

Pi from the sky – A null test of general relativity from a population of gravitational wave observations

Carl-Johan Haster*

LIGO Laboratory, Massachusetts Institute of Technology, 185 Albany St, Cambridge, MA 02139, USA and
Department of Physics and Kavli Institute for Astrophysics and Space Research,
Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139, USA

Our understanding of observed Gravitational Waves (GWs) comes from matching data to known signal models describing General Relativity (GR). These models, expressed in the post-Newtonian formalism, contain the mathematical constant π . Allowing π to vary thus enables a strong, universal and generalisable null test of GR. From a population of 22 GW observations, we make an astrophysical measurement of $\pi = 3.115_{-0.088}^{+0.048}$, and prefer GR as the correct theory of gravity with a Bayes factor of 321. We find the variable π test robust against simulated beyond-GR effects.

INTRODUCTION

Observations of gravitational waves (GWs) from compact binary coalescences (CBCs) by the LIGO [1] and Virgo [2] detectors have brought tests of General Relativity (GR) in the strong-field regime to hitherto unachievable levels [3–6]. This is fundamentally dependent on the detailed knowledge about the structure of the GWs emitted from a binary of compact objects (COs) (black holes (BHs) or neutron stars (NSs)) stemming from decades of analytical [7–16] and numerical [17–20] studies of GWs from binary systems of COs.

So far, the majority of theories of gravity beyond GR are unable to construct predictions for GWs emitted by coalescing binaries with generic COs (but see [21–34] for the status of current efforts), hence tests of GR are generically formulated as consistency tests only, where the primary approach is to introduce *ad-hoc* modifications of the GR waveforms. This can be constructed to test different regions and functional dependencies of the overall waveform [4, 35–39], such as deviations from the analytical coefficients of the post-Newtonian (PN) expansion [7–10], which has been successful when investigating constraints on each included PN-order separately [3–6, 40, 41]. These constraints can later be mapped onto bounds on specific alternate theories of gravity [42–45], something which in turn highlights a potential flaw of this approach. Since the PN-coefficients themselves depend on the specific properties of the source’s COs, like their masses, it would be reasonable to also assume any deviations from the GR-predicted values to also be source dependent. If a hypothetical theory modifies BH-spin behaviour, but not any mass parameters, a general PN-deviation would be different for two binaries with the same BH spin magnitudes but different mass ratios. This is not accounted for in most current analyses [3–5, 40, 41] (but see [6] for a more general approach) and could lead to misinterpreted inference if any deviation from GR was observed [46, 47]. In addition, the strength of this class of tests is reduced when more than one PN-term is simultaneously allowed to vary, where the addition of a

large number of unconstraining degrees of freedom generates an overall null gain in information about any of the included terms (cf. Fig 7 of [3]).

In this letter we implement a null test of GR, probing the validity of the current knowledge about GR, and specifically its nonlinear behaviour originating from GW tail effects [48–54], with the mathematical constant π treated as a variable. π can here be considered as a universal parameter across all GW observations of CBCs [46], and simultaneously tests 4 (out of the included 8) PN-orders. This enables an unprecedentedly powerful test, as it is both theory-agnostic and conceptually generalisable to probe a population of GWs through a quantity that is formally consistent across independent observations (while also being comparatively inexpensive computationally). Throughout this letter we denote the true value of π as $\pi^T = 3.141592653\dots$ [55], a number which has been independently evaluated through several methods [56–61]. We assume $G = c = 1$.

METHOD

The GW signal from a CBC can be generally expressed in the form

$$\tilde{h}(\theta, f) = A(\theta, f)e^{i\Psi(\theta, f)}, \quad (1)$$

where $\tilde{h}(f)$ is the emitted GW strain in the frequency domain, with amplitude $A(\theta, f)$ and phase $\Psi(\theta, f)$ being functions of the source parameters θ , e.g. CO masses $m_{1,2}$, spin vectors $\vec{S}_{1,2}$ and tidal deformabilities $\Lambda_{1,2}$ (we fix $\Lambda_{1,2} = 0$ for BHs). When the two COs are sufficiently separated, for an orbital velocity $u \ll 1$ with $u = \pi^T M f$ and M being the binary’s total mass, Eq. (1) can be described accurately through a post-Newtonian expansion in u . Under the stationary phase approximation (SPA) [62–67], $\Psi(\theta, f)$ is given for the TAYLORF2 (TF2)

model [62, 67–70] as

$$\begin{aligned} \Psi_{\text{TF2}} = & 2\pi^{\text{T}}ft_c - \varphi_c - \pi^{\text{T}}/4 \\ & + \frac{3}{128\eta}u^{-5/3}\sum_{i=0}^7(\varphi_i + \log(u)\varphi_i^l)u^{i/3}, \end{aligned} \quad (2)$$

where t_c and φ_c are the overall time and phase defined at coalescence and $\eta = m_1m_2/(m_1 + m_2)^2$ is the symmetric mass ratio. The expansion coefficients φ_i and φ_i^l are then given as functions of θ [10, 62, 67, 71–75]. In this formalism, multiples of π appear in $\varphi_3, \varphi_5, \varphi_5^l, \varphi_6$ and φ_7 . To also capture the post-inspiral PN description of a CBC signal, we employ the IMRPHENOMPv2 model, where the analytical inspiral description from TF2 is smoothly extended with a phenomenological description of a binary black hole (BBH) merger-ringdown section [75, 76] together with an effective-precession treatment [77]. For the binary neutron star (BNS) events, this is further extended with a description of NS matter effects [78, 79]. Neither extension depends on the variable π , as the extensions are phenomenological rather than analytical in their nature.

We note that π is included in the orbital velocity u , originating in a conversion from angular to linear orbital frequencies. From Eq. (2), u is already strongly constrained at the leading-order phase term and can be taken as known to sufficiently high precision in GR. Since we are here interested in specifically probing the post-Newtonian formalism, expressed through the φ_i and φ_i^l coefficients, we fix $\pi = \pi^{\text{T}}$ in u throughout. Similarly, the π in $2\pi^{\text{T}}ft_c$, originating from a Fourier transform of the time-domain GW signal, and the factor $\pi^{\text{T}}/4$ appears in Eq. (2) out of convention. As both describe an overall phase shift, perfectly degenerate with the variables t_c and φ_c respectively, we fix those two $\pi = \pi^{\text{T}}$ in this analysis.

It is important to note that the appearance of π in the PN-coefficients follows purely from definitions of mathematics itself [54, 80], e.g. through the use of known identities to evaluate integrals of a specific form (cf. Eq. 5.4 of [54]), and does not depend on the specific assumptions of GR as a theory of gravity. This formally justifies treating π in all φ_i and φ_i^l coefficients as fundamentally the same quantity, and also treating it as a universal parameter across multiple independent CBC observations [46].

We also note that the PN-orders where π appears are primarily describing so called GW tail effects [48–54], where the outgoing GWs backscatter off the (approximately) static spacetime of the CBC source. Tail effects are an inherently nonlinear behaviour present in GR, hence the use of a variable π can directly probe the validity of the nonlinear terms expressed through the PN-representation of GR itself.

While the analysis with a variable π formally is an extension of GR, we do not argue that results presented here are direct suggestions for alternative theories of gravity. Instead, we interpret this study primarily as a

strong null test, validating the current understanding of GR through a multi-order probe of the PN-formalism.

Finally, we acknowledge that the GW detectors, as well as the data they record, are constructed and calibrated for $\pi = \pi^{\text{T}}$ only.

Bayesian methods

We explore the parameter space θ defined by the CBC models using Bayes’ theorem to infer the posterior probability density function (PDF):

$$p(\theta|d, H) = \frac{p(\theta|H)p(d|\theta, H)}{p(d|H)}, \quad (3)$$

where $p(\theta|H)$ is the prior PDF of θ given the model H , $p(d|\theta, H)$ is the likelihood of observing the data d assuming θ and $p(d|H) = \int p(\theta|H)p(d|\theta, H)d\theta$ is the evidence for H . We perform Bayesian inference using the LALINFERENCE package [81–83], following the analysis configuration from [84] which includes a fixed noise power spectral density (PSD) (defined for the analysed data d and generated as a median PSD using BAYESWAVE [85–90]) and marginalisation over uncertainties in the calibration of d [88–92]. All GW events are analysed using publicly available data [93–98].

We assume prior choices consistent with those used in [84, 99–101]. For the two BNSs, we perform only analyses with $|\bar{S}_{1,2}| \leq 0.05$, and parametrize the NS tidal deformability following the equation of state independent relations from [102]. We assume a prior distribution for π that is uniform between $-20 \leq \pi \leq 20$.

As π can be considered a formally universal parameter, as defined by [46], it is trivial to evaluate joint constraints on π from a set of N individual observations by multiplying the 1D likelihood distributions (marginalised over all other parameters), dividing by one instance of the common prior and normalising the resulting posterior PDF.

Finally, as GR is nested inside the model which allows for a variable π it is possible to compute a Bayes factor (BF) in favour of GR, more directly where $\pi = \pi^{\text{T}}$, using the Savage-Dickey density ratio [103, 104] as

$$\text{BF} = \frac{p(\pi = \pi^{\text{T}}|d, H)}{p(\pi = \pi^{\text{T}}|H)}, \quad (4)$$

i.e. the ratio of the posterior and prior PDFs evaluated at $\pi = \pi^{\text{T}}$.

ASTROPHYSICAL MEASUREMENT OF π

The LIGO/Virgo Collaboration (LVC) has so far, from its first observation run (O1), second observation run (O2) and third observation run (O3) [105], confirmed 13 GW observations, 2 BNSs and 11 BBHs [84, 100,

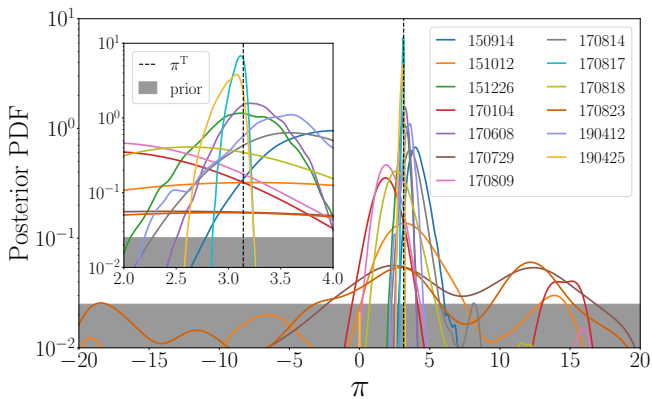


FIG. 1. Individual posterior PDFs of π for the thirteen LVC GW observations [84, 100, 101]. The dashed line indicates π^T , the true value of π . The shaded gray region indicates the prior PDF.

101]. Whereas other studies restrict themselves to high-significance events only [5, 6], primarily due to computational restrictions, the analysis presented here is easily extendable to and informed by all available GW observations. The individual-event posterior PDFs for π are shown in Fig. 1, visualised through kernel density estimators (KDEs). All GW events support the region near π^T , with the strongest constraints coming from the two BNSs and the lowest-mass BBHs (GW151226 and GW170806). This agrees with prior expectations as lower-mass CBC signals are dominated by the binary inspiral, described by the PN-series, in turn constrained by this analysis. Apart from a general broadening of the recovered posterior PDFs in other source parameters, consistent with the addition of a new degree of freedom, we note no general degeneracies between π and other parameters. This is especially noticeable as φ_3 contains the leading-order terms for both π and the effects from CO-spins. As both π and spin-parameters however appear jointly at higher PN-orders, with different interdependences than in φ_3 , the potentially strong degeneracy is thus broken in this analysis.

The chronological progression of the joint posterior PDF of π , from the population of CBCs reported by the LVC is shown in Fig. 2, again highlighting the significant contribution of the four lowest-mass events. Together, these 13 events give a maximum a posteriori value, with associated 90% credible interval (CI), of $\pi = 3.113^{+0.049}_{-0.091}$. For this set of events, the Bayes factor in favour of GR being an accurate description of strong-field gravity is 301.

In addition to the eleven BBHs reported by the LVC in [84], independent analyses (Zackay *et al.* [106], Venumadhav *et al.* [107], Nitz *et al.* [108], Zackay *et al.* [109] hereafter collectively labelled ZVNZ) have claimed an additional nine BBH observations, whose posterior PDFs of π are shown in Fig. 3. It should be noted that

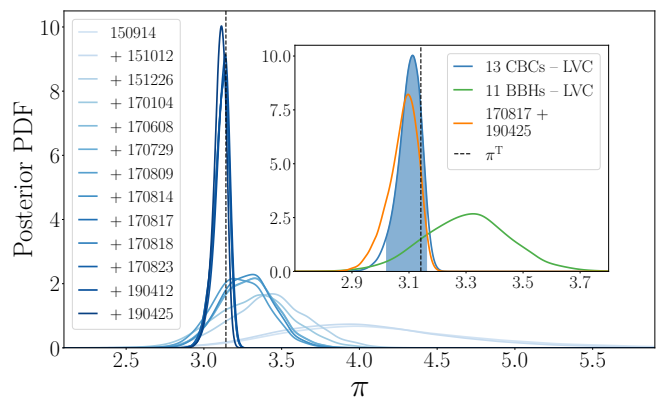


FIG. 2. Joint posterior PDFs of π for the thirteen LVC GW observations [84, 100, 101]. The observations are added chronologically, corresponding to the light to dark blue transition. The dashed line indicates π^T , the true value of π . *Inset*: The joint posterior PDFs for the eleven BBHs (green), the two BNSs (orange) and all 13 LVC CBCs. The shaded blue region in the inset corresponds to the 90% CI.

out of all 22 included CBCs, only GW151216 [106] and GW170304 [107] recover π disfavouring π^T with single-event BFs in support of GR of 1/2 and 1/5 respectively. These two events have previously been identified as especially sensitive to overall prior and analysis choices [110, 111]. The population of 22 CBC observations gives a measurement of $\pi = 3.115^{+0.048}_{-0.088}$, and a BF in favour of GR of 321. This constitutes the strongest constraints on the validity of the positive PN-order coefficients to date [112], with a fractional width of the joint π 90% CI < 0.04 , more than a factor of 2 improvement over previous single-PN-order variability results [4, 5, 113]. We also note a more significant improvement when comparing against the constraints on φ_3 , the lowest PN-order directly probed by this analysis. This can be attributed to a combination of the inherent multi-order nature of the variable π analysis and the inclusion of a larger population of CBC observations than previous studies, thus together enabling a stronger constraint on the validity of the tested theory.

BBH-LIKE NOISE TRANSIENTS – π ESTIMATION

In order to test the reliability of this analysis against spurious false-positives we analyse a set of background triggers, where sections of real data from LIGO [93, 98] have been offset in time by longer than the light-travel time between sites. This time-shifted data is thus guaranteed to not contain any real coincident GW events, and primarily represent noise-transients from the LIGO instruments. We select 10 high-significance BBH background triggers produced by the PyCBC search

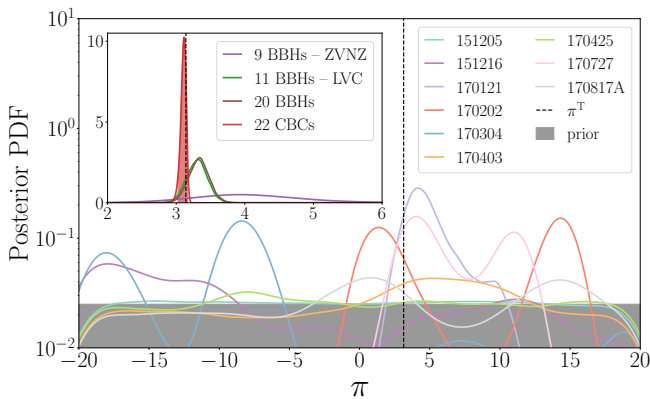


FIG. 3. Individual posterior PDFs of π for the nine ZVNZ BBHs observations [106–109]. The dashed line indicates π^T , the true value of π . The shaded gray region indicates the prior PDF. *Inset*: The joint posterior PDFs for the nine ZVNZ BBHs (purple), the eleven LVC BBHs (green, cf. Fig. 2), all 20 BBHs (brown) and all 22 CBCs (red). The shaded red region in the inset corresponds to the 90% CI.

pipeline [114–117] and used by [118]. Following the same procedure as above, we recover their individual and joint posterior PDFs on π , shown in Fig. 4. When the analysed signals do not correspond to GR, as is the case for these noise-transients, there is significant scatter of the recovered posterior PDFs in π . The apparent strong constraint from the joint analysis of the 10 triggers stems primarily from the narrow region hosting the black curve in Fig. 4 being the only range where $p(\pi|d, H) \neq 0$ for all triggers, and that the joint posterior PDF has a unit area. The corresponding BF $\lesssim 1/10^{300}$ for $\pi = \pi^T$, further highlighting the variable π analysis correctly identifying the non-GR features present in these noise triggers. We conclude that the analysis appears stable against generating false-positive results when exposed to even a small population of 10 known non-GR (background noise) signals.

PRESENCE OF MASSIVE GRAVITON

To show that this analysis can reveal the presence of realistic beyond-GR effects, we simulate three BBH systems with parameters consistent with the three BBHs detected by the LVC during O1 [40, 84, 88]. We modify the φ_2 PN coefficients of the simulated signals to mimic a massive graviton with a Compton wavelength λ_G [46, 119]. We choose λ_G to be in the range between 2.48×10^{13} km, consistent with the current GW observational lower bound from [5], and 10^{12} km, a value already ruled-out observationally. We also simulate “pure” GR, with $\lambda_G = \infty$. It should be noted that λ_G enters at a PN-order where π is not present.

In Fig. 5 we show the joint posterior PDFs on π from the three BBH signals for each value of λ_G . Given that

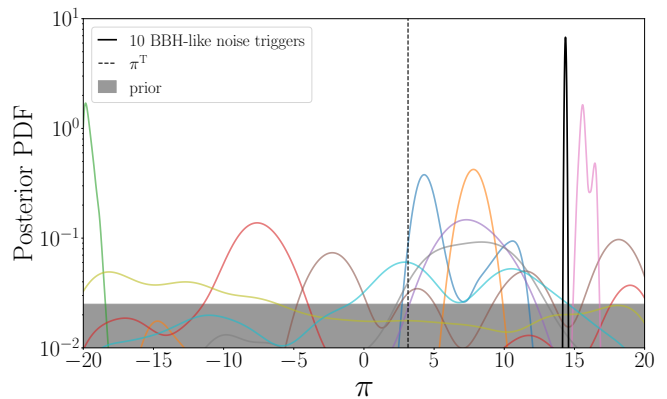


FIG. 4. Individual posterior PDFs (coloured) of π for ten BBH-like noise triggers from a background distribution. The dashed line indicates π^T , the true value of π . The shaded gray region indicates the prior PDF. The black line corresponds to the joint posterior PDFs for the background triggers.

$\lambda_G \geq 2.48 \times 10^{13}$ km is not ruled out by current GW observations [5], it is not surprising that an analysis using this bound yields a posterior PDF in agreement with GR. For $\lambda_G = 10^{12}$ km, more than an order of magnitude below the current lower bound, the recovered π posterior PDF is biased away from π^T but only marginally informative over the assumed prior. This indicates that a strong beyond-GR effect, acting partially orthogonal to the changes to the signal from a varying π , can be sufficient to saturate the constraining power of this test. Namely, if no allowed value of π is able to sufficiently “correct” for the beyond-GR modification present in the signal, the variable π degree of freedom becomes uninformative. It is instead the case in between these extremes that is the most illustrative, where a presence of a marginal beyond-GR effect induces a clear bias in the recovered π and a BF $\sim 1/10^{15}$ for $\pi = \pi^T$. Hence, a detection of $\pi \neq \pi^T$ in a population of real observations can be interpreted as first indication of the presence of beyond-GR behaviour, with the variable π test being especially powerful from its generalisable and multi-PN-order nature. The identification of $\pi \neq \pi^T$ does itself not guide what beyond-GR effect is present. Such questions can only be answered by performing theory-specific model comparison analyses [4, 5, 43, 44] over the population of observations for which $\pi \neq \pi^T$.

DISCUSSION

In the post-Newtonian formalism of General Relativity, the mathematical constant π presents a powerful null test of our currently preferred theory of gravity. With π simultaneously probing four PN-orders, fundamentally describing the same conceptual quantity in all instances, doing so in a theory-agnostic way that is also

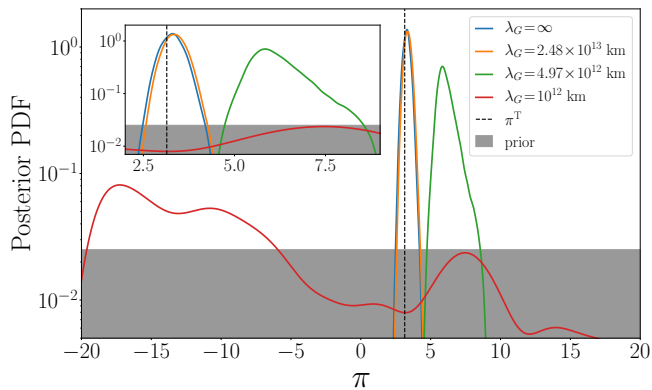


FIG. 5. Joint distributions of π shown for four different instances of λ_G , for GW signals consistent with the three O1 events

generalisable and universal across independent GW observations, it provides an unmatched capability for validating our understanding of GR. Using the current set of 22 CBC observations in data from LIGO and Virgo, identified by both the LVC and independent researchers [84, 100, 101, 106–109], we achieve an astrophysical measurement of $\pi = 3.115^{+0.048}_{-0.088}$, consistent with the accepted π^T value. This is the most stringent constraint on the positive-order PN-series to date [112], and the first viable multi-PN-order constraint from GW observations. The analysis also allows, through the construction of a Bayes factor, direct validation of GR with $\text{BF}=321$ in support of it as the currently favoured theory of gravity. We have shown the analysis to be robust when exposed to non-signal, but high-significance, GW triggers as well as being able to indicate the presence of beyond-GR effects in the case where such signals were to exist. The method presented in this letter is easily extended to future GW observations, of both CBCs and other modelled sources, such as the quasi-monochromatic GWs emitted by spinning NSs [120–122], and capable of accommodating observations from across the GW spectrum [23, 123–134].

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* haster@mit.edu

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