THE GINI INDEX OF AN INTEGER PARTITION

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Abstract. The Gini index is a number that attempts to measure how equitably a resource is distributed throughout a population, and is commonly used in economics as a measurement of inequality of wealth or income. The Gini index is often defined as the area between the Lorenz curve of a distribution and the line of equality, normalized to be between zero and one. In this fashion, we define a Gini index on the set of integer partitions and show that it is closely related to the second elementary symmetric polynomial, and the dominance order on partitions. We conclude with a generating function for the Gini index, and discuss how it can be used to find lower bounds on the width of the dominance lattice.

1. Introduction

In part one of his 1912 book "Variabilità e Mutabilità" (Variability and Mutability), the statistician Corrado Gini formulated a number of different summary statistics; among which was what is now known as the *Gini index* - a measure that quantifies how equitably a resource is distributed throughout a population. Referring to "the" Gini index can be misleading, as no fewer than thirteen formulations of his famous index appeared in the original publication [4]. Since then, many others have appeared in a variety of different fields.

The Gini index is usually defined using the *Lorenz Curve*. In "Methods of Measuring the Concentration of Wealth", Lorenz defined this curve in the following fashion. Consider a population of people amongst whom is distributed some fixed amount of wealth. Let $L(x)$ be the percentage of total wealth possessed by the poorest x percent of the population. The graph $y = L(x)$ is the Lorenz curve of the population [8].

It is clear from this definition that $L(0) = 0$ (I.E., none of the people have none of the wealth), $L(1) = 1$ (all of the people have all of the wealth), and L is non-decreasing. Since any population of people must have finite size n, the function $L(x)$ as defined above would appear to be a discrete function on the set $\{\frac{k}{n} : k \in \mathbb{Z} \text{ and } 0 \le k \le n\}$. However, in practice L is often made

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continuous on $[0,1]$ by linear interpolation [6].

If each person possesses the same amount of wealth, then the Lorenz curve for this distribution is the line $y = x$, which we call the "line of equality". The area between the line of equality and the Lorenz curve of a wealth distribution provides a measurement of the wealth inequality in that population.

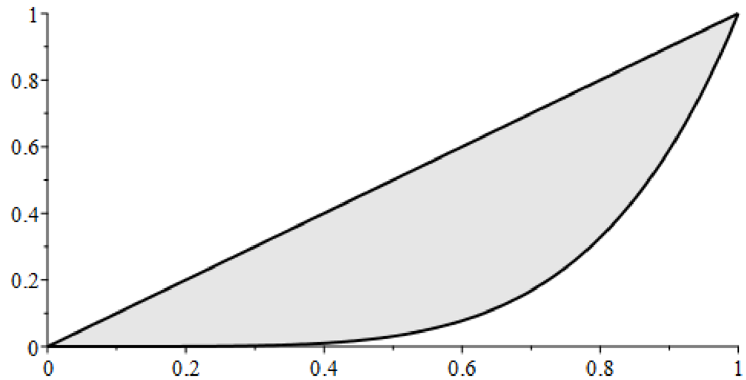


FIGURE 1. Area between the line of equality and a typical Lorenz curve

The maximum possible area of $\frac{1}{2}$ arises from the distribution in which one person controls all of the wealth ($L(1) = 1$, and $L(x) = 0$ for all $x \neq 1$). The Gini index of a distribution is then defined by calculating the area between the line of equality and Lorenz curve of the distribution, and normalizing this area to be between zero and one:

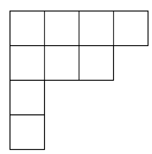
$$G = 2\int_0^1 \big(x - L(x)\big)dx.$$

In this paper we consider distributions of a discrete indivisible resource in a finite population, where the amount of that resource is equal to the number of people in the population. There is a natural one-to-one correspondence between the set of such distributions with n people, and the set of partitions of n. We will then define the Gini index of a partition in a similar fashion as above.

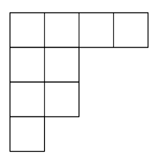
2. Preliminaries

2.1. **Partitions and Young Diagrams.** A *partition*, λ, of a positive integer n (denoted $\lambda \vdash n$) is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $\ell \leq n$ non-increasing non-negative integers such that $\sum_{i=1}^{\ell} \lambda_i = n$. The λ_i $(1 \leq i \leq \ell)$ are called the "parts of λ". To avoid repeating parts, it is sometimes useful to write a partition as $(\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_\ell^{a_\ell})$ to represent λ_i repeating a_i times. In this case we have that $\sum_{i=1}^{\ell} a_i\lambda_i = n$, and $\lambda_i \neq \lambda_j$ for all $i \neq j$. This notation will be used in the proof of Proposition 4.1. In order to make the length of λ (the number of parts) equal to n, one can "pad out" the partition by adding $n - \ell$ zeros to the end. For example, the partition $(4, 3, 1, 1)$ of 9 is equivalent to $(4, 3, 1, 1, 0, 0, 0, 0, 0)$. This technique will be used when defining the Lorenz curve of a partition.

A *Young diagram* is a finite collection of boxes arranged in left-justified rows, with a weakly decreasing number of boxes in each row [7]. Integer partitions are in one to one correspondence with Young diagrams in the following way: if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of n then the Young diagram of shape λ has λ_1 boxes in its first row, λ_2 boxes in its second row, etc. For example, if $\lambda = (4, 3, 1, 1)$, then the Young diagram of shape λ is


.

The *conjugate partition* $\widetilde{\lambda}$ of λ is the partition of n obtained by reflecting the Young diagram of λ across its main diagonal. As in the previous example, if $\lambda = (4, 3, 1, 1)$, then the Young Diagram of $\widetilde{\lambda}$ is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hence $\widetilde{\lambda} = (4, 2, 2, 1)$. Conjugation is clearly a bijection on the partitions of n.

The *dominance order* is a partial order on the set of partitions of n. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ are partitions of n, then $\mu \preceq \lambda$ if

$$\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i$$

for all $k \geq 1$. It is well known that conjugation of partitions is an antiautomorphism on the dominance lattice of partitions of n [3]. In other words, if $\mu \preceq \lambda$, then $\widetilde{\lambda} \preceq \widetilde{\mu}$. We will write $\mu \prec \lambda$ if $\mu \preceq \lambda$ and $\mu \neq \lambda$, and will denote by P_n the partially ordered set of partitions of n with respect to dominance.

For a fixed positive integer n, an *antichain* in P_n is a subset of P_n in which all partitions are pairwise incomparable. A *maximum antichain* is an antichain of maximal cardinality. The length of the maximum antichain is also known as the *width of the lattice*. The width of P_n (A076269 on OEIS) is currently an open problem.

2.2. **The Second Elementary Symmetric Polynomial.** The *second elementary symmetric polynomial*, e_2, in n variables, x_1, $x_2, \dots x_n$, is defined

$$e_2(x_1, x_2, \dots, x_n) = \sum_{1\le i<j\le n} x_i x_j.$$

For example, if $\lambda = (4, 3, 1, 1)$ is a partition of 9, then

$$e_2(\lambda) = \Big(4(3+1+1) + 3(1+1) + 1(1)\Big) = 27.$$

We will make use of the following result.

Lemma 2.1. *If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition of a positive integer n, then*

$$e_2(\lambda) = \binom{n+1}{2} - \sum_{i=1}^{\ell} \binom{\lambda_i + 1}{2}.$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition of n. Note that $\sum_{i=1}^{\ell} \lambda_i = n$. Then

$$\begin{aligned}
e_2(\lambda) &= \sum_{1\le i<j\le \ell} \lambda_i\lambda_j \\
&= \binom{n+1}{2} - \left(\sum_{1\le i<j\le\ell} (-\lambda_i\lambda_j) + \binom{n+1}{2} \right) \\
&= \binom{n+1}{2} - \frac{1}{2}\left(\sum_{1\le i<j\le\ell} (-2\lambda_i\lambda_j) + n(n+1) \right) \\
&= \binom{n+1}{2} - \frac{1}{2}\left(\sum_{1\le i<j\le\ell} (-2\lambda_i\lambda_j) + \left(\sum_{i=1}^{l} \lambda_i\right)\left(\sum_{j=1}^{l} \lambda_j + 1\right) \right) \\
&= \binom{n+1}{2} - \frac{1}{2}\left(\sum_{1\le i<j\le\ell} (-2\lambda_i\lambda_j) + \sum_{i=1}^{l} (\lambda_i^2 + \lambda_i) + \sum_{1\le i<j\le l} (2\lambda_i\lambda_j) \right) \\
&= \binom{n+1}{2} - \frac{1}{2}\left(\sum_{i=1}^{\ell} \lambda_i(\lambda_i+1) \right) \\
&= \binom{n+1}{2} - \left(\sum_{i=1}^{\ell} \binom{\lambda_i+1}{2} \right).
\end{aligned}$$

□

3. The Gini Index of an Integer Partition

As previously stated, we restrict our study of the Gini index to finite populations where the amount of a discrete indivisible resource is equal to the size of the population. In other words, there is one of said resource available for each person. The distributions of n of such a resource amongst n people is in one-to-one correspondence with the integer partitions of n.

For example, if there are 4 dollars in a population of 4 people, then the partition $(3, 1)$ of 4 would correspond to one person having 3 dollars, one person having 1 dollar, and the two remaining people having nothing. Whereas the partitions $(1, 1, 1, 1)$ and (4) correspond to completely equitable and completely inequitable distributions, respectively.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a positive integer n (padded with zeros on the tail, if necessary), the *Lorenz curve of* λ, $L_\lambda : [0, n] \longrightarrow [0, n]$, is defined as $L_\lambda(0) = 0$, and $L_\lambda(x) = \sum_{i=n-k+1}^{n} \lambda_i$, where $1 \leq k \leq n$ is the unique positive integer such that $x \in (k-1, k]$. In other words, for k from 1 to n, the Lorenz curve of λ on the interval $(k-1, k]$ is the sum of the last k parts of λ, $\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-k+1}$. Since total equality corresponds to the flat partition (1^n), using the above definition for the Lorenz curve of a partition, we find that the line of equality is given by $y = \lceil x \rceil$.

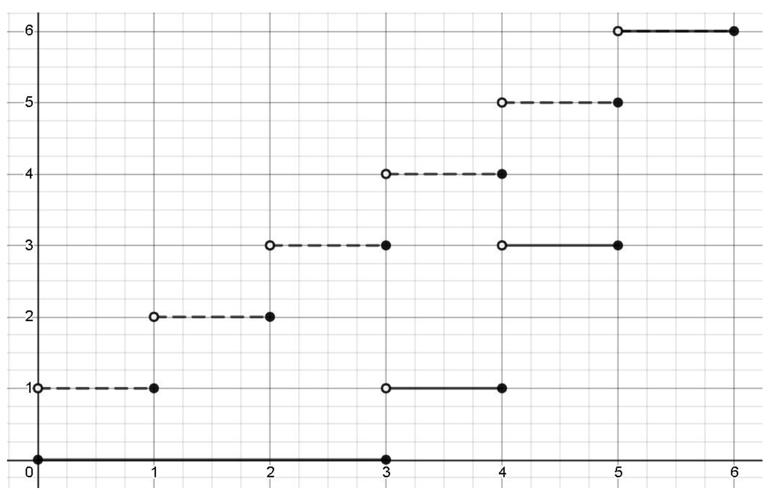

Figure 2. The line of equity (dashed) and the Lorenz curve of the partition (3,2,1) of 6 (solid).

The standard Gini index is calculated by finding the area between the line of equality and the Lorenz curve, and normalizing. In a similar fashion we define the Gini index, g, of a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of n by

$$g(\lambda) = \int_0^n (\lceil x \rceil - L_\lambda(x))\, dx$$
$$= \binom{n+1}{2} - \sum_{i=1}^{n} i\lambda_i.$$

The ordinary Gini index is normalized to be between zero and one. For a fixed value of n, the function g attains its maximum value of $\binom{n}{2}$ on the partition (n) of n. So the Gini index of a partition λ of n can be normalized by dividing $g(\lambda)$ by $\binom{n}{2}$. As long as n, and $g(\lambda)$ are both known, the normalized Gini index of λ can always be calculated in this fashion. With this in mind, we may disregard the normalization, and view g itself as the integer valued Gini index of a partition.

Our construction of g in conjunction with Lemma 2.1 yields some interesting results:

Proposition 3.1. *If λ is an integer partition, then $g(\lambda) = e_2(\widetilde{\lambda})$, where $\widetilde{\lambda}$ is the conjugate partition of λ.*

Proposition 3.2. *Let λ and μ be partitions of n. If $\mu \prec \lambda$ then $g(\mu) < g(\lambda)$ and $e_2(\lambda) < e_2(\mu)$.*

The normalized Gini index on $\mathbb{R}^n$ restricted to P_n is equal to $\frac{2g}{n^2}$. It is known that this function is strictly Schur convex, so Proposition 3.2 follows from this fact (see [2]). A complete proof of Proposition 3.2 that does not utilize these facts will be given in section 5.

The converse of Proposition 3.2 does not hold, in general. However, the contrapositive provides us with an easily calculated lower bound on the width of P_n. For if $\lambda, \mu \in P_n$ are distinct partitions such that $g(\lambda) = g(\mu)$, then the partitions λ and μ are incomparable. Such lower bounds can be calculated using the generating function of g given in the following section.

4. Generating Functions

It is often useful in Algebraic Combinatorics to record a discrete data set in the coefficients or powers of a formal power series. We call these power series "generating functions" for the data set. By "formal" we mean that the convergence of the series is immaterial. Any variables appearing in the series are taken as indeterminates rather than numbers. Alternatively, one may consider a formal power series as an ordinary power series that converges only at zero.

We define a generating function for the Gini index $g(\lambda)$ of an integer partition λ by

$$G(q,x)=\sum_{n=1}^{\infty}\sum_{\lambda\vdash n} q^{\left(\binom{n+1}{2}-g(\lambda)\right)}x^n.$$

Perhaps the most widely known example of a generating function is that of the integer partition function $P(n)$, which counts the number of partitions of the integer n. For example, $n=4$ has partitions

$$(1,1,1,1),\ (2,1,1),\ (2,2),\ (3,1),\ \text{and}\ (4),$$

so $P(4)=5$. It is well known (see [1]) that $P(n)$ has generating function

$$\prod_{n=1}^{\infty}\frac{1}{1-x^n}=\sum_{n=0}^{\infty}P(n)x^n,$$

Where $P(0)$ is defined to be 1.

In light of our previous results, we obtain a similar equality for $G(q,x)$.

Proposition 4.1.

$$\prod_{n=1}^{\infty}\frac{1}{1-q^{\binom{n+1}{2}}x^n}-1=\sum_{n=1}^{\infty}\sum_{\lambda\vdash n} q^{\left(\binom{n+1}{2}-g(\lambda)\right)}x^n$$

The proof is provided in the following section.

We can use $G(q,x)$ to find lower bounds on the width of P_n by calculating the sizes of the maximum level sets of g on P_n. In particular, the size of these level sets will be the largest coefficient on the powers of q that form the coefficient of x^n. Expanding $G(q,x)$ yields

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{\lambda\vdash n} q^{\left(\binom{n+1}{2}-g(\lambda)\right)}x^n &= qx+(q^2+q^3)x^2+(q^3+q^4+q^6)x^3\\ &+(q^4+q^5+q^6+q^7+q^{10})x^4\\ &+(q^5+q^6+q^7+q^8+q^9+q^{11}+q^{15})x^5\\ &+(\cdots+2q^9+\cdots)x^6+(\cdots+2q^{10}+\cdots)x^7\\ &+(\cdots+2q^{11}+\cdots)x^8+(\cdots+3q^{15}+\cdots)x^9\\ &+\cdots.\end{aligned}$$

So the size of the maximal level sets of g are 1, 1, 1, 1, 1, 2, 2, 2, and 3, on P_1 through P_9, respectively.

Denote by $b(n)$ the size of the maximal level set of g on P_n. Proposition 3.2 implies that $b(n)\leq a(n)$ for all positive integers n, where $a(n)$ is the size of the maximum antichain in P_n. In [5], Early proved that

$\Omega\left(n^{-5/2}e^{\pi\sqrt{2n/3}}\right) \leq a(n)$. It is currently unknown how $b(n)$ relates asymptotically to $a(n)$ or $n^{-5/2}e^{\pi\sqrt{2}}$. However, we conjecture that $b(n) \leq O\left(n^{-5/2}e^{\pi\sqrt{2n/3}}\right)$.

5. Proofs of Main Results

5.1. Proof of Proposition 3.1.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of a positive integer n, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We can calculate $g(\lambda)$ by filling the Young diagram of shape λ with numbers, where the entry in any box counts the number of boxes in that column that are weakly above it. For example, for the partition $(4, 3, 1, 1)$, we would have

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,.$$

Then the sums of the values in each row are

$$\begin{aligned} \sum \big(\text{Entries in row } 1\big) &= \lambda_1, \\ \sum \big(\text{Entries in row } 2\big) &= 2\lambda_2, \\ \sum \big(\text{Entries in row } 3\big) &= 3\lambda_3, \\ &\ \vdots \\ \sum \big(\text{Entries in row } \ell\big) &= \ell\lambda_\ell. \end{aligned}$$

Summing all values in the Young diagram of λ yields $\sum_{i=1}^{\ell} i\lambda_i$. By subtracting this from $\binom{n+1}{2}$ we have

$$\binom{n+1}{2} - \sum \big(\text{Entries in Young Diagram } i\big) = \binom{n+1}{2} - \sum_{i=1}^{\ell} i\lambda_i = g(\lambda).$$

We can calculate $e_2(\lambda)$ similarly by forming a Young diagram of shape λ where each each box's entry counts the number of boxes in the same row that are weakly to the left of its own. Again using $(4, 3, 1, 1)$ as an example, we would have

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\,.$$

In general, the i^{th} row of the diagram for λ will be of the form

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \multicolumn{1}{c|}{\ \ldots\ } & \lambda_i - 1 & \lambda_i \\ \hline \end{array}\,,$$

so the sum of the boxes in the i^{th} row will be $\binom{\lambda_i+1}{2}$. Summing all of the entries in the Young diagram of λ and subtracting this from $\binom{n+1}{2}$ yields

$$\binom{n+1}{2} - \sum_{i=1}^{\ell} (\text{Entries in row } i) = \binom{n+1}{2} - \sum_{i=1}^{\ell} \binom{\lambda_i+1}{2} = e_2(\lambda),$$

where the last equality is by Lemma 2.1. Since $g(\lambda)$ is calculated by counting boxes in the columns of the Young diagram of λ, and $e_2(\lambda)$ is calculated by counting boxes in the rows, it follows that $g(\lambda) = e_2(\widetilde{\lambda})$.

□

5.2. **Proof of Proposition 3.2.**

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be partitions of n (padded with zeros in their tails, if necessary). Suppose that λ covers μ, I.E. there is no partition ρ of n such that $\mu \prec \rho \prec \lambda$. Now λ covers μ if and only if

$$\begin{aligned}
\lambda_i &= \mu_i + 1,\\
\lambda_k &= \mu_k - 1, \mathit{and}\\
\lambda_j &= \mu_j,
\end{aligned}$$

for all $j \neq i$ or k, and either $k = i+1$ or $\mu_i = \mu_k$ [3]. In other words, λ covers μ if and only if all but two of the rows (row i and k, with $i < k$) in the Young diagrams of λ and μ are of the same length, and the diagram of λ can be obtained from that of μ by removing the last box from the k^{th} row, and appending it to end of the i^{th} row.

Begin with the Young diagram of μ and, as in the proof of Proposition 3.1, fill the diagram with numbers so that each box's entry counts the number of boxes weakly to the left of it.

$$\begin{array}{ccccccc}
\boxed{1} & \boxed{2} & \cdots & \boxed{\mu_1-3} & \boxed{\mu_1-2} & \boxed{\mu_1-1} & \boxed{\mu_1}\\
\vdots & & & & & & \\
\boxed{1} & \boxed{2} & \cdots & \boxed{\mu_i-1} & \boxed{\mu_i} & & \\
\vdots & & & & & & \\
\boxed{1} & \boxed{2} & \cdots & \boxed{\mu_k-1} & \boxed{\mu_k} & & \\
\vdots & & & & & & \\
\boxed{1} & \boxed{2} & \cdots & \boxed{\mu_n} & & &
\end{array}$$

From row k we remove the box containing μ_k and append it to the end of row i to obtain a diagram of shape λ.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 2 & \cdots & \mu_1-3 & \mu_1-2 & \mu_1-1 & \mu_1 \\
\hline
\vdots & & & & & & \\
\hline
1 & 2 & \cdots & \mu_i-1 & \mu_i & \mu_k & \\
\hline
\vdots & & & & & & \\
\hline
1 & 2 & \cdots & \mu_k-1 & & & \\
\hline
\vdots & & & & & & \\
\hline
1 & 2 & \cdots & \mu_n & & & \\
\hline
\end{array}
$$

But $i < k$, hence $\mu_k \leq \mu_i$, and the corresponding filling of the diagram for λ would have the last cell in row i containing $\mu_i + 1$, which is strictly greater than μ_k. Thus the sum of all numbers in the diagram for λ is

$$\sum_{\substack{j=1 \\ j\neq i,k}}^{n} \binom{\mu_j+1}{2} + \binom{\mu_i+2}{2} + \binom{\mu_k}{2},$$

and the sum of all numbers in the diagram for μ is

$$\sum_{j=1}^{n} \binom{\mu_j+1}{2}.$$

By Lemma 2.1, we have

$$
\begin{aligned}
e_2(\mu) - e_2(\lambda) &= \binom{n+1}{2} - \sum_{j=1}^{n} \binom{\mu_j+1}{2} - \\
&\quad \left(\binom{n+1}{2} - \left(\sum_{\substack{j=1 \\ j\neq i,k}}^{n} \binom{\mu_j+1}{2} + \binom{\mu_i+2}{2} + \binom{\mu_k}{2} \right) \right) \\
&= \binom{\mu_i+2}{2} + \binom{\mu_k}{2} - \binom{\mu_i+1}{2}\binom{\mu_k+1}{2} \\
&= \frac{(\mu_i+1)(\mu_i+2-\mu_i) + (\mu_k)(\mu_k-1-\mu_k)}{2} \\
&= \frac{2\mu_i+2-\mu_k}{2} \\
&> 0.
\end{aligned}
$$

So $e_2(\mu) > e_2(\lambda)$. Moreover $\mu \prec \lambda$ if and only if $\widetilde{\lambda} \prec \widetilde{\mu}$. Hence $\widetilde{\mu}$ covers $\widetilde{\lambda}$, and by Proposition 3.1, $g(\mu) < g(\lambda)$. The general case follows by transitivity. $\square$

5.3. **Proof of Proposition 4.1.**

Proof. We will show that the power series about $x = 0$ of the product

$$\prod_{n=1}^{\infty} \frac{1}{1 - q^{\binom{n+1}{2}} x^n} - 1 \tag{1}$$

has as its general coefficient

$$\sum_{\lambda \vdash n} q^{\left(\binom{n+1}{2} - g(\lambda)\right)}.$$

Considering each factor of the product as a geometric series, we have

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{1}{1 - q^{\binom{n+1}{2}} x^n} =& \frac{1}{\left(1 - q^{\binom{2}{2}} x\right)} \cdot \frac{1}{\left(1 - q^{\binom{3}{2}} x^2\right)} \cdot \frac{1}{\left(1 - q^{\binom{4}{2}} x^3\right)} \cdot \frac{1}{\left(1 - q^{\binom{5}{2}} x^4\right)} \cdot \dots \\
=& \left(1 + q^{\binom{2}{2}} x + q^{2\binom{2}{2}} x^2 + q^{3\binom{2}{2}} x^3 + q^{4\binom{2}{2}} x^4 + \dots\right) \cdot \\
& \left(1 + q^{\binom{3}{2}} x^2 + q^{2\binom{3}{2}} x^4 + q^{3\binom{3}{2}} x^6 + q^{4\binom{3}{2}} x^8 + \dots\right) \cdot \\
& \left(1 + q^{\binom{4}{2}} x^3 + q^{2\binom{4}{2}} x^6 + q^{3\binom{4}{2}} x^9 + q^{4\binom{4}{2}} x^{12} + \dots\right) \cdot \\
& \left(1 + q^{\binom{5}{2}} x^4 + q^{2\binom{5}{2}} x^8 + q^{3\binom{5}{2}} x^{12} + q^{4\binom{5}{2}} x^{16} + \dots\right) \cdot \dots .
\end{aligned}$$

If we distribute and simplify, for example, the coefficient of x^4, we see that it is

$$q^{4\binom{2}{2}} + q^{2\binom{3}{2}} + q^{\binom{2}{2} + \binom{4}{2}} + q^{2\binom{2}{2} + \binom{3}{2}} + q^{\binom{5}{2}},$$

where each of the terms correspond to the partitions

$$(1, 1, 1, 1),\ (2, 2),\ (3, 1),\ (2, 1, 1),\ \text{and}\ (4),$$

respectively, by

$$(\lambda_1, \lambda_2, \dots, \lambda_l) \mapsto q^{\left(\sum_{i=1}^{l} \binom{\lambda_i + 1}{2}\right)}.$$

This is true, in general, for the coefficient of x^n, for all positive integers n. To see this, consider the coefficient on x^n in the power series expansion of (1). If we set $q = 1$ in the product of (1), we obtain the generating function of $P(n)$ (the number of partitions of n). Hence there are $P(n)$ different ways to obtain a power of x^n. So the x^n term in (1) will be of the form

$$\sum_{j=1}^{P(n)} \prod_{i=1}^{m_j} q^{\left(a_{j,i} \binom{\lambda_{j,i} + 1}{2}\right)} x^{(a_{j,i} \lambda_{j,i})},$$

where $a_{j,i}, \lambda_{j,i} > 0$, and $\sum_{i=1}^{m_j} a_{j,i}\lambda_{j,i} = n$. Thus the coefficient on x^n will be

$$\sum_{j=1}^{P(n)} q^{\left(\sum_{i=1}^{m_j} a_{j,i}\binom{\lambda_{j,i}+1}{2}\right)}. \tag{2}$$

Since each $\binom{\lambda_{j,i}+1}{2}$ in (2) comes from a different term of the product in (1), we have that $\lambda_{j,i} \neq \lambda_{j,k}$ whenever $i \neq k$. Therefore, by reordering, we may choose the power $a_{j,i}\binom{\lambda_{j,i}+1}{2}$ on q so that $\lambda_{j,i} > \lambda_{j,i+1} > 0$, for $1 \leq i < m_j$. It follows that $(\lambda_{j,1}^{a_{j,1}}, \lambda_{j,2}^{a_{j,2}}, \ldots, \lambda_{j,m_j}^{a_{j,m_j}})$ is a partition of n, where $\lambda_{j,i}$ is repeated $a_{j,i}$ times.

Again, using the generating function for $P(n)$, the ways of writing x^n as a product $\prod_i x^{(a_{j,i}\lambda_{j,i})}$ (where $a_{j,i}\lambda_{j,i} > 0$) is in bijection with the partitions of n. Since each of the sums $\sum_{i=1}^{m_j} a_{j,i}\lambda_{j,i} = n$ have distinct summands for all $1 \leq j \leq P(n)$, it follows that the sums $\sum_{i=1}^{m_j} a_{j,i}\binom{\lambda_{j,i}+1}{2}$ are all distinct for different values of j. In other words, every partition $(\lambda_{j,1}^{a_{j,1}}, \ldots, \lambda_{j,m_j}^{a_{j,m_j}})$ of n appears as a power in (2). Hence (2) is equal to

$$\sum_{\lambda \vdash n} q^{\left(\sum_{i=1}^{n}\binom{\lambda_i+1}{2}\right)}.$$

By Lemma 2.1, $e_2(\lambda) = \binom{n+1}{2} - \sum_{i=1}^{n}\binom{\lambda_i+1}{2}$, thus

$$\sum_{\lambda \vdash n} q^{\left(\sum_{i=1}^{n}\binom{\lambda_i+1}{2}\right)} = \sum_{\lambda \vdash n} q^{\left(\sum_{i=1}^{n}\binom{n+1}{2} - e_2(\lambda)\right)}.$$

Finally, by Proposition 3.1, we have that the general coefficient on x^n in the power series expansion of (1) is

$$\sum_{\lambda \vdash n} q^{\left(\sum_{i=1}^{n}\binom{n+1}{2} - g(\lambda)\right)}.$$

$\square$

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