp)\neq\{0\}$. Note that this subspace K'' is obtained by deleting $k-(n-d)$ equations in the defining condition of K', e.g. in the condition in (6.2). The previous case yields that if such a subspace K'' exists, the general position assumption (B1) is not satisfied, which is contradiction. $\square$

The analogous result holds for the general position assumptions (A1) and (A2) of the A_{n-1}-case, which we will restate here. We say that $y_1,\dots,y_n\in\mathbb{R}^d$ satisfy the *general position assumptions* (A1) or (A2) if the following holds.

(A1) For every $\sigma\in\mathcal{S}_n$ the vectors $y_{\sigma(1)}-y_{\sigma(2)},y_{\sigma(2)}-y_{\sigma(3)},\dots,y_{\sigma(n-1)}-y_{\sigma(n)}$ are in general position.

(A2) The linear subspace $L^\perp$ has dimension d and is in general position with respect to the hyperplane arrangement $\mathcal{A}(A_{n-1})$, where $L:=\{\beta\in\mathbb{R}^n:\beta_1y_1+\dots+\beta_ny_n=0\}$.

Proof of Theorem 4.2. This is proven in the same way as Theorem 3.2. Therefore we will only give a short sketch of the proof. Suppose (A1) is not satisfied, then there is a $\sigma\in\mathcal{S}_n$, such that $y_{\sigma(1)}-y_{\sigma(2)},\dots,y_{\sigma(n-1)}-y_{\sigma(n)}$ are not in general position. Consider the case $\sigma(i)=i$, for all $i=1,\dots,n$. Following the proof of Theorem 3.2 and replacing k by $k-1$, this implies that there is a linearly dependent subset of the form

$$\underbrace{y_1-y_2,\dots,y_{i_1-1}-y_{i_1}}_{\text{group 1}},\underbrace{y_{i_1+1}-y_{i_2+1},\dots,y_{i_2-1}-y_{i_2}}_{\text{group 2}},\dots,\underbrace{y_{i_{k-1}+1}-y_{i_{k-1}+2},\dots,y_{n-1}-y_n}_{\text{group }k}$$

for a $k\geq n-d$ and suitable indices $1\leq i_1<i_2<\ldots<i_{k-1}<n$. This holds if and only if $K^\perp\cap L\neq\{0\}$, for the k-dimensional subspace

$$K=\{\beta\in\mathbb{R}^n:\beta_1=\ldots=\beta_{i_1},\beta_{i_1+1}=\ldots=\beta_{i_2},\dots,\beta_{i_{k-1}+1}=\ldots=\beta_n\}.$$

Since K is the intersection of hyperplanes from the reflection arrangement $\mathcal{A}(A_{n-1})$, this implies that $L^\perp$ is not in general position to $\mathcal{A}(A_{n-1})$, which contradicts (A2). The general case $\sigma\in\mathcal{S}_n$ follows in the same way.

We already saw in the proof of Theorem 3.2 that (A1) implies $\dim L^\perp=d$. Now, suppose $L^\perp$ is not in general position to $\mathcal{A}(A_{n-1})$. Then there is a k-dimensional linear subspace K' that can be represented as the intersection of hyperplanes from $\mathcal{A}(A_{n-1})$, such that

$$\dim(K'\cap L^\perp)\neq\max\{0,d-n+k\}.\tag{6.3}$$

Since in the equation defining K' the coordinates $\beta_1,\dots,\beta_n$ are decomposed into k non-empty groups and are required to be equal inside each group, there is a suitable reflection $g_\sigma\in\mathcal{G}(A_{n-1})$, such that $g_\sigma K'$ is given by

$$\{\beta\in\mathbb{R}^n:\beta_1=\ldots=\beta_{i_1},\beta_{i_1+1}=\ldots=\beta_{i_2},\dots,\beta_{i_{k-1}+1}=\ldots=\beta_n\}$$

for some $1\leq i_1<\ldots<i_{k-1}<n$. In both cases, $k\leq n-d$ and $k>n-d$, (6.3) implies that $y_{\sigma(1)}-y_{\sigma(2)},\dots,y_{\sigma(n-1)}-y_{\sigma(n)}$ are not in general position, which contradicts (A1). This is proven in the same way as in the proof of Theorem 3.2. $\square$

6.2. **Proof of Lemma 5.2.** Let $Y_1,\dots,Y_n$ be random vectors in $\mathbb{R}^d$ having a joint μ^n-density f on $(\mathbb{R}^d)^n$, where μ denotes a σ-finite measure on $\mathbb{R}^d$ that assigns measure zero to each affine hyperplane. Our aim is to prove that (B1) and (B2) are satisfied a.s.

Proof of Lemma 5.2. The conditions (B1) and (B2) are equivalent, thus, we only need to prove (B1). Since $Y_1,\dots,Y_n$ have a joint density function with respect to μ^n, so does $\varepsilon_1Y_{\sigma(1)},\dots,\varepsilon_nY_{\sigma(n)}$, for each $\varepsilon\in\{\pm1\}^n$ and $\sigma\in\mathcal{S}_n$. Therefore, it suffices to prove that $Y_1-Y_2,\dots,Y_{n-1}-Y_n,Y_n$ are in general position a.s., or equivalently, that they are not in general position with probability 0. In order to do this, suppose there is a subset of $n-k\leq d$ linearly dependent vectors. Recalling the proof of Theorem 3.2, this set is of the form

$$Y_1-Y_2,\dots,Y_{i_1-1}-Y_{i_1},Y_{i_1+1}-Y_{i_1+2},\dots,Y_{i_2-1}-Y_{i_2},\dots,Y_{i_k+1}-Y_{i_k+2},\dots,Y_{n-1}-Y_n,Y_n$$

for suitable indices $1 \le i_1 < \ldots < i_k \le n$. Thus, we are able to find numbers λ_i with $i \in \{1,\ldots,n\}\backslash\{i_1,\ldots,i_k\}$ that do not vanish simultaneously and such that

$$\begin{aligned}\lambda_1 Y_1 = (\lambda_1 - \lambda_2)Y_2 + \ldots + (\lambda_{i_1-2} - \lambda_{i_1-1})Y_{i_1-1} + \lambda_{i_1-1}Y_{i_1}\\ + (-\lambda_{i_1+1})Y_{i_1+1} + (\lambda_{i_1+1} - \lambda_{i_1+2})Y_{i_1+2} + \ldots + (\lambda_{i_2-2} - \lambda_{i_2-1})Y_{i_2-1} + \lambda_{i_2-1}Y_{i_2}\\ + \cdots + (-\lambda_{i_k+1})Y_{i_k+1} + (\lambda_{i_k+1} - \lambda_{i_k+2})Y_{i_k+2} + \ldots + (\lambda_{n-1} - \lambda_n)Y_n\end{aligned} \tag{6.4}$$

holds true (see (6.1) solved for $\lambda_1 Y_1$). Without loss of generality, we may assume that $\lambda_1 \neq 0$ (Otherwise, choose the smallest i, such that $\lambda_i \neq 0$ and solve for $\lambda_i Y_i$). Divide 6.4 by λ_1. The possible values of the first line coincide with the affine hull of $Y_2,\ldots,Y_{i_1}$ denoted by $\operatorname{aff}\{Y_2,\ldots,Y_{i_1}\}$, since the coefficients of the Y_i's satisfy the relation

$$\frac{\lambda_1 - \lambda_2}{\lambda_1} + \ldots + \frac{\lambda_{i_1-2} - \lambda_{i_1-1}}{\lambda_1} + \frac{\lambda_{i_1-1}}{\lambda_1} = 1.$$

The dimension of this affine subspace is at most $i_1 - 2$. The possible values of the second line of (6.4), divided by λ_1, define the linear subspace

$$L_1 := \{\beta_{i_1+1}Y_{i_1+1} + \ldots + \beta_{i_2}Y_{i_2} : \beta_{i_1+1} + \ldots + \beta_{i_2} = 0\},$$

since the coefficients satisfy the relation

$$\frac{-\lambda_{i_1+1}}{\lambda_1} + \frac{\lambda_{i_1+1} - \lambda_{i_1+2}}{\lambda_1} + \ldots + \frac{\lambda_{i_2-2} - \lambda_{i_2-1}}{\lambda_1} + \frac{\lambda_{i_2-1}}{\lambda_1} = 0.$$

Similarly, the subsequent lines, except the last one, define linear subspaces $L_2,\ldots,L_{k-1}$. The dimension of the linear subspaces $L_1,\ldots,L_{k-1}$ is at most $i_2 - i_1 - 1,\ldots,i_k - i_{k-1} - 1$, respectively. Thus, (6.4) implies that

$$Y_1 \in L(Y_2,\ldots,Y_n) := \operatorname{aff}\{Y_2,\ldots,Y_{i_1}\} + L_1 + \ldots + L_{k-1} + \operatorname{lin}\{Y_{i_k+1},\ldots,Y_n\},$$

and the dimension of the affine subspace $L(Y_2,\ldots,Y_n)$ is at most

$$(i_1 - 2) + (i_2 - i_1 - 1) + \ldots + (i_k - i_{k-1} - 1) + (n - i_k) = n - k - 1 < d.$$

It remains to show that the event $Y_1 \in L(Y_2,\ldots,Y_n)$ has probability 0. Now, since $(Y_1,\ldots,Y_n)$ has a joint μ^n-density, the conditional μ-density of Y_1 conditioned on the event that $(Y_2,\ldots,Y_n) = (y_2,\ldots,y_n)$ exists and we will denote it by $f(y_1|y_2,\ldots,y_n)$. Following the above reasoning, we see that

$$\begin{aligned}&\mathbb{P}(Y_1 \in L(Y_2,\ldots,Y_n))\\ &\quad= \int_{(\mathbb{R}^d)^{n-1}} \mathbb{P}\big(Y_1 \in L(y_2,\ldots,y_n)\big|(Y_2,\ldots,Y_n) = (y_2,\ldots,y_n)\big)\,\mu^{n-1}(\mathrm{d}(y_2,\ldots,y_n))\\ &\quad= \int_{(\mathbb{R}^d)^{n-1}} \int_{L(y_2,\ldots,y_n)} f(y_1|y_2,\ldots,y_n)\,\mu(\mathrm{d}y_1)\,\mu^{n-1}(\mathrm{d}(y_2,\ldots,y_n))\\ &\quad= 0,\end{aligned}$$

since $\dim L(y_2,\ldots,y_n) < d$ and μ assigns measure to each affine hyperplane. $\square$

References

[1] D. Amelunxen and M. Lotz. Intrinsic volumes of polyhedral cones: A combinatorial perspective. *Discrete & Computational Geometry*, 58(2):371–409, jul 2017.
[2] E. Arbeiter and M. Zähle. Geometric measures for random mosaics in spherical spaces. *Stochastics Stochastics Rep.*, 46(1-2):63–77, 1994.
[3] I. Bárány, D. Hug, M. Reitzner, and R. Schneider. Random points in halfspheres. *Random Structures Algorithms*, 50(1):3–22, 2017.
[4] T. Cover and B. Efron. Geometrical probability and random points on a hypersphere. *Ann. Math. Stat*, 38:213–220, 1967.
[5] D. Hug and R. Schneider. Random conical tessellations. *Discrete & Computational Geometry*, 56(2):395–426, may 2016.
[6] D. Hug and C. Thäle. Splitting tessellations in spherical spaces. *Electron. J. Probab.*, 24:Paper No. 24, 60, 2019.
[7] Z. Kabluchko, A. Marynych, D. Temesvari, and C. Thäle. Cones generated by random points on half-spheres and convex hulls of Poisson point processes. *Probab. Theory Related Fields*, 175(3-4):1021–1061, 2019.
[8] Z. Kabluchko, D. Temesvari, and C. Thäle. A new approach to weak convergence of random cones and polytopes., 2020. Preprint at arXiv: 2003.04001.
[9] Z. Kabluchko and C. Thäle. The typical cell of a Voronoi tessellation on the sphere., 2019. Preprint at arXiv: 1911.07221.
[10] Z. Kabluchko, V. Vysotsky, and D. Zaporozhets. Convex hulls of random walks: expected number of faces and face probabilities. *Adv. Math.*, 320:595–629, 2017.
[11] Z. Kabluchko, V. Vysotsky, and D. Zaporozhets. Convex hulls of random walks, hyperplane arrangements, and Weyl chambers. *Geom. Funct. Anal.*, 27(4):880–918, 2017.
[12] Z. Kabluchko, V. Vysotsky, and D. Zaporozhets. A multidimensional analogue of the arcsine law for the number of positive terms in a random walk. *Bernoulli*, 25(1):521–548, feb 2019.
[13] R. E. Miles. Random points, sets and tessellations on the surface of a sphere. *Sankhyā Ser. A*, 33:145–174, 1971.
[14] L. Schlfli. *Gesammelte Mathematische Abhandlungen I.* Verlag Birkhauser, 1950.
[15] R. Schneider. Intersection probabilities and kinematic formulas for polyhedral cones. *Acta Math. Hungar.*, 155(1):3–24, 2018.
[16] R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer Berlin Heidelberg, 2008.
[17] N. J. A. Sloane (editor). The On-Line Encyclopedia of Integer Sequences. https://oeis.org.
[18] R. Suter. Two analogues of a classical sequence. *J. Integer Seq.*, 3(1):Article 00.1.8, 1 HTML document, 2000.
[19] V. Vysotsky and D. Zaporozhets. Convex hulls of multidimensional random walks. *Trans. Amer. Math. Soc.*, 370(11):7985–8012, 2018.
[20] J. Wendel. A problem in geometric probability. *Math. Scandinavica*, 11:109–111, 1962.

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