

MONOBRICK, A UNIFORM APPROACH TO TORSION-FREE CLASSES AND WIDE SUBCATEGORIES

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ABSTRACT. For a length abelian category, we show that all torsion-free classes can be classified by using only the information on bricks, including non functorially-finite ones. The idea is to consider the set of simple objects in a torsion-free class, which has the following property: it is a set of bricks where every non-zero map between them is an injection. We call such a set a monobrick. In this paper, we provide a uniform method to study torsion-free classes and wide subcategories via monobricks. We show that monobricks are in bijection with left Schur subcategories, which contains all subcategories closed under extensions, kernels and images, thus unifies torsion-free classes and wide subcategories. Then we show that torsion-free classes bijectively correspond to cofinally closed monobricks. Using monobricks, we deduce several known results on torsion(-free) classes and wide subcategories (e.g. finiteness result and bijections) in length abelian categories, without using τ -tilting theory. For Nakayama algebras, left Schur subcategories are the same as subcategories closed under extensions, kernels and images, and we show that its number is related to the large Schröder number.

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1. INTRODUCTION

For a finite-dimensional algebra Λ , several classes of subcategories of $\text{mod } \Lambda$ have been investigated in the representation theory of algebras. Among them, *torsion classes* and *torsion-free classes* have been central in the representation theory of algebras, together with their connection to the tilting theory and various triangulated categories.

Recently, Adachi-Iyama-Reiten's paper [AIR] made a major breakthrough in classification of these subcategories. They show that functorially finite torsion-free classes can be classified using *support τ^- -tilting modules*. Their method is to consider Ext-injective objects.

In this paper, we take a different approach, which enables us to classify *all* torsion-free classes in *any length abelian categories*. Our method is to consider *simple objects* (Definition 2.2). We observe that every simple object in a torsion-free class is a brick (a module with division endomorphism rings), and we classify torsion-free classes using only the information on bricks:

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Theorem A (= Theorem 3.15). *Let \mathcal{A} be a length abelian category. Then we have a bijection between the following sets.*

- (1) *The set of all torsion-free classes \mathcal{F} in \mathcal{A} .*
- (2) *$\{\mathcal{M} \mid \mathcal{M} \text{ is a set of bricks in } \mathcal{A} \text{ satisfying the following two conditions: }\}$*
 - (MB) *Every non-zero map between objects in \mathcal{M} is injective.*
 - (CC) *If there is an injection $N \hookrightarrow M$ for a brick $N \notin \mathcal{M}$ and $M \in \mathcal{M}$, then there is a non-zero non-injection $N \rightarrow M'$ for some $M' \in \mathcal{M}$.*

The map from (1) to (2) is given by the set $\text{sim } \mathcal{F}$ of simple objects in \mathcal{F} , and from (2) to (1) is given by taking the extension closure $\text{Filt } \mathcal{M}$ of \mathcal{M} .

Example 1.1 (= Example 7.4). Let Λ be any Nakayama algebra whose quiver is given by $1 \rightleftharpoons 2$. Then there are 4 bricks in $\text{mod } \Lambda$, namely, $1, 2, \frac{1}{2}, \frac{2}{1}$. Only from the information on (non-)injections between them, we can combinatorially list up a set \mathcal{M} satisfying (MB) and (CC) above, namely, $\emptyset, \{1\}, \{2\}, \{1, \frac{2}{1}\}, \{2, \frac{1}{2}\}, \{1, 2\}$. Hence there are 6 torsion-free classes in $\text{mod } \Lambda$, namely, $0 = \text{Filt } \emptyset, \text{Filt}\{1\}, \text{Filt}\{2\}, \text{Filt}\{1, \frac{2}{1}\}, \text{Filt}\{2, \frac{1}{2}\}, \text{Filt}\{1, 2\} = \text{mod } \Lambda$.

In general, we call a set \mathcal{M} of bricks satisfying (MB) a *monobrick*. A well-known example is a semibrick, a pairwise Hom-orthogonal set of bricks. It is classical that semibricks in \mathcal{A} are in bijection with wide subcategories in \mathcal{A} by the same maps as in Theorem A (c.f. [Rin1, 1.2]). The aim of this paper is to provide a uniform theory to study monobricks and several kinds of subcategories including torsion-free classes and wide subcategories, thereby giving a systematic framework for studying these subcategories.

Our starting point is the bijection between $\text{mbrick } \mathcal{A}$, the set of monobricks in \mathcal{A} , and $\text{Schur}_L \mathcal{A}$, the set of left Schur subcategories. A left Schur subcategory is a category whose simple objects satisfy the one-sided Schur's lemma (see Definition 2.5). The class of left Schur subcategories contains any subcategories of \mathcal{A} which are closed under extensions and kernels and images, thus unifies torsion-free classes and wide subcategories.

Theorem B (= Theorems 2.11, 3.15, 4.5). *Let \mathcal{A} be a length abelian category. Then we have bijections between the set of left Schur subcategories and monobricks in \mathcal{A} :*

$$\text{Schur}_L \mathcal{A} \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} \text{mbrick } \mathcal{A}$$

Moreover, this bijection restricts to the following bijections:

- $\text{wide } \mathcal{A} \rightleftharpoons \text{sbrick } \mathcal{A}$ *between the set of wide subcategories of \mathcal{A} and semibricks in \mathcal{A} , and*
- $\text{torf } \mathcal{A} \rightleftharpoons \text{mbrick}_{\text{c.c.}} \mathcal{A}$ *between the set of torsion-free classes in \mathcal{A} and cofinally closed monobricks in \mathcal{A} (monobricks satisfying (CC) in Theorem A).*

In the case of Nakayama algebras, we show that left Schur subcategories are precisely categories closed under extensions, kernels and images (Theorem 6.1).

We establish Theorem B by using two natural maps $W: \text{Schur}_L \mathcal{A} \rightarrow \text{wide } \mathcal{A}$ and $F: \text{Schur}_L \mathcal{A} \rightarrow \text{torf } \mathcal{A}$, where $W(\mathcal{E})$ is the same as in [IT, MS] (Definition 4.3) and $F(\mathcal{E})$ is the smallest torsion-free class containing \mathcal{E} . We describe these maps in terms of a natural *poset structure* of monobricks \mathcal{M} , namely, $L \leq M$ in \mathcal{M} if there is an injection $L \hookrightarrow M$. Then semibricks and cofinally closed monobricks can be characterized by this poset structure (see Proposition 4.2 and Definition 3.2). Now the maps W and F are easily described in terms of the poset structure as follows.

Theorem C (= Theorems 3.15, 4.5). *Let \mathcal{A} be a length abelian category.*

- (1) *The following diagram commutes, and the horizontal maps are bijections.*

$$\begin{array}{ccc} \text{Schur}_L \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{mbrick } \mathcal{A} \\ \downarrow W & & \downarrow \max \\ \text{wide } \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{sbrick } \mathcal{A} \end{array}$$

Here $\max \mathcal{M}$ for a monobrick \mathcal{M} is the set of maximal element of \mathcal{M} .

(2) The following diagram commutes, and the horizontal maps are bijections.

$$\begin{array}{ccc}
 \text{Schur}_L \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{mbrick } \mathcal{A} \\
 \downarrow \text{F} & & \downarrow (\overline{\quad}) \\
 \text{torf } \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{mbrick}_{\text{c.c.}} \mathcal{A}
 \end{array}$$

Here $\overline{\mathcal{M}}$ for a monobrick \mathcal{M} is the cofinal closure of \mathcal{M} (Definition 3.5).

As an application, we can quickly prove the finiteness result in [DIJ]: $\#\text{torf } \mathcal{A}$ is finite if and only if there are only finitely many bricks (Theorem 5.5). In addition, we can easily deduce the following bijections between $\text{torf } \mathcal{A}$ and $\text{wide } \mathcal{A}$ using only some combinatorial observation on posets, by considering its monobrick counterpart. This was proved in [MS] in the case of finite-dimensional algebras by using τ -tilting theory.

Corollary D (= Corollary 5.4). *Let \mathcal{A} be a length abelian category. Suppose that $\text{torf } \mathcal{A}$ is a finite set. Then the maps $W: \text{torf } \mathcal{A} \rightleftharpoons \text{wide } \mathcal{A}: F$ are mutually inverse bijections.*

Comparison to τ -tilting theory. For the convenience of the reader, we summarize advantages and disadvantages of monobricks compared to τ -tilting theory.

(Advantages)

- τ -tilting theory uses Ext-projectives, while we use simple objects. This enables us to work with any length abelian categories, where there may not be any projective objects.
- τ -tilting theory cannot classify non-functorially finite cases, while monobricks can. This is because non-functorially finite subcategories may not have Ext-projectives.
- Using monobricks, we can study both wide subcategories and torsion-free classes in the same framework, and the relation between them become more transparent.
- Left Schur subcategories, or subcategories closed under extensions, kernels and images, seem to be new objects to study. Our enumerative result on Nakayama algebras via the Schröder number suggests that there are more hidden combinatorics in other algebras.

(Disadvantages)

- In general, left Schur subcategories are complicated to deal with. Actually, there are left Schur subcategories which are not even closed under direct summands (Example 7.2).
- One of the benefits of τ -tilting theory is a *mutation*, which provides a way to create various torsion-free classes starting from $\text{mod } \Lambda$. So far, we have no such theory for monobricks.
- We cannot investigate functorially finiteness by monobricks. More precisely, two isomorphic monobricks (as posets) can correspond to functorially finite and non-functorially finite torsion-free classes (Example 7.5).

Finally, we should mention the relation of this paper to [Asa], where a bijection between functorially finite torsion-free classes and semibricks satisfying some conditions was established. Although we cannot reprove his result (due to the last disadvantage), his map can be easily described via monobricks: $\mathcal{F} \mapsto \max(\text{sim } \mathcal{F})$. See Remark 4.6 for the more detail.

Organization. This paper is organized as follows. In Section 2, we study basic properties of left Schur subcategories and monobricks, and establish a bijection between them. In Section 3, we study the cofinal closure $\overline{\mathcal{M}}$ and show its relation to torsion-free classes. In Section 4, we study the map W and its relation to $\max \mathcal{M}$. In Section 5, we apply the previous results to show results on torsion-free classes and wide subcategories. In Section 6, we give a combinatorial classification of monobricks over Nakayama algebras, and enumerate its number. In Section 7, we show some examples of the classification of monobricks and the computation of $\overline{\mathcal{M}}$ and $\max \mathcal{M}$.

Conventions and notation. Throughout this paper, we assume that all categories are skeletally small, that is, the isomorphism classes of objects form a set. In addition, all subcategories are assumed to be full and closed under isomorphisms. We often identify an isomorphism class in a category with its representative. We always denote by \mathcal{A} a skeletally small length abelian category, that is, an abelian category in which every object has finite length. For a collection \mathcal{C} of objects

in \mathcal{A} , we denote by $\mathbf{add}\mathcal{C}$ the subcategory of \mathcal{A} consisting of direct summands of finite direct sums of objects in \mathcal{C} . For a finite-dimensional algebra Λ , we denote by $\mathbf{mod}\Lambda$ the category of finitely generated right Λ -modules. For a set A , we denote by $\#A$ its cardinality.

2. BIJECTION BETWEEN MONOBICKS AND LEFT SCHUR SUBCATEGORIES

First we introduce a *monobrick* in a length abelian category \mathcal{A} . Recall that a *brick* in \mathcal{A} is an object M such that $\mathrm{End}_{\mathcal{A}}(M)$ is a division ring.

Definition 2.1. Let \mathcal{M} be a set of isomorphism classes of bricks in \mathcal{A} .

- (1) \mathcal{M} is called a *monobrick* if every morphism between elements of \mathcal{M} is either zero or an injection in \mathcal{A} . We denote by $\mathbf{mbrick}\mathcal{A}$ the set of monobricks in \mathcal{A} .
- (2) \mathcal{M} is called a *semibrick* if every morphism between elements of \mathcal{M} is either zero or an isomorphism. We denote by $\mathbf{sbrick}\mathcal{A}$ the set of semibricks in \mathcal{A} .

Note the assumption that a monobrick \mathcal{M} consists of bricks is automatically satisfied if we require the above property, since every non-zero endomorphism of M in \mathcal{M} should be an injection, thus an isomorphism since \mathcal{A} is length.

Clearly every semibrick is a monobrick, thus $\mathbf{sbrick}\mathcal{A} \subset \mathbf{mbrick}\mathcal{A}$ holds. Next we introduce *left Schur subcategories* of \mathcal{A} . Roughly speaking, it is an extension-closed subcategory of \mathcal{A} such that the ‘‘one-sided Schur’s lemma’’ holds. Let us define some notations.

Definition 2.2. Let \mathcal{E} be a subcategory of \mathcal{A} .

- (1) \mathcal{E} is *closed under extensions* or *extension-closed* in \mathcal{A} if it satisfies the following condition: for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in \mathcal{A} , if X and Z belong to \mathcal{E} , then so does Y .

- (2) Suppose that \mathcal{E} is extension-closed in \mathcal{A} . Then a non-zero object M in \mathcal{E} is a *simple object* in \mathcal{E} if there is *no* exact sequence of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in \mathcal{A} satisfying $L, M, N \in \mathcal{E}$ and $L, N \neq 0$. We denote by $\mathbf{sim}\mathcal{E}$ the set of isomorphism classes of simple objects in \mathcal{E} .

Clearly $\mathbf{sim}\mathcal{A}$ is nothing but the set of the usual simple objects in an abelian category \mathcal{A} . Thus $\mathbf{sim}\mathcal{E}$ is an analogue of simple objects *inside* \mathcal{E} . Simple objects can be considered as one of the invariants of extension-closed subcategories, and has an application such as a characterization of the Jordan-Hölder type property inside subcategories [Eno1].

To define and study left Schur subcategories, the following terminology is useful.

Definition 2.3. Let \mathcal{C} be a collection of objects in \mathcal{A} . Then a non-zero object $M \in \mathcal{A}$ is *left Schurian for \mathcal{C}* if every morphism $M \rightarrow C$ with $C \in \mathcal{C}$ is either zero or an injection in \mathcal{A} .

Note that we do not require that M belongs to \mathcal{C} . It is clear that a collection \mathcal{M} of non-zero objects in \mathcal{A} is a monobrick if and only if every object in \mathcal{M} is left Schurian for \mathcal{M} . Simple objects in \mathcal{A} (in the usual sense) are left Schurian for any collection \mathcal{C} .

The fundamental relation between left Schurian objects and simple objects is as follows.

Proposition 2.4. *Let \mathcal{E} be an extension-closed subcategory of \mathcal{A} . Then the following hold.*

- (1) $\{M \in \mathcal{E} \mid M \text{ is left Schurian for } \mathcal{E}\}$ is a monobrick.
- (2) If M in \mathcal{E} is left Schurian for \mathcal{E} , then M is simple in \mathcal{E} .

Proof. (1) Suppose that M and N in \mathcal{E} are left Schurian for \mathcal{E} . Then clearly every morphism $M \rightarrow N$ is either zero or an injection in \mathcal{A} by definition. Thus the assertion holds.

- (2) Suppose that M in \mathcal{E} is left Schurian for \mathcal{E} , and take an exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$

in \mathcal{A} with $L, N \in \mathcal{E}$. Then π should be either zero or an injection. In the former case, we have $N = 0$, and in the latter, we have $L = 0$. Thus M is simple in \mathcal{E} . \square

Then we can define a left Schur subcategory as follows.

Definition 2.5. A subcategory \mathcal{E} of \mathcal{A} is *left Schur* if it satisfies the following conditions:

- (1) \mathcal{E} is closed under extensions in \mathcal{A} .
- (2) Every simple object in \mathcal{E} is left Schurian for \mathcal{E} , that is, for a simple object M in \mathcal{E} , every morphism $M \rightarrow X$ with $X \in \mathcal{E}$ is either zero or an injection in \mathcal{A} .

We denote by $\text{Schur}_L \mathcal{A}$ the set of left Schur subcategories of \mathcal{A} .

The following immediately follows from Proposition 2.4:

Corollary 2.6. *Let \mathcal{E} be a left Schur subcategory of \mathcal{A} and M an object in \mathcal{E} . Then M is simple in \mathcal{E} if and only if M is left Schurian for \mathcal{E} . Moreover, $\text{sim } \mathcal{E}$ is a monobrick.*

We will see that all wide subcategories and torsion-free classes are left Schur. Let us recall the definitions of these subcategories.

Definition 2.7. Let \mathcal{E} be a subcategory of \mathcal{A} .

- (1) \mathcal{E} is *closed under kernels (resp. cokernels, images)* if for every morphism $X \rightarrow Y$ in \mathcal{E} , we have $\text{Ker } f$ (resp. $\text{Coker } f, \text{Im } f$) belongs to \mathcal{E} .
- (2) \mathcal{E} is *closed under subobjects (resp. quotients)* if every subobject (resp. quotient object) of X belongs to \mathcal{E} for every X in \mathcal{E} .
- (3) \mathcal{E} is a *torsion-free class (resp. torsion class)* in \mathcal{A} if it is closed under extensions and subobjects (resp. extensions and quotients) in \mathcal{A} . We denote by $\text{torf } \mathcal{A}$ the set of torsion-free classes in \mathcal{A} .
- (4) \mathcal{E} is a *wide subcategory* of \mathcal{A} if it is closed under extensions, kernels and cokernels. We denote by $\text{wide } \mathcal{A}$ the set of wide subcategories of \mathcal{A} .

It can be easily shown that every wide subcategory or every torsion-free class in \mathcal{A} is closed under extensions, kernels and images. We prove that this condition implies left Schur, thus $\text{wide } \mathcal{A} \subset \text{Schur}_L \mathcal{A}$ and $\text{torf } \mathcal{A} \subset \text{Schur}_L \mathcal{A}$ hold.

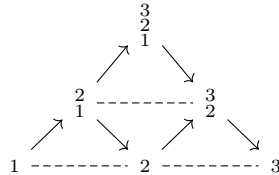
Proposition 2.8. *Let \mathcal{E} be a subcategory of \mathcal{A} which is closed under extensions, kernels and images in \mathcal{A} . Then \mathcal{E} is a left Schur subcategory of \mathcal{A} .*

Proof. Let M be a simple object in \mathcal{E} and $f: M \rightarrow X$ be a morphism with $X \in \mathcal{E}$. Then we have the following exact sequence in \mathcal{A} :

$$0 \longrightarrow \text{Ker } f \longrightarrow M \longrightarrow \text{Im } f \longrightarrow 0$$

Since \mathcal{E} is closed under kernels and images, we have $\text{Ker } f, \text{Im } f \in \mathcal{E}$. Thus either $\text{Ker } f = 0$ or $\text{Im } f = 0$ since M is simple in \mathcal{E} . In the former case, we have that f is an injection in \mathcal{A} , and in the latter, we have $f = 0$. Thus M is left Schurian for \mathcal{E} , hence \mathcal{E} is left Schur. \square

Example 2.9. Let k be a field and Q be a quiver $1 \leftarrow 2 \leftarrow 3$. Then the Auslander-Reiten quiver of $\text{mod } kQ$ is as follows:



Now the subcategory $\mathcal{E}_1 = \text{add}\{1, \frac{2}{1}, \frac{3}{1}, 2\}$ is a torsion-free class, and $\text{sim } \mathcal{E}_1 = \{1, 2, \frac{3}{1}\}$. It can be checked that every simple object in \mathcal{E}_1 is left Schurian for \mathcal{E}_1 , thus is a left Schur subcategory (this follows also from Proposition 2.8). On the other hand, consider $\mathcal{E}_2 = \text{add}\{\frac{2}{1}, \frac{3}{1}, 2\}$. This subcategory is closed under extensions, and all the three indecomposables are simple objects in \mathcal{E}_2 . However, we have a non-zero non-injection $\frac{2}{1} \rightarrow 2$. Thus \mathcal{E}_2 is not a left Schur subcategory.

For a left Schur subcategory \mathcal{E} of \mathcal{A} , we have a monobrick $\text{sim } \mathcal{E}$ by Corollary 2.6. Conversely, for a given monobrick \mathcal{M} , we will construct a left Schur subcategory whose simples are \mathcal{M} . To do this, we will use the following operation.

Definition 2.10. Let \mathcal{C} be a collection of objects in \mathcal{A} . Then $\text{Filt } \mathcal{C}$ denotes the subcategory of \mathcal{A} consisting of objects X such that there is a chain

$$0 = X_0 < X_1 < \cdots < X_n = X$$

of subobjects of X such that X_i/X_{i-1} is in \mathcal{C} for each i . We call such a chain a \mathcal{C} -filtration of X , and n a length of this \mathcal{C} -filtration.

It follows from the Noether isomorphism theorem that $\text{Filt } \mathcal{C}$ is extension-closed, and it is obvious from the construction that it is the smallest extension-closed subcategory of \mathcal{A} containing \mathcal{C} .

Now we can state our first main result of this paper.

Theorem 2.11. *Let \mathcal{A} be a length abelian category. Then sim and Filt give mutually inverse bijections between left Schur subcategories of \mathcal{A} and monobricks in \mathcal{A} , which extends the bijection between wide subcategories and semibricks:*

$$\begin{array}{ccc} \text{sim}: \text{Schur}_L \mathcal{A} & \xrightarrow{\sim} & \text{mbrick } \mathcal{A} : \text{Filt} \\ \uparrow & & \uparrow \\ \text{wide } \mathcal{A} & \xrightarrow{\sim} & \text{sbrick } \mathcal{A} \end{array}$$

We need some preparation to prove it. For two collections \mathcal{C} and \mathcal{D} of objects in \mathcal{A} , we denote by $\mathcal{C} * \mathcal{D}$ the subcategory of \mathcal{A} consisting of objects X such that there is an exact sequence

$$0 \longrightarrow C \longrightarrow X \longrightarrow D \longrightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. As for this, the following lemma in [Eno2, Lemma 3.10] is useful. We give a proof for the convenience of the reader.

Lemma 2.12. *Let M be an object in \mathcal{A} . If M is left Schurian for two collections \mathcal{C} and \mathcal{D} of objects in \mathcal{A} , then M is left Schurian also for $\mathcal{C} * \mathcal{D}$.*

Proof. Take any short exact sequence in \mathcal{A}

$$0 \longrightarrow C \xrightarrow{\iota} X \xrightarrow{\pi} D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let $\varphi: M \rightarrow X$ be any morphism, and we will prove that φ is either zero or an injection in \mathcal{A} . Consider the following commutative diagram.

$$\begin{array}{ccccccc} & & & M & & & \\ & & & \downarrow \varphi & & & \\ & & \swarrow \bar{\varphi} & & \searrow & & \\ 0 & \longrightarrow & C & \xrightarrow{\iota} & X & \xrightarrow{\pi} & D \longrightarrow 0 \end{array}$$

Since M is left Schurian for \mathcal{D} , either $\pi\varphi$ is an injection or $\pi\varphi = 0$. In the former case, φ is an injection, so suppose the latter. Then there exists a morphism $\bar{\varphi}: M \rightarrow C$ with $\varphi = \bar{\varphi}\iota$. Since M is left Schurian for \mathcal{C} , we have that $\bar{\varphi}$ is either zero or injective. Thus φ is either zero or injective respectively. \square

Now we are ready to prove Theorem 2.11.

Proof of Theorem 2.11. For a left Schur subcategory \mathcal{E} of \mathcal{A} , we have $\text{sim } \mathcal{E} \in \text{mbrick } \mathcal{A}$ by Corollary 2.6, thus we have a map $\text{sim}: \text{Schur}_L \mathcal{A} \rightarrow \text{mbrick } \mathcal{A}$.

For the converse direction, let \mathcal{M} be a monobrick in \mathcal{A} . We will prove that $\text{Filt } \mathcal{M}$ is a left Schur subcategory of \mathcal{A} . Then $\text{Filt } \mathcal{M}$ is closed under extensions in \mathcal{A} . We show the following claim:

(Claim): Let \mathcal{M} be a monobrick. Then the following are equivalent for $M \in \text{Filt } \mathcal{M}$.

- (1) M is simple in $\text{Filt } \mathcal{M}$.
- (2) M is in \mathcal{M} .
- (3) M is left Schurian for $\text{Filt } \mathcal{M}$.

Proof of (Claim).

(1) \Rightarrow (2): By the construction of $\text{Filt } \mathcal{M}$ and the definition of a simple object, every simple object in $\text{Filt } \mathcal{M}$ should belong to \mathcal{M} .

(2) \Rightarrow (3): Let $M \in \mathcal{M}$. Since \mathcal{M} is a monobrick, M is left Schurian for \mathcal{M} . Then by using Lemma 2.12 repeatedly, M is left Schurian for $\text{Filt } \mathcal{M}$.

(3) \Rightarrow (1): This follows from Proposition 2.4. \blacksquare

In particular, the implication (1) \Rightarrow (3) implies that $\text{Filt } \mathcal{M}$ is left Schur. Therefore we obtain a map $\text{Filt}: \text{mbrick } \mathcal{A} \rightarrow \text{Schur}_L \mathcal{A}$. We will prove that these maps are mutually inverse to each other. Since \mathcal{A} is length, it can be easily shown by induction on lengths that $\mathcal{E} = \text{Filt}(\text{sim } \mathcal{E})$ holds for any extension-closed subcategory \mathcal{E} of \mathcal{A} . Thus $\text{Filt} \circ \text{sim}: \text{Schur}_L \mathcal{A} \rightarrow \text{Schur}_L \mathcal{A}$ is the identity. Conversely, (Claim) implies $\text{sim}(\text{Filt } \mathcal{M}) = \mathcal{M}$ for a monobrick \mathcal{M} . Therefore, we have mutually inverse bijections $\text{sim}: \text{Schur}_L \mathcal{A} \rightleftarrows \text{mbrick } \mathcal{A}: \text{Filt}$.

Finally, we claim that this bijections $\text{sim}: \text{Schur}_L \mathcal{A} \xrightarrow{\sim} \text{mbrick } \mathcal{A}: \text{Filt}$ restricts to bijections $\text{wide } \mathcal{A} \leftrightarrow \text{sbrick } \mathcal{A}$. Ringel's result [Rin1, 1.2] implies that $\text{Filt } \mathcal{M}$ is a wide subcategory of \mathcal{A} if \mathcal{M} is a semibrick. Conversely, let \mathcal{E} be a wide subcategory of \mathcal{A} . Then \mathcal{E} is an abelian category, and it is easy to see that simple objects in \mathcal{E} coincides with usual simple objects in an abelian category \mathcal{E} . Thus the Schur's lemma in \mathcal{E} implies that $\text{sim } \mathcal{E}$ is a semibrick. \square

3. MAPS TO TORSION-FREE CLASSES AND COFINALLY CLOSED MONOBRICKS

In this section, we will show that a left Schur subcategory \mathcal{E} of \mathcal{A} is a torsion-free class if and only if $\text{sim } \mathcal{E}$ is a *cofinally closed monobrick*. Then we construct a map from $\text{mbrick } \mathcal{A}$ to the set of cofinally closed monobricks, taking the *cofinal closure*, which corresponds to the map $F(\mathcal{E})$ which sends \mathcal{E} to the smallest torsion-free class containing \mathcal{E} .

3.1. Cofinal extension and cofinal closure of monobricks. First we observe that each monobrick has a natural poset structure, which will play a central role of this paper.

Definition 3.1. Let \mathcal{M} be a monobrick in \mathcal{A} . For M, N in \mathcal{M} , we define $M \leq N$ if there is an injection $M \rightarrow N$ in \mathcal{A} . Since \mathcal{A} is length, it is easily checked that \leq is actually a partial order on \mathcal{M} . We call this order the *submodule order on \mathcal{M}* .

We introduce a notion of *cofinal extension* between monobricks, and *cofinally closed* monobricks.

Definition 3.2. Let \mathcal{M} and \mathcal{N} be two monobricks in \mathcal{A} . Then we say that \mathcal{N} is a *cofinal extension of \mathcal{M}* if it satisfies the following conditions:

- (1) $\mathcal{M} \subset \mathcal{N}$ holds.
- (2) For every $N \in \mathcal{N}$, there exists $M \in \mathcal{M}$ satisfying $N \leq M$ in \mathcal{N} .

We say that a monobrick \mathcal{M} is *cofinally closed* if there is no proper cofinal extension of \mathcal{M} . We denote by $\text{mbrick}_{c.c.} \mathcal{A}$ the set of cofinally closed monobricks in \mathcal{A} .

Note that this is a purely poset theoretical notion, and has nothing to do with the actual module structure of each brick.

Cofinal extensions of \mathcal{M} is closed under unions in the following sense:

Proposition 3.3. *Let \mathcal{M} be a monobrick in \mathcal{A} . Suppose that $\{\mathcal{N}_i \mid i \in I\}$ is a family of cofinal extensions of \mathcal{M} . Then $\bigcup_{i \in I} \mathcal{N}_i$ is a cofinal extension of \mathcal{M} (in particular, it is a monobrick).*

Proof. Clearly, we only have to see that $\mathcal{N} := \bigcup_{i \in I} \mathcal{N}_i$ is actually a monobrick. Take N_1 and N_2 in \mathcal{N} with $N_1 \in \mathcal{N}_{i_1}$ and $N_2 \in \mathcal{N}_{i_2}$, and let $f: N_1 \rightarrow N_2$ be any map. Since \mathcal{N}_{i_2} is a cofinal extension of \mathcal{N} , there is an injection $\iota: N_2 \hookrightarrow M$ with $M \in \mathcal{M}$. Then the composition $\iota f: N_1 \rightarrow M$ is a map between elements in \mathcal{N}_{i_1} , thus it should be either zero or an injection since \mathcal{N}_{i_1} is a monobrick. Then the injectivity of ι implies that f is either zero or an injection. \square

This immediately implies the existence of the *largest cofinal extension* of a given monobrick, which is cofinally closed.

Corollary 3.4. *Let \mathcal{M} be a monobrick in \mathcal{A} . Then the union $\overline{\mathcal{M}}$ of all cofinal extensions of \mathcal{M} satisfies the following conditions:*

- (1) $\overline{\mathcal{M}}$ is a cofinal extension of \mathcal{M} .
- (2) For every cofinal extension \mathcal{N} of \mathcal{M} , we have $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$.
- (3) $\overline{\mathcal{M}}$ is cofinally closed. Moreover, if \mathcal{N} is a cofinal extension of \mathcal{M} which is cofinally closed, then $\mathcal{N} = \overline{\mathcal{M}}$ holds.

Proof. (1), (2) Clear from Proposition 3.6 and the definition of $\overline{\mathcal{M}}$.

(3) Let \mathcal{M}' be a cofinal extension of $\overline{\mathcal{M}}$. Then it is easy to see that \mathcal{M}' is also a cofinal extension of \mathcal{M} . Thus (2) implies $\mathcal{M}' \subset \overline{\mathcal{M}}$, thus $\mathcal{M}' = \overline{\mathcal{M}}$. Therefore, $\overline{\mathcal{M}}$ is cofinally closed. On the other hand, let \mathcal{N} be a cofinal extension of \mathcal{M} which is cofinally closed. Then we have $\mathcal{M} \subset \mathcal{N} \subset \overline{\mathcal{M}}$ holds by (2). It is obvious that $\overline{\mathcal{M}}$ is a cofinal extension of \mathcal{N} , thus we have $\mathcal{N} = \overline{\mathcal{M}}$ by the definition of the cofinally closedness. \square

Definition 3.5. Let \mathcal{M} be a monobrick. We denote by $\overline{\mathcal{M}}$ the union of all cofinal extensions of \mathcal{M} , and call it the *cofinal closure* of \mathcal{M} . Then $\overline{\mathcal{M}}$ is the unique cofinal extension of \mathcal{M} which is cofinally closed by Corollary 3.4.

Taking the cofinal closure defines a map $(\overline{\quad}) : \mathbf{mbrick} \mathcal{A} \rightarrow \mathbf{mbrick}_{c.c.} \mathcal{A}$, which is the identity on $\mathbf{mbrick}_{c.c.} \mathcal{A}$ by the definition of cofinally closedness. Similarly, we can check that a monobrick \mathcal{M} is cofinally closed if and only if $\overline{\mathcal{M}} = \mathcal{M}$.

Next, we will characterize the cofinal closure as in the theory of integral extensions of commutative rings: an integral closure of a ring is given by the set of all elements which is integral over the base ring. To this purpose, we will introduce the following notion.

Definition 3.6. Let \mathcal{M} be a monobrick in \mathcal{A} . We say that a brick N in \mathcal{A} is *cofinal over* \mathcal{M} if it satisfies the following conditions:

- (1) There exist $M \in \mathcal{M}$ and an injection $N \hookrightarrow M$ in \mathcal{A} .
- (2) Every map $N \rightarrow M'$ with $M' \in \mathcal{M}$ is either zero or an injection.

Proposition 3.7. Let \mathcal{M} be a monobrick in \mathcal{A} and N a brick in \mathcal{A} . Then N is cofinal over \mathcal{M} if and only if $\mathcal{M} \cup \{N\}$ is a cofinal extension of \mathcal{M} .

Proof. The “if” part is clear. Conversely, suppose that N is cofinal over \mathcal{M} , and we claim that $\mathcal{M} \cup \{N\}$ is a cofinal extension of \mathcal{M} . Obviously it suffices to show that $\mathcal{M} \cup \{N\}$ is a monobrick.

Clearly we only have to show that every map $f: M \rightarrow N$ with $M \in \mathcal{M}$ is either zero or an injection. By the assumption, there is an injection $\iota: N \hookrightarrow M'$ with $M' \in \mathcal{M}$. Then the composition $\iota f: M \rightarrow M'$ is a map between elements in \mathcal{M} , thus is either zero or an injection. Since ι is injective, this implies that f is either zero or an injection. \square

Now we can describe a cofinal extension of a monobrick in terms of elements:

Corollary 3.8. Let \mathcal{M} be a monobrick in \mathcal{A} and \mathcal{N} a set of bricks in \mathcal{A} satisfying $\mathcal{M} \subset \mathcal{N}$ (we do not require that \mathcal{N} is a monobrick). Then the following are equivalent:

- (1) \mathcal{N} is a cofinal extension of \mathcal{M} (in particular, \mathcal{N} is a monobrick).
- (2) Every element in \mathcal{N} is cofinal over \mathcal{M} .

Proof. (1) \Rightarrow (2): It follows immediately from definition.

(2) \Rightarrow (1): By Proposition 3.7, we have that $\mathcal{M} \cup \{N\}$ is a cofinal extension of \mathcal{M} for each $N \in \mathcal{N}$. Since we have $\mathcal{N} = \bigcup_{N \in \mathcal{N}} (\mathcal{M} \cup \{N\})$, Proposition 3.6 implies that \mathcal{N} is a cofinal extension of \mathcal{M} . \square

Similarly, we have the following description of the cofinal closure.

Corollary 3.9. Let \mathcal{M} be a monobrick in \mathcal{A} . Then we have

$$\overline{\mathcal{M}} = \{N \in \mathbf{brick} \mathcal{A} \mid N \text{ is cofinal over } \mathcal{M}\}.$$

In particular, \mathcal{M} is cofinally closed if and only if the following condition is satisfied:

- (CC) If a brick N is a submodule of some $M' \in \mathcal{M}$ and $N \notin \mathcal{M}$, then there is some non-zero non-injection $N \rightarrow M$ to $M \in \mathcal{M}$.

Proof. Since $\overline{\mathcal{M}}$ is a cofinal extension of \mathcal{M} , every element in $\overline{\mathcal{M}}$ is cofinal over \mathcal{M} . Conversely, suppose that a brick N is cofinal over \mathcal{M} . Then $\mathcal{M} \cup \{N\}$ is a cofinal extension of \mathcal{M} by Proposition 3.7. Thus we have $\mathcal{M} \cup \{N\} \subset \overline{\mathcal{M}}$ by Corollary 3.4, hence $N \in \overline{\mathcal{M}}$. \square

3.2. Torsion-free classes and cofinally closed monobricks. In this subsection, we will show that the map $F: \text{Schur}_L \mathcal{A} \rightarrow \text{torf } \mathcal{A}$ corresponds to the map $(\overline{}): \text{mbrick } \mathcal{A} \rightarrow \text{mbrick}_{c.c.} \mathcal{A}$ defined in the previous subsection.

First of all, we can construct a torsion-free class from any collection of objects in \mathcal{A} as follows.

Definition 3.10. Let \mathcal{C} be a collection of objects in \mathcal{A} .

- We denote by $\text{sub } \mathcal{C}$ the collection of all subobjects of objects in \mathcal{C} (where subobjects mean the usual subobjects in an abelian category \mathcal{A}).
- We denote by $F(\mathcal{C}) := \text{Filt}(\text{sub } \mathcal{C})$.

Lemma 3.11. *Let \mathcal{C} be a collection of objects in \mathcal{A} . Then $F(\mathcal{C})$ is the smallest torsion-free class containing \mathcal{C} .*

Proof. Although this is well-known (e.g. [MS, Lemma 3.1]), we give a proof here for the convenience. Clearly it suffices to show that $F(\mathcal{C})$ is a torsion-free class in \mathcal{A} . Since $F(\mathcal{C}) = \text{Filt}(\text{sub } \mathcal{C})$ is extension-closed in \mathcal{A} , it is enough to show that $F(\mathcal{C})$ is closed under subobjects.

Take any $M \in F(\mathcal{C})$ and its subobject $X \hookrightarrow M$. We will show $X \in F(\mathcal{C})$ by the induction of a $(\text{sub } \mathcal{C})$ -filtration length n of M . If $n = 1$, then $M \in \text{sub } \mathcal{C}$ holds, thus M is a subobject of some $C \in \mathcal{C}$. Then it follows that X is also a subobject of C , which proves $X \in \text{sub } \mathcal{C} \subset F(\mathcal{C})$.

Now suppose $n > 1$. Then there is a short exact sequence $0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} N \rightarrow 0$ with $L, N \in F(\mathcal{C})$ such that L and N has $(\text{sub } \mathcal{C})$ -filtrations of length smaller than n . We can obtain the following exact commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \cap X & \longrightarrow & X & \longrightarrow & \pi(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{\iota} & M & \xrightarrow{\pi} & N & \longrightarrow & 0 \end{array}$$

where all the horizontal maps are injections. By the induction hypothesis, we have $L \cap X, \pi(X) \in F(\mathcal{C})$. Thus $X \in F(\mathcal{C})$ holds since $F(\mathcal{C})$ is extension-closed. \square

The following basic observation is used later.

Lemma 3.12. *Let \mathcal{C} be a collection of objects in \mathcal{A} . Then $F(\mathcal{C}) = F(\text{Filt } \mathcal{C})$ holds.*

Proof. Since $\mathcal{C} \subset \text{Filt } \mathcal{C} \subset F(\text{Filt } \mathcal{C})$, we have $F(\mathcal{C}) \subset F(\text{Filt } \mathcal{C})$ by the minimality of $F(\mathcal{C})$. On the other hand, $F(\mathcal{C}) = \text{Filt}(\text{sub } \mathcal{C}) \supset \text{Filt } \mathcal{C}$ holds, thus $F(\mathcal{C}) \supset F(\text{Filt } \mathcal{C})$. \square

Thus we have the following commutative diagram.

$$\begin{array}{ccc} \text{mbrick } \mathcal{A} & \xrightarrow[\text{Filt}]{\sim} & \text{Schur}_L \mathcal{A} \\ & \searrow F & \downarrow F \\ & & \text{torf } \mathcal{A} \\ & & \downarrow \\ \text{mbrick } \mathcal{A} & \xleftarrow[\text{sim}]{\sim} & \text{Schur}_L \mathcal{A} \end{array}$$

The following claims that the dotted map is nothing but taking the cofinal closure.

Proposition 3.13. *Let \mathcal{M} be a monobrick in \mathcal{A} . Then we have $\overline{\mathcal{M}} = \text{sim } F(\mathcal{M})$.*

Proof. First, we will prove $\mathcal{M} \subset \text{sim } F(\mathcal{M})$, which is equivalent to that every object in \mathcal{M} is left Schurian for $F(\mathcal{M})$ by Corollary 2.6. Take any $M \in \mathcal{M}$. Then M is left Schurian for \mathcal{M} since \mathcal{M} is a monobrick. Since every object in $\text{sub } \mathcal{M}$ admits an injection into some object in \mathcal{M} , it is easily checked that M is left Schurian also for $\text{sub } \mathcal{M}$. Then Lemma 2.12 implies that M is left Schurian for $\text{Filt}(\text{sub } \mathcal{M}) = F(\mathcal{M})$.

Next, we will prove that $\mathcal{M} \subset \text{sim F}(\mathcal{M})$ is a cofinal extension. Let X be a simple object in $\text{F}(\mathcal{M}) = \text{Filt}(\text{sub } \mathcal{M})$. Then clearly we must have $X \in \text{sub } \mathcal{M}$. It follows that there is an injection $X \hookrightarrow M$ with $M \in \mathcal{M}$. This shows that $\text{sim } \mathcal{F}$ is a cofinal extension of \mathcal{M} .

Therefore, we have $\text{sim F}(\mathcal{M}) \subset \overline{\mathcal{M}}$ by Corollary 3.4. On the other hand, since $\overline{\mathcal{M}}$ is a monobrick, every object in $\overline{\mathcal{M}}$ is left Schurian for $\overline{\mathcal{M}}$, thus so is for \mathcal{M} . Then the same argument as the first part implies $\overline{\mathcal{M}} \subset \text{sim F}(\mathcal{M})$. Hence $\text{sim F}(\mathcal{M}) = \overline{\mathcal{M}}$ holds. \square

As an corollary, we have the following description of simple objects in $\text{F}(\mathcal{E})$ for a left Schur category \mathcal{E} .

Corollary 3.14. *Let \mathcal{E} be a left Schur subcategory of \mathcal{A} . Then $\text{sim F}(\mathcal{E})$ consists of bricks N in \mathcal{A} which satisfy the following conditions:*

- (1) *There is an injection $N \hookrightarrow M$ with $M \in \text{sim } \mathcal{E}$.*
- (2) *Every map $N \rightarrow M'$ with $M' \in \text{sim } \mathcal{E}$ is either zero or an injection.*

Proof. This immediately follows from Corollary 3.9, since we have $\text{sim F}(\mathcal{E}) = \overline{\text{sim } \mathcal{E}}$ by Proposition 3.13 and Theorem 2.11. \square

Now we can state our characterization of torsion-free classes via monobricks.

Theorem 3.15. *Let \mathcal{A} be a length abelian category. Then we have the following commutative diagram, and all the horizontal maps are bijective.*

$$\begin{array}{ccc}
 \text{torf } \mathcal{A} & \xrightleftharpoons[\text{Filt}]{\text{sim}} & \text{mbrick}_{\text{c.c.}} \mathcal{A} \\
 \downarrow & & \downarrow \\
 \text{Schur}_L \mathcal{A} & \xrightleftharpoons[\text{Filt}]{\text{sim}} & \text{mbrick } \mathcal{A} \\
 \downarrow \text{F} & & \downarrow (-) \\
 \text{torf } \mathcal{A} & \xrightleftharpoons[\text{Filt}]{\text{sim}} & \text{mbrick}_{\text{c.c.}} \mathcal{A}
 \end{array}$$

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Proof. Clearly it suffices to show that the following are equivalent for a monobrick \mathcal{M} in \mathcal{A} :

- (1) \mathcal{M} is cofinally closed.
- (2) $\text{Filt } \mathcal{M}$ is a torsion-free class.

(1) \Rightarrow (2): If \mathcal{M} is cofinally closed, then $\overline{\mathcal{M}} = \mathcal{M}$ holds. Therefore, we have $\text{Filt } \mathcal{M} = \text{Filt } \overline{\mathcal{M}} = \text{Filt}(\text{sim F}(\mathcal{M})) = \text{F}(\mathcal{M})$ by Proposition 3.13, thus $\text{Filt } \mathcal{M}$ is a torsion-free class in \mathcal{A} .

(2) \Rightarrow (1): Since $\text{Filt } \mathcal{M}$ is a torsion-free class, we have $\text{F}(\mathcal{M}) = \text{F}(\text{Filt } \mathcal{M}) = \text{Filt } \mathcal{M}$ holds by Lemma 3.12. Thus we have $\overline{\mathcal{M}} = \text{sim F}(\mathcal{M}) = \text{sim}(\text{Filt } \mathcal{M}) = \mathcal{M}$ by Proposition 3.13. Therefore \mathcal{M} is cofinally closed. \square

We can obtain the following characterization of left Schur subcategories:

Corollary 3.16. *Let \mathcal{E} be a subcategory of \mathcal{A} . Then \mathcal{E} is left Schur if and only if there exist a torsion-free class \mathcal{F} of \mathcal{A} and a subset \mathcal{M} of $\text{sim } \mathcal{F}$ such that $\mathcal{E} = \text{Filt } \mathcal{M}$ holds.*

Proof. Since $\text{sim } \mathcal{F}$ (and its subset) is a monobrick for a torsion-free class \mathcal{F} , the “if” part is clear. Conversely, let \mathcal{E} be a left Schur subcategory of \mathcal{A} . Then we have $\text{sim } \mathcal{E} \subset \overline{\text{sim } \mathcal{E}} = \text{sim F}(\mathcal{E})$ by Proposition 3.13. Thus $\mathcal{F} := \text{F}(\mathcal{E})$ satisfies the condition. \square

As a similar result, we can prove the following.

Corollary 3.17. *Let \mathcal{M} be a set of isomorphism classes of bricks in \mathcal{A} . Then \mathcal{M} is a monobrick if and only if there exist a cofinally closed monobrick \mathcal{N} of \mathcal{A} such that \mathcal{M} is a subset of \mathcal{N} .*

Proof. The “if” part is clear, and take $\mathcal{N} := \overline{\mathcal{M}}$ for the “if” part. \square

4. MAPS TO SEMIBRICKS AND WIDE SUBCATEGORIES

We construct maps $\max: \mathbf{mbrick} \mathcal{A} \rightarrow \mathbf{sbrick} \mathcal{A}$ and $W: \mathbf{Schur}_L \mathcal{A} \rightarrow \mathbf{wide} \mathcal{A}$, which is an identity if restricted to $\mathbf{sbrick} \mathcal{A}$ and $\mathbf{wide} \mathcal{A}$ respectively. These maps correspond to each other under Theorem 2.11.

First we consider the following operation on monobricks.

Proposition 4.1. *Let \mathcal{M} be a monobrick in \mathcal{A} . Then the set $\max \mathcal{M}$ of maximal elements for the submodule order on \mathcal{M} is a semibrick.*

Proof. It suffices to show that every map $f: M \rightarrow N$ with $M, N \in \max \mathcal{M}$ is zero if $M \neq N$. Suppose that f is non-zero. Then f should be an injection in \mathcal{A} since \mathcal{M} is a monobrick. It follows that $M \leq N$, which contradicts to the maximality of M . \square

Thus we obtain the map $\max: \mathbf{mbrick} \mathcal{A} \rightarrow \mathbf{sbrick} \mathcal{A}$ for a length abelian category \mathcal{A} . Actually we have the following characterization of a semibrick in terms of the poset structure.

Proposition 4.2. *Let \mathcal{S} be a monobrick in \mathcal{A} . Then the following are equivalent:*

- (1) \mathcal{S} is a semibrick.
- (2) \mathcal{S} is a discrete poset, that is, $M \leq N$ in \mathcal{S} implies $M = N$.
- (3) $\max \mathcal{S} = \mathcal{S}$ holds.

Proof. This is immediate from definitions. \square

Next we consider introduce a map $W: \mathbf{Schur}_L \mathcal{A} \rightarrow \mathbf{wide} \mathcal{A}$. This extends the map $W: \mathbf{torf} \mathcal{A} \rightarrow \mathbf{wide} \mathcal{A}$ defined by Marks-Šťovíček [MS].

Definition 4.3. Let \mathcal{E} be a left Schur subcategory of \mathcal{A} . Then $W(\mathcal{E})$ is a subcategory of \mathcal{E} consisting of objects $W \in \mathcal{E}$ satisfying the following condition: For every map $f: W \rightarrow X$ with $X \in \mathcal{E}$, we have $\text{Coker } f \in \mathcal{E}$, where $\text{Coker } f$ denotes the cokernel of f in \mathcal{A} .

The following is a key lemma to show that $W(\mathcal{E})$ is actually a wide subcategory.

Lemma 4.4. *Let \mathcal{E} be a left Schur subcategory of \mathcal{A} . Then the following holds.*

- (1) For $M \in \mathbf{sim} \mathcal{E}$, the following are equivalent:
 - (a) M belongs to $W(\mathcal{E})$.
 - (b) M is maximal in the submodule order on $\mathbf{sim} \mathcal{E}$.
 - (c) For every non-zero morphism $f: M \rightarrow X$ is an injection in \mathcal{A} satisfying $\text{Coker } f \in \mathcal{E}$.
- (2) If we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

in \mathcal{A} with $L, M, N \in \mathcal{E}$, then M is in $W(\mathcal{E})$ if and only if both L and N are in $W(\mathcal{E})$.

Proof. (1) Let M be a simple object in \mathcal{E} .

(a) \Rightarrow (b): Suppose that M is not maximal in $\mathbf{sim} \mathcal{E}$. Then we have a proper injection $\iota: M \hookrightarrow M'$ with $M' \in \mathbf{sim} \mathcal{E}$. Consider the following exact sequence in \mathcal{A} :

$$0 \longrightarrow M \xrightarrow{\iota} M' \longrightarrow \text{Coker } \iota \longrightarrow 0$$

Since ι is not an isomorphism, $\text{Coker } \iota$ is non-zero. Then $\text{Coker } \iota$ does not belong to \mathcal{E} , since otherwise M' would not be simple in \mathcal{E} . This implies that M does not belong to $W(\mathcal{E})$.

(b) \Rightarrow (c) Let $f: M \rightarrow X$ be a non-zero map with $X \in \mathcal{E}$. Then f is an injection in \mathcal{A} since \mathcal{E} is left Schur. We will prove $\text{Coker } f \in \mathcal{E}$ by induction on a $(\mathbf{sim} \mathcal{E})$ -filtration length of X .

If X belongs to $\mathbf{sim} \mathcal{E}$, then the maximality of M clearly implies that f should be an isomorphism. Suppose that X has $(\mathbf{sim} \mathcal{E})$ -filtration of length $n > 1$. Then we have a short exact sequence $0 \rightarrow L \xrightarrow{\iota} X \xrightarrow{\pi} N \rightarrow 0$ with $L \in \mathbf{sim} \mathcal{E}$ and N has $(\mathbf{sim} \mathcal{E})$ -filtration of length $n - 1$ (in particular, $L, N \in \mathcal{E}$). Consider the following diagram:

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow f & & & & \\ 0 & \longrightarrow & L & \xrightarrow{\iota} & X & \xrightarrow{\pi} & N \longrightarrow 0 \end{array}$$

We consider two cases.

(Case 1): $\pi f = 0$. In this case, there is a map $\bar{f}: M \rightarrow L$ satisfying $f = \iota \bar{f}$. By the induction hypothesis, \bar{f} is either zero or an injection with $\text{Coker } \bar{f} \in \mathcal{E}$. In the former case, we have $f = 0$, which is a contradiction. Thus the latter holds. Then we obtain the following exact commutative diagram.

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & M & \xlongequal{\quad} & M & & & & \\
& & \downarrow \bar{f} & & \downarrow f & & & & \\
0 & \longrightarrow & L & \xrightarrow{\quad \iota \quad} & X & \xrightarrow{\quad \pi \quad} & N & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \text{Coker } \bar{f} & \longrightarrow & \text{Coker } f & \longrightarrow & N & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}$$

Since $\text{Coker } \bar{f}$ and N belong to \mathcal{E} , so does $\text{Coker } f$ since \mathcal{E} is extension-closed.

(Case 2): $\pi f \neq 0$. In this case, πf is an injection with $\text{Coker}(\pi f) \in \mathcal{E}$ by the induction hypothesis. Then we obtain the following exact commutative diagram.

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & M & \xlongequal{\quad} & M & & & & \\
& & \downarrow f & & \downarrow \pi f & & & & \\
0 & \longrightarrow & L & \xrightarrow{\quad \iota \quad} & X & \xrightarrow{\quad \pi \quad} & N & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \text{Coker } f & \longrightarrow & \text{Coker}(\pi f) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}$$

Since L and $\text{Coker}(\pi f)$ belong to \mathcal{E} , so does $\text{Coker } f$ since \mathcal{E} is extension-closed.

(c) \Rightarrow (a): Clear from the definition of $\mathcal{W}(\mathcal{E})$.

(2) Suppose that L and N belong to $\mathcal{W}(\mathcal{E})$, and we will prove $M \in \mathcal{W}(\mathcal{E})$. Take any map $f: M \rightarrow X$ with $X \in \mathcal{E}$. Then we obtain the following exact commutative diagram, where \bar{f} is a map induced from the universality of the cokernel N of i .

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\quad i \quad} & M & \xrightarrow{\quad p \quad} & N & \longrightarrow & 0 \\
& & \parallel & & \downarrow f & & \downarrow \bar{f} & & \\
& & L & \xrightarrow{\quad fi \quad} & X & \longrightarrow & \text{Coker}(fi) & \longrightarrow & 0
\end{array}$$

Since L is in $\mathcal{W}(\mathcal{E})$ and X is in \mathcal{E} , we have $\text{Coker}(fi) \in \mathcal{E}$. Therefore, we have $\text{Coker } \bar{f} \in \mathcal{E}$ since N is in $\mathcal{W}(\mathcal{E})$. On the other hand, it can be shown that the right square is a pushout diagram. Thus $\text{Coker } f \cong \text{Coker } \bar{f}$ holds, which proves $\text{Coker } f \in \mathcal{E}$. Therefore $M \in \mathcal{W}(\mathcal{E})$ holds.

Conversely, suppose that M belongs to $\mathcal{W}(\mathcal{E})$, and we will show that L and N belong to $\mathcal{W}(\mathcal{E})$. First we will prove $L \in \mathcal{W}(\mathcal{E})$. Take any map $f: L \rightarrow X$ with $X \in \mathcal{E}$. Then by taking the pushout, we obtain the following exact commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\quad i \quad} & M & \xrightarrow{\quad p \quad} & N & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow \bar{f} & & \parallel & & \\
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Since the left square is pushout, we have $\text{Coker } f \cong \text{Coker } \bar{f}$. On the other hand, we have $E \in \mathcal{E}$ since \mathcal{E} is extension-closed and $X, N \in \mathcal{E}$. Thus $\text{Coker } \bar{f} \in \mathcal{E}$ holds by $M \in \mathbf{W}(\mathcal{E})$. Therefore $\text{Coker } f \in \mathcal{E}$, which proves $L \in \mathbf{W}(\mathcal{E})$.

Next we will prove $N \in \mathbf{W}(\mathcal{E})$. Take any map $f: N \rightarrow X$ with $X \in \mathcal{E}$. Then since p is a surjection, $\text{Coker } f \cong \text{Coker}(fp)$ holds. On the other hand, $\text{Coker}(fp) \in \mathcal{E}$ holds by $M \in \mathbf{W}(\mathcal{E})$ and $X \in \mathcal{E}$. Therefore $\text{Coker } f \in \mathcal{E}$, which proves $N \in \mathbf{W}(\mathcal{E})$. \square

Now we are ready to prove the main result in this section.

Theorem 4.5. *Let \mathcal{A} be a length abelian category. Then the following hold.*

- (1) $\mathbf{W}(\mathcal{W}) = \mathcal{W}$ holds for a wide subcategory \mathcal{W} of \mathcal{A} .
- (2) $\mathbf{W}(\mathcal{E})$ is a wide subcategory of \mathcal{A} for a left Schur subcategory \mathcal{E} of \mathcal{A} .
- (3) The following diagram commutes, and all the horizontal maps are bijective.

$$\begin{array}{ccc}
 \text{wide } \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{sbrick } \mathcal{A} \\
 \downarrow & & \downarrow \\
 \text{Schur}_L \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{mbrick } \mathcal{A} \\
 \downarrow \mathbf{W} & & \downarrow \text{max} \\
 \text{wide } \mathcal{A} & \begin{array}{c} \xrightarrow{\text{sim}} \\ \xleftarrow{\text{Filt}} \end{array} & \text{sbrick } \mathcal{A}
 \end{array}$$

(A large curved arrow labeled '1' encircles the entire diagram.)

Proof. (1) Clear from the definition of $\mathbf{W}(\mathcal{W})$ since \mathcal{W} is closed under cokernels.

(2), (3) Let \mathcal{E} be a left Schur subcategory of \mathcal{A} and put $\mathcal{M} := \text{sim } \mathcal{E}$. By Theorem 2.11, clearly it suffices to show that $\mathbf{W}(\mathcal{E}) = \text{Filt}(\text{max } \mathcal{M})$, since $\text{max } \mathcal{M}$ is a semibrick by Proposition 4.1. By Lemma 4.4 (1), we have $\text{max } \mathcal{M} \subset \mathbf{W}(\mathcal{E})$. Since \mathcal{E} is extension-closed, it can be easily checked that Lemma 4.4 (2) implies $\text{Filt}(\text{max } \mathcal{M}) \subset \mathbf{W}(\mathcal{E})$.

Conversely, take any M in $\mathbf{W}(\mathcal{E})$. We will prove $M \in \text{Filt}(\text{max } \mathcal{M})$ by the induction on a \mathcal{M} -filtration length n of X .

If $n = 1$, we have $M \in \mathcal{M} = \text{sim } \mathcal{E}$. Thus Lemma 4.4 (1) implies $X \in \text{max } \mathcal{M}$, hence in particular $M \in \text{Filt}(\text{max } \mathcal{M})$. Suppose $n > 1$, then there is a short exact sequence in \mathcal{A}

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

such that L and N has \mathcal{M} -filtrations of length smaller than n . In particular, we have $L, N \in \mathcal{E}$. Then Lemma 4.4 (2) implies that $L, N \in \mathbf{W}(\mathcal{E})$ by $M \in \mathbf{W}(\mathcal{E})$. By the induction hypothesis, we have $L, N \in \text{Filt}(\text{max } \mathcal{M})$, which shows $M \in \text{Filt}(\text{max } \mathcal{M})$. \square

Remark 4.6. Let \mathcal{F} be a torsion-free class in \mathcal{A} . Then $\text{max}(\text{sim } \mathcal{F}) = \text{sim } \mathbf{W}(\mathcal{F})$ holds by Theorem 4.5. In [AP, Proposition 6.5], it is shown that this coincides with the set of *brick labels of arrows* starting at \mathcal{F} , which is introduced in [DIRRT]. Therefore, in our context, considering the brick labels of \mathcal{F} is nothing but taking the maximal element of the simple objects in \mathcal{F} . In [Asa], the bijection between functorially finite torsion-free classes and semibricks satisfying some condition was established, and its bijection is given by brick labels, thus coincides with $\text{max}(\text{sim } \mathcal{F})$.

5. APPLICATIONS

In this section, we give an application of the theory of monobricks to torsion-free classes and wide subcategories. We give new proofs of several results on these subcategories, such as Demonet-Iyama-Jasso's finiteness results [DIJ] and Marks-Šťovíček's bijection [MS], and make the relation between torsion-free classes and wide subcategories more transparent which can be applied to any length abelian categories, without using any τ -tilting theory.

5.1. Maps between wide \mathcal{A} and $\text{torf } \mathcal{A}$ via monobricks. In this section, we consider the restrictions of our maps \mathbf{W} and \mathbf{F} to \mathbf{W} : $\text{torf } \mathcal{A} \rightarrow \text{wide } \mathcal{A}$ and $\mathbf{F}: \text{wide } \mathcal{A} \rightarrow \text{torf } \mathcal{A}$. By using monobricks, we can reprove and generalize a Marks-Šťovíček's result by using only the easy poset theoretical argument.

Proposition 5.1. *Let \mathcal{A} be a length abelian category. Then the following hold.*

- (1) Let \mathcal{M} be a monobrick in \mathcal{A} and \mathcal{N} a cofinal extension of \mathcal{M} . Then $\max \mathcal{M} = \max \mathcal{N}$ holds. In particular, we have $\max \overline{\mathcal{M}} = \max \mathcal{M}$ holds.
- (2) Let \mathcal{S} be a semibrick. Then we have $\max \mathcal{S} = \mathcal{S}$. Thus the composition $\max \circ (\overline{\quad}) : \text{sbrick } \mathcal{A} \rightarrow \text{sbrick } \mathcal{A}$ is the identity.
- (3) [MS, Proposition 3.3] The composition $\text{W} \circ \text{F} : \text{wide } \mathcal{A} \rightarrow \text{wide } \mathcal{A}$ is the identity.

Proof. (1) Let M be a maximal element of \mathcal{M} . If M is not maximal in \mathcal{N} , then there is some $N \in \mathcal{N}$ with $M < N$. However, since \mathcal{N} is cofinal in \mathcal{M} , there is some $M' \in \mathcal{M}$ with $N \leq M'$, which implies $M < M'$. This is a contradiction, thus we have $\max \mathcal{M} \subset \max \mathcal{N}$. Conversely, let N be a maximal element of \mathcal{M} . Then since \mathcal{N} is cofinal in \mathcal{M} , there is some $M \in \mathcal{M}$ with $N \leq M$. Then the maximality implies $N = M \in \mathcal{M}$, thus $N \in \max \mathcal{M}$ holds.

(2) Obvious from (1).

(3) This follows from (2) and Theorems 3.15 and 4.5. \square

In general, the map $\text{F} : \text{wide } \mathcal{A} \rightarrow \text{torf } \mathcal{A}$ is not a surjection, and its image is studied in [AP]. We give a description of its image in terms of monobricks.

Proposition 5.2. *Let \mathcal{F} be a torsion-free class in \mathcal{A} . Then the following are equivalent.*

- (1) $\mathcal{F} = \text{F}(\text{W}(\mathcal{F}))$ holds.
- (2) There is a wide subcategory \mathcal{W} satisfying $\mathcal{F} = \text{F}(\mathcal{W})$.
- (3) There is a semibrick \mathcal{S} satisfying $\text{sim } \mathcal{F} = \overline{\mathcal{S}}$.
- (4) $\text{sim } \mathcal{F}$ is a cofinal extension of some semibrick.
- (5) $\max(\text{sim } \mathcal{F})$ is cofinal in $\text{sim } \mathcal{F}$, that is, for every element M in $\text{sim } \mathcal{F}$, there is an element $S \in \text{sim } \mathcal{F}$ such that $M \leq S$ holds and S is maximal in $\text{sim } \mathcal{F}$.

Proof. (1) \Rightarrow (2): Obvious.

(2) \Rightarrow (3): Clear from Theorem 4.5 and Theorem 3.15.

(3) \Rightarrow (4): This is clear since $\text{sim } \mathcal{F} = \overline{\mathcal{S}}$ is a cofinal extension of \mathcal{S} .

(4) \Rightarrow (5): Let \mathcal{S} be a semibrick such that $\text{sim } \mathcal{F}$ is a cofinal extension of \mathcal{S} . Then we have $\mathcal{S} = \max \mathcal{S} = \max(\text{sim } \mathcal{F})$ holds by Propositions 4.2 and 5.1. Thus $\max(\text{sim } \mathcal{F})$ is cofinal in $\text{sim } \mathcal{F}$, which is nothing but (5).

(5) \Rightarrow (1): (5) implies that $\text{sim } \mathcal{F}$ is a cofinal extension of $\max(\text{sim } \mathcal{F})$, and $\text{sim } \mathcal{F}$ is cofinally closed by Theorem 3.15. Thus we have $\overline{\max(\text{sim } \mathcal{F})} = \text{sim } \mathcal{F}$ holds by Corollary 3.4. This is nothing but (1) under the bijections in Theorems 3.15 and Theorem 4.5. \square

Example 5.3. Let us consider the 2-Kronecker case, see Example 7.5 for the detail and notations. By checking which cofinally closed monobrick \mathcal{M} satisfies that $\max \mathcal{M}$ is cofinal in \mathcal{M} , any monobricks except case (M2) satisfy this. Thus any torsion-free classes except (M2) (torsion-free classes consisting of all preprojective modules) belong to the image of $\text{F} : \text{wide } \mathcal{A} \rightarrow \text{torf } \mathcal{A}$.

As a corollary, we can quickly prove a Marks-Šťovíček's bijection (c.f. [MS, Corollary 3.11]).

Corollary 5.4. *Let \mathcal{A} be a length abelian category. Then the maps $\text{F} : \text{wide } \mathcal{A} \rightleftarrows \text{torf } \mathcal{A} : \text{W}$ induces a bijection between $\text{wide } \mathcal{A}$ and $\text{F}(\text{wide } \mathcal{A})$. If \mathcal{A} has only finitely many torsion-free classes, then $\text{F}(\text{wide } \mathcal{A}) = \text{torf } \mathcal{A}$ holds, thus F and W are mutually inverse bijections between $\text{wide } \mathcal{A}$ and $\text{torf } \mathcal{A}$.*

Proof. Since the composition $\text{wide } \mathcal{A} \xrightarrow{\text{F}} \text{torf } \mathcal{A} \xrightarrow{\text{W}} \text{wide } \mathcal{A}$ is the identity by Proposition 5.1, it suffices to show the last assertion. Suppose that \mathcal{A} has finitely many torsion-free classes, and it suffices to prove that $\max \mathcal{M}$ is cofinal in \mathcal{M} for every (cofinally closed) monobricks by Proposition 5.2. We will see in Theorem 5.5 that there are only finitely many bricks in \mathcal{A} up to isomorphism. Therefore, every monobrick \mathcal{M} is a finite poset, thus clearly $\max \mathcal{M}$ is cofinal in \mathcal{M} . \square

5.2. Finiteness conditions. In this subsection, we study several finiteness conditions on monobricks. First we consider when $\text{torf } \mathcal{A}$ or $\text{wide } \mathcal{A}$ or $\text{mbrick } \mathcal{A}$ is finite. We denote by $\text{brick } \mathcal{A}$ the set of isomorphism classes of bricks in \mathcal{A} .

Theorem 5.5. *Let \mathcal{A} be a length abelian category. Then the following are equivalent:*

- (1) $\text{brick } \mathcal{A}$ is finite, that is, there are only finitely many bricks in \mathcal{A} up to isomorphisms.

- (2) $\text{mbrick } \mathcal{A}$ is finite.
- (2)' $\text{Schur}_L \mathcal{A}$ is finite.
- (3) $\text{mbrick}_{\text{c.c.}} \mathcal{A}$ is finite.
- (3)' $\text{torf } \mathcal{A}$ is finite.
- (4) $\text{sbrick } \mathcal{A}$ is finite.
- (4)' $\text{wide } \mathcal{A}$ is finite.
- (5) There are only finitely many subcategories of \mathcal{A} which are closed under extensions, kernels and images.

Proof. First, note that (i) and (i)' are equivalent for $i = 2, 3, 4$ by Theorems 2.11 and 3.15.

- (1) \Rightarrow (2): This is clear since $\text{mbrick } \mathcal{A}$ is a subset of $2^{\text{brick } \mathcal{A}}$, the power set of $\text{brick } \mathcal{A}$.
- (2) \Rightarrow (3): This is clear by $\text{mbrick}_{\text{c.c.}} \mathcal{A} \subset \text{mbrick } \mathcal{A}$.
- (3) \Rightarrow (4): This is clear by an injection $(\overline{-}): \text{sbrick } \mathcal{A} \hookrightarrow \text{mbrick}_{\text{c.c.}} \mathcal{A}$ shown in Proposition 5.1.
- (4) \Rightarrow (1): A map $\text{brick } \mathcal{A} \rightarrow \text{sbrick } \mathcal{A}$ defined by $S \mapsto \{S\}$ is clearly injective.
- (2)' \Rightarrow (5): This is clear since every subcategory of \mathcal{A} closed under extensions, kernels and images is left Schur.
- (5) \Rightarrow (3)': This is clear since every torsion-free class in \mathcal{A} is closed under extensions, kernels and images. \square

Remark 5.6. In the case $\mathcal{A} = \text{mod } \Lambda$ for a finite-dimensional algebra Λ , the equivalence of (1) and (3)' is a particular case of [DIJ, Theorems 3.8, 4.2], and call such an algebra τ -tilting finite. Actually it was shown in [DIJ] that $\text{brick } \Lambda$ is finite if and only if there are only finitely many functorially finite torsion-free classes, a little stronger result than ours.

Definition 5.7. We call a length abelian category \mathcal{A} *brick-finite* if it satisfies the equivalent conditions of Theorem 5.5.

Next we consider when a monobrick consists of finitely many bricks. We begin with the following general observation on posets. A subset X of a poset P is called a *chain* if X is totally ordered, and an *antichain* if every two distinct elements in X are incomparable. For an element m of a poset P , we put $\downarrow m := \{x \in P \mid x \leq m\}$.

Lemma 5.8. *Let P be a poset such that every chain in $\downarrow m$ is finite for every m in P . Then P is finite if and only if it satisfies the following conditions.*

- (1) $\max P$ is cofinal in P , that is, every element is below some maximal element.
- (2) Every antichain of P is a finite set.

Proof. If P is finite, then it clearly satisfies (1) and (2).

Conversely, suppose that P is an infinite set. Since $\max P$ is an antichain of P , it is a finite set by (2). By (1), we have $P = \bigcup_{m \in \max P} (\downarrow m)$. Since P is infinite and $\max P$ is finite, we may assume that $\downarrow m_1$ is an infinite set for $m_1 \in \max P$.

Put $P_1 := (\downarrow m_1) \setminus \{m_1\} = \{x \in P \mid x < m_1\}$. Clearly P_1 also satisfies (2), since every antichain of P_1 is also an antichain of P . Suppose that there is an element $x \in P_1$ which is not below any maximal elements in P_1 . Since x is not maximal in P_1 , there is some $x < x_1$ with $x_1 \in P_1$, and x_1 is not below any maximal elements in P_1 . By iterating this, we obtain an infinite chain inside $P_1 \subset (\downarrow m_1)$, which is a contradiction. Thus P_1 satisfies (1).

Now we can repeat the same process as P to find an element $m_2 \in P_1$ such that $\downarrow m_2$ is infinite. Then apply the same process to $P_2 := \{x \in P \mid x < m_2\}$. We can iterate this procedure, and obtain an infinite chain $m_1 > m_2 > m_3 > \dots$ in $\downarrow m_1$. This is a contradiction. \square

By using this, we can prove the following criterion of the finiteness of a monobrick.

Proposition 5.9. *Let \mathcal{M} be a monobrick in \mathcal{A} . Then \mathcal{M} is a finite set if and only if it satisfies the following conditions:*

- (1) Every element in \mathcal{M} is below some maximal element in \mathcal{M} .
- (2) Every semibrick \mathcal{S} with $\mathcal{S} \subset \mathcal{M}$ is a finite set.

Proof. First we will check that a poset \mathcal{M} satisfies the assumption and the conditions in Lemma 5.8. Take any $M \in \mathcal{M}$ and consider $\downarrow M$. Clearly $0 \leq l(X) < l(M)$ holds for every element X in $\downarrow M$, where $l(-)$ denotes the lengths of objects in \mathcal{A} . If $X < X'$ in \mathcal{M} , then $l(X) < l(X')$ holds. Thus clearly $\downarrow M$ cannot contain any infinite chains.

The condition (1) and (2) in Lemma 5.8 is nothing but (1) and (2) in this proposition respectively. Moreover, we can check that a subset \mathcal{S} of \mathcal{M} is an antichain if and only if \mathcal{S} is a semibrick. Thus the assertion holds. \square

As an application, we have the following criterion of the finiteness of simple objects in a given torsion-free class.

Corollary 5.10. *Let \mathcal{F} be a torsion-free class in \mathcal{A} . Then $\text{sim } \mathcal{F}$ is a finite set if and only if \mathcal{F} satisfies the following conditions.*

- (1) $\mathcal{F} = \text{F}(\mathcal{W})$ holds for some wide subcategory \mathcal{W} of \mathcal{A} .
- (2) Every semibrick \mathcal{S} satisfying $\mathcal{S} \subset \text{sim } \mathcal{F}$ is finite.

The following is the fundamental relation between the finiteness of $\text{mbrick } \mathcal{A}$ and the finiteness of each monobrick.

Theorem 5.11. *Let \mathcal{A} be a length abelian category. Then the following are equivalent.*

- (1) Every monobrick in \mathcal{A} is a finite set.
- (2) Every cofinally closed monobrick in \mathcal{A} is a finite set.
- (2)' $\#\text{sim } \mathcal{F}$ is finite for every torsion-free class \mathcal{F} in \mathcal{A} .
- (3) Every semibrick in \mathcal{A} is a finite set, and the map $\text{F}: \text{wide } \mathcal{A} \hookrightarrow \text{torf } \mathcal{A}$ is surjective.

Moreover, if $\mathcal{A} = \text{mod } \Lambda$ for a finite-dimensional algebra Λ , then the following are also equivalent.

- (4) $\text{mod } \Lambda$ is brick-finite, that is, there are only finitely many bricks in $\text{mod } \Lambda$.

Proof. (1) \Rightarrow (2): Obvious.

(2) \Rightarrow (1): Let \mathcal{M} be a monobrick in \mathcal{A} . Then $\mathcal{M} \subset \overline{\mathcal{M}}$ holds for the cofinal closure of \mathcal{M} . Since $\overline{\mathcal{M}}$ is cofinally closed, it is finite by (2), thus so is \mathcal{M} .

(2) \Leftrightarrow (2)': Clear from Theorem 3.15.

(1) + (2)' \Rightarrow (3): The surjectivity of the map $\text{F}: \text{wide } \mathcal{A} \hookrightarrow \text{torf } \mathcal{A}$ follows from Corollary 5.10. Since every semibrick is a monobrick, it is finite by (1).

(3) \Rightarrow (2)': Clear from Corollary 5.10.

Now we have shown the equivalence of (1), (2), (2)' and (3). From now on, suppose that Λ is a finite-dimensional algebra and $\mathcal{A} = \text{mod } \Lambda$.

(2)' \Rightarrow (4): Suppose that $\text{mod } \Lambda$ is not brick-finite. Then by [DIJ, Theorem 3.8], there is a torsion-free class \mathcal{F} in $\text{mod } \Lambda$ which is not functorially finite. Put $\mathcal{F}_0 := 0 \in \text{torf}(\text{mod } \Lambda)$. Then [DIJ, Theorem 3.1] implies that there is a functorially finite torsion-free class \mathcal{F}_1 satisfying $\mathcal{F}_0 \subsetneq \mathcal{F}_1 \subset \mathcal{F}$. Since \mathcal{F} is not functorially finite, we have $\mathcal{F}_1 \subsetneq \mathcal{F}$. By repeating this process, we obtain a strictly ascending chain $0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \dots$. Put $\mathcal{G} := \bigcup_{i \geq 0} \mathcal{F}_i$. Then it is clearly a torsion-free class, and $\text{sim } \mathcal{G}$ is finite by (2)'. Therefore, there is some i such that $\text{sim } \mathcal{G} \subset \mathcal{F}_i$ holds. Since \mathcal{F}_i is extension-closed, this would imply $\mathcal{G} = \text{Filt}(\text{sim } \mathcal{G}) \subset \mathcal{F}_i \subset \mathcal{G}$, thus $\mathcal{F}_i = \mathcal{F}_{i+1} = \dots = \mathcal{G}$, which is a contradiction.

(4) \Rightarrow (1): Clear. \square

We propose the following conjecture related to this, which is of interest in its own.

Conjecture 5.12. *Let Λ be a finite-dimensional algebra. If every semibrick in $\text{mod } \Lambda$ is a finite set, then $\text{mod } \Lambda$ is brick-finite, that is, Λ is τ -tilting finite.*

Roughly speaking, Proposition 5.9 and Theorem 5.11 says that in order to show brick-finiteness, we have to show the finiteness of antichains (semibricks) and chains of bricks. Thus this conjecture is roughly equivalent to the following question: if every monobrick has *finite width* (finite antichains), then does every monobrick have a *finite height*?

Regarding this, recently it was shown in [ST, Theorem 1.1] that the finiteness of *height* implies brick-finiteness. More precisely, it was shown that if there is an upper bound on the lengths of bricks, then $\text{mod } \Lambda$ is brick-finite.

6. MONOBLOCKS OVER NAKAYAMA ALGEBRAS

For a finite-dimensional algebra Λ , we put $\mathbf{mbrick} \Lambda := \mathbf{mbrick}(\mathbf{mod} \Lambda)$ and so on. In this section, we investigate monoblocks and left Schur subcategories of $\mathbf{mod} \Lambda$ for a Nakayama algebra Λ . For the details on Nakayama algebras, we refer the reader to standard text such as [ASS, V.3].

First of all, we show that left Schur subcategories are precisely categories closed under extensions, kernels and images.

Theorem 6.1. *Let Λ be a Nakayama algebra and \mathcal{E} a subcategory of $\mathbf{mod} \Lambda$. Then \mathcal{E} is left Schur if and only if \mathcal{E} is closed under extensions, kernels and images. In particular, we have the bijection between the following two sets:*

- (1) $\mathbf{mbrick} \Lambda$, the set of monoblocks in $\mathbf{mod} \Lambda$.
- (2) The set of subcategories of $\mathbf{mod} \Lambda$ closed under extensions, kernels and images.

The map is given by \mathbf{Filt} and \mathbf{sim} .

Proof. We use the result in [Eno1, Corollary 5.19]: every torsion-free class in $\mathbf{mod} \Lambda$ satisfies the Jordan-Hölder type property. We refer the reader to [Eno1] for the details on this property.

By Proposition 2.8, we only have to show that every left Schur subcategory \mathcal{E} of $\mathbf{mod} \Lambda$ is closed under kernels and images. By Theorem 2.11, there is a monoblock \mathcal{M} satisfying $\mathcal{E} = \mathbf{Filt} \mathcal{M}$. Consider the cofinal closure $\overline{\mathcal{M}}$ of \mathcal{M} and put $\mathcal{F} := \mathbf{F}(\mathcal{E})$. Then we have $\overline{\mathcal{M}} = \mathbf{sim} \mathcal{F} \supset \mathcal{M}$ by Proposition 3.13. Let $f: X \rightarrow Y$ be a map in \mathcal{E} . Since \mathcal{F} is closed under kernels and images in $\mathbf{mod} \Lambda$, we obtain the following short exact sequence in \mathcal{F} :

$$0 \longrightarrow \mathbf{Ker} f \longrightarrow X \longrightarrow \mathbf{Im} f \longrightarrow 0.$$

Since \mathcal{F} satisfies the Jordan-Hölder property, we can speak of *composition factors inside \mathcal{F}* . Since X is in $\mathcal{E} = \mathbf{Filt} \mathcal{M}$, all the composition factors of X inside \mathcal{F} belongs to \mathcal{M} by $\mathcal{M} \subset \mathbf{sim} \mathcal{F}$. Therefore, all the composition factors of $\mathbf{Im} f$ and $\mathbf{Ker} f$ must be in \mathcal{M} , since the above short exact sequence is a conflation in \mathcal{F} . This implies $\mathbf{Im} f$ and $\mathbf{Ker} f$ belong to $\mathbf{Filt} \mathcal{M} = \mathcal{E}$. \square

As an immediate corollary, we have the following classification theorem of subcategories.

Corollary 6.2. *For a Nakayama algebra Λ , there is a bijection between the following two sets:*

- (1) $\mathbf{mbrick} \Lambda$, the set of monoblocks in $\mathbf{mod} \Lambda$.
- (2) The set of subcategories of $\mathbf{mod} \Lambda$ which is closed under extensions, kernels and images.

The maps are given by $\mathcal{M} \mapsto \mathbf{Filt} \mathcal{M}$ for (1) to (2) and $\mathcal{E} \mapsto \mathbf{sim} \mathcal{E}$ for (2) to (1).

Our next aim is to give a combinatorial classification of monoblocks for Nakayama algebras. The following basic observation on quotient algebras and monoblocks is useful. Recall that for a two-sided ideal I of a finite-dimensional algebra Λ , we have the natural fully faithful functor $\mathbf{mod}(\Lambda/I) \hookrightarrow \mathbf{mod} \Lambda$, and its essential image consists of Λ -modules M satisfying $MI = 0$. By this, we may identify $\mathbf{mod}(\Lambda/I)$ with the subcategory of $\mathbf{mod} \Lambda$ consisting of such modules.

Proposition 6.3. *Let Λ be a finite-dimensional algebra and I its two-sided ideal. Then by identifying $\mathbf{mod}(\Lambda/I)$ with the subcategory of $\mathbf{mod} \Lambda$, we have*

$$\mathbf{mbrick}(\Lambda/I) = \mathbf{mod}(\Lambda/I) \cap \mathbf{mbrick} \Lambda$$

Proof. This follows from the fact that the natural functor $\mathbf{mod}(\Lambda/I) \hookrightarrow \mathbf{mod} \Lambda$ is fully faithful and that a morphism in $\mathbf{mod}(\Lambda/I)$ is an injection in $\mathbf{mod}(\Lambda/I)$ if and only if so is in $\mathbf{mod} \Lambda$. \square

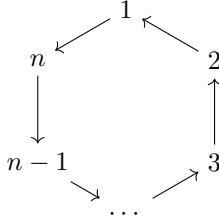
By this, the classification of monoblocks over Λ/I is reduced to that of Λ in principle. Keeping this in mind, it suffices to consider the following two Nakayama algebras.

Definition 6.4. Let n be a positive integer. Then we define two algebras A_n and B_n as follows:

- (1) A_n is the path algebra of the following quiver.

$$1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow n$$

- (2) B_n is the quotient of the path algebra of the following by the ideal generated by all the paths of lengths n .



Note that we have the natural identification $B_n/\langle e_n \rangle \cong A_{n-1}$, where e_n is the primitive idempotent of B_n corresponding to n .

The following shows that to classify monobricks over Nakayama algebras, it suffices to consider A_n and B_n .

Proposition 6.5. *Let Λ be a basic connected Nakayama algebra with $\#\text{sim}(\text{mod } \Lambda) = n$.*

- (1) *If the quiver of Λ is acyclic, then $\Lambda \cong A_n/I$ for some I , thus $\text{mbrick } \Lambda \subset \text{mbrick } A_n$ holds.*
- (2) *If the quiver of Λ is cyclic, then there exist a Nakayama algebra B' and two algebra surjections $B' \twoheadrightarrow \Lambda$ and $B' \twoheadrightarrow B_n$ such that $\text{mbrick } \Lambda \subset \text{mbrick } B_n$ holds inside $\text{mbrick } B'$.*

Proof. (1) This is well-known, e.g. [ASS, Theorem V.3.2]

(2) The existence of a Nakayama algebra B' such that Λ and B_n are quotients of B' is obvious (consider the path algebra of the cyclic quiver and annihilate sufficiently large paths), thus it suffices to see that every brick M in $\text{mod } B'$ is contained in $\text{mod } B_n$. This is clear since if indecomposable module M does not belong to B_n , then it is easily checked that M has non-zero endomorphism which is not an isomorphism. \square

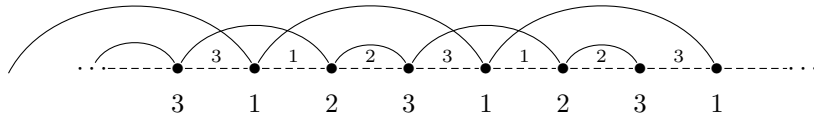
To deal with modules over A_n and B_n , we will use the following combinatorial description.

Definition 6.6. Let n be a positive integer.

- We put $[n] := \{1, 2, \dots, n\}$.
- For two elements i, j in $[n]$, we calculate $i + j$ and $i - j$ modulo n , for example, $n + 1 = 1$ and $1 - 1 = n$.
- An *arc on $[n]$* is an element of $[n] \times [n]$.
- An *admissible arc on $[n]$* is an arc (i, j) satisfying $i < j$.
- For an arc $\alpha = (i, j)$ on $[n]$, we call i its *starting point* and j its *ending point*.
- The *socle series* of an arc $\alpha = (i, j)$ on $[n]$ is a sequence of elements in $[n]$ defined by $(i, i + 1, \dots, j - 1)$.
- An *arc diagram \mathcal{D} on $[n]$* is a set of arcs, that is, a subset of $[n] \times [n]$.
- An arc diagram \mathcal{D} is *admissible* if every arc in \mathcal{D} is admissible.

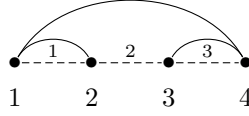
We represent arcs and arc diagrams on $[n]$ as follows: Consider the Euclidean plane \mathbb{R}^2 and put i on $(i, 0) + \mathbb{Z}(n, 0)$ for each $i \in [n]$. Then for an arc $\alpha = (i, j)$, we draw “arcs” in the upper half-plane which connect each i and the first j which appear right to this i .

For example, the following is an arc diagram $\mathcal{D} = \{(1, 1), (2, 3), (3, 2)\}$ on $[3]$. These three arcs have socle series $(1, 2, 3)$, (2) , $(3, 1)$ respectively.



As this figure, it is convenient to draw a dashed line on the x -axis and label each line segment as above, so that the socle series of α is the sequence of labels surrounded by α . Also, we often draw an *admissible* arc diagram by omitting the repeated part, for example, the following is a picture

of an admissible arc diagram $\{(1, 2), (1, 4), (3, 4)\}$ on $[4]$.



We say that a sequence (n_1, \dots, n_a) is a *partial sequence* of a sequence (m_1, \dots, m_b) if there is some integer i with $1 \leq i \leq b - a + 1$ satisfying $n_1 = m_i, n_2 = m_{i+1}, \dots, n_a = m_{i+a-1}$ holds. For example, $(3, 1), (2)$ and $(2, 3, 1)$ are subsequences of $(2, 3, 1)$, but $(1, 2), (2, 1)$ and $(3, 1, 2)$ are not.

Definition 6.7. We say that a pair $\{\alpha, \beta\}$ of two different arcs α and β on $[n]$ is a *weakly non-crossing pair* if either of the following conditions is satisfied:

- The socle series of α is a partial sequence of that of β .
- The socle series of β is a partial sequence of that of α .
- The socle series of α and β are disjoint, that is, there exists no element in $[n]$ which appears in the both socle series.

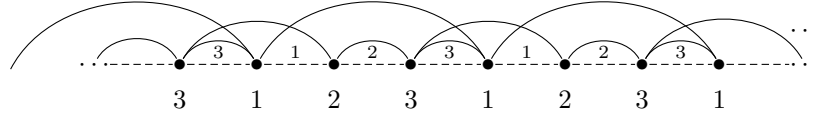
Moreover, for a weakly non-crossing pair $\{\alpha, \beta\}$, we define the following.

- (1) It is a *mono-crossing pair* if α and β have the same starting point.
- (2) It is a *epi-crossing pair* if α and β have the same ending point.
- (3) It is a *non-crossing pair* if it is neither mono-crossing nor epi-crossing.

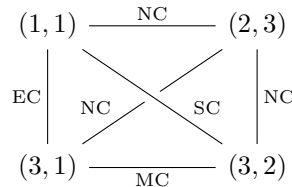
We say that $\{\alpha, \beta\}$ is a *strictly crossing pair* if they are not weakly non-crossing.

Intuitively, a pair of two arcs is weakly non-crossing if they do not *cross* in the half-plane model except at their starting points or ending points, and is non-crossing if in addition they do not share neither the starting points nor the ending points.

Example 6.8. Consider an arc diagram $\{(1, 1), (2, 3), (3, 1), (3, 2)\}$ on $[3]$:



Then the crossing relation between four arcs are as follows:



Here NC, EC, MC and SC means non-crossing, epi-crossing, mono-crossing and strictly crossing respectively.

Remark 6.9. Suppose that $\alpha = (a, b)$ and $\beta = (c, d)$ are distinct *admissible* arcs. Then it is straightforward to see that $\{\alpha, \beta\}$ is strictly crossing if and only if $a < c < b < d$ or $c < a < d < b$.

Definition 6.10. Let n be a positive integer and \mathcal{D} an arc diagram on $[n]$.

- (1) \mathcal{D} is *non-crossing* if every distinct pair of arcs in \mathcal{D} is a non-crossing pair.
- (2) \mathcal{D} is *mono-crossing* if every distinct pair of arcs in \mathcal{D} is either a mono-crossing or a non-crossing pair.

Now let us return to the algebraic side.

Definition 6.11. Let n be a positive integer and $\alpha = (i, j)$ be an arc on $[n]$. Then we denote by M_α the unique indecomposable B_n -module satisfying $\text{soc } M_\alpha = i$ and $\text{top } M_\alpha = j - 1$. If α is admissible, that is, $i < j$, then we often regard M_α as an A_{n-1} -module by the surjection $B_n \twoheadrightarrow B_n/\langle e_n \rangle \cong A_{n-1}$.

Now by the standard description of indecomposable modules and morphisms between them over Nakayama algebras (e.g. [ASS, Theorem V.3.5]), it is easy to show the following.

Proposition 6.12. *Let n be a positive integer. Then the assignment $\alpha \mapsto M_\alpha$ induces a bijection between the set of arcs on $[n]$ and $\mathbf{brick} B_n$, and a bijection between the set of admissible arcs on $[n]$ and $\mathbf{brick} A_{n-1}$. Moreover, the following hold for two arcs α and β on $[n]$.*

- (1) $\{\alpha, \beta\}$ is a non-crossing pair if and only if $\{M_\alpha, M_\beta\}$ is a semibrick.
- (2) $\{\alpha, \beta\}$ is a mono-crossing pair if and only if $\{M_\alpha, M_\beta\}$ is a monobrick and not a semibrick.

Therefore, we have the following bijections:

$$\begin{array}{ccc} \{\text{mono-crossing arc diagrams on } [n]\} & \xrightarrow[\sim]{\mathcal{M}_{(-)}} & \mathbf{mbrick} B_n \\ \uparrow & & \uparrow \\ \{\text{non-crossing arc diagrams on } [n]\} & \xrightarrow{\sim} & \mathbf{sbrick} B_n \end{array}$$

and

$$\begin{array}{ccc} \{\text{mono-crossing admissible arc diagrams on } [n]\} & \xrightarrow[\sim]{\mathcal{M}_{(-)}} & \mathbf{mbrick} A_{n-1} \\ \uparrow & & \uparrow \\ \{\text{non-crossing admissible arc diagrams on } [n]\} & \xrightarrow{\sim} & \mathbf{sbrick} A_{n-1} \end{array}$$

where $\mathcal{M}_{\mathcal{D}} := \{M_\alpha \mid \alpha \in \mathcal{D}\}$ for an arc diagram \mathcal{D} on $[n]$.

By combining this with Corollary 6.2, the problem of classifying all the subcategories closed under extensions, kernels and images is reduced to purely combinatorial problem, namely, classifying all the mono-crossing (admissible) arc diagrams on $[n]$.

In the rest of this section, we give an explicit formula on $\#\mathbf{mbrick} A_n$ and $\#\mathbf{mbrick} B_n$. Note that the formula on $\#\mathbf{sbrick} A_n$ and $\#\mathbf{sbrick} B_n$ is given by Asai [Asa, Lemmas 3.4, 3.7]:

$$\begin{aligned} \#\mathbf{sbrick} A_{n-1} &= \frac{1}{n+1} \binom{2n}{n} && \text{(the } n\text{-th Catalan number, [OEIS, A000108])} \\ \#\mathbf{sbrick} B_n &= \binom{2n}{n} && \text{([OEIS, A000984])} \end{aligned}$$

We can also compute $\#\mathbf{sbrick} A_{n-1}$ using Proposition 6.12. Non-crossing admissible arc diagrams on $[n]$ clearly correspond to the classical *non-crossing partition on $[n]$* (see [Rin2, N.4.1] for the detail). Therefore, its number is equal to the number of non-crossing partitions, which is well-known to be equal to the Catalan number.

Our enumeration of monobricks is based on the same idea: find a bijection between the set of mono-crossing arc diagrams and some combinatorial sets, whose number has already been computed by combinatorialists.

The following is our enumerative result.

Theorem 6.13. *Let n be a positive integer. Then the following equalities hold.*

$$\#\mathbf{mbrick} A_n = [\text{OEIS, A006318}](n) = \sum_{i=0}^n \frac{1}{i+1} \binom{n}{i} \binom{n+i}{i}, \text{ the } n\text{-th large Schröder number.} \quad (6.1)$$

$$\#\mathbf{mbrick} B_n = [\text{OEIS, A002003}](n) = 2 \sum_{i=0}^n \binom{n-1}{i} \binom{n+i}{i}. \quad (6.2)$$

Proof. By Proposition 6.12, it suffices to count the numbers of

- (1) mono-crossing admissible arc diagrams on $[n+1]$, and
- (2) mono-crossing arc diagrams on $[n]$.

(1) We will show that the number of mono-crossing admissible arc diagrams on $[n]$ is equal to the $(n-1)$ -the large Schröder number. The outline of the enumeration is as follows: we will show that mono-crossing admissible arc diagrams are in bijection with *non-crossing linked partitions* introduced in [Dyk], whose number was shown to be the large Schröder number.

A *non-crossing linked partition* of $[n]$ is a set π of non-empty subsets of $[n]$ satisfying the following conditions.

- (NCL1) $[n] = \bigcup_{E \in \pi} E$ holds.
- (NCL2) For every $E, F \in \pi$ with $E \neq F$, there exists no $a < b < c < d$ satisfying $a, c \in E$ and $b, d \in F$.
- (NCL3) We have $\#(E \cap F) \leq 1$ for every distinct $E, F \in \pi$, and if $j \in E \cap F$, then either $j = \min E$, $\#E > 1$ and $j \neq \min F$ hold, or the converse $j = \min F$, $\#F > 1$ and $j \neq \min E$ hold. In particular, if $E \cap F \neq \emptyset$, then $\#E, \#F > 1$ holds.

Then the number of non-crossing linked partitions of $[n]$ is equal to the $(n-1)$ -the large Schröder number by [Dyk]. We will finish the proof of (6.1) by constructing a bijection from the set of non-crossing linked partitions of $[n]$ to the set of mono-crossing admissible arc diagrams. The construction we will give is essentially the same as the graphical presentation given in [CWY].

Let π be a non-crossing linked partition of $[n]$. For each $E \in \pi$ and $j \in E$ with $j \neq \min E$, we draw an arc $(\min E, j)$. By this, we obtain an admissible arc diagram \mathcal{D}_π .

We claim that \mathcal{D}_π is actually a mono-crossing arc diagram. Let $\{\alpha, \beta\}$ be a pair of arcs in \mathcal{D}_π with $\alpha \neq \beta$. By (NCL2) and Remark 6.9, this pair is weakly non-crossing. Thus it suffices to show $\{\alpha, \beta\}$ is not epi-crossing. Assume that $\{\alpha, \beta\}$ is epi-crossing, then α and β have the same ending point, that is, we can write as $\alpha = (i_\alpha, j)$ and $\beta = (i_\beta, j)$ with $i_\alpha \neq i_\beta$. By the construction of \mathcal{D}_π , there is E and F in π satisfying $\{i_\alpha, j\} \subset E$, $\{i_\beta, j\} \subset F$, $i_\alpha = \min E$ and $i_\beta = \min F$. Then $E \neq F$ holds by $i_\alpha \neq i_\beta$. Now we have $j \in E \cap F$ but $j \neq \min E, \min F$, which contradicts to (NCL3). Therefore, $\{\alpha, \beta\}$ is not epi-crossing, thus \mathcal{D}_π is a mono-crossing arc diagram.

Conversely, let \mathcal{D} be a mono-crossing admissible arc diagram on $[n]$. For each i in $[n]$, define $E_i \subset [n]$ as follows:

$$E_i := \begin{cases} \{i\} \cup \{j \mid (i, j) \in \mathcal{D}\} & \text{if there is some arc starting at } i, \\ \{i\} & \text{if there is no arc either starting or ending at } i, \\ \emptyset & \text{otherwise} \end{cases}$$

Note that $i = \min E_i$ holds if $E_i \neq \emptyset$, thus E_i 's are pairwise distinct.

Put $\pi_{\mathcal{D}} := \{E_i \mid 1 \leq i \leq n, E_i \neq \emptyset\}$. We claim that $\pi_{\mathcal{D}}$ is a non-crossing linked partition of $[n]$. Clearly $\pi_{\mathcal{D}}$ satisfies (NCL1). Assume that $\pi_{\mathcal{D}}$ does not satisfy (NCL2), that is, there is some $E_{i_1}, E_{i_2} \in \pi_{\mathcal{D}}$ and $a, c \in E_{i_1}$, $b, d \in E_{i_2}$ satisfying $a < b < c < d$. Then we have $i_1 < b < c < d$ by $i_1 = \min E_{i_1}$. We consider two cases $i_1 < i_2$ and $i_2 < i_1$.

If $i_1 < i_2$, then $i_1 < i_2 \leq b < c < d$ holds by $i_2 = \min E_{i_2}$. Now $c \in E_{i_1}$ and $d \in E_{i_2}$ imply $(i_1, c) \in \mathcal{D}$ and $(i_2, d) \in \mathcal{D}$. From this, (i_1, c) and (i_2, d) is strictly crossing by $i_1 < i_2 < c < d$, which is a contradiction. If $i_2 < i_1$, then $i_2 < i_1 < b < c$ holds. Now $c \in E_{i_1}$ and $b \in E_{i_2}$ imply $(i_1, c) \in \mathcal{D}$ and $(i_2, b) \in \mathcal{D}$. Since these two arcs are strictly crossing, this is a contradiction. Thus (NCL2) holds.

Next we will show that $\pi_{\mathcal{D}}$ satisfies (NCL3). Suppose that $E_{i_1} \cap E_{i_2} \neq \emptyset$ for $i_1 \neq i_2$ and take $j \in E_{i_1} \cap E_{i_2}$. If $\#E_{i_1} = 1$, then $j = i_1$ and there is no arc either starting or ending at i_1 . However, $i_1 \in E_{i_2}$ and $i_1 \neq i_2$ implies that $(i_2, i_1) \in \mathcal{D}$, which is a contradiction. Thus $\#E_{i_1} \geq 2$ and $\#E_{i_2} \geq 2$ hold. Now if $j \neq i_1$ and $j \neq i_2$, then $(i_1, j), (i_2, j) \in \mathcal{D}$ holds. This is a contradiction since these two arcs are epi-crossing. Thus either $j = i_1$ or $j = i_2$ holds. In the former case, we have $j = i_1 = \min E_{i_1}$, $j = i_1 \neq i_2 = \min E_{i_2}$, and in the latter we have $j = \min E_{i_2}$ and $j \neq \min E_{i_1}$. Therefore (NCL3) is satisfied.

Now we have shown that $\pi_{\mathcal{D}}$ is a non-crossing linked partition of $[n]$ if \mathcal{D} is a mono-crossing admissible arc diagram. It is quite straightforward to see that $\mathcal{D} = \mathcal{D}_{\pi_{\mathcal{D}}}$ holds for a mono-crossing admissible arc diagram, so we omit the proof.

Finally we show that $\pi = \pi_{\mathcal{D}_\pi}$ holds for a non-crossing linked partition π of $[n]$. Let $E \in \pi$, and we will show $E = E_i$ for $i := \min E$. We consider two cases:

(Case 1): $E = \{i\}$. Suppose that there is some arc (i, j) in \mathcal{D}_π . By the construction of \mathcal{D}_π , there is some $F \in \pi$ with $i = \min F$ and $j \in F$. This contradicts to (NCL3) since $i \in E \cap F$ and $\#E = 1$. It follows that there is no arc starting at i . Similarly, suppose that there is some arc (j, i) in \mathcal{D}_π . Then there is some $F \in \pi$ with $j = \min F$ and $i \in F$. This contradicts to (NCL3) by $i \in E \cap F$ and $\#E = 1$. Therefore, there is no arc either starting or ending at i , hence $E_i = \{i\} = E$ holds.

(Case 2): $\#E \geq 2$. In this case, there is some arc starting at i in \mathcal{D}_π . By construction, $E \subset E_i$ holds. Conversely, take $j \in E_i$. Then $(i, j) \in \mathcal{D}_\pi$ holds, thus there is some $F \in \pi$ with $i = \min F$ and $j \in F$. Since $i \in E \cap F$ and $i = \min E = \min F$, we must have $E = F$ by (NCL3). Thus $j \in F = E$ holds, hence $E = E_i$.

We have shown $\pi \subset \pi_{\mathcal{D}_\pi}$. Conversely, take $E_i \in \pi_{\mathcal{D}_\pi}$. We consider two cases.

(Case 1): $E_i = \{i\}$. In this case, by construction, there is no arc either starting or ending at i in \mathcal{D}_π . This means that there is no $F \in \pi$ with $\#F > 1$ which contains i . Thus $\{i\} \in \pi$ should hold by (NCL1), that is, $E_i \in \pi$.

(Case 2): $\#E_i \geq 2$. By construction, there is some arc (i, j) in \mathcal{D}_π , thus there is some $E \in \pi$ satisfying $i = \min E$ and $j \in E$. It suffices to show $E = E_i$. If $j' \in E$ with $j' \neq i$, then $(i, j') \in \mathcal{D}_\pi$ holds by construction. Thus $j' \in E_i$ holds. Hence we obtain $E \subset E_i$. Conversely, suppose $j' \in E_i$ with $j' \neq i$. Then $(i, j') \in \mathcal{D}_\pi$, so there is some $E' \in \pi$ with $i = \min E'$ and $j' \in E'$. Then $i \in E \cap E'$ satisfies $i = \min E = \min E'$, which implies $E = E'$ by (NCL3). Thus $j' \in E' = E$ holds. Therefore, we have $E_i = E \in \pi$.

Hence we obtain $\pi = \pi_{\mathcal{D}_\pi}$, which completes the proof.

(2) We will give a proof of (6.2) by calculating the generating function using (6.1). Put $a_n := \#\text{mbrick } A_n$, $b_n := \#\text{mbrick } B_n$ and consider the following generating functions.

$$f(t) := \sum_{n=1}^{\infty} a_n t^n,$$

$$g(t) := \sum_{n=1}^{\infty} b_n t^n.$$

By using (6.1), it is known that the following hold (see e.g. [Bru, Theorem 8.5.7]):

$$f(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2}$$

To compute $g(t)$, we claim the following relation between a_n and b_n .

(Claim): *The equality $b_n = a_n + \sum_{i=1}^n i a_i a_{n+1-i}$ holds.*

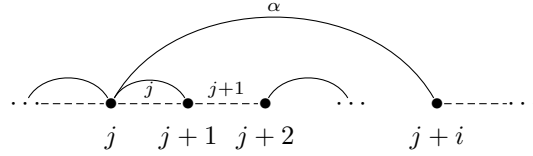
Proof of (Claim): Let MD_n be the set of mono-crossing arc diagram on $[n]$, so $\#\text{MD}_n = b_n$. For $0 \leq i \leq n$, we define $\text{MD}_n(i)$ as follows: $\text{MD}_n(0)$ consists of $\mathcal{D} \in \text{MD}_n$ such that there is no arc in \mathcal{D} whose socle series contains n , and for $1 \leq i \leq n$, $\text{MD}_n(i)$ consists of $\mathcal{D} \in \text{MD}_n$ there is an arc in \mathcal{D} whose socle series contains n , and the minimum length of such arcs is i . Here the length of the arc (i, j) is defined to be $j - i \in [n]$. Then clearly we have the following decomposition, and we will count the number of $\text{MD}_n(i)$.

$$\text{MD}_n = \bigsqcup_{i=0}^n \text{MD}_n(i),$$

For $i = 0$, clearly elements in $\text{MD}_n(0)$ are precisely mono-crossing *admissible* arc diagrams on $[n]$. Thus $\#\text{MD}_n(0) = a_n$ holds.

Let $1 \leq i \leq n$. There are i arcs whose socle series contain n and whose lengths are i , that is, $(n - i + 1, 1), (n - i + 2, 2), \dots, (n, n + i)$. Elements in $\text{MD}_n(i)$ contains precisely one such arc, since any two such arcs are strictly crossing. Fix one such arc $\alpha = (j, j + i)$, and we will count the number of elements in $\text{MD}_n(i)$ which contains α . Let \mathcal{D} be such an element. Then consider the restriction of \mathcal{D} to the part $\{j, j + 1, \dots, j + i\}$, more precisely, consider the set of arcs whose

socle series are partial sequences of that of α .



By shifting $-i$, these arcs except α gives mono-crossing admissible arc diagram on $[i]$ (note that it is not $[i-1]$, since arcs cannot share the endpoint with α). Conversely, any mono-crossing admissible arc diagram on $[i]$ can be occur in this way by shifting $+i$.

In a similar way, consider the set of arcs whose socle series are disjoint from that of α . By shifting $-(j+i)$, these arcs gives mono-crossing admissible arc diagram on $[n-i+1]$, and vice versa. Therefore, there are $i \cdot a_i \cdot a_{n-i+1}$ possible arc diagrams in $\text{MD}_n(i)$. \blacksquare

Now, by using (Claim), we obtain the following equality.

$$\begin{aligned}
 g(t) &= \sum_{n=1}^{\infty} b_n t^n \\
 &= (a_1 + 1a_1 a_1)t + (a_2 + 1a_1 a_2 + 2a_2 a_1)t^2 + (a_3 + 1a_1 a_3 + 2a_2 a_2 + 3a_3 a_1)t^3 + \dots \\
 &= (a_1 t + a_2 t^2 + a_3 t^3 + \dots) \cdot (1 + a_1 + 2a_2 t + 3a_3 t^2 + \dots) \\
 &= f(t) \cdot \left(1 + \frac{df(t)}{dt}\right) \\
 &= \frac{1-t-\sqrt{1-6t+t^2}}{2} \cdot \frac{1}{2} \left(\frac{3-t}{\sqrt{1-6t+t^2}} + 1\right) \\
 &= \frac{1}{2} \left(\frac{1+t}{\sqrt{1-6t+t^2}} - 1\right)
 \end{aligned}$$

Since this coincides with the generating function of [OEIS, A002003], we have done. \square

Remark 6.14. In the paper [Eno3], we will compute the number of monobricks in $\text{mod } A_n$ by using the completely different method. In fact, the number of subcategories in $\text{mod } kQ$ closed under extensions, kernels and images (thus left Schur) is determined in [Eno3] for a Dynkin quiver Q . If we choose Q to be an A_n quiver with the linear orientation, A_n is nothing but kQ which is Nakayama, thus the number of monobricks in $\text{mod } A_n$ is equal to the number of such subcategories by Corollary 6.2. Moreover, in [Eno3], it is shown that the number appeared in the right hand side of (6.1) for a fixed i is equal to the number of monobricks with i elements.

7. EXAMPLES OF COMPUTATIONS

In what follows, we fix an algebraically closed field k . For several finite-dimensional algebra Λ , we show the lists of monobricks and left Schur subcategories, and the behavior of the maps $W: \text{Schur}_L \Lambda \rightarrow \text{wide } \Lambda$ and $F: \text{Schur}_L \Lambda \rightarrow \text{torf } \Lambda$ in terms of their counterparts $\text{max}: \text{mbrick } \Lambda \rightarrow \text{sbrick } \Lambda$ and $(\overline{-}): \text{mbrick } \Lambda \rightarrow \text{mbrick}_{\text{c.c.}} \Lambda$.

Example 7.1. Let Q be a quiver $1 \leftarrow 2 \leftarrow 3$, then the AR quiver of $\text{mod } kQ$ is given in Table 1. By Theorem 6.13, we have $\# \text{mbrick } kQ = 22$, the third Schröder number. There are $1+6$ monobricks \mathcal{M} with $\#\mathcal{M} \leq 1$, namely, an empty set, and a singleton consisting of each indecomposable kQ -modules.

In Table 2, we list the remaining 15 monobricks, together with their submodule order. For example, the notation $1 < \frac{2}{1}, 3$ means that this poset consists of the disjoint union of two chains $1 < \frac{2}{1}$ and 3 . For each monobrick \mathcal{M} , we write the corresponding left Schur subcategory $\text{Filt } \mathcal{M}$ in the AR quiver, where the black vertices are \mathcal{M} , and the white are the rest. If $\text{Filt } \mathcal{M}$ is not a wide subcategory, then we write the monobrick corresponding to $W(\text{Filt } \mathcal{M})$, which is equal to $\text{max } \mathcal{M}$ by Theorem 4.5. Similarly, if $\text{Filt } \mathcal{M}$ is not a torsion-free class, then we write the monobrick corresponding to $F(\text{Filt } \mathcal{M})$, which is equal to the cofinal closure $\overline{\mathcal{M}}$ by Theorem 3.15.

TABLE 1. The Auslander-Reiten quiver of $\text{mod } k[1 \leftarrow 2 \leftarrow 3]$

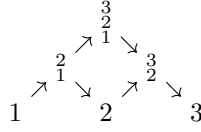


TABLE 2. Monobricks \mathcal{M} over $k[1 \leftarrow 2 \leftarrow 3]$ with $\#\mathcal{M} \geq 2$

\mathcal{M} (as a poset)	left Schur subcats	wide?	$\max \mathcal{M}$	torsion-free?	$\overline{\mathcal{M}}$
$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$		No	$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$	Yes	itself
$1 < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$		No	$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$	No	$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$
$1, 2$		Yes	itself	Yes	itself
$1, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$		Yes	itself	No	$1, 2 < \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$
$1, 3$		Yes	itself	Yes	itself
$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$		No	$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$	No	$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$
$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 3$		Yes	itself	No	$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 3$
$2, \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$		Yes	itself	No	$1 < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, 2$
$2 < \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$		No	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	Yes	itself
$2, 3$		Yes	itself	Yes	itself
$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$		No	$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$	Yes	itself
$1 < \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 3$		No	$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 3$	Yes	itself
$1 < \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, 2$		No	$2, \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$	Yes	itself
$1, 2 < \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$		No	$1, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$	Yes	itself
$1, 2, 3$		Yes	itself	Yes	itself

Now let us see some specific examples of computation of $\max \mathcal{M}$ and $\overline{\mathcal{M}}$. For a given monobrick \mathcal{M} , it is easy to describe its poset structure (we have $L \leq M$ in \mathcal{M} whenever there is a non-zero map $L \rightarrow M$). Thus its maximal element $\max \mathcal{M}$ can be easily computed. For example, consider $\mathcal{M} = \{1, \frac{2}{1}, 3\}$. Then since we have an injection $1 \hookrightarrow \frac{2}{1}$ and there are no other non-zero homomorphism, its poset structure is $1 < \frac{2}{1}, 3$, hence $\max \mathcal{M} = \{\frac{2}{1}, 3\}$.

The computation of $\overline{\mathcal{M}}$ is a little harder than $\max \mathcal{M}$, but not so difficult. Recall from Corollary 3.9 that $\overline{\mathcal{M}}$ consists of all bricks N which satisfies the following two conditions:

- (1) N is a submodule of some $M \in \mathcal{M}$.
- (2) Every map $N \rightarrow M'$ with $M' \in \mathcal{M}$ is either zero or an injection.

Thus, to compute $\overline{\mathcal{M}}$, first list up all submodules of elements in \mathcal{M} which are bricks and not in \mathcal{M} , then check whether the condition (2) above holds. For example, let $\mathcal{M} = \{2, \frac{3}{1}\}$. Then proper submodules which are bricks are exactly $\frac{2}{1}$ and 1. However, there is a non-zero non-injection $\frac{2}{1} \rightarrow 2$, thus we exclude $\frac{2}{1}$. In this way we obtain $\overline{\mathcal{M}} = \mathcal{M} \cup \{1\}$.

Next let us consider the path algebra of an A_3 quiver with another orientation.

Example 7.2. Let Q be a quiver $1 \rightarrow 2 \leftarrow 3$. There are $1 + 6$ monobricks \mathcal{M} with $\#\mathcal{M} \leq 1$, namely, an empty set, and a singleton consisting of each indecomposable kQ -modules. It turns out that $\#\text{mbrick } \Lambda = 26$, which is different from that of $1 \leftarrow 2 \leftarrow 3$. This means that the number of left Schur subcategories (or monobricks) depends on the orientation of the quiver.

In Table 4, we list the remaining 19 monobricks and their maximal elements and cofinal closures. Wide subcategories are categories in which $\max \mathcal{M}$ is “itself,” and torsion-free classes are categories in which $\overline{\mathcal{M}}$ is “itself.” In this case, there are several examples which are not closed under direct summands, kernels or images. Subcategories \mathcal{E} with (*) are not closed under direct summands (hence is closed under neither images nor kernels), and the white vertices in \mathcal{E} indicate indecomposables of $\text{add } \mathcal{E}$ which does not belong to \mathcal{E} . The only one category with (**) is closed under images, thus closed under direct summands, but is not closed under kernels. The remaining categories are all closed under kernels and images, and there are 22 such subcategories. This number coincides with the previous example, and see the next remark for the explanation.

Remark 7.3. In [Eno3], it is shown that the number of subcategories of $\text{mod } kQ$ which are closed under extensions, kernels and images does not depend on the orientation of the underlying graph for a Dynkin quiver Q , although the number of monobricks does depend as we have seen. In particular, if Q is of type A_n , then the number of such subcategories is equal to the n -th large Schröder number by Theorem 6.13.

The next example is non-hereditary case, which already appeared in the introduction.

Example 7.4. Let Λ be *any* Nakayama algebra whose quiver is $1 \rightleftarrows 2$. Then there are four bricks in $\text{mod } \Lambda$, namely, $\text{brick } \Lambda = \{1, 2, \frac{1}{2}, \frac{2}{1}\}$. By using this (and without any consideration of other modules), we obtain the list of monobricks Table 5.

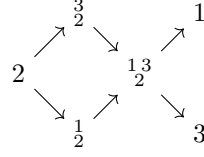
Finally, we consider representation-infinite case.

Example 7.5. Let Q be a 2-Kronecker quiver, that is, $Q = [1 \rightleftarrows 2]$. Then the complete classification of indecomposable kQ -module is known, see e.g. [ARS, Section VIII.7]. By using this, we obtain the following three classes of bricks.

- (1) Indecomposable preprojective modules $\{P_1, P_2, P_3, \dots\}$.
- (2) Regular simple modules $\{R_\lambda\}_{\lambda \in \mathbb{P}^1(k)}$.
- (3) Indecomposable preinjective modules $\{I_1, I_2, I_3, \dots\}$.

Here $P_1 = P(1), P_2 = P(2), P_3 = \tau^- P_1, P_4 = \tau^- P_2, P_5 = \tau^- P_3, \dots$ and $I_1 = I(2), I_2 = I(1), I_3 = \tau I_1, I_4 = \tau I_2, \dots$, where $P(i)$ (resp. $I(i)$) is the indecomposable projective (resp. injective) module corresponding to the vertex i , and τ is the Auslander-Reiten translation.

To classify monobricks over kQ , we need to know the lists of pairs (B_1, B_2) of bricks such that there is a non-zero non-injection from B_1 to B_2 , and pairs such that there is an injection but no non-zero non-injection from B_1 to B_2 . This is summarized in Figure 1, where $B_1 \rightsquigarrow B_2$ (resp. $B_1 \hookrightarrow B_2$) indicates that there is a non-zero non-injection $B_1 \rightarrow B_2$ (resp. an injection).

TABLE 3. The Auslander-Reiten quiver of $\text{mod } k[1 \rightarrow 2 \leftarrow 3]$ TABLE 4. Monobricks \mathcal{M} over $k[1 \rightarrow 2 \leftarrow 3]$ with $\#\mathcal{M} \geq 2$

\mathcal{M}	\mathcal{E}	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$2 < \frac{1}{2}$		$\frac{1}{2}$	itself
$2 < \frac{3}{2}$		$\frac{3}{2}$	itself
$2 < \frac{1^3}{2}$		$\frac{1^3}{2}$	\mathcal{N}
$2, 1$		itself	itself
$2, 3$		itself	itself
$\frac{1}{2}, \frac{3}{2}$		itself	$\frac{3}{2} > 2 < \frac{1}{2}$
$\frac{1}{2} < \frac{1^3}{2}$		$\frac{1^3}{2}$	itself
$\frac{3}{2} < \frac{1^3}{2}$		$\frac{1^3}{2}$	itself
$\frac{1}{2}, 3$		itself	$2 < \frac{1}{2}, 3$
$\frac{3}{2}, 1$		itself	$2 < \frac{3}{2}, 1$
$\frac{3}{2} < \frac{1^3}{2} > \frac{1}{2}$		$\frac{1^3}{2}$	\mathcal{N}
$1, 2, 3$		itself	itself
$\mathcal{N} = 2 \begin{matrix} \nearrow \frac{3}{2} \\ \searrow \frac{1^3}{2} \\ \nearrow \frac{1}{2} \end{matrix}$		$\frac{1^3}{2}$	itself

TABLE 5. Monobricks over cyclic Nakayama algebras with 2 simples

\mathcal{M}	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
\emptyset	itself	itself
1	itself	itself
2	itself	itself
$\frac{1}{2}$	itself	$2 < \frac{1}{2}$

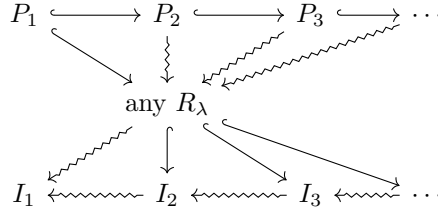
\mathcal{M}	$\max \mathcal{M}$	$\overline{\mathcal{M}}$
$\frac{2}{1}$	itself	$1 < \frac{2}{1}$
$1 < \frac{2}{1}$	$\frac{2}{1}$	itself
$2 < \frac{1}{2}$	$\frac{1}{2}$	itself
1, 2	itself	itself

Any other pairs can be deduced from the composition of arrows in Figure 1. Since there are lots of monobricks, we only consider cofinally closed monobricks. This is enough for classifying monobricks since a set of bricks is a monobrick if and only if it is a subset of some cofinally closed monobricks by Corollary 3.17.

The following are the list of all cofinally closed monobricks, or the list of simple objects in all torsion-free classes.

(M1) $\{P_1, P_2, \dots, P_i\}$ for some i .

FIGURE 1. Structures of bricks in $\text{mod } k[1 \Leftarrow 2]$



- (M2) $\{P_1, P_2, P_3, \dots\}$.
- (M3) $\{P_1\} \cup \{R_\lambda\}_{\lambda \in X}$ for any non-empty subset $X \subset \mathbb{P}^1(k)$.
- (M4) $\{P_1\} \cup \{R_\lambda\}_{\lambda \in \mathbb{P}^1(k)} \cup \{I_i\}$ for $i \geq 2$.

In this list, finite monobricks are (M1) and (M2) for a finite set X . The poset structure is as follows.

(M1)	(M2)	(M3)	(M4)
$P_1 < P_2 < \dots < P_i$	$P_1 < P_2 < \dots$	$ \begin{array}{c} \begin{array}{ c } \hline R_\lambda \\ \hline \vdots \\ \hline R_{\lambda'} \\ \hline \end{array} \\ \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \\ P_1 \\ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \\ X \end{array} $	$ \begin{array}{c} \begin{array}{ c } \hline R_\lambda \\ \hline \vdots \\ \hline R_{\lambda'} \\ \hline \end{array} \\ \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \\ P_1 \\ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \\ \mathbb{P}^1(k) \\ \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \\ I_i \end{array} $

TABLE 6. The poset structure of each monobricker

Using this, we can easily compute $W(\mathcal{F})$ for each torsion-free class, since $W(\mathcal{F})$ is equal to $\text{Filt}(\max(\text{sim } \mathcal{F}))$ by Theorem 4.5. Moreover, $\max(\text{sim } \mathcal{F})$ is nothing but the brick labels starting at \mathcal{F} (Remark 4.6), we can compute the brick labels (c.f. [DIRRT, Example 3.6]). This can be summarized as follows.

(M1)	(M2)	(M3)	(M4)
$\{P_i\}$	\emptyset	$\{R_\lambda\}_{\lambda \in X}$	$\{I_i\}$

TABLE 7. The maximal element of each monobricker, or all the semibricks

Since $\max: \text{mbrick}_{\text{c.c.}} \Lambda \rightarrow \text{sbrick } \Lambda$ is surjective by Proposition 5.1, this table can also be seen as a table of all semibricks.

We remark that if X consists of one element in (M3), then the monobricker is isomorphic to $P_1 < P_2$ as posets, although the former corresponds to non-functorially finite torsion-free class but the latter to functorially finite.

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