# Unconditional applicability of the Lehmer's measure to the two-term Machin-like formula for pi

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**Abstract**

Lehmer in his publication [6] defined a measure

$$\mu = \sum_{j=1}^{J} \frac{1}{\log_{10}\left(|\beta_j|\right)},$$

where $\beta_j$ is a set of the constants that may be either integers or rational numbers in the Machin-like formula for pi. At $\beta_j \in \mathbb{Z}$ the Lehmer's measure can be used to determine computational efficiency of the given Marchin-like formula for pi. However, as a result of complexities in computation it remains unclear if the Lehmer's measure is applicable when at least one of the constants from the set $\beta_j$ is a rational number. In this work we present a new algorithm for the two-term Machin-like formula for pi as an example for unconditional applicability of the Lehmer's measure. This approach does not involve any irrational numbers and may be promising for rapid convergence to pi by the Newton–Raphson iteration method for the tangent function.

**Keywords:** constant pi; Machin-like formula; Lehmer's measure; Newton–Raphson iteration

## 1 Introduction

In 1706 the English astronomer and mathematician John Machin discovered a two-term formula for pi [1–3]

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right), \tag{1}$$

that was later named in his honor. This formula for pi appeared to be most efficient than any others known by that time. In particular, due to relatively rapid convergence of equation (1) he was able to calculate 100 decimal digits of pi [1]. Nowadays, the identities in form

$$\frac{\pi}{4} = \sum_{j=1}^{J} \alpha_j \arctan\left(\frac{1}{\beta_j}\right), \tag{2}$$

where $\alpha_j$ and $\beta_j$ are either integers or rational numbers, are regarded as the Machin-like formulas for pi. Consequently, the two-term Machin-like formula for pi is given by

$$\frac{\pi}{4} = \alpha_1 \arctan\left(\frac{1}{\beta_1}\right) + \alpha_2 \arctan\left(\frac{1}{\beta_2}\right). \tag{3}$$

If in equation (3) the constants $\alpha_1$ and $\beta_1$ are some positive integers and $\alpha_2 = 1$, then the unknown value $\beta_2$ can be found as [4, 5]

$$\frac{1}{\beta_2} = \frac{2}{\left[(\beta_1 + i)\,/\,(\beta_1 - i)\right]^{\alpha_1} + i} + i \Leftrightarrow \beta_2 = \frac{2}{\left[(\beta_1 + i)\,/\,(\beta_1 - i)\right]^{\alpha_1} - i} - i. \tag{4}$$

Furthermore, since we assumed that $\alpha_1$ and $\beta_1$ are some positive integers, from equation (4) it immediately follows that $\beta_2$ must be either an integer or a rational number.

In 1938 Lehmer in his paper [6] introduced a measure (see also [7])

$$\mu = \sum_{j=1}^{J} \frac{1}{\log_{10}\left(|\beta_j|\right)}, \tag{5}$$

showing how much labor is required for a specific Machin-like formula for pi in computation. In particular, when the measure (5) for some given Machin-like formula for pi is smaller, then less computational efforts are required and, consequently, the computational efficiency of this formula is higher. The Lehmer's measure is smaller if number of the summation terms $J$ is less and the constants $\beta_j$ are larger in magnitude. For more efficient computation the constants $\beta_j$ should be larger by absolute value since it is easier to approximate the arctangent function when its argument tends to zero (see [7] for more details).

It is also important to emphasize that in the same paper [6] Lehmer presented few Machin-like formulas where some constants from the set $\beta_j$ are not integers but rational numbers. This signifies that Lehmer assumed that his measure (5) remains valid regardless whether some constants from the set $\beta_j$ are integers or rational numbers.

In 2002 Kanada using the following self-checking pair of the Machin-like formulas for pi

$$\frac{\pi}{4} = 44 \arctan\left(\frac{1}{57}\right) + 7 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{682}\right) + 24 \arctan\left(\frac{1}{12943}\right)$$

and

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{49}\right) + 32 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) + 12 \arctan\left(\frac{1}{110443}\right)$$

computed more than 1 trillion digits of pi [8]. These two examples show that the Machin-like formulas have colossal potential in computation of decimal digits of pi.

In 1997 Chien-Lih showed a remarkable formula [9]

$$\begin{aligned}\frac{\pi}{4} =& 183 \arctan\left(\frac{1}{239}\right) + 32 \arctan\left(\frac{1}{1023}\right) - 68 \arctan\left(\frac{1}{5832}\right) \\ &+ 12 \arctan\left(\frac{1}{110443}\right) - 12 \arctan\left(\frac{1}{4841182}\right) - 100 \arctan\left(\frac{1}{6826318}\right)\end{aligned}$$

with $\mu \approx 1.51244$. According to Weisstein [10], this Lehmer's measure is the smallest known value for the set $\beta_k$ consisting of integers only. Later Chien-Lih in his work [11] showed how the Lehmer's measure can be reduced even further by using the Euler's-type identity in an iteration for generating the two-term Machin-like formulas of kind (3) such that $\beta_1$ and $\beta_2$ are rational numbers.

In our previous publication [12] we obtained the following simple identity (see also [13])

$$\frac{\pi}{4} = 2^{k-1} \arctan\left(\frac{\sqrt{2 - c_{k-1}}}{c_k}\right), \qquad k = \{2, 3, 4, \ldots\}, \tag{6}$$

where $c_1 = \sqrt{2}$ and $c_k = \sqrt{2 + c_{k-1}}$, and described how using this identity another efficient method for generating the two-term Machin-like formula for pi

$$\frac{\pi}{4} = 2^{k-1} \arctan\left(\frac{1}{\beta_1}\right) + \arctan\left(\frac{1}{\beta_2}\right), \tag{7}$$

with small Lehmer's measure can be developed. In this approach the constant $\beta_1$ can be chosen as a positive integer such that

$$\beta_1 = \left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor \tag{8}$$

and, in accordance with equation (4), the constant $\beta_2$ in equation (7) can be found from

$$\beta_2 = \frac{2}{\left[(\beta_1 + i) \,/\, (\beta_1 - i)\right]^{2^{k-1}} - i} - i. \tag{9}$$

It is not reasonable to solve equation (9) directly for determination of the rational number $\beta_2$ as its solution becomes tremendously difficult with increasing integer $k$. However, this problem can be effectively resolved by using a very simple two-step iteration procedure that will be discussed in the next section. Therefore, our approach in generating the two-term Machin-like formula (7) for pi with small Lehmer's measure is much easier than the Chien-Lih's method [11].

Wetherfield in his paper [7] provides detailed explanation clarifying significance of the Lehmer's measure that shows how much computational labor is required for some given Machin-like formulas for pi when all constants in the set $\beta_j \in \mathbb{Z}$. However, it is unclear if this paradigm is also applicable when at least one number from the set $\beta_j$ is rational. More specifically, the problem that occurs in computation of the two-term Machin-like formula for pi in Chien-Lih's [11] and our [4] iteration methods is related to the rapidly growing number of digits in numerator and denominator in the constants $\beta_1$ and/or $\beta_2$. This rapid increase of digits occurs simultaneously with an attempt to decrease the Lehmer's measure. As a result of the increasing number of the digits that undergo subsequent exponentiation in the conventional algorithms, the computation becomes inefficient in determination of decimal digits of pi. Therefore, the applicability of the Lehmer's measure for the Machin-like formula for pi for the case $\beta_j \in \mathbb{Q}$ is questionable. For example, the Lehmer's measure may be small, say less than 1. This means that less computational efforts are needed to calculate pi. However, due to large number of the digits in the numerator and denominator in the constants $\beta_1$ and/or $\beta_2$, a computer performs more intense arithmetic operations that make the run-time significantly longer and consume additional hardware resources. Consequently, we have a contradiction and a question may be asked: "Is the Lehmer's measure still applicable when at least one constant from the set $\beta_j$ is not an integer but a rational number?".

Motivated by an interesting paper in regards to equation (7) that was recently published by contributor(s) from Wolfram Mathematica [14], we develop further our previous works [4, 5]. In this work we propose a new algorithm showing how unconditional applicability of the Lehmer's measure for the two-term Machin-like formula (7) for pi can be achieved. We also describe how linear and quadratic convergence to pi can be implemented by using the Mathematica codes.

## 2 Preliminaries

As it has been mentioned above, the number of the summation terms $J$ in equation (2) should be reduced in order to minimize the Lehmer measure. Since at $J = 1$ there exists only one Machin-like formula for pi

$$\frac{\pi}{4} = \arctan(1)\,, \tag{10}$$

we have to consider the case $J = 2$. Therefore, we attempted to find a method that can be used to generate the two-term Machin-like formula for pi with small Lehmer's measure. One of the efficient ways to generate the two-term Machin-like formula for pi with small Lehmer measure is to obtain it in form of equation (7).

In fact, the original Machin formula (1) for pi appears quite naturally from the equations (7), (8) and (9) at $k = 3$. Specifically, equation (8) provides

$$\beta_1 = \left\lfloor \frac{c_3}{\sqrt{2 - c_{2-1}}} \right\rfloor = \left\lfloor \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} \right\rfloor = 5.$$

Substituting these integers into equation (9) results in

$$\frac{1}{\beta_2} = \frac{2}{\left[(5+i)\,/\,(5-i)\right]^{2^{3-1}} + i} + i = -\frac{1}{239}.$$

Consequently, at $k = 3$ we obtain the following constants $2^{k-1} = 4$, $\beta_1 = 5$ and $\beta_2 = -239$ in equation (7). Since arctangent function is odd, we can represent it such that

$$\arctan\left(\frac{1}{-239}\right) = -\arctan\left(\frac{1}{239}\right).$$

Therefore, the constants for equation (3) can be rearranged as $\alpha_1 = 4$, $\alpha_2 = -1$, $\beta_1 = 5$ and $\beta_2 = 239$. This corresponds to the original Machin formula (1) for pi.

**Theorem 2.1.** *There are only four possible cases for the two-term Machin-like formula* (3) *for pi when all four constants $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are integers such that*
*1) $\alpha_1 = 4$, $\alpha_2 = -1$, $\beta_1 = 5$ and $\beta_2 = 239$ (Machin)*
*2) $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 2$ and $\beta_2 = 3$ (Euler)*
*3) $\alpha_1 = 2$, $\alpha_2 = -1$, $\beta_1 = 2$ and $\beta_2 = 7$ (Hermann)*
*4) $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 3$ and $\beta_2 = 7$ (Hutton).*

The proof for Theorem 2.1 can be found in [10].

**Lemma 2.2.** *If in equation* (7) *$\beta_1 \in \mathbb{Z}$, then $\beta_2 \notin \mathbb{Z}$ at any integer $k > 3$.*

*Proof.* As we can see from the four cases given in Theorem 2.1, the largest possible value for $\alpha_1 = 2^{k-1}$ is 4 and it occurs at $k = 3$ (see example above). Therefore, it follows that for any integer $\beta_1$ at integer $k > 3$, the constant $\beta_2$ in the equation (7) cannot be an integer. □

**Theorem 2.3.**

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k \text{ square roots}} = 2.$$

*Proof.* This is a simplest kind of the Ramanujan nested radical and its proof is straightforward. Denote $X$ as unknown. Then,

$$\lim_{k\to\infty} c_k = X$$

or

$$\lim_{k\to\infty} \sqrt{2 + c_{k-1}} = \sqrt{2 + \lim_{k\to\infty} c_{k-1}} = \sqrt{2 + \lim_{k\to\infty} c_k} = X.$$

Consequently, squaring on both sides leads to

$$2 + \lim_{k\to\infty} c_k = X^2 \Leftrightarrow 2 + X = X^2.$$

Solving this quadratic equation results in two possible solutions $X_1 = -1$ and $X_2 = 2$. Since for any index $k$ the value $c_k$ is always positive, we must exclude the solution $X_1 = -1$ from consideration. □

**Lemma 2.4.**

$$\lim_{k\to\infty} \sqrt{2 - c_{k-1}} = \lim_{k\to\infty} \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{k-1 \text{ square roots}}} = 0.$$

*Proof.* The proof follows immediately from the Theorem 2.3 since

$$\lim_{k\to\infty} \sqrt{2 - c_{k-1}} = \lim_{k\to\infty} \sqrt{2 - c_k} = \sqrt{2 - \lim_{k\to\infty} c_k} = \sqrt{2 - 2} = 0.$$

□

**Lemma 2.5.**

$$\lim_{k\to\infty} \frac{\left(\frac{c_k}{\sqrt{2 - c_{k-1}}}\right)}{\beta_1} = 1.$$

*Proof.* Using equation (8) that defines $\beta_1$ by the floor function, the limit above can be rewritten as

$$\lim_{k\to\infty} \frac{\left(\frac{c_k}{\sqrt{2 - c_{k-1}}}\right)}{\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor} = 1. \tag{11}$$

By definition, the fractional part given by the difference

$$\text{frac}\left(\frac{c_k}{\sqrt{2 - c_{k-1}}}\right) = \frac{c_k}{\sqrt{2 - c_{k-1}}} - \left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor,$$

cannot be greater than unity. Therefore, the limit (11) can be rewritten in form

$$\lim_{k\to\infty} \frac{\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor + \text{frac}\left(\frac{c_k}{\sqrt{2 - c_{k-1}}}\right)}{\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor} = \lim_{k\to\infty} \frac{\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor}{\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor} + 0 = 1.$$

□

**Theorem 2.6.** *The Lehmer's measure* (5) *may be vanishingly small.*

*Proof.* The proof is not difficult. From equation (9) it is not evident if the following limit

$$\lim_{k\to\infty}\frac{1}{\beta_2}=\lim_{k\to\infty}\frac{2}{\left[\left(\beta_1+i\right)/\left(\beta_1-i\right)\right]^{2^{k-1}}+i}+i=0$$

is true to claim that

$$\lim_{k\to\infty}|\beta_2|=\infty. \tag{12}$$

Therefore, in order to prove the limit (12) we start with limit (11) from Lemma 2.5. Since the limit (11) is equal to unity, reciprocation of its numerator and denominator must also give us unity

$$\lim_{k\to\infty}\frac{\left(\frac{c_k}{\sqrt{2-c_{k-1}}}\right)^{-1}}{\frac{1}{\beta_1}}=\lim_{k\to\infty}\frac{\left(\frac{\sqrt{2-c_{k-1}}}{c_k}\right)}{\frac{1}{\beta_1}}=1. \tag{13}$$

From Theorem 2.3 and Lemma 2.4 it immediately follows that

$$\lim_{k\to\infty}\left(\frac{\sqrt{2-c_{k-1}}}{c_k}\right)=\frac{\lim\limits_{k\to\infty}\sqrt{2-c_{k-1}}}{\lim\limits_{k\to\infty}c_k}=\frac{0}{2}=0$$

and

$$\lim_{k\to\infty}\frac{1}{\beta_1}=\lim_{k\to\infty}\frac{1}{\left[\frac{c_k}{\sqrt{2-c_{k-1}}}\right]}=\frac{1}{\left[\frac{\lim\limits_{k\to\infty}c_k}{\lim\limits_{k\to\infty}\sqrt{2-c_{k-1}}}\right]}=\frac{1}{\infty}=0. \tag{14}$$

Since both, the numerator and denominator, in the limit (13) tends to zero when integer $k$ tends to infinity and since according to the Maclaurin series expansion

$$\arctan\left(x\right)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=x+O\left(x^3\right)\Rightarrow\arctan\left(x\right)\to x\,\text{at}\,x\to 0,$$

we can rewrite equation (13) as

$$\lim_{k\to\infty}\frac{\arctan\left(\frac{\sqrt{2-c_{k-1}}}{c_k}\right)}{\arctan\left(\frac{1}{\beta_1}\right)}=1$$

or

$$\lim_{k\to\infty} \frac{2^{k-1}\arctan\left(\frac{\sqrt{2-c_{k-1}}}{c_k}\right)}{2^{k-1}\arctan\left(\frac{1}{\beta_1}\right)} = 1. \tag{15}$$

Since equation (6) is valid at any arbitrarily large $k$, the limit (15) implies that

$$\lim_{k\to\infty} 2^{k-1}\arctan\left(\frac{\sqrt{2-c_{k-1}}}{c_k}\right) = \lim_{k\to\infty} 2^{k-1}\arctan\left(\frac{1}{\beta_1}\right) = \frac{\pi}{4}. \tag{16}$$

However, we also have

$$\lim_{k\to\infty}\left[2^{k-1}\arctan\left(\frac{1}{\beta_1}\right) + \arctan\left(\frac{1}{\beta_2}\right)\right] = \frac{\pi}{4} \tag{17}$$

since equation (7) is also valid at any arbitrarily large integer $k$. Comparing now the limits (16) and (17) with each other we get

$$\lim_{k\to\infty} 2^{k-1}\arctan\left(\frac{1}{\beta_1}\right) = \lim_{k\to\infty}\left[2^{k-1}\arctan\left(\frac{1}{\beta_1}\right) + \arctan\left(\frac{1}{\beta_2}\right)\right] = \frac{\pi}{4}. \tag{18}$$

The equation (18) is valid if and only if

$$\lim_{k\to\infty}\arctan\left(\frac{1}{\beta_2}\right) = 0.$$

Consequently, we can infer that the limit (12) is true.

Return to the limit (14). This limit signifies that

$$\lim_{k\to\infty}\beta_1 = \infty. \tag{19}$$

Finally, from the limits (12) and (19) we can conclude that at $k \to \infty$, the values $\beta_1 \to \infty$ and $|\beta_2| \to \infty$. Thus, according to equation (5) the Lehmer's measure $\mu$ for the two-term Machin-like formula (7) for pi tends to zero with increasing integer $k$. □

**Theorem 2.7.** *If equation* (8) *holds, then the constant* $\beta_2$ *is always negative.*

*Proof.* There is only one single-term Machin-like formula (10) for pi such that the constants $\alpha_1$ and $\beta_1$ are both integers. Therefore, in equation (7) at any integer $k \geq 2$ the argument of the arctangent function $\sqrt{2 - c_{k-1}}/c_k$ cannot be represented as a reciprocated integer. This signifies that $c_k/\sqrt{2 - c_{k-1}}$ is not an integer. Therefore, the fractional part

$$\mathrm{frac}\left(\frac{c_k}{\sqrt{2 - c_{k-1}}}\right) > 0$$

and the following inequality

$$\frac{c_k}{\sqrt{2 - c_{k-1}}} > \left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor, \qquad k = \{2, 3, 4, \ldots\},$$

holds. From this inequality it follows that

$$\arctan\left(\frac{\sqrt{2 - c_{k-1}}}{c_k}\right) < \arctan\left(1/\left\lfloor \frac{c_k}{\sqrt{2 - c_{k-1}}} \right\rfloor\right)$$

or

$$\arctan\left(\frac{\sqrt{2 - c_{k-1}}}{c_k}\right) < \arctan\left(\frac{1}{\beta_1}\right)$$

or

$$2^{k-1} \arctan\left(\frac{\sqrt{2 - c_{k-1}}}{c_k}\right) < 2^{k-1} \arctan\left(\frac{1}{\beta_1}\right). \tag{20}$$

In order to transfer the inequality (20) into equality, we have to add the error term $\varepsilon < 0$ such that

$$2^{k-1} \arctan\left(\frac{\sqrt{2 - c_{k-1}}}{c_k}\right) = 2^{k-1} \arctan\left(\frac{1}{\beta_1}\right) + \varepsilon.$$

Defining the constant $\beta_2$ in accordance with equation (9) we can find the error term as

$$\varepsilon = \arctan\left(\frac{1}{\beta_2}\right). \tag{21}$$

Since the error term $\varepsilon$ is negative, from equation (21) it follows that the constant $\beta_2$ is also negative. □

# 3 Iteration methods

## 3.1 Arctangent function

Since in equation (7) the constant $\beta_1$ is an integer, the first arctangent function term can be found by any existing methods of computation. For example, we can use the Euler's formula for the arctangent function

$$\arctan\left(x\right) = \sum_{m=0}^{\infty} \frac{2^{2m}\left(m!\right)^2}{\left(2m+1\right)!} \frac{x^{2m+1}}{\left(1+x^2\right)^{m+1}}. \tag{22}$$

Chien-Lih used this Euler's formula in development of his iteration method for generating the two-term Machin-like formula for pi [11] and later he found an elegant derivation of this formula [15].

Another series expansion of the arctangent function has been reported in our publications [5, 12]

$$\arctan\left(x\right) = i \sum_{m=1}^{\infty} \frac{1}{2m-1}\left(\frac{1}{\left(1+2i/x\right)^{2m-1}} - \frac{1}{\left(1-2i/x\right)^{2m-1}}\right). \tag{23}$$

It interesting to note that generalizing the derivation method that was used to obtain equation (23), we can find by induction the identity

$$\arctan\left(x\right) = \sum_{m=1}^{M} \arctan\left(\frac{Mx}{M^2 + \left(m-1\right)mx^2}\right)$$

that yields simple approximations like

$$\arctan\left(x\right) \approx \sum_{m=1}^{M} \frac{Mx}{M^2 + \left(m-1\right)mx^2}$$

and

$$\arctan\left(x\right) \approx \sum_{m=1}^{M} \left(\frac{Mx}{M^2 + \left(m-1\right)mx^2} - \frac{1}{3}\left(\frac{Mx}{M^2 + \left(m-1\right)mx^2}\right)^3\right)$$

since from the Maclaurin series expansion it follows that $\arctan\left(x\right) = x + O\left(x^3\right)$ and $\arctan\left(x\right) = x - x^3/3 + O\left(x^5\right)$, respectively.

The representation (23) of the arctangent function is not optimal for algorithmic implementation since it deals with complex numbers. Fortunately, as we have shown in [4] this series expansion can be significantly simplified as

$$\arctan\left(x\right) = 2\sum_{m=1}^{\infty}\frac{1}{2m-1}\frac{g_m\left(x\right)}{g_m^2\left(x\right)+h_m^2\left(x\right)}, \tag{24}$$

where the expansion coefficients are computed by iteration procedure as

$$g_1\left(x\right) = 2/x, \;\; h_1\left(x\right) = 1,$$

$$g_m\left(x\right) = g_{m-1}\left(x\right)\left(1-4/x^2\right) + 4h_{m-1}\left(x\right)/x,$$
$$h_m\left(x\right) = h_{m-1}\left(x\right)\left(1-4/x^2\right) - 4g_{m-1}\left(x\right)/x.$$

Both series expansions (22) and (24) are rapid in convergence and need no undesirable irrational numbers in computation of pi. However, the computational test we performed shows that the series expansion (24) is more rapid in convergence by many orders of the magnitude than the Euler's formula (22) (see Figs. 2 and 3 in [4]). Therefore, the series expansion (24) is more advantageous and can be taken for computation of the first arctangent function term from the two-term Machin-like formula (7) for pi.

The second arctangent function term in equation (7) should not be computed by straightforward substitution of the constant $\beta_2$ into equation (24). As it has been mentioned above, due to exponentiation of the ratio consisting of large number of digits in the numerator and denominator causes computational complexities that should be avoided. Instead, the second arctangent function term in equation (7) can be computed by using the Newton–Raphson iteration.

## 3.2 Rational number

Once the value of the integer $k$ is chosen, then it is not difficult to determine the integer $\beta_1$ by using equation (8) with help of a Computer Algebra System (CAS). However, an actual problem occurs with determination of the second constant $\beta_2$. As it has been mentioned already, solving equation (9) becomes very difficult as integer $k$ increases and CAS is simply unable to find the solution beyond some threshold value of $k$. In order to overcome such a problem we have proposed another method for determination of the constant

$\beta_2$ [4]. Particularly, defining a very simple two-step iteration procedure such that

$$\begin{cases} u_n = u_{n-1}^2 - v_{n-1}^2 \\ v_n = 2u_{n-1}v_{n-1}, \end{cases} \qquad n = \{2, 3, 4, \ldots, k\},$$

where

$$u_1 = \frac{\beta_1^2 - 1}{\beta_1^2 + 1},$$

and

$$v_1 = \frac{2\beta_1}{\beta_1^2 + 1},$$

we can find it as

$$\beta_2 = \frac{2u_k}{u_k^2 + (v_k - 1)^2}. \tag{25}$$

At $k > 3$ the second arctangent function term in the equation (7) deals only with a rational number $\beta_2$. As it has been pointed out above, increase of $k$ increases the number of the digits in the numerator and denomination of the constant $\beta_2$. For example, at $k = 6$ equation (8) yields

$$\beta_1 = \left\lfloor \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} \right\rfloor = 40$$

and using the iteration-based formula (25) we can find that

$$\begin{aligned}\beta_2 &= -\frac{2634699316100146880926635665506082395762836079845121}{38035138859000075702655846657186322249216830232319} \\ &= -69.27013796024857670135\ldots\,(\text{rational})\,.\end{aligned}$$

Consequently, the two-term Machin-like formula (7) for pi is generated as

$$\begin{aligned}\frac{\pi}{4} =& 32\arctan\left(\frac{1}{40}\right) \\ &- \arctan\left(\frac{38035138859000075702655846657186322249216830232319}{2634699316100146880926635665506082395762836079845121}\right),\end{aligned}$$

where $32 = 2^{6-1}$. The corresponding Lehmer's measure for this two-term Machin-like formula for pi is $\mu \approx 1.16751$. However, if we take $k = 27$, then

$$\beta_1 = \left\lfloor \frac{\overbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}^{27\text{ square roots}}}{\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{26\text{ square roots}}}} \right\rfloor = 85445659$$

and using iteration-based formula (25) we obtain

$$\beta_2 = -\frac{\overbrace{2368557598\ldots9903554561}^{522,185,816\text{ digits}}}{\underbrace{9732933578\ldots4975692799}_{522,185,807\text{ digits}}}$$

$$= -2.43354953523904089818\ldots\times 10^8(\text{rational}).$$

The corresponding two-term Machin like formula for pi

$$\frac{\pi}{4} = 67108864\arctan\left(\frac{1}{85445659}\right) - \arctan\left(\frac{\overbrace{9732933578\ldots4975692799}^{522,185,807\text{ digits}}}{\underbrace{2368557598\ldots9903554561}_{522,185,816\text{ digits}}}\right)$$

provides Lehmer's measure $\mu \approx 0.245319$ only. Such a large number of the digits in the numerator and denominator within second arctangent function may look somehow unusual. It should be noted, however, that some formulas for pi obtained from the Borwein integrals involving sinc function can also result in ratios containing large number of digits in numerator and denominator. For example, Bäsel and Baillie reported a formula for pi that uses a quotient with $453,130,145$ digits in numerator and $453,237,170$ digits in denominator [16]. Interested readers can download a file with all digits of the constant $\beta_2$ from [17].

As we can see from these examples, the Lehmer's measure decreases with increasing $k$. However, decrease of the Lehmer's measure occurs simultaneously with rapid increase of the digits in the numerator and denominator of the constant $\beta_2$. As a result, a continuous exponentiation of the large number of the digits strongly decelerates the computation. Therefore, this causes some doubts if the Lehmer's measure (5) is indeed applicable for some given Machin-like formula for pi when at least one coefficient from the set $\beta_j$ is a rational number.

In order to resolve this problem we have considered already the application of the Newton–Raphson iteration [18]. Specifically, we have shown that each consecutive iteration doubles number of the digits in the second term of the arctangent function in equation (7). This method is based on the following iteration formula [18]

$$y_{p+1} = y_p - \left(1 - \sin^2\left(y_p\right)\right)\left(\tan\left(y_p\right) - \frac{1}{\beta_2}\right), \tag{26}$$

such that

$$\lim_{n \to \infty} y_p = \arctan\left(\frac{1}{\beta_2}\right).$$

The most essential advantage of the iteration formula (26) is that the rational number $1/\beta_2$ is not involved in computation of the trigonometric functions. As we can see from the Newton–Raphson iteration-based formula (26), this rational number is no longer problematic because it does not undergo any exponentiation as it is not within computation of the sine or the arctangent functions that consumes most part of the run-time. Instead, it is actually applied in a single subtraction only. Therefore, we do not have exponentially growing number of the digits that could appear otherwise by using direct substitution of this "troublesome" rational number $1/\beta_2$ into series expansion (24). This single arithmetic operation of subtraction that can be implemented with changing precision practically takes a negligibly small amount of time as compared to time for computation of $\sin\left(y_p\right)$ and $\tan\left(y_p\right)$ functions.

In order to reduce number of trigonometric functions from two to one, it is convenient to represent equation (26) in form

$$y_{p+1} = y_p - \left(1 - \left(\frac{2\tan\left(\frac{y_p}{2}\right)}{1 + \tan^2\left(\frac{y_p}{2}\right)}\right)^2\right)\left(\frac{2\tan\left(\frac{y_p}{2}\right)}{1 - \tan^2\left(\frac{y_p}{2}\right)} - \frac{1}{\beta_2}\right), \tag{27}$$

since

$$\sin\left(y_p\right) = \frac{2\tan\left(\frac{y_p}{2}\right)}{1 + \tan^2\left(\frac{y_p}{2}\right)}$$

and

$$\tan\left(y_p\right) = \frac{2\tan\left(\frac{y_p}{2}\right)}{1 - \tan^2\left(\frac{y_p}{2}\right)}.$$

The tangent function can be found, for example, by using equation

$$\tan\left(\frac{y_p}{2}\right) = \frac{\sin\left(\frac{y_p}{2}\right)}{\cos\left(\frac{y_p}{2}\right)} = \frac{\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{y_p}{2}\right)^{2n+1}}{\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{y_p}{2}\right)^{2n}}$$

representing the ratio based on the Maclaurin series expansions for sine and cosine functions. Alternatively, the following series expansion

$$\tan\left(\frac{y_p}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}\left(2^{2n} - 1\right) B_{2n}}{(2n)!}\left(\frac{y_p}{2}\right)^{2n-1},$$

where $B_{2n}$ is a set of the Bernoulli numbers, can also be used. There are several other equations like continued fractions [19, 20] available for computation of the tangent function. One of the efficient ways to compute the tangent function is to use the Newton–Raphson iteration again [21]. Perhaps the argument reduction method for the tangent function can also be used to improve accuracy. However, we did not implement the argument reduction method in order to built the algorithm as simple as possible.

## 3.3 Tangent function

The equation (27) is based on the Newton–Raphson iteration method. However, this iteration formula contains the tangent function. We have shown in our previous publication [18] that once the first arctangent term is computed, the number of correct digits in $\pi$ can be doubled at each consecutive step of iteration as it supposed to be with the Newton–Raphson iteration method. In order to approximate the tangent function with high accuracy, we can apply the Newton–Raphson iteration again [21]. The derivation of the iteration-based equation for the tangent function is not difficult. Denoting $Y$ and $z$ as an argument of tangent function and as an unknown, respectively, we have

$$\tan\left(Y\right) = z \Leftrightarrow Y = \arctan\left(z\right)$$

or

$$\arctan\left(z\right) - Y = 0.$$

Substituting this equation into Newton–Raphson iteration formula

$$z_{q+1} = z_q - \frac{f(z_q)}{f'(z_q)},$$

where

$$f(z) = \arctan(z) - Y \Rightarrow f'(z) = \frac{d}{dy}(\arctan(z) - Y) = \frac{1}{1+z^2},$$

we obtain

$$z_{q+1} = z_q - \left(1 + z_q^2\right)(\arctan(z_q) - Y) \tag{28}$$

such that

$$\tan(Y) = \lim_{q \to \infty} z_q.$$

Substituting $Y = y_p/2$ into equation (28) yields

$$z_{q+1} = z_q - \left(1 + z_q^2\right)\left(\arctan(z_q) - \frac{y_p}{2}\right). \tag{29}$$

Iteration-based expansion series (24) for the arctangent function is very rapid in convergence. Therefore, we can apply it for computation of the arctangent function in equation (29).

# 4 Implementation

## 4.1 Linear convergence

The series expansion (24) of the arctangent function can be computed by entering the following command lines

```
atan[x_,M_] := atan[x, M] = 2*Sum[(1/(2*m - 1))*(g[m,x]/(g[m,x]^2 +
    h[m, x]^2)),{m,1,M}];

g[1,x_] := g[1,x] = 2/x;
h[1,x_] := h[1,x] = 1;
g[m_,x_] := g[m,x] = g[m - 1,x]*(1 - 4/x^2) + 4*h[m - 1, x]/x;
h[m_,x_] := h[m,x] = h[m - 1,x]*(1 - 4/x^2) - 4*g[m - 1, x]/x;
```

Next we define the nested radicals consisting of square roots of twos

```
c[1] := Sqrt[2];
c[k_] := Sqrt[2 + c[k - 1]];
```

The command lines below are to compute the constants $\beta_1$ and $\beta_2$ for the two-term Machin-like formula (7) for pi at $k = 6$

```
\[Beta]1 = Floor[c[6]/Sqrt[2 - c[6 - 1]]];

u[1] := (\[Beta]1^2 - 1)/(\[Beta]1^2 + 1);
v[1] := 2*\[Beta]1/(\[Beta]1^2 + 1);
u[n_] := u[n - 1]^2 - v[n - 1]^2;
v[n_] := 2*u[n - 1]*v[n - 1];

\[Beta]2 = 2*u[6]/(u[6]^2 + (v[6] - 1)^2);
```

Note that for the constant $\beta_2$ we use the iteration-based formula (25) instead of equation (9). Now we can display the values for the integer $\beta_1$ and rational number $\beta_2$

```
Print["Constant \[Beta]1 = ", \[Beta]1, "\nConstant \[Beta]2 = ",
    \[Beta]2];
```

The output for this print command is displayed as follows

Constant $\beta 1 = 40$

Constant $\beta 2 = -\frac{2634699316100146880926635665506082395762836079845121}{3803513885900007570265584665718632224921683023 2319}$

Accuracy of computation improves with each iteration. Therefore, we do not need to use the highest accuracy at each step of iteration. This can be achieved by using the command `SetPrecision`. At $k = 6$ the Newton–Raphson iteration-based formula (29) provides 4 to 5 correct digits of the tangent function at each step of iteration. Therefore, it is reasonable to take a parameter $5*q+2$, where 2 is taken to minimize the rounding or truncation errors. As an initial guess for the Newton–Raphson iteration we can choose $x = -1443/100000$ since this value is close to the actual value of the tangent function $\tan(y_p/2)$. The main part of computation is run by the following command lines

```
x := SetPrecision[-1443/100000,5]
z[1,x_] := z[1,x] = x;
z[q_,x_] := z[q,x] = SetPrecision[z[q - 1,x] - (1 + z[q - 1,x]^2)*
    (atan[z[q - 1,x],q] - x),5*q + 2];
```

This part of the program performing computation of the tangent function takes most of the run-time. The computation of the tangent function is performed by the Newton–Raphson iteration built on the basis of series expansion (24) of the arctangent function (see (29)).

In this command line we define a reciprocated value of the second constant num $= 1/\beta_2$

```
(* Reciprocated rational number *)
num = 1/\[Beta]2;
```

The Newton–Raphson iteration formula (27) is coded as

```
y[1] := y[1] = x;
y[p_] := y[p] = SetPrecision[y[p - 1],5*p + 2] -
    (1 - ((2*z[p,y[p - 1]/2])/(1 + z[p,y[p - 1]/2]^2))^2)*
        ((2*z[p,y[p - 1]/2])/(1 - z[p,y[p - 1]/2]^2) - num);
```

This part of the program invokes tangent function value $\tan(y_p/2)$ and performs just few arithmetic operations. It is important to note that the reciprocated number num $= 1/\beta_2$ containing large number of the digits in the numerator and denominator is not involved in computation of the tangent function that takes most of the run-time. There is only a minor portion of time required for subtraction of this number (see equation (27)). Consequently, from this example we can see that applicability of the Lehmer's measure is unconditional.

In order to see how computation of the arctangent function is performed at each iteration step with changing precision, we type the following command line

```
Table[{p - 1, y[p]}, {p, 2, 11}] // TableForm
```

The corresponding output is given by

| | |
|---|---|
| 1 | -0.0144354827911 |
| 2 | -0.014435232407997704 |
| 3 | -0.01443523240799679443925 |
| 4 | -0.0144352324079967944392951115 |
| 5 | -0.014435232407996794439295110969614 |
| 6 | -0.01443523240799679443929511096961893161 |
| 7 | -0.0144352324079967944392951109696189315443963 |
| 8 | -0.014435232407996794439295110969618931544397010224 |
| 9 | -0.01443523240799679443929511096961893154439701021536520 |
| 10 | -0.0144352324079967944392951109696189315443970102153652922241 |

Finally, if we want to see the convergence rate, we just put down the following command lines

```
For[p = 2,p <= 11,p++,Print[p - 1,If[p == 2," iteration, ",
    " iterations, "],Abs[MantissaExponent[N[\[Pi] - 4*(2^(6 - 1)*
        atan[1/\[Beta]1,1000] + y[p]),75]][[2]]]," digits of Pi"]]
```

The corresponding output is

```
1 iteration, 5 digits of Pi
2 iterations, 14 digits of Pi
3 iterations, 21 digits of Pi
4 iterations, 26 digits of Pi
5 iterations, 31 digits of Pi
6 iterations, 36 digits of Pi
7 iterations, 41 digits of Pi
8 iterations, 46 digits of Pi
9 iterations, 51 digits of Pi
10 iterations, 56 digits of Pi
```

As we can see, only the first iteration adds 9 correct digits of pi. In all other iterations the number of the added correct digits of pi is 5 per iteration.

The convergence rate increases with decreasing the Lehmer's measure. Such a tendency can be readily confirmed by increasing the integer $k$ and adjusting correspondingly `SetPrecision` parameter in this algorithm. This method of computation does not require any undesirable irrational numbers. Furthermore, since the Lehmer's measure may be vanishingly small, there is no upper bound in convergence rate per iteration.

## 4.2 Quadratic convergence

Consider another variation of the algorithm based on the Newton–Raphson iterations for the tangent function that can be implemented to obtain a quadratic convergence to pi. Assume that only 50 decimal digits of pi are known at the beginning. The following code assigns the value 50 for the initial decimal digits `decD` of pi and provides corresponding value for the variable $y$

```
Clear[y, z]

decD = 50;
```

```
y = Floor[SetPrecision[Pi/4 - 32*atan[1/40,Ceiling[2*decD/4]],
    2*decD]*10^decD]/10^decD;

Print["decD = ",decD,"\n","y = ",y]
```

The two outputs are

$$\text{decD} = 50$$
$$y = -\frac{1443523240799679443929511096961893154439701021537}{100000000000000000000000000000000000000000000000000}$$

We determined experimentally that at $k = 6$ the equation (24) provides 4 or 5 correct digits of pi at each increment. Therefore, it is sufficient to take a number of the terms equal `decD/4`. However, if we want to exclude the truncation or rounding errors, we should increase this value by some factor, say 2. Thus, we use `2*decD/4` terms for the arctangent function.

The following command line defines the variable `numb[q]` that we can use to determine the accuracy of computation and for summation terms in approximation (24) of the arctangent function

```
numb[q_] := If[4*2^q < 2*decD,4*2^q,2*decD + 10];
```

The multiplier 4 to $2^q$ was found experimentally. When the integer $4 \times 2^q$ exceeds `2*decD`, we can restrict its rapid growth by `2*decD + 10` as we do not need extra accuracy at this stage. The integer 10 is added just as a precaution to exclude the truncation or rounding errors.

These command lines are to perform main computation of the tangent function given by approximation (29)

```
z[1] := -0.007;
z[q_] := z[q] = SetPrecision[(z[q - 1] - (1 + z[q - 1]^2)*
    (atan[z[q - 1],Ceiling[numb[q - 1]/4]] - y/2)),numb[q]];
```

We can take initial value for $z = -0.07$.

Once the tangent function is computed, we need to substitute it into equation (27). This can be done by using the following code

```
(* Computation of value q *)
q = 1; While[numb[q] < 2*decD,q++]; q++;

(* Final values zD and yD with doubled accuracy *)
zD = z[q];
yD = y - (1 - (2*zD/(1 + zD^2))^2)*(2*zD/(1 - zD^2) - 1/\[Beta]2);
```

Note that for doubling decimal digits in `yD` we need to iterate only a single time.

The following lines show the approximated values for the arctangent function by iteration and Mathematica built-in function

```
N[yD,2*decD]
N[ArcTan[1/\[Beta]2],2*decD]
```

As we can see, there is a complete match between two values

$$-0.0144352324079967944392951109696189315443970102153652922229287402060\ldots\\4313374848838008024955210206070687 4$$

$$-0.01443523240799679443929511096961893154439701021536529222292874020 6\ldots\\43133748488380080249552102060706874$$

Since we obtained the value of `yD` with doubled accuracy, it can be used now to compute pi with significantly improved accuracy

```
piAppD = 4*(32*atan[1/40,Ceiling[4*decD/4]] + yD)
```

Note that the number of the terms in arctangent function is increased by two as the decimal digits of pi are doubled.

The corresponding output of pi approximation is

$$3.14159265358979323846264338327950288419716939937510582097494459230781 6\ldots\\4062862089986280348253421170679808682758 70$$

In order to see quadratic convergence, define the digits of pi

```
(* Digits of Pi *)
dPi = Abs[MantissaExponent[N[Pi - piAppD, 10000]][[2]]]
```

The output shows that the number of decimal digits of pi is increased from 50 to 101 (by factor of 2)

```
101
```

This can be observed explicitly by running the following code

```
piAppD=SetPrecision[piAppD,dPi]
N[\[Pi], dPi]
```

The outputs show a complete match between computed approximation of pi and that of provided by Mathematica

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170680

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170680

The first arctangent function in the two-term Machin-like formula (7) for pi can also be found by using same algorithm based on the Newton–Raphson iteration. Consequently, this method results in quadratic convergence to pi. However, unlike the Brent–Salamin algorithm (also known as the Gauss–Brent–Salamin algorithm) with quadratic convergence to pi [2], this approach does not involve any irrational numbers. The number of summation terms in equation (24) and the number of iteration cycles for computation of the tangent function (29) decrease with increasing $k$. This can be confirmed by using the code described above. To the best of our knowledge, this is the first algorithm showing feasibility of quadratic convergence to pi without irrational numbers involved in computation.

# 5 Conclusion

In this work we propose a new algorithm for computation of the two-term Machin-like formula (7) for pi and show an example where the condition $\beta_j \in \mathbb{Z}$ is not necessary in order to validate the Lehmer's measure (5). Since this algorithmic implementation enables us to avoid subsequent exponentiation of the second constant $\beta_2$, this approach may be promising for more rapid computation of pi without irrational numbers.

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