On the Partitions into Distinct Parts and Odd Parts

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Abstract

In this paper, we show that the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n satisfies Euler's recurrence relation for the partition function p(n)when n is odd. A decomposition of this difference in terms of the total number of parts in all the partitions of n is also derived. In this context, we conjecture that for k > 0, the series

$$(q^2; q^2)_{\infty} \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} {n-1 \brack k-1}$$

has non-negative coefficients.

Keywords: partitions, truncated theta series

MSC 2010: 11P81, 05A17

1 Introduction

A partition of a positive integer n is a sequence of positive integers whose sum is n. The order of the summands is unimportant when writing the partitions of n, but for consistency, a partition of n will be written with the summands in a nonincreasing order [1]. As usual, we denote by p(n) the number of the partitions of n. For example, we have p(5) = 7 because the partitions of 5 are given as:

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [7, 8].

One of the well-known theorems in the partition theory is Euler's pentagonal number theorem, i.e.,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_{\infty}.$$

Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_0 = 1,$$
 $(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$ $(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$

Because the infinite product $(a; q)_{\infty}$ diverges when $a \neq 0$ and $|q| \ge 1$, whenever $(a; q)_{\infty}$ appears in a formula, we shall assume that |q| < 1. Euler's pentagonal number theorem gives an easy linear recurrence relation for p(n), namely

$$\sum_{j=-\infty}^{\infty} (-1)^j p(n-j(3j-1)/2) = \delta_{0,n}, \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta function and p(n) = 0 if n < 0.

A famous theorem of Euler asserts that there are as many partitions of n into distinct parts as there are partitions into odd parts [1, p. 5. Cor. 1.2]. For instance, the odd partitions of 5 are:

5,
$$3+1+1$$
 and $1+1+1+1+1$,

while the distinct partitions of 5 are:

5,
$$4+1$$
 and $3+2$.

We recall Euler's bijective proof of this result [5]: A partition into distinct parts can be written as

$$n = d_1 + d_2 + \dots + d_k$$

Each integer d_i can be uniquely expressed as a power of 2 times an odd number, i.e.,

$$n = 2^{\alpha_1} o_1 + 2^{\alpha_2} o_2 + \dots + 2^{\alpha_k} o_k$$

where each o_i is an odd number. Grouping together the odd numbers, we get the following expression

$$n = t_1 \cdot 1 + t_3 \cdot 3 + t_5 \cdot 5 + \cdots,$$

where $t_i \ge 0$. If d_i is odd, then we have $\alpha_i = 0$. For d_i even, it is clear that $\alpha_i > 0$. So we deduce that

$$(t_1 + t_3 + t_5 + \cdots) - k \ge 0,$$

for any positive integer n. In other words, the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n is nonnegative. A combinatorial interpretation of this difference has been conjectured recently by George Beck [12, A090867, Apr 22 2017].

Conjecture 1.1. The difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n equals the number of partitions of n in which the set of even parts has only one element.

A few days later, George E. Andrews [2, Theorem 1] provides a solution for this Beck's problem and introduces a new combinatorial interpretation for the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n.

Theorem 1.2. For all $n \ge 1$, a(n) = b(n) = c(n), where:

- a(n) is the number of partitions of n in which the set of even parts has only one element;
- b(n) is the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n;
- c(n) is the number of partitions of n in which exactly one part is repeated.

For example, a(5) = 4 because the four partitions in question are:

4+1, 3+2, 2+2+1 and 2+1+1+1.

We have already seen there are 9 parts in the odd partitions of 5 and 5 parts in the distinct partitions of 5 with the difference b(5) = 4. On the other hand, we have c(5) = 4 where the relevant partitions are:

$$3+1+1$$
, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$.

In this paper, inspired by Andrews's proof of Theorem 1.2, we provide new properties for the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n considering two factorizations for the generating function of b(n).

This paper is organized as follows. In Section 2 we will show that the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n satisfies Euler's recurrence relation (1) when n is odd. In Section 3 we will provide a decomposition of b(n) in terms of the total number of parts in all the partitions of n. A linear homogeneous inequality for the difference b(n) are conjectured in Section 4 in analogy with the linear homogeneous inequality for Euler's partition function p(n) provided by Andrews and Merca in [3].

2 A pentagonal number recurrence for b(n)

In this section we consider s(n) to be the difference between the number of parts in all the partitions of n into odd number of distinct parts and the number of parts in all the partitions of n into even number of distinct parts. For instance, considering the partitions of 5 into distinct parts, we see that

$$s(5) = 1 - 2 - 2 = -3.$$

In [3], Andrews and Merca defined $M_k(n)$ to be the number of partitions of n in which k is the least positive integer that is not a part and there are more parts > k than there are parts < k. If n = 18 and k = 3 then we have $M_3(18) = 3$ because the three partitions in question are:

5+5+5+2+1, 6+5+4+2+1, and 7+4+4+2+1.

We have the following result.

Theorem 2.1. Let k and n be positive integers. The partition functions b(n), s(n) and $M_k(n)$ are related by

$$(-1)^{k-1} \left(\sum_{j=-(k-1)}^{k} (-1)^{j} b \left(n - j(3j-1)/2 \right) - \frac{1 + (-1)^{n}}{2} s \left(\frac{n}{2} \right) \right)$$
$$= \sum_{j=1}^{\lfloor n/2 \rfloor} s(j) M_{k}(n-j).$$

Proof. As we can see in [2], the proof of Theorem 1.2 invokes the equality of the generating functions for a(n), b(n) and c(n). So we consider the following factorization of Andrews for the generating function of b(n):

$$\sum_{n=0}^{\infty} b(n)q^n = (-q;q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}}.$$
(2)

On the other hand, the identity

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \frac{1}{(q;q)_{\infty}} \sum_{n=1}^{\infty} s(n)q^n.$$

is a specialization of the Lambert series factorization theorem [10, Theorem 1.2]. A proof of this relation via logarithmic differentiation can be seen in [9, Theorem 1].

We have

$$\begin{split} \sum_{n=0}^{\infty} b(n)q^n &= \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \\ &= \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}} \cdot \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} s(n)q^{2n} \\ &= \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} s(n)q^{2n}. \end{split}$$
(3)

In [3], the authors considered Euler's pentagonal number theorem and proved the following truncated form:

$$\frac{(-1)^{k-1}}{(q;q)_{\infty}} \sum_{n=-(k-1)}^{k} (-1)^n q^{n(3n-1)/2} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}} + (k+1)n}{(q;q)_n} \begin{bmatrix} n-1\\k-1 \end{bmatrix}, \quad (4)$$

where

is the q-binomial coefficient.

Multiplying both sides of (4) by $\sum_{n=1}^{\infty} s(n)q^{2n}$, we obtain

$$(-1)^{k-1} \left(\left(\sum_{n=1}^{\infty} b(n)q^n \right) \left(\sum_{n=-(k-1)}^k (-1)^n q^{n(3n-1)/2} \right) - \sum_{n=1}^{\infty} s(n)q^{2n} \right) \\ = \left(\sum_{n=1}^{\infty} s(n)q^{2n} \right) \left(\sum_{n=0}^{\infty} M_k(n)q^n \right),$$

where we have invoked the generating function for $M_k(n)$ [3],

$$\sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}} + (k+1)n}{(q;q)_n} \begin{bmatrix} n-1\\k-1 \end{bmatrix}.$$

The proof follows easily considering Cauchy's multiplication of two power series. $\hfill\square$

The limiting case $k \to \infty$ of Theorem 2.1 provides the following linear recurrence relation for b(n) involving the generalized pentagonal numbers.

Corollary 2.2. For $n \ge 0$,

$$\sum_{k=-\infty}^{\infty} (-1)^k b (n - k(3k - 1)/2) = \begin{cases} s(n/2), & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

Theorem 2.1 can be seen as a truncated form of Corollary 2.2. Considering again the relation (3), we remark the following convolution identity.

Corollary 2.3. For $n \ge 0$,

$$b(n) = \sum_{j=0}^{\lfloor n/2 \rfloor} s(j)p(n-2j).$$

3 A decomposition of b(n)

Let us define S(n) to be the total number of parts in all the partitions of n. For example, we have

$$S(5) = 1 + 2 + 2 + 3 + 3 + 4 + 5 = 20.$$

Andrews and Merca [4] defined $MP_k(n)$ to be the number of partitions of n in which the first part larger than 2k - 1 is odd and appears exactly k times. All other odd parts appear at most once. For example, $MP_2(19) = 10$, and the partitions in question are:

$$\begin{array}{l}9+9+1, \ 9+5+5, \ 8+5+5+1, \ 7+7+3+2, \ 7+7+2+2+1, \\7+5+5+2, \ 6+5+5+3, \ 6+5+5+2+1, \ 5+5+3+2+2+2, \\5+5+2+2+2+2+2+1.\end{array}$$

We have the following result.

Theorem 3.1. Let k and n be positive integers. The partition functions b(n), S(n) and $MP_k(n)$ are related by

$$b(n) - \sum_{j=0}^{2k-1} S(n/2 - j(j+1)/4) = (-1)^k \sum_{j=0}^n (-1)^j b(n-j) M P_k(j),$$

where S(x) = 0 if x is not a positive integer

Proof. First we want the generating function for partitions where z keeps track of the number of parts equal to k. This is

$$\frac{1}{1 - zq^k} \prod_{\substack{n=1\\n \neq k}}^{\infty} \frac{1}{1 - q^n} = \frac{1}{(q;q)_{\infty}} \cdot \frac{1 - q^k}{1 - zq^k}.$$

Let $S_k(n)$ denote the total number of k's in all the partitions of n. Hence

$$\sum_{n=0}^{\infty} S_k(n)q^n = \frac{d}{dz}\Big|_{z=1} \frac{(1-q^k)}{(q;q)_{\infty}(1-zq^k)} = \frac{q^k}{1-q^k} \cdot \frac{1}{(q;q)_{\infty}}.$$

Thus, we deduce the following generating function for S(n):

$$\sum_{n=0}^{\infty} S(n)q^n = \frac{1}{(q;q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

So we can write

$$\begin{split} \sum_{n=0}^{\infty} b(n)q^n &= \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \\ &= \frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}(q^2;q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \\ &= \frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}} \sum_{n=0}^{\infty} S(n)q^{2n}. \end{split}$$

This identity can be written as follows:

$$\frac{(q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \sum_{n=0}^{\infty} b(n)q^n = \sum_{n=0}^{\infty} S(n)q^{2n}.$$
(5)

In [4], the authors considered the following theta identity of Gauss

$$\sum_{n=0}^{\infty} (-q)^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}}$$
(6)

and proved the following truncated form:

$$\frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \sum_{j=0}^{2k-1} (-q)^{j(j+1)/2}$$

= 1 + (-1)^{k-1} \frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}} \sum_{j=0}^{\infty} \frac{q^{k(2k+2j+1)}(-q^{2k+2j+3};q^2)_{\infty}}{(q^{2k+2j+2};q^2)_{\infty}}.

By this relation, with q replaced by -q, we obtain

$$\frac{(q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} \sum_{j=0}^{2k-1} q^{j(j+1)/2} = 1 + (-1)^{k-1} \sum_{n=0}^{\infty} (-1)^n M P_k(n) q^n, \tag{7}$$

where we have invoked the generating function for $MP_k(n)$ [4],

$$\sum_{n=0}^{\infty} MP_k(n)q^n = \frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}} \sum_{j=0}^{\infty} \frac{q^{k(2k+2j+1)}(-q^{2k+2j+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty}$$

Multiplying both sides of (7) by $\sum_{n=0}^{\infty} b(n)q^n$, we obtain

$$(-1)^{k-1} \left(\left(\sum_{n=1}^{\infty} S(n)q^{2n} \right) \left(\sum_{n=0}^{2k-1} q^{n(n+1)/2} \right) - \sum_{n=1}^{\infty} b(n)q^n \right)$$
$$= \left(\sum_{n=1}^{\infty} b(n)q^n \right) \left(\sum_{n=0}^{\infty} (-1)^n M P_k(n)q^n \right).$$

The proof follows easily considering Cauchy's multiplication of two power series.

The limiting case $k \to \infty$ of Theorem 3.1 provides the following decomposition of the difference b(n) in terms of S(n).

Corollary 3.2. For $n \ge 0$,

$$b(n) = \sum_{k=0}^{\infty} S(n/2 - k(k+1)/4),$$

with S(x) = 0 if x is not a positive integer.

More explicitly, this corollary can be rewritten as:

$$b(2n) = \sum_{k=-\infty}^{\infty} S(n - k(4k - 1))$$

and

$$b(2n+1) = \sum_{k=-\infty}^{\infty} S(n-k(4k-3)).$$

Combinatorial proofs of these identities would be very interesting. On the other hand, the relation (5) allows us to remark that

$$S(n) = \sum_{k=0}^{2n} (-1)^k e(k)b(2n-k),$$

where e(n) is the number of partitions of n in which each even part occurs with even multiplicity and there is no restriction on the odd parts [12, A006950]. Other properties for S(n) can be found in [6].

As a consequence of Theorem 3.1, we remark the following infinite families of inequalities involving the partition functions b(n) and $MP_k(n)$.

Corollary 3.3. Let k and n be positive integers. Then

$$(-1)^k \sum_{j=0}^n (-1)^j b(n-j)MP_k(j) \ge 0.$$

Proof. We take into account that

$$b(n) - \sum_{j=0}^{1} S(n/2 - j(j+1)/4) \ge b(n) - \sum_{j=0}^{3} S(n/2 - j(j+1)/4)$$
$$\ge \dots \ge b(n) - \sum_{j=0}^{\infty} S(n/2 - j(j+1)/4) = 0.$$

Relevant to Theorem 3.1, it would be very appealing to have combinatorial interpretations of

$$(-1)^k \sum_{j=0}^n (-1)^j b(n-j) M P_k(j).$$

4 Open problems

Linear homogeneous inequalities involving Euler's partition function p(n) have been the subject of recent studies [3, 4, 7, 11]. In [7], the author proved the inequality

$$p(n) - p(n-1) - p(n-2) + p(n-5) \le 0, \quad n > 0,$$

in order to provide the fastest known algorithm for the generation of the partitions of n. Subsequently, Andrews and Merca [3] proved more generally that, for k > 0,

$$(-1)^{k-1} \sum_{j=-(k-1)}^{k} (-1)^{j} p(n-j(3j-1)/2) \ge 0,$$
(8)

with strict inequality if $n \ge k(3k+1)/2$. In other words, for k > 0, the coefficients of q^n in the series

$$(-1)^{k-1} \left(\frac{1}{(q;q)_{\infty}} \sum_{j=-(k-1)}^{k} (-1)^{j} q^{j(3j-1)/2} - 1 \right)$$

are all zero for $0 \le n < k(3k+1)/2$, and for $n \ge k(3k+1)/2$ all the coefficients are positive. Related to this result on truncated pentagonal number series, we remark that there is a substantial amount of numerical evidence to conjecture a stronger result.

Conjecture 4.1. For k > 0, the coefficients of q^n in the series

$$(-1)^{k-1} \left(\frac{1}{(q;q)_{\infty}} \sum_{j=-(k-1)}^{k} (-1)^{j} q^{j(3j-1)/2} - 1 \right) (q^{2};q^{2})_{\infty}$$

are all zero for $0 \leq n < k(3k+1)/2$, and for $n \geq k(3k+1)/2$ all the coefficients are positive.

Let Q(n) be the number of partitions of n into odd parts. It is well known that the generating function for Q(n) is $1/(q;q^2)_{\infty}$. Assuming Conjecture 4.1, we immediately deduce that the partition functions p(n) and Q(n), share a common infinite family of linear inequalities of the form (8) when n is odd. In addition, considering Theorem 2.1, we easily deduce that the partition function b(n) satisfies the following infinite families of linear inequalities.

Conjecture 4.2. For k > 0,

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k} (-1)^{j} b\left(n-j(3j-1)/2\right) - \frac{1+(-1)^{n}}{2} s\left(\frac{n}{2}\right)\right) \ge 0,$$

with strict inequalities if $n \ge 2 + k(3k+1)/2$.

In this context, relevant to Theorem 2.1, it would be very appealing to have combinatorial interpretations of

$$\sum_{j=1}^{\lfloor n/2 \rfloor} s(j) M_k(n-j).$$

5 Concluding remarks

New properties for the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n have been introduced in this paper.

Surprisingly, when n is odd, Euler's partition function p(n) and the difference b(n) share two common linear homogeneous recurrence relations. As we can see in Corollary 2.2, the first recurrence relation involves the generalized pentagonal numbers:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - p(n-26) + \cdots,$$

and

$$b(n) = b(n-1) + b(n-2) - b(n-5) - b(n-7) + b(n-12) + b(n-15) - b(n-22) - b(n-26) + \cdots$$

The second recurrence relation combines the partition function p(n) and the difference b(n) with the triangular numbers, as follows:

$$p(n) = p(n-1) + p(n-3) - p(n-6) - p(n-10) + p(n-15) + p(n-21) - p(n-28) - p(n-36) + \cdots,$$

and

$$b(n) = b(n-1) + b(n-3) - b(n-6) - b(n-10) + b(n-15) + b(n-21) - b(n-28) - b(n-36) + \cdots$$

These relations can be easily derived considering again the theta identity of Gauss (6) and the following two identities:

$$\frac{(q^2;q^2)_{\infty}}{(-q;q^2)_{\infty}}\sum_{n=0}^{\infty}p(n)q^n = (-q^2;q^2)_{\infty},$$

and

$$\frac{(q^2;q^2)_{\infty}}{(-q;q^2)_{\infty}}\sum_{n=0}^{\infty}b(n)q^n = (q^4;q^4)_{\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}$$

Finally, we want to thank Professor George E. Andrews for his valuable comments on the first version of this paper.

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