# On the Partitions into Distinct Parts and Odd Parts 

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#### Abstract

In this paper, we show that the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ satisfies Euler's recurrence relation for the partition function $p(n)$ when $n$ is odd. A decomposition of this difference in terms of the total number of parts in all the partitions of $n$ is also derived. In this context, we conjecture that for $k>0$, the series


$$
\left(q^{2} ; q^{2}\right)_{\infty} \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1) n}}{(q ; q)_{n}}\left[\begin{array}{l}
n-1 \\
k-1
\end{array}\right]
$$

has non-negative coefficients.
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## 1 Introduction

A partition of a positive integer $n$ is a sequence of positive integers whose sum is $n$. The order of the summands is unimportant when writing the partitions of $n$, but for consistency, a partition of $n$ will be written with the summands in a nonincreasing order [1]. As usual, we denote by $p(n)$ the number of the partitions of $n$. For example, we have $p(5)=7$ because the partitions of 5 are given as:

$$
5,4+1,3+2,3+1+1,2+2+1,2+1+1+1,1+1+1+1+1
$$

The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [7, 8.

One of the well-known theorems in the partition theory is Euler's pentagonal number theorem, i.e.,

$$
\sum_{n=-\infty}^{\infty}(-1)^{n} q^{n(3 n-1) / 2}=(q ; q)_{\infty}
$$

Here and throughout this paper, we use the following customary $q$-series notation:

$$
(a ; q)_{0}=1, \quad(a ; q)_{n}=\prod_{k=0}^{n-1}\left(1-a q^{k}\right), \quad(a ; q)_{\infty}=\prod_{k=0}^{\infty}\left(1-a q^{k}\right)
$$

Because the infinite product $(a ; q)_{\infty}$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a ; q)_{\infty}$ appears in a formula, we shall assume that $|q|<1$. Euler's pentagonal number theorem gives an easy linear recurrence relation for $p(n)$, namely

$$
\begin{equation*}
\sum_{j=-\infty}^{\infty}(-1)^{j} p(n-j(3 j-1) / 2)=\delta_{0, n} \tag{1}
\end{equation*}
$$

where $\delta_{i, j}$ is the Kronecker delta function and $p(n)=0$ if $n<0$.
A famous theorem of Euler asserts that there are as many partitions of $n$ into distinct parts as there are partitions into odd parts [1, p. 5. Cor. 1.2]. For instance, the odd partitions of 5 are:

$$
5, \quad 3+1+1 \quad \text { and } \quad 1+1+1+1+1
$$

while the distinct partitions of 5 are:

$$
5, \quad 4+1 \text { and } 3+2
$$

We recall Euler's bijective proof of this result [5]: A partition into distinct parts can be written as

$$
n=d_{1}+d_{2}+\cdots+d_{k} .
$$

Each integer $d_{i}$ can be uniquely expressed as a power of 2 times an odd number, i.e.,

$$
n=2^{\alpha_{1}} o_{1}+2^{\alpha_{2}} o_{2}+\cdots+2^{\alpha_{k}} o_{k}
$$

where each $o_{i}$ is an odd number. Grouping together the odd numbers, we get the following expression

$$
n=t_{1} \cdot 1+t_{3} \cdot 3+t_{5} \cdot 5+\cdots,
$$

where $t_{i} \geqslant 0$. If $d_{i}$ is odd, then we have $\alpha_{i}=0$. For $d_{i}$ even, it is clear that $\alpha_{i}>0$. So we deduce that

$$
\left(t_{1}+t_{3}+t_{5}+\cdots\right)-k \geqslant 0
$$

for any positive integer $n$. In other words, the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ is nonnegative. A combinatorial interpretation of this difference has been conjectured recently by George Beck [12, A090867, Apr 22 2017].

Conjecture 1.1. The difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ equals the number of partitions of $n$ in which the set of even parts has only one element.

A few days later, George E. Andrews [2] Theorem 1] provides a solution for this Beck's problem and introduces a new combinatorial interpretation for the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$.
Theorem 1.2. For all $n \geqslant 1, a(n)=b(n)=c(n)$, where:

- $a(n)$ is the number of partitions of $n$ in which the set of even parts has only one element;
$-b(n)$ is the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$;
- $c(n)$ is the number of partitions of $n$ in which exactly one part is repeated.

For example, $a(5)=4$ because the four partitions in question are:

$$
4+1, \quad 3+2, \quad 2+2+1 \quad \text { and } \quad 2+1+1+1
$$

We have already seen there are 9 parts in the odd partitions of 5 and 5 parts in the distinct partitions of 5 with the difference $b(5)=4$. On the other hand, we have $c(5)=4$ where the relevant partitions are:

$$
3+1+1, \quad 2+2+1, \quad 2+1+1+1 \quad \text { and } \quad 1+1+1+1+1 .
$$

In this paper, inspired by Andrews's proof of Theorem [1.2, we provide new properties for the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ considering two factorizations for the generating function of $b(n)$.

This paper is organized as follows. In Section 2 we will show that the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ satisfies Euler's recurrence relation (1) when $n$ is odd. In Section 3 we will provide a decomposition of $b(n)$ in terms of the total number of parts in all the partitions of $n$. A linear homogeneous inequality for the difference $b(n)$ are conjectured in Section 4 in analogy with the linear homogeneous inequality for Euler's partition function $p(n)$ provided by Andrews and Merca in (3).

## 2 A pentagonal number recurrence for $b(n)$

In this section we consider $s(n)$ to be the difference between the number of parts in all the partitions of $n$ into odd number of distinct parts and the number of parts in all the partitions of $n$ into even number of distinct parts. For instance, considering the partitions of 5 into distinct parts, we see that

$$
s(5)=1-2-2=-3 .
$$

In [3], Andrews and Merca defined $M_{k}(n)$ to be the number of partitions of $n$ in which $k$ is the least positive integer that is not a part and there are more parts $>k$ than there are parts $<k$. If $n=18$ and $k=3$ then we have $M_{3}(18)=3$ because the three partitions in question are:

$$
5+5+5+2+1, \quad 6+5+4+2+1, \quad \text { and } \quad 7+4+4+2+1
$$

We have the following result.
Theorem 2.1. Let $k$ and $n$ be positive integers. The partition functions $b(n)$, $s(n)$ and $M_{k}(n)$ are related by

$$
\begin{aligned}
& (-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^{j} b(n-j(3 j-1) / 2)-\frac{1+(-1)^{n}}{2} s\left(\frac{n}{2}\right)\right) \\
& =\sum_{j=1}^{\lfloor n / 2\rfloor} s(j) M_{k}(n-j) .
\end{aligned}
$$

Proof. As we can see in [2, the proof of Theorem 1.2 invokes the equality of the generating functions for $a(n), b(n)$ and $c(n)$. So we consider the following factorization of Andrews for the generating function of $b(n)$ :

$$
\begin{equation*}
\sum_{n=0}^{\infty} b(n) q^{n}=(-q ; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \tag{2}
\end{equation*}
$$

On the other hand, the identity

$$
\sum_{n=1}^{\infty} \frac{q^{n}}{1-q^{n}}=\frac{1}{(q ; q)_{\infty}} \sum_{n=1}^{\infty} s(n) q^{n}
$$

is a specialization of the Lambert series factorization theorem 10, Theorem 1.2]. A proof of this relation via logarithmic differentiation can be seen in (9) Theorem 1].

We have

$$
\begin{align*}
\sum_{n=0}^{\infty} b(n) q^{n} & =\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{(q ; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \\
& =\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{(q ; q)_{\infty}} \cdot \frac{1}{\left(q^{2} ; q^{2}\right)_{\infty}} \sum_{n=1}^{\infty} s(n) q^{2 n} \\
& =\frac{1}{(q ; q)_{\infty}} \sum_{n=1}^{\infty} s(n) q^{2 n} \tag{3}
\end{align*}
$$

In [3] the authors considered Euler's pentagonal number theorem and proved the following truncated form:

$$
\frac{(-1)^{k-1}}{(q ; q)_{\infty}} \sum_{n=-(k-1)}^{k}(-1)^{n} q^{n(3 n-1) / 2}=(-1)^{k-1}+\sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1) n}}{(q ; q)_{n}}\left[\begin{array}{l}
n-1  \tag{4}\\
k-1
\end{array}\right]
$$

where

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]= \begin{cases}\frac{(q ; q)_{n}}{(q ; q)_{k}(q ; q)_{n-k}}, & \text { if } 0 \leqslant k \leqslant n \\
0, & \text { otherwise }\end{cases}
$$

is the $q$-binomial coefficient.
Multiplying both sides of (4) by $\sum_{n=1}^{\infty} s(n) q^{2 n}$, we obtain

$$
\begin{aligned}
& (-1)^{k-1}\left(\left(\sum_{n=1}^{\infty} b(n) q^{n}\right)\left(\sum_{n=-(k-1)}^{k}(-1)^{n} q^{n(3 n-1) / 2}\right)-\sum_{n=1}^{\infty} s(n) q^{2 n}\right) \\
& =\left(\sum_{n=1}^{\infty} s(n) q^{2 n}\right)\left(\sum_{n=0}^{\infty} M_{k}(n) q^{n}\right)
\end{aligned}
$$

where we have invoked the generating function for $M_{k}(n)$ 3],

$$
\sum_{n=0}^{\infty} M_{k}(n) q^{n}=\sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1) n}}{(q ; q)_{n}}\left[\begin{array}{l}
n-1 \\
k-1
\end{array}\right] .
$$

The proof follows easily considering Cauchy's multiplication of two power series.

The limiting case $k \rightarrow \infty$ of Theorem 2.1 provides the following linear recurrence relation for $b(n)$ involving the generalized pentagonal numbers.

Corollary 2.2. For $n \geqslant 0$,

$$
\sum_{k=-\infty}^{\infty}(-1)^{k} b(n-k(3 k-1) / 2)= \begin{cases}s(n / 2), & \text { for } n \text { even } \\ 0, & \text { for } n \text { odd }\end{cases}
$$

Theorem 2.1 can be seen as a truncated form of Corollary 2.2. Considering again the relation (3), we remark the following convolution identity.

Corollary 2.3. For $n \geqslant 0$,

$$
b(n)=\sum_{j=0}^{\lfloor n / 2\rfloor} s(j) p(n-2 j)
$$

## 3 A decomposition of $b(n)$

Let us define $S(n)$ to be the total number of parts in all the partitions of $n$. For example, we have

$$
S(5)=1+2+2+3+3+4+5=20
$$

Andrews and Merca [4 defined $M P_{k}(n)$ to be the number of partitions of $n$ in which the first part larger than $2 k-1$ is odd and appears exactly $k$ times. All other odd parts appear at most once. For example, $M P_{2}(19)=10$, and the partitions in question are:

$$
\begin{aligned}
& 9+9+1,9+5+5,8+5+5+1,7+7+3+2,7+7+2+2+1 \\
& 7+5+5+2,6+5+5+3,6+5+5+2+1,5+5+3+2+2+2 \\
& 5+5+2+2+2+2+1
\end{aligned}
$$

We have the following result.
Theorem 3.1. Let $k$ and $n$ be positive integers. The partition functions $b(n)$, $S(n)$ and $M P_{k}(n)$ are related by

$$
b(n)-\sum_{j=0}^{2 k-1} S(n / 2-j(j+1) / 4)=(-1)^{k} \sum_{j=0}^{n}(-1)^{j} b(n-j) M P_{k}(j)
$$

where $S(x)=0$ if $x$ is not a positive integer
Proof. First we want the generating function for partitions where $z$ keeps track of the number of parts equal to $k$. This is

$$
\frac{1}{1-z q^{k}} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{1}{1-q^{n}}=\frac{1}{(q ; q)_{\infty}} \cdot \frac{1-q^{k}}{1-z q^{k}}
$$

Let $S_{k}(n)$ denote the total number of $k$ 's in all the partitions of $n$. Hence

$$
\sum_{n=0}^{\infty} S_{k}(n) q^{n}=\left.\frac{d}{d z}\right|_{z=1} \frac{\left(1-q^{k}\right)}{(q ; q)_{\infty}\left(1-z q^{k}\right)}=\frac{q^{k}}{1-q^{k}} \cdot \frac{1}{(q ; q)_{\infty}}
$$

Thus, we deduce the following generating function for $S(n)$ :

$$
\sum_{n=0}^{\infty} S(n) q^{n}=\frac{1}{(q ; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{n}}{1-q^{n}}
$$

So we can write

$$
\begin{aligned}
\sum_{n=0}^{\infty} b(n) q^{n} & =\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{(q ; q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \\
& =\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{\left(q ; q^{2}\right)_{\infty}\left(q^{2} ; q^{2}\right)_{\infty}} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}} \\
& =\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{\left(q ; q^{2}\right)_{\infty}} \sum_{n=0}^{\infty} S(n) q^{2 n}
\end{aligned}
$$

This identity can be written as follows:

$$
\begin{equation*}
\frac{\left(q ; q^{2}\right)_{\infty}}{\left(q^{2} ; q^{2}\right)_{\infty}} \sum_{n=0}^{\infty} b(n) q^{n}=\sum_{n=0}^{\infty} S(n) q^{2 n} \tag{5}
\end{equation*}
$$

In 4], the authors considered the following theta identity of Gauss

$$
\begin{equation*}
\sum_{n=0}^{\infty}(-q)^{n(n+1) / 2}=\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{\left(-q ; q^{2}\right)_{\infty}} \tag{6}
\end{equation*}
$$

and proved the following truncated form:

$$
\begin{aligned}
& \frac{\left(-q ; q^{2}\right)_{\infty}}{\left(q^{2} ; q^{2}\right)_{\infty}} \sum_{j=0}^{2 k-1}(-q)^{j(j+1) / 2} \\
& \quad=1+(-1)^{k-1} \frac{\left(-q ; q^{2}\right)_{k}}{\left(q^{2} ; q^{2}\right)_{k-1}} \sum_{j=0}^{\infty} \frac{q^{k(2 k+2 j+1)}\left(-q^{2 k+2 j+3} ; q^{2}\right)_{\infty}}{\left(q^{2 k+2 j+2} ; q^{2}\right)_{\infty}}
\end{aligned}
$$

By this relation, with $q$ replaced by $-q$, we obtain

$$
\begin{equation*}
\frac{\left(q ; q^{2}\right)_{\infty}}{\left(q^{2} ; q^{2}\right)_{\infty}} \sum_{j=0}^{2 k-1} q^{j(j+1) / 2}=1+(-1)^{k-1} \sum_{n=0}^{\infty}(-1)^{n} M P_{k}(n) q^{n} \tag{7}
\end{equation*}
$$

where we have invoked the generating function for $M P_{k}(n)$ 4],

$$
\sum_{n=0}^{\infty} M P_{k}(n) q^{n}=\frac{\left(-q ; q^{2}\right)_{k}}{\left(q^{2} ; q^{2}\right)_{k-1}} \sum_{j=0}^{\infty} \frac{q^{k(2 k+2 j+1)}\left(-q^{2 k+2 j+3} ; q^{2}\right)_{\infty}}{\left(q^{2 k+2 j+2} ; q^{2}\right)_{\infty}}
$$

Multiplying both sides of (17) by $\sum_{n=0}^{\infty} b(n) q^{n}$, we obtain

$$
\begin{aligned}
& (-1)^{k-1}\left(\left(\sum_{n=1}^{\infty} S(n) q^{2 n}\right)\left(\sum_{n=0}^{2 k-1} q^{n(n+1) / 2}\right)-\sum_{n=1}^{\infty} b(n) q^{n}\right) \\
& =\left(\sum_{n=1}^{\infty} b(n) q^{n}\right)\left(\sum_{n=0}^{\infty}(-1)^{n} M P_{k}(n) q^{n}\right) .
\end{aligned}
$$

The proof follows easily considering Cauchy's multiplication of two power series.

The limiting case $k \rightarrow \infty$ of Theorem 3.1 provides the following decomposition of the difference $b(n)$ in terms of $S(n)$.
Corollary 3.2. For $n \geqslant 0$,

$$
b(n)=\sum_{k=0}^{\infty} S(n / 2-k(k+1) / 4)
$$

with $S(x)=0$ if $x$ is not a positive integer.

More explicitly, this corollary can be rewritten as:

$$
b(2 n)=\sum_{k=-\infty}^{\infty} S(n-k(4 k-1))
$$

and

$$
b(2 n+1)=\sum_{k=-\infty}^{\infty} S(n-k(4 k-3))
$$

Combinatorial proofs of these identities would be very interesting. On the other hand, the relation (5) allows us to remark that

$$
S(n)=\sum_{k=0}^{2 n}(-1)^{k} e(k) b(2 n-k)
$$

where $e(n)$ is the number of partitions of $n$ in which each even part occurs with even multiplicity and there is no restriction on the odd parts [12, A006950]. Other properties for $S(n)$ can be found in (6).

As a consequence of Theorem 3.1] we remark the following infinite families of inequalities involving the partition functions $b(n)$ and $M P_{k}(n)$.

Corollary 3.3. Let $k$ and $n$ be positive integers. Then

$$
(-1)^{k} \sum_{j=0}^{n}(-1)^{j} b(n-j) M P_{k}(j) \geqslant 0
$$

Proof. We take into account that

$$
\begin{aligned}
b(n)-\sum_{j=0}^{1} S(n / 2-j(j+1) / 4) & \geqslant b(n)-\sum_{j=0}^{3} S(n / 2-j(j+1) / 4) \\
& \geqslant \cdots \geqslant b(n)-\sum_{j=0}^{\infty} S(n / 2-j(j+1) / 4)=0
\end{aligned}
$$

Relevant to Theorem 3.1, it would be very appealing to have combinatorial interpretations of

$$
(-1)^{k} \sum_{j=0}^{n}(-1)^{j} b(n-j) M P_{k}(j)
$$

## 4 Open problems

Linear homogeneous inequalities involving Euler's partition function $p(n)$ have been the subject of recent studies [3, 4, 7, 11. In 7], the author proved the inequality

$$
p(n)-p(n-1)-p(n-2)+p(n-5) \leqslant 0, \quad n>0
$$

in order to provide the fastest known algorithm for the generation of the partitions of $n$. Subsequently, Andrews and Merca 3] proved more generally that, for $k>0$,

$$
\begin{equation*}
(-1)^{k-1} \sum_{j=-(k-1)}^{k}(-1)^{j} p(n-j(3 j-1) / 2) \geqslant 0 \tag{8}
\end{equation*}
$$

with strict inequality if $n \geqslant k(3 k+1) / 2$. In other words, for $k>0$, the coefficients of $q^{n}$ in the series

$$
(-1)^{k-1}\left(\frac{1}{(q ; q)_{\infty}} \sum_{j=-(k-1)}^{k}(-1)^{j} q^{j(3 j-1) / 2}-1\right)
$$

are all zero for $0 \leqslant n<k(3 k+1) / 2$, and for $n \geqslant k(3 k+1) / 2$ all the coefficients are positive. Related to this result on truncated pentagonal number series, we remark that there is a substantial amount of numerical evidence to conjecture a stronger result.

Conjecture 4.1. For $k>0$, the coefficients of $q^{n}$ in the series

$$
(-1)^{k-1}\left(\frac{1}{(q ; q)_{\infty}} \sum_{j=-(k-1)}^{k}(-1)^{j} q^{j(3 j-1) / 2}-1\right)\left(q^{2} ; q^{2}\right)_{\infty}
$$

are all zero for $0 \leqslant n<k(3 k+1) / 2$, and for $n \geqslant k(3 k+1) / 2$ all the coefficients are positive.

Let $Q(n)$ be the number of partitions of $n$ into odd parts. It is well known that the generating function for $Q(n)$ is $1 /\left(q ; q^{2}\right)_{\infty}$. Assuming Conjecture 4.1 we immediately deduce that the partition functions $p(n)$ and $Q(n)$, share a common infinite family of linear inequalities of the form (8) when $n$ is odd. In addition, considering Theorem 2.1, we easily deduce that the partition function $b(n)$ satisfies the following infinite families of linear inequalities.

Conjecture 4.2. For $k>0$,

$$
(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^{j} b(n-j(3 j-1) / 2)-\frac{1+(-1)^{n}}{2} s\left(\frac{n}{2}\right)\right) \geqslant 0,
$$

with strict inequalities if $n \geqslant 2+k(3 k+1) / 2$.
In this context, relevant to Theorem[2.1 it would be very appealing to have combinatorial interpretations of

$$
\sum_{j=1}^{\lfloor n / 2\rfloor} s(j) M_{k}(n-j) .
$$

## 5 Concluding remarks

New properties for the difference between the number of parts in the odd partitions of $n$ and the number of parts in the distinct partitions of $n$ have been introduced in this paper.

Surprisingly, when $n$ is odd, Euler's partition function $p(n)$ and the difference $b(n)$ share two common linear homogeneous recurrence relations. As we can see in Corollary 2.2, the first recurrence relation involves the generalized pentagonal numbers:

$$
\begin{aligned}
p(n)=p & (n-1)+p(n-2)-p(n-5)-p(n-7) \\
& +p(n-12)+p(n-15)-p(n-22)-p(n-26)+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
b(n)=b & (n-1)+b(n-2)-b(n-5)-b(n-7) \\
& +b(n-12)+b(n-15)-b(n-22)-b(n-26)+\cdots
\end{aligned}
$$

The second recurrence relation combines the partition function $p(n)$ and the difference $b(n)$ with the triangular numbers, as follows:

$$
\begin{aligned}
p(n)=p & (n-1)+p(n-3)-p(n-6)-p(n-10) \\
& +p(n-15)+p(n-21)-p(n-28)-p(n-36)+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
b(n)=b & (n-1)+b(n-3)-b(n-6)-b(n-10) \\
& +b(n-15)+b(n-21)-b(n-28)-b(n-36)+\cdots
\end{aligned}
$$

These relations can be easily derived considering again the theta identity of Gauss (6) and the following two identities:

$$
\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{\left(-q ; q^{2}\right)_{\infty}} \sum_{n=0}^{\infty} p(n) q^{n}=\left(-q^{2} ; q^{2}\right)_{\infty}
$$

and

$$
\frac{\left(q^{2} ; q^{2}\right)_{\infty}}{\left(-q ; q^{2}\right)_{\infty}} \sum_{n=0}^{\infty} b(n) q^{n}=\left(q^{4} ; q^{4}\right)_{\infty} \sum_{n=1}^{\infty} \frac{q^{2 n}}{1-q^{2 n}}
$$

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## References

[1] Andrews, G.E., The Theory of Partitions, Addison-Wesley Publishing, 1976.
[2] Andrews, G.E., Euler's Partition Identity and Two Problems of George Beck, Math. Student 86 (2017) 115-119.
[3] Andrews, G.E., Merca, M., The truncated pentagonal number theorem, J. Comb. Theory A 119 (2012) 1639-1643.
[4] Andrews, G.E., Merca, M., Truncated theta series and a problem of Guo and Zeng, J. Comb. Theory A 154 (2018) 610-619.
[5] Glaisher, J.W.L., A theorem in partitions, Messenger of Math. 12 (1883) 158-170.
[6] Knopfmacher, A., Robbins, N., Identities for the total number of parts in partitions of integers, Util. Math. 67 (2005) 9-18.
[7] Merca, M., Fast algorithm for generating ascending compositions, J. Math. Model. Algorithms 11 (2012) 89-104.
[8] Merca, M., Binary diagrams for storing ascending compositions, Comput. J. 56(11) (2013) 1320-1327.
[9] Merca, M., Combinatorial interpretations of a recent convolution for the number of divisors of a positive integer, J. Number Theory 160 (2016) 60-75.
[10] Merca, M., The Lambert series factorization theorem, Ramanujan J 44 (2017) 417-435.
[11] Merca, M., Katriel, J., A general method for proving the non-trivial linear homogeneous partition inequalities, Ramanujan $J$ 51(2) (2020) 245-266.
[12] Sloane, N.J.A., The On-Line Encyclopedia of Integer Sequences, 2020. Published electronically at http://oeis.org.

