CONICAL INTRINSIC VOLUMES OF WEYL CHAMBERS

THOMAS GODLAND AND ZAKHAR KABLUCHKO

ABSTRACT. We give a new, direct proof of the formulas for the conical intrinsic volumes of the Weyl chambers of types A_{n-1}, B_n and D_n. These formulas express the conical intrinsic volumes in terms of the Stirling numbers of the first kind and their B- and D-analogues. The proof involves an explicit determination of the internal and external angles of the faces of the Weyl chambers.

1. INTRODUCTION

A polyhedral cone in the Euclidean space $\mathbb{R}^n$ is a set of solutions to a finite system of linear homogeneous inequalities. That is, a polyhedral cone $C \subseteq \mathbb{R}^n$ can be represented as

$$C = \{\beta \in \mathbb{R}^n : \langle \beta, x_i \rangle \leq 0 \text{ for all } i = 1, \ldots, m\}$$

for some finite collection of vectors $x_1, \ldots, x_m \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. The fundamental Weyl chambers of types A_{n-1}, B_n and D_n are the polyhedral cones defined by

$$\begin{aligned}
\mathcal{C}(A_{n-1}) &:= \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n\}, \\
\mathcal{C}(B_n) &:= \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq 0\}, \\
\mathcal{C}(D_n) &:= \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{n-1} \geq |\beta_n|\},
\end{aligned}$$

where $\beta = (\beta_1, \ldots, \beta_n)$ is the coordinate representation of $\beta \in \mathbb{R}^n$.

In this paper, we shall be interested in the conical intrinsic volumes of the Weyl chambers. The conical intrinsic volumes of cones are analogues of the classical Euclidean intrinsic volumes of convex bodies in the setting of conical or spherical geometry. A recent increase of interest to conical intrinsic volumes is due to their relevance in convex optimization [4, 1, 2]. Let us briefly define the conical intrinsic volumes, referring to Section 2.1 for more details and to [15, Section 6.5] and [4, 3] for an extensive account of the theory. Given some point $x \in \mathbb{R}^n$, the Euclidean projection $\Pi_C(x)$ of x to a polyhedral cone $C \subseteq \mathbb{R}^n$ is the unique vector $y \in C$ minimizing the Euclidean distance $\|x - y\|$. For $k \in \{0, \ldots, n\}$, the k-th conical intrinsic volume $\upsilon_k(C)$ of C is defined as the probability that the Euclidean projection $\Pi_C(g)$ of an n-dimensional standard Gaussian random vector g on C lies in the relative interior of a k-dimensional face of C.

The conical intrinsic volumes of the Weyl chambers are given by the following theorem.

Theorem 1.1. *For all $n \in \{1, 2, \ldots\}$ and all $k \in \{0, \ldots, n\}$ we have*

$$\upsilon_k(\mathcal{C}(A_{n-1})) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!}, \quad \upsilon_k(\mathcal{C}(B_n)) = \frac{B(n,k)}{2^n n!} \quad \text{and} \quad \upsilon_k(\mathcal{C}(D_n)) = \frac{D(n,k)}{2^{n-1} n!}, \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 52A22, 60D05. Secondary: 52A55, 11B73, 51F15, 52A39.

Key words and phrases. Stochastic geometry, polyhedral cones, Weyl chambers, conical intrinsic volumes, external and internal angles, Stirling numbers, random walks and bridges.

where the $\left[{n \atop k}\right]$*'s denote the Stirling numbers of the first kind, the* $B(n,k)$*'s their* B*-analogues and the* $D(n,k)$*'s their* D*-analogues defined as the coefficients of the following polynomials:*

$$t(t+1)\cdot \ldots \cdot (t+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k,$$

$$(t+1)(t+3)\cdot \ldots \cdot (t+2n-1) = \sum_{k=0}^{n} B(n,k)t^k,$$

$$(t+1)(t+3)\cdot \ldots \cdot (t+2n-3)(t+n-1) = \sum_{k=0}^{n} D(n,k)t^k.$$

The theorem is known, see [11, Theorem 4.2]. In [11], the formulas (1.1) were proven by relating the conical intrinsic volumes of the Weyl chambers to the coefficients of the characteristic polynomial of the hyperplane arrangements that generate the Weyl chambers. A general formula relating conical intrinsic volumes of chambers to the characteristic polynomial of a hyperplane arrangement was first conjectured by Drton and Klivans [7] and proven by Klivans and Swartz [12]; see also [11, Theorem 4.1] and [14]. Given this interpretation, it remains to compute the characteristic polynomials of the hyperplane arrangements generating the Weyl chambers. This can be elegantly done by the finite field method; see Section 5.1 in [18] or Section 1.7.4 in [6]. Overall, the method just described relies on the results from the theory of hyperplane arrangements. It is therefore natural to ask whether there is a more direct proof of Theorem 1.1.

In the present paper, we want to prove Theorem 1.1 by computing the internal and the external angles of the faces of the Weyl chambers. The determination of the external angles is closely related to a result of Gao and Vitale [9] who computed the classical intrinsic volumes of the Schläfli orthoscheme which is defined as the simplex in $\mathbb{R}^n$ with the vertices 0 and $e_1+\ldots+e_i$, $1\leq i\leq n$, where $e_1,\ldots,e_n$ denotes the standard orthonormal basis in $\mathbb{R}^n$, Equivalently, this simplex is given by

$$\{\beta\in\mathbb{R}^n : 1\geq \beta_1\geq\beta_2\geq\ldots\geq\beta_n\geq 0\}.$$

Later, Gao [8] computed the intrinsic volumes of the simplex with the vertices $e_1+\ldots+e_i-\frac{i}{n}(e_1+\ldots+e_n)$, $1\leq i\leq n$, which can be also given by

$$\{\beta\in\mathbb{R}^n : \beta_1\geq\beta_2\geq\ldots\geq\beta_n, \beta_1+\ldots+\beta_n=0, \beta_n-\beta_1\leq 1\}.$$

These simplices are closely related to the Weyl chambers of types B_n and A_{n-1}, respectively. Both the computation of the external angles of the faces of the Weyl chambers (which follows the method of Gao and Vitale [9] and Gao [8]), and the computation of the internal angles proceed by re-arranging the solid angles under interest in such a way that they cover the whole space, from which we conclude that the sum of the angles is 1.

The rest of the paper is mostly devoted to the proof of Theorem 1.1.

2. Preliminaries

2.1. **Conical intrinsic volumes and solid angles.** In this section we collect some information on polyhedral cones (called just cones, for simplicity). A supporting hyperplane for a cone $C\subseteq\mathbb{R}^n$ is a linear hyperplane H with the property that C lies entirely in one of the closed half-spaces H^+ and H^- induced by H. A *face* of C is a set of the form $F=C\cap H$, for a supporting hyperplane H, or the cone C itself. We denote by $\mathcal{F}_k(C)$ the set of all k-dimensional faces of C, for $k\in\{0,\ldots,n\}$. Note that the dimension of a face F is defined as the dimension of its linear hull, i.e. $\dim F=\dim \operatorname{lin}(F)$.

Equivalently, the faces of C are obtained by replacing some of the half-spaces, whose intersection defines the polyhedral cone, by their boundaries and taking the intersection.

The *dual cone* (or the *polar cone*) of a cone $C \subseteq \mathbb{R}^n$ is defined as

$$C^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 0 \ \forall y \in C\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. For example, if $C = L$ is a linear subspace, then $C^\circ = L^\perp$ is its orthogonal complement. There is a one-to-one correspondence between the sets $\mathcal{F}_k(C)$ and the $(n-k)$-faces $\mathcal{F}_{n-k}(C^\circ)$ via the bijective mapping

$$\begin{array}{ccc} \mathcal{F}_k(C) & \to & \mathcal{F}_{n-k}(C^\circ) \\ F & \mapsto & N(F,C) \end{array},$$

where $N(F, C) := (\operatorname{lin} F)^\perp \cap C^\circ$ is called the *normal face* (of F with respect to C).

The positive hull of a finite set $\{x_1, \dots, x_m\} \subseteq \mathbb{R}^n$ is defined as the smallest cone containing this set, that is

$$\operatorname{pos}\{x_1, \dots, x_m\} = \{\lambda_1 x_1 + \ldots + \lambda_m x_m : \lambda_1 \ge 0, \dots, \lambda_m \ge 0\}.$$

We will repeatedly make use of the known duality relations

$$\operatorname{pos}\{x_1, \dots, x_m\}^\circ = \bigcap_{i=1}^m x_i^- \quad \text{and} \quad \operatorname{pos}\{x_1, \dots, x_m\} = \left(\bigcap_{i=1}^m x_i^-\right)^\circ \tag{2.1}$$

for $x_1, \dots, x_m \in \mathbb{R}^n$ and $x_i^- := \{v \in \mathbb{R}^n : \langle v, x_i\rangle \le 0\}$, $i = 1, \dots, n$.

Now, we define the conical intrinsic volumes. The definition and further properties are taken from [3, Section 2.2] and [10, Section 2].

Definition 2.1. Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone, and g be an n-dimensional standard Gaussian random vector. Then, for $k \in \{0, \dots, n\}$, the k-th *conical intrinsic volume* (or, for simplicity, just *intrinsic volume*) of C is defined by

$$\upsilon_k(C) := \sum_{F \in \mathcal{F}_k(C)} \upsilon_F(C),$$

where for a face $F \in \mathcal{F}_k(C)$, we put

$$\upsilon_F(C) := \mathbb{P}(\Pi_C(g) \in \operatorname{relint}(F)).$$

Here, Π_C denotes the orthogonal projection on C, that is $\Pi_C(x)$ is the vector in C minimizing the Euclidean distance to $x \in \mathbb{R}^n$. Also, $\operatorname{relint}(F)$ denotes the interior of F taken with respect to its linear hull $\operatorname{lin}(F)$ as an ambient space.

The Moreau decomposition of a point $x \in \mathbb{R}^n$ is the representation

$$x = \Pi_C(x) + \Pi_{C^\circ}(x),$$

and yields the product formula

$$\upsilon_F(C) = \upsilon_k(F)\upsilon_{d-k}(N(F, C)).$$

Thus, we can express the intrinsic volumes of a cone C as follows:

$$\upsilon_k(C) = \sum_{F \in \mathcal{F}_k(C)} \upsilon_k(F)\upsilon_{d-k}(N(F, C)). \tag{2.2}$$

The *solid angle* (or just angle) of an n-dimensional cone $C \subseteq \mathbb{R}^n$, denoted by $\alpha(C)$, is defined as the Gaussian volume of C, i.e. the probability that an n-dimensional standard Gaussian vector g lies in C. Equivalently, the solid angle of C is the relative spherical volume of $C \cap \mathbb{S}^{n-1}$, i.e.

$$\alpha(C) = \frac{\sigma_{n-1}(C \cap \mathbb{S}^{n-1})}{\omega_n},$$

where σ_{n-1} denotes the $(n-1)$-dimensional spherical Lebesgue measure. Here, ω_k denotes the normalizing constant

$$\omega_k := \sigma_{k-1}(\mathbb{S}^{k-1}) = \frac{2\pi^{k/2}}{\Gamma(k/2)}.$$

This definition can be extended to lower-dimensional cones. For a cone C with $\dim C = k \in \{0, \ldots, n\}$, we define

$$\alpha(C) := \frac{\sigma_{k-1}(C \cap \mathbb{S}^{n-1})}{\omega_k},$$

where σ_{k-1} denotes the $(k-1)$-dimensional spherical Hausdorff measure. Equivalently, we obtain the solid angle of a cone C with $\dim C = k$ as the probability that a random vector having a standard Gaussian distribution on the ambient linear subspace $\operatorname{lin} C$ lies in C.

For a k-dimensional cone $C \subseteq \mathbb{R}^n$, $k \in \{1, \ldots, n\}$, the k-th conical intrinsic volume coincides with the solid angle of C, that is

$$\upsilon_k(C) = \alpha(C).$$

The *external angle* of a cone $C \subseteq \mathbb{R}^n$ at a face F is defined as the solid angle of the normal face of F with respect to C, that is the external angle is $\alpha(N(F, C))$. The *internal angle* of a face F at 0 is defined as $\alpha(F)$. Together with (2.2), this yields a formula for the conical intrinsic volumes in terms of the internal and external angles:

$$\upsilon_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(F)\alpha(N(F, C)), \qquad k \in \{0, \ldots, n\}. \tag{2.3}$$

The main result of this paper is a proof of the known formula for the intrinsic volumes of Weyl chambers by evaluating the internal and external angles of their faces.

2.2. **Stirling numbers of the first kind and their generating functions.** In this section we recall some facts on the Stirling numbers and their B- and D-analogues. These numbers, well known in combinatorics, appear in the formulas for the intrinsic volumes of the Weyl chambers. The (unsigned) Stirling numbers of the first kind are denoted by $\left[{n \atop k}\right]$ and defined as the coefficients of the polynomial

$$t(t+1) \cdot \ldots \cdot (t+n-1) = \sum_{k=1}^{n} \left[{n \atop k}\right] t^k. \tag{2.4}$$

By convention, $\left[{n \atop k}\right] = 0$ for $k \notin \{1, \ldots, n\}$. Equivalently, $\left[{n \atop k}\right]$ can be defined as the number of permutations of the set $\{1, \ldots, n\}$ having exactly k cycles. Other representations of the Stirling numbers of the first kind are known, e.g.

$$\left[{n \atop k}\right] = \frac{n!}{k!} \sum_{\substack{i_1, \ldots, i_k \in \mathbb{N} \\ i_1 + \ldots + i_k = n}} \frac{1}{i_1 i_2 \cdot \ldots \cdot i_k}; \tag{2.5}$$

see [13, (1.9) and (1.13)]. We shall also need the generating functions of the Stirling numbers of the first kind:

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^n}{n!} = \frac{(-\log(1-t))^k}{k!} \quad \text{and} \quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^n}{n!} y^k = (1-t)^{-y}, \tag{2.6}$$

for all complex t such that $|t| < 1$ and all $y \in \mathbb{C}$.

The B-analogues of the (signless) Stirling numbers of the first kind are denoted by $B(n,k)$ and defined as the coefficients of the polynomial

$$(t+1)(t+3)\cdot\ldots\cdot(t+2n-1) = \sum_{k=0}^{n} B(n,k)t^k. \tag{2.7}$$

Again, by convention, we put $B(n,k) = 0$ for $k \notin \{0,\ldots,n\}$. These numbers appear as entry A028338 (or A039757 for the signed version) in the On-Line Encyclopedia of Integer Sequences [16] and were studied in detail by Suter [19]. In Entry A028338 of [16] the following explicit formula for the number $B(n,k)$ in terms of the Stirling numbers of the first kind was stated by F. Woodhouse without a proof.

Proposition 2.2. *The B-analogues $B(n,k)$ of the Stirling numbers of the first kind are explicitly given by*

$$B(n,k) = \sum_{i=k}^{n} 2^{n-i} \binom{i}{k} \begin{bmatrix} n \\ i \end{bmatrix}, \quad k \in \{0,\ldots,n\}. \tag{2.8}$$

Proof. We want to check whether the numbers on the right-hand side of (2.8) coincide with the coefficients of the polynomial in (2.7). We have

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{i=k}^{n} 2^{n-i}\binom{i}{k}\begin{bmatrix} n \\ i \end{bmatrix} t^k &= 2^n \sum_{i=0}^{n} 2^{-i} \begin{bmatrix} n \\ i \end{bmatrix} \left(\sum_{k=0}^{i} \binom{i}{k} t^k\right) \\
&= 2^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \left(\frac{t+1}{2}\right)^i \\
&= 2^n \left(\frac{t+1}{2}\right)\left(\frac{t+1}{2}+1\right)\cdot\ldots\cdot\left(\frac{t+1}{2}+n-1\right) \\
&= (t+1)(t+3)\cdot\ldots\cdot(t+2n-1)
\end{aligned}$$

using the Binomial Theorem and (2.4). This completes the proof. □

As follows easily from their definition, the numbers $B(n,k)$ satisfy the recurrence relation

$$B(n,k) = (2n-1)B(n-1,k) + B(n-1,k-1); \tag{2.9}$$

see [11, Section 2.2]. The following lemma gives a formula for the generating function of the B-analogues of the Stirling numbers.

Proposition 2.3. *The generating function of the array $(B(n,k))_{n,k\geq 0}$ is given by*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(n,k)\frac{t^n}{n!}y^k = (1-2t)^{-\frac{1}{2}(y+1)}$$

for all complex $|t| < 1/2$ and $y \in \mathbb{C}$.

Proof. We use the explicit representation (2.8) of $B(n,k)$, and obtain

$$\begin{aligned}\sum_{k=0}^{n} B(n,k)y^k &= \sum_{k=0}^{n}\sum_{i=k}^{n} 2^{n-i}\binom{i}{k}\begin{bmatrix} n \\ i \end{bmatrix} y^k \\ &= \sum_{i=0}^{n} 2^n \begin{bmatrix} n \\ i \end{bmatrix}\Big(\sum_{k=0}^{i}\binom{i}{k}\Big(\frac{1}{2}\Big)^i y^k\Big) \\ &= \sum_{i=0}^{n} 2^n \begin{bmatrix} n \\ i \end{bmatrix}\Big(\sum_{k=0}^{i}\binom{i}{k}\Big(\frac{1}{2}\Big)^{i-k}\Big(\frac{y}{2}\Big)^k\Big).\end{aligned}$$

Using the Binomial Theorem, we get

$$\sum_{k=0}^{n} B(n,k)y^k = \sum_{i=0}^{n} 2^n \begin{bmatrix} n \\ i \end{bmatrix}\Big(\frac{y}{2}+\frac{1}{2}\Big)^i = \sum_{i=0}^{n} 2^n \begin{bmatrix} n \\ i \end{bmatrix}\Big(\frac{y+1}{2}\Big)^i.$$

Thus, the generating function is given by

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(n,k)\frac{t^n}{n!}y^k = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}\Big(\frac{y+1}{2}\Big)^i\frac{(2t)^n}{n!} = (1-2t)^{-\frac{1}{2}(y+1)},$$

where we used (2.6) in the last step. □

The D-analogues of the (signless) Stirling numbers of the first kind are denoted by $D(n,k)$ and defined as the coefficients of the polynomial

$$(t+1)(t+3)\cdot\ldots\cdot(t+2n-3)(t+n-1) = \sum_{k=0}^{n} D(n,k)t^k. \tag{2.10}$$

By convention, $D(n,k) = 0$ for $k \notin \{0,\ldots,n\}$. The signed version of these numbers appears as entry A039762 in [16] and they can be expressed through the B-analogues by

$$D(n,k) = (n-1)B(n-1,k) + B(n-1,k-1);$$

see [11, Section 2.2]. Together with (2.9) this yields

$$D(n,k) = B(n,k) - nB(n-1,k). \tag{2.11}$$

3. Conical intrinsic volumes of Weyl chambers

This section is dedicated to proving the formulas for the intrinsic volumes of the Weyl chambers mentioned in (1.1). For this, we want to use the formula (2.3) and evaluate the internal and external angle of the faces of the Weyl chambers. The computation of the external angles relies on the ideas of Gao and Vitale [9] who derived the classic intrinsic volumes of a related simplex. ???

3.1. **Type A_{n-1}.** We start with the simpler A_{n-1} case. Recall the definition of the fundamental Weyl chamber of type A_{n-1}:

$$C^A := \mathcal{C}(A_{n-1}) = \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n\}.$$

Then our first result is the following.

Theorem 3.1. *For $k = 0, 1, \ldots, n$ the k-th conical intrinsic volume of the Weyl chamber of type A_{n-1} is given by*

$$\upsilon_k(\mathcal{C}(A_{n-1})) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!},$$

where the Stirling numbers $\left[{n \atop k}\right]$ are defined as in Section 2.2.

In order to prove Theorem 3.1, we want to evaluate the internal and external angles of the faces of C^A. Observe that the faces of C^A can be obtained by replacing some of the inequalities defining C^A by the corresponding equalities. It follows that for $2 \le k \le n$, the k-faces of C^A are determined by the collection of indices $1 \le l_1 < \ldots < l_{k-1} < l_k := n$ and given by

$$C^A(l_1, \ldots, l_{k-1}) = \{\beta \in \mathbb{R}^n : \beta_1 = \ldots = \beta_{l_1} \ge \beta_{l_1+1} = \ldots = \beta_{l_2} \ge \ldots \ge \beta_{l_{k-1}+1} = \ldots = \beta_n\}.$$

Then, (2.3) yields the formula

$$\begin{aligned} \upsilon_k(C^A) &= \sum_{F \in \mathcal{F}_k(C^A)} \alpha(F)\alpha(N(F, C^A)) \\ &= \sum_{1 \le l_1 < \ldots < l_{k-1} \le n-1} \alpha\big(C^A(l_1, \ldots, l_{k-1})\big)\alpha\big(N(C^A(l_1, \ldots, l_{k-1}), C^A)\big). \end{aligned}$$

For $k = 1$, the only 1-face is $F_1 := \{\beta \in \mathbb{R}^n : \beta_1 = \ldots = \beta_n\}$ and we have

$$\upsilon_1(C^A) = \alpha(F_1)\alpha(N(F_1, C^A)).$$

Since C^A has no 0-dimensional faces, we have

$$\upsilon_0(C^A) = 0 = \begin{bmatrix} n \\ 0 \end{bmatrix}.$$

External Angles. Let $2 \le k \le n$. Take some $1 \le l_1 < \ldots < l_{k-1} < l_k := n$. We start by computing the normal face $N(C^A(l_1, \ldots, l_{k-1}), C^A)$. Our method of proof partly relies on the approach of Gao and Vitale [9]. By definition, we have

$$N(C^A(l_1, \ldots, l_{k-1}), C^A) = (\operatorname{lin} C^A(l_1, \ldots, l_{k-1}))^\perp \cap (C^A)^\circ.$$

Using (2.1), we obtain

$$\begin{aligned} (C^A)^\circ &= \{\beta \in \mathbb{R}^n : \beta_1 \ge \beta_2 \ge \ldots \ge \beta_n\}^\circ \\ &= \operatorname{pos}\{(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1), (-1, \ldots, -1)\}^\circ \\ &= \{x \in \mathbb{R}^n : \langle x, (1, 0, \ldots, 0)\rangle \le 0, \langle x, (1, 1, 0, \ldots, 0)\rangle \le 0, \\ &\qquad \ldots, \langle x, (1, \ldots, 1)\rangle \le 0, \langle x, (-1, \ldots, -1)\rangle \le 0\} \\ &= \{x \in \mathbb{R}^n : x_1 \le 0, x_1 + x_2 \le 0, \ldots, x_1 + \ldots + x_{n-1} \le 0, x_1 + \ldots + x_n = 0\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} (\operatorname{lin} C^A(l_1, \ldots, l_{k-1}))^\perp &= \{\beta \in \mathbb{R}^n : \beta_1 = \ldots = \beta_{l_1}, \beta_{l_1+1} = \ldots = \beta_{l_2}, \ldots, \beta_{l_{k-1}+1} = \ldots = \beta_n\}^\perp \\ &= \operatorname{lin}\{e_2 - e_1, \ldots, e_{l_1} - e_{l_1-1}, e_{l_1+2} - e_{l_1+1}, \ldots, e_{l_2} - e_{l_2-1}, \\ &\qquad \ldots, e_{l_{k-1+2}} - e_{l_{k-1}+1}, \ldots, e_n - e_{n-1}\}. \end{aligned}$$

Here, $e_1, \ldots, e_n$ denotes the standard Euclidean orthonormal basis of $\mathbb{R}^n$. Now, suppose $x \in (\operatorname{lin} C^A(l_1, \ldots, l_{k-1}))^\perp$. Then there exist numbers $\lambda_i \in \mathbb{R}$ with $i \in \{1, \ldots, n\} \backslash \{l_1, \ldots, l_{k-1}, n\}$, such that

$$x = \big(-\lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_{l_1-2} - \lambda_{l_1-1}, \lambda_{l_1-1}, -\lambda_{l_1+1}, \lambda_{l_1+1} - \lambda_{l_1+2}, \ldots, \lambda_{l_2-2} - \lambda_{l_2-1}, \lambda_{l_2-1}, \\ \ldots, -\lambda_{l_{k-1}+1}, \lambda_{l_{k-1}+1} - \lambda_{l_{k-1}+2}, \ldots, \lambda_{n-2} - \lambda_{n-1}, \lambda_{n-1}\big). \quad (3.1)$$

We observe that the coordinates of $x = (x_1, \ldots, x_n)$ satisfy $x_1 + \ldots + x_i = 0$ for all $i \in J := \{l_1, \ldots, l_{k-1}, n\}$, and $x_1 + \ldots + x_j = -\lambda_j$ for all $j \in \{1, \ldots, n\} \backslash J$. If we additionally suppose $x \in (C^A)^\circ$, we have that x is of the above form seen in (3.1) satisfying additionally $\lambda_j \geq 0$ for all $j \in \{1, \ldots, n\} \backslash J$. Thus, it is easy to see that

$$\begin{aligned} N(C^A(l_1, \ldots, l_{k-1}), C^A) &= \mathrm{pos}\{e_2 - e_1, \ldots, e_{l_1} - e_{l_1-1}, e_{l_1+2} - e_{l_1+1}, \ldots, e_{l_2} - e_{l_2-1}, \\ &\qquad \ldots, e_{l_{k-1+2}} - e_{l_{k-1}+1}, \ldots, e_n - e_{n-1}\} \\ &= \mathrm{pos}\{u_j : j \notin J\}, \end{aligned}$$

where $u_i = e_{i+1} - e_i$ is the n-dimensional vector whose i-th coordinate is -1, whose $(i+1)$-st coordinate is 1, and the other coordinates are zeros. For $i = n$, we put $u_n = (0, \ldots, 0, -1)$.

To evaluate the solid angle of the normal face $N(C^A(l_1, \ldots, l_k), C^A) = \mathrm{pos}\{u_j : j \notin J\}$, we arrange the vectors $u_j, j \notin J$ as rows in the following $(n-k) \times n$ Matrix

$$\left[\begin{array}{cccccccccccccccccc}
-1 & 1 & & & & & & & & & & & & & & & & \\
 & -1 & 1 & & & & & & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & & & & & & \\
 & & & -1 & 1 & & & & & & & & & & & & & \\
\hline
 & & & & & -1 & 1 & & & & & & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & & & & & & \\
 & & & & & & & -1 & 1 & & & & & & & & & \\
\hline
 & & & & & & & & & \ddots & & & & & & & & \\
\hline
 & & & & & & & & & & -1 & 1 & & & & & & \\
 & & & & & & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & & & & & & -1 & 1 & & & & \\
\hline
 & & & & & & & & & & & & & & -1 & 1 & & \\
 & & & & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & & & & -1 & 1 \\
\hline
\end{array}\right],$$

The horizontal lines denote the missing vectors $u_j, j \in J$. This matrix contains k small submatrices, denoted by $A_1, \ldots, A_k$. They are of the same form but possibly of a different size. This matrix emphasizes the product structure of the normal face $N(C^A(l_1, \ldots, l_{k-1}), C^A)$, since the row vectors of the matrix belonging to different submatrices are orthogonal. This corresponds to the fact that we can write the normal face as the orthogonal product of k cones:

$$N(C^A(l_1, \ldots, l_{k-1}), C^A) = D_{l_1} \times D_{l_2 - l_1} \times \ldots \times D_{l_k - l_{k-1}},$$

where

$$D_i := \{x \in \mathbb{R}^i : x_1 \leq 0, x_1 + x_2 \leq 0, \ldots, x_1 + \ldots + x_{i-1} \leq 0, x_1 + \ldots + x_i = 0\}.$$

The cones D_i have the same structure and differ only by dimension. Thus, it suffices to compute the solid angle for one of these cones. In the following, we consider the cone D_{l_1} spanned by the rows of the matrix A_1.

Let $v_1, \ldots, v_{l_1-1}$ denote the l_1-dimensional row vectors of the $(l_1-1)\times l_1$-matrix A_1. We want to evaluate the angle of $\operatorname{pos}\{v_1, \ldots, v_{l_1-1}\}$. For the sake of simplicity we will often refer to it as the angle of A_1 or just $\alpha(A_1)$. In order to do this, we want to produce i_1 matrices, denoted by $A_1^0 := A_1, A_1^1, \ldots, A_1^{l_1-1}$ that satisfy the following two properties:

(i) For each $1 \le i \le l_1 - 1$, A_1^i has the same angle as A_1, i.e for each matrix the positive hull of its row vectors has the same angle as $\operatorname{pos}\{v_1, \ldots, v_{l_1-1}\}$.
(ii) The positive hulls of the row vectors of each matrix $A_1^0, A_1^1, \ldots, A_1^{l_1-1}$ form a conical mosaic of $\mathbb{R}^{l_1-1}$ (or rather of a (l_1-1)-dimensional subspace or $\mathbb{R}^{l_1}$), i.e. they have no common interior points and their union equals the whole space $\mathbb{R}^{l_1-1}$.

The construction of these matrices is as follows. For $1 \le i \le l_1 - 1$ delete the i-th row of A_1 and add the l_1-dimensional vector $v_{l_1} = (1, 0, \ldots, 0, -1)$ at the bottom. Then move the first $i-1$ rows to the bottom and call the obtained matrix A_1^l. We will display the matrices:

$$A_1 = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix},$$

$$A_1^1 = \begin{bmatrix} 0 & -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 1 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix},$$

$$A_1^2 = \begin{bmatrix} 0 & 0 & -1 & 1 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & -1 \\ -1 & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$\vdots$$

$$A_1^{l_1-1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \ldots & -1 & 1 & 0 \end{bmatrix}.$$

Let us give an interpretation of the cones spanned by the rows of the matrices A_1^i. First of all, the cone defined by A_1^i can be represented as

$$\{(x_1, \ldots, x_{l_1}) \in \mathbb{R}^{l_1} : x_{i+1} \le 0, x_{i+1} + x_{i+2} \le 0, \ldots, x_{i+1} + \ldots + x_{l_1} + x_1 + \ldots + x_{i-1} \le 0,$$
$$x_{i+1} + \ldots + x_{l_1} + x_1 + \ldots + x_i = 0\}.$$

Consider some vector $(x_1,\dots,x_{l_1})\in\mathbb{R}^{l_1}$ satisfying $x_1+\ldots+x_{l_1}=0$ and the corresponding partial sums:

$$x_1, x_1+x_2,\dots,x_1+\ldots+x_{n-1},x_1+\ldots+x_n=0.$$

The cone generated by the rows of A_1 consists of all such vectors for which all partial sums are non-positive (i.e., they form a "bridge" staying below zero). More generally, the cone generated by the rows of A_1^i consists of all vectors $(x_1,\dots,x_n)$ for which a cyclic permutation $(x_{i+1},\dots,x_{l_1},x_1,\dots,x_i)$ satisfies the same non-positivity condition. In the following, it will be shown that ignoring the vectors on the boundaries of the cones, for each vector there is exactly one cyclic permutation satisfying the non-positivity condition. A probabilistic version of this reasoning is due to Sparre Andersen [17] who proved that the probability that a random bridge (satisfying certain minor conditions) of length l_1 stays below zero is $1/l_1$.

We observe that each matrix A_1^i is a column permutation of A_1 and that we can also construct the matrix A^{l_1} which coincides with A_1. Thus, there is a permutation matrix $\mathcal{O}=\mathcal{O}_i$, such that $A_1^i=A_1\mathcal{O}^{-1}$, and therefore,

$$(A_1^i)^T=\mathcal{O}(A_1)^T=(\mathcal{O}v_1,\dots,\mathcal{O}v_{l_1-1}),$$

where the last notation refers to the matrix consisting of the columns $\mathcal{O}v_i$, $i=1,\dots,l_1-1$. Thus, we see that the angle of A_1^i coincides with the angle of A_1, since for all $l=1,\dots l_1-1$

$$\alpha(A_1^i)=\alpha(\operatorname{pos}\{\mathcal{O}v_1,\dots,\mathcal{O}v_{l_1-1}\})=\alpha(\mathcal{O}\operatorname{pos}\{v_1,\dots,v_{l_1-1}\})=\alpha(\operatorname{pos}\{v_1,\dots,v_{l_1-1}\})=\alpha(A_1),$$

where we used the rotation invariance of the spherical Lebesgue measure.

Now, it is left to prove that $A_1^1,\dots,A_1^{l_1-1},A_1^{l_1}$, or rather the positive hulls of its respective rows, form a conical mosaic of the (l_1-1)-dimensional subspace $H_{l_1}:=\{(x_1,\dots,x_{l_1}):x_1+\ldots+x_{l_1}=0\}\subset\mathbb{R}^{l_1}$. We are going to show that the positive hulls have no common interior points, where the notion of interior is always understood in the sense of the ambient space H_{l_1}. To this end, we take indices $1\le i<j\le l_1$ and show that the interiors of the positive hulls of the rows of A_1^i and A_1^j are disjoint. Note that the rows of A_1^i are given by $v_1,\dots,v_{i-1},v_{i+1},\dots,v_{l_1}$ and the rows of A_1^j by $v_1,\dots,v_{j-1},v_{j+1},\dots,v_{l_1}$. If $x=(x_1,\dots,x_{l_1})\in\operatorname{int}(\operatorname{pos}\{v_1,\dots,v_{i-1},v_{i+1},\dots,v_{l_1}\})$, then there exist numbers $\alpha_k>0$, $k\in\{1,\dots,l_1\}\backslash\{i\}$ such that

$$x=\alpha_1v_1+\ldots+\alpha_{i-1}v_{i-1}+\alpha_{i+1}v_{i+1}+\ldots+\alpha_{l_1}v_{l_1},$$

which uses the linear independence of the v_i's. Thus,

$$\begin{aligned}x_{i+1}+x_{i+2}+\ldots+x_j&=-\alpha_{i+1}+(\alpha_{i+1}-\alpha_{i+2})+\ldots+(\alpha_{j-1}-\alpha_j)\\&=-\alpha_j<0.\end{aligned}$$

If also $x\in\operatorname{int}(\operatorname{pos}\{v_1,\dots,v_{j-1},v_{j+1},\dots,v_{l_1}\})$, then there exist numbers $\beta_k>0$, $k\in\{1,\dots,l_1\}\backslash\{j\}$, such that

$$x=\beta_1v_1+\ldots+\beta_{j-1}v_{j-1}+\beta_{j+1}v_{j+1}+\ldots+\beta_{l_1}v_{l_1}$$

Thus,

$$\begin{aligned}x_{i+1}+x_{i+2}+\ldots+x_j&=(\beta_i-\beta_{i+1})+\ldots+(\beta_{j-2}-\beta_{j-1})+\beta_{j-1}\\&=\beta_i>0,\end{aligned}$$

which is a contradiction.

Finally, we claim that the union of the positive hulls defined by the rows of $A_1^0,\dots,A_1^{l_1-1}$, respectively, equals the (l_1-1)-dimensional subspace $H_{l_1}=\{(x_1,\dots,x_{l_1}):x_1+\ldots+x_{l_1}=0\}\subset\mathbb{R}^{l_1}$. Since the vectors $v_1,\dots,v_{l_1-1}$ are linearly independent and $v_1,\dots,v_{l_1}\in H_{l_1}$, we have that

$H_{l_1} = \text{lin}\{v_1, \ldots, v_{l_1}\}$. Thus, H_{l_1} contains the positive hulls defined by the rows of $A_1^0, \ldots, A_1^{l_1-1}$, respectively. On the other hand, we know that for each $x \in H_{l_1}$ there exist numbers $c_1, \ldots, c_{l_1} \in \mathbb{R}$, such that

$$x = c_1 v_1 + \ldots + c_{l_1} v_{l_1}.$$

Then there is a permutation σ of the set $\{1, \ldots, l_1\}$ with $c_{\sigma(1)} \leq \ldots \leq c_{\sigma(l_1)}$. Thus, we have

$$\begin{aligned} x &= c_{\sigma(1)}(v_1 + \ldots + v_{l_1}) + (c_{\sigma(2)} - c_{\sigma(1)})v_{\sigma(2)} + \ldots + (c_{\sigma(l_1)} - c_{\sigma(1)})v_{\sigma(l_1)} \\ &= (c_{\sigma(2)} - c_{\sigma(1)})v_{\sigma(2)} + \ldots + (c_{\sigma(l_1)} - c_{\sigma(1)})v_{\sigma(l_1)} \in \text{pos}\{v_{\sigma(2)}, \ldots, v_{\sigma(l_1)}\}, \end{aligned}$$

proving that the positive hulls defined by $A_1^0, \ldots, A_1^{l_1-1}$ form a conical mosaic of H_{l_1}.

Summarizing, we proved that $\alpha(A_1) = 1/l_1$. Similarly, one proves that $\alpha(A_2) = 1/(l_2 - l_1), \ldots, A_k = 1/(n - l_{k-1})$. Since the cone $\alpha\big(N(C^A(l_1, \ldots, l_{k-1}), C^A)\big)$ is the orthogonal product of the k cones spanned by the rows of the matrices $A_1, \ldots, A_k$, it follows that

$$\alpha\big(N(C^A(l_1, \ldots, l_{k-1}), C^A)\big) = \alpha(A_1)\alpha(A_2) \cdot \ldots \cdot \alpha(A_k) = \frac{1}{l_1(l_2 - l_1) \cdot \ldots \cdot (n - l_{k-1})}. \tag{3.2}$$

The case $k = 1$ follows in the same way as above. For the only 1-face $F_1 = \{\beta \in \mathbb{R}^n : \beta_1 = \ldots = \beta_n\}$ of C^A, we have

$$\begin{aligned} N(F_1, C^A) &= (F_1)^\perp \cap (C^A)^\circ \\ &= \{\beta \in \mathbb{R}^n : \beta_1 \leq 0, \beta_1 + \beta_2 \leq 0, \ldots, \beta_1 + \ldots + \beta_{n-1} \leq 0, \beta_1 + \ldots + \beta_n = 0\} \\ &= \text{pos}\{u_i, i \notin J := \{n\}\}. \end{aligned}$$

Thus, we can apply the same arguments as for $k \geq 2$ and obtain

$$N(F_1, C^A) = \frac{1}{n}.$$

Internal Angles. Now, we need to compute the internal angles of the faces of C^A. Let $2 \leq k \leq n$. In order to do this, we consider the linear hull of $C^A(l_1, \ldots, l_{k-1})$ which forms a k-dimensional linear subspace of $\mathbb{R}^n$. Recall that for $1 \leq l_1 < \ldots < l_{k-1} \leq n-1$ the linear hull of $C^A(l_1, \ldots, l_{k-1})$ is given by

$$\text{lin}\, C^A(l_1, \ldots, l_{k-1}) = \{\beta \in \mathbb{R}^n : \beta_1 = \ldots = \beta_{l_1}, \ldots, \beta_{l_{k-1}+1} = \ldots = \beta_n\}.$$

Thus, the vectors x_i given by

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{l_1}}(\overbrace{1, \ldots, 1}^{1,\ldots,l_1}, 0, \ldots, 0), x_2 = \frac{1}{\sqrt{l_2 - l_1}}(0, \ldots, 0, \overbrace{1, \ldots, 1}^{l_1+1,\ldots,l_2}, 0, \ldots, 0), \ldots, \\ x_{k-1} &= \frac{1}{\sqrt{l_{k-1} - l_{k-2}}}(0, \ldots, 0, \overbrace{1, \ldots, 1}^{l_{k-2}+1,\ldots,l_{k-1}}, 0, \ldots, 0), x_k = \frac{1}{\sqrt{n - l_{k-1}}}(0, \ldots, 0, \overbrace{1, \ldots, 1}^{l_{k-1}+1,\ldots,n}) \end{aligned}$$

form an orthonormal basis of $\text{lin}\, C^A(l_1, \ldots, l_{k-1})$. For independent and standard normal distributed random variables $\xi_1, \ldots, \xi_k$, the random vector $N := \xi_1 x_1 + \cdots + \xi_k x_k$ is k-dimensional standard normal distributed on the linear subspace $\text{lin}\, C^A(l_1, \ldots, l_{k-1})$. Thus, we obtain for the angle of $C^A(l_1, \ldots, l_{k-1})$ the following formula:

$$\alpha(C^A(l_1, \ldots, l_{k-1})) = \mathbb{P}(N \in C^A(l_1, \ldots, l_{k-1}))$$

$$= \mathbb{P}\Big(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2 - l_1}} \geq \ldots \geq \frac{\xi_{k-1}}{\sqrt{l_{k-1} - l_{k-2}}} \geq \frac{\xi_k}{\sqrt{n - l_{k-1}}}\Big). \tag{3.3}$$

Finally, we are able to prove Theorem 3.1.

Proof of Theorem 3.1. Let $2 \leq k \leq n$. Using the formulas (2.3) and (3.2) for the internal and external angles, we obtain

$$\begin{aligned}
\upsilon_k(C^A) &= \sum_{1 \leq l_1 < \ldots < l_{k-1} \leq n-1} \alpha(C^A(l_1, \ldots, l_{k-1}))\alpha(N(C^A(l_1, \ldots, l_{k-1}), C^A)) \\
&= \sum_{1 \leq l_1 < \ldots < l_{k-1} \leq n-1} \frac{\mathbb{P}\Big(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2 - l_1}} \geq \ldots \geq \frac{\xi_{k-1}}{\sqrt{l_{k-1} - l_{k-2}}} \geq \frac{\xi_k}{\sqrt{n - l_{k-1}}}\Big)}{l_1(l_2 - l_1) \cdot \ldots \cdot (l_{k-1} - l_{k-2})(n - l_{k-1})}.
\end{aligned}$$

Defining $i_1 = l_1, i_2 = l_2 - l_1, \ldots, i_{k-1} = l_{k-1} - l_{k-2}, i_k = n - l_{k-1}$, we can change the summation of the above sum and obtain

$$\sum_{\substack{i_1, \ldots, i_k \in \mathbb{N} \\ i_1 + \ldots + i_k = n}} \frac{1}{i_1 i_2 \cdot \ldots \cdot i_k} \mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \ldots \geq \frac{\xi_{k-1}}{\sqrt{i_{k-1}}} \geq \frac{\xi_k}{\sqrt{i_k}}\Big). \tag{3.4}$$

Now, fix a permutation $\pi \in \mathcal{S}_k$. For each tuple $(i_1, \ldots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \ldots + i_k = n$, the tuple $(i_{\pi(1)}, \ldots, i_{\pi(k)})$ also satisfies $i_{\pi(1)} + \ldots + i_{\pi(k)} = n$. Thus, the sum in (3.4) does not change if we replace the tuple $(i_1, \ldots, i_k)$ by $(i_{\pi(1)}, \ldots, i_{\pi(k)})$ inside the sum. It follows that

$$\begin{aligned}
\upsilon_k(C^A) &= \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} \sum_{\substack{i_1, \ldots, i_k \in \mathbb{N} \\ i_1 + \ldots + i_k = n}} \frac{1}{i_{\pi(1)} i_{\pi(2)} \cdot \ldots \cdot i_{\pi(k)}} \mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_2}{\sqrt{i_{\pi(2)}}} \geq \ldots \geq \frac{\xi_k}{\sqrt{i_{\pi(k)}}}\Big) \\
&= \frac{1}{k!} \sum_{\substack{i_1, \ldots, i_k \in \mathbb{N} \\ i_1 + \ldots + i_k = n}} \frac{1}{i_1 i_2 \cdot \ldots \cdot i_k} \sum_{\pi \in \mathcal{S}_k} \mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_2}{\sqrt{i_{\pi(2)}}} \geq \ldots \geq \frac{\xi_k}{\sqrt{i_{\pi(k)}}}\Big) \\
&= \frac{1}{k!} \sum_{\substack{i_1, \ldots, i_k \in \mathbb{N} \\ i_1 + \ldots + i_k = n}} \frac{1}{i_1 i_2 \cdot \ldots \cdot i_k}.
\end{aligned}$$

In the last step we used that for a given composition $(i_1, \ldots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \ldots + i_k = n$, we have

$$\sum_{\pi \in \mathcal{S}_k} \mathbb{P}\Big(\frac{\xi_{\pi(1)}}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_{\pi(2)}}{\sqrt{i_{\pi(2)}}} \geq \ldots \geq \frac{\xi_{\pi(k)}}{\sqrt{i_{\pi(k)}}}\Big) = 1,$$

and thus, also

$$\sum_{\pi \in \mathcal{S}_k} \mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_2}{\sqrt{i_{\pi(2)}}} \geq \ldots \geq \frac{\xi_k}{\sqrt{i_{\pi(k)}}}\Big) = 1$$

since $\xi_1, \ldots, \xi_k$ are independent and identically distributed, in particular exchangeable. Using the representation (2.5) of the Stirling numbers of the first kind, we obtain

$$\upsilon_k(C^A) = \frac{1}{k!} \begin{bmatrix} n \\ k \end{bmatrix} \frac{k!}{n!} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!}.$$

For $k=1$, the only 1-face of C^A is $F_1=\{\beta\in\mathbb{R}^n:\beta_1=\ldots=\beta_n\}$ and we have

$$\upsilon_1(C^A)=\alpha(F_1)\alpha(N(F_1,C^A))=1\cdot\frac{1}{n}=\begin{bmatrix}n\\1\end{bmatrix}\frac{1}{n!}.$$

For $k=0$, we already saw that $\upsilon_0(C^A)=0=\left[{n \atop 0}\right]$, which completes the proof. □

3.2. **Type B_n.** Now, we proceed with the B_n case. Recall the definition of the fundamental Weyl chamber of type B_n:

$$C^B:=\mathcal{C}(B_n)=\{\beta\in\mathbb{R}^n:\beta_1\geq\beta_2\geq\ldots\geq\beta_n\geq 0\}.$$

Then the analogue of Theorem 3.1 is the following.

Theorem 3.2. *For $k=0,1,\ldots,n$ the k-th conical intrinsic volume of the Weyl chamber $\mathcal{C}(B_n)$ is given by*

$$\upsilon_k(\mathcal{C}(B_n))=\frac{B(n,k)}{2^n n!}.$$

Again, to obtain the faces of C^B we have to replace some of the inequalities defining C^B by the corresponding equalities. It follows that for $1\leq k\leq n$ any k-face of C^B is determined by a collection of indices $1\leq l_1<\ldots<l_k\leq n$ and given by

$$C^B(l_1,\ldots,l_k)=\{\beta\in\mathbb{R}^n:\beta_1=\ldots=\beta_{l_1}\geq\ldots\geq\beta_{l_{k-1}+1}=\ldots=\beta_{l_k}\geq\ldots\geq\beta_{l_k+1}=\ldots=\beta_n=0\}.$$

Then, (2.3) yields the formula

$$\begin{aligned}\upsilon_k(C^B)&=\sum_{F\in\mathcal{F}_k(C^B)}\alpha(F)\alpha(N(F,C^B))\\&=\sum_{1\leq l_1<\ldots<l_k\leq n}\alpha\big(C^B(l_1,\ldots,l_k)\big)\alpha\big(N(C^B(l_1,\ldots,l_k),C^B)\big).\end{aligned}$$

For $k=0$, the only 0-dimensional face of C^B is $\{0\}$ and we have

$$\upsilon_0(C^B)=\alpha(\{0\})\alpha(N(\{0\},C^B))=\alpha(N(\{0\},C^B)).$$

External Angles. The computation of the external angles for the B_n case is similar to the A_{n-1} case. Take some $1\leq l_1<\ldots<l_k\leq n$. We start by computing the normal face $N(C^B(l_1,\ldots,l_k),C^B)$ of $C^B(l_1,\ldots,l_k)$. First of all, we have

$$\begin{aligned}(C^B)^\circ&=\operatorname{pos}\{(1,0,\ldots,0),(1,1,0,\ldots,0),\ldots,(1,\ldots,1)\}^\circ\\&=\{x\in\mathbb{R}^n:x_1\leq 0,x_1+x_2\leq 0,\ldots,x_1+\ldots+x_n\leq 0\}.\end{aligned}$$

Since

$$\begin{aligned}(\operatorname{lin}C^B(l_1,\ldots,l_k))^\perp&=\{x\in\mathbb{R}^n:x_1=\ldots=x_{l_1},x_{l_1+1}=\ldots=x_{l_2},\ldots,x_{l_k+1}=\ldots=x_n=0\}^\perp\\&=\operatorname{lin}\{e_1-e_2,\ldots,e_{l_1-1}-e_{l_1},e_{l_1+1}-e_{l_1+2},\ldots,e_{l_2-1}-e_{l_2},\\&\qquad\ldots,e_{l_k+1}-e_{l_k+2},\ldots,e_{n-1}-e_n,e_n\},\end{aligned}$$

it follows that

$$\begin{aligned}N(C^B(l_1,\ldots,l_k),C^B)&=(\operatorname{lin}C^B(l_1,\ldots,l_k))^\perp\cap(C^B)^\circ\\&=\operatorname{pos}\{u_i:i\notin J\}\end{aligned}$$

with $J = \{l_1, \dots, l_k\}$ by a similar argument as in the A_{n-1} case. Again, $u_i = e_{i+1} - e_i$ is the n-dimensional vector whose i-th coordinate is -1, whose $(i+1)$-st coordinate is 1, and the other coordinates are zeros, for $i \in \{1, \dots, n-1\}$. For $i = n$, we put $u_n = -e_n = (0, \dots, 0, -1)$.

To evaluate the solid angle of the normal face $N(C^B(l_1, \dots, l_k), C^B) = \operatorname{pos}\{u_i : i \notin J\}$, we arrange the vectors $u_i, i \notin J$, similarly to the A_{n-1} case, as rows in the following $(n-k) \times n$-matrix

$$
\left[\begin{array}{cccccccccccccccccc}
-1 & 1 & & & & & & & & & & & & & & & & \\
 & -1 & 1 & & & & & & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & & & & & & \\
 & & & -1 & 1 & & & & & & & & & & & & & \\
\hline
 & & & & & -1 & 1 & & & & & & & & & & & \\
 & & & & & & \ddots & \ddots & & & & & & & & & & \\
 & & & & & & & -1 & 1 & & & & & & & & & \\
\hline
 & & & & & & & & & \ddots & & & & & & & & \\
\hline
 & & & & & & & & & & -1 & 1 & & & & & & \\
 & & & & & & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & & & & & & -1 & 1 & & & & \\
\hline
 & & & & & & & & & & & & & & -1 & 1 & & \\
 & & & & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & & & & -1 & 1 \\
 & & & & & & & & & & & & & & & & & -1
\end{array}\right],
$$

where the horizontal lines indicate the missing vectors $u_i, i \in J$. This matrix contains $k+1$ small submatrices, denoted by $B_1, \dots, B_{k+1}$. The first k small matrices $B_1, \dots, B_k$ are all of the same form but possibly of a different size. Additionally, these small matrices $B_1, \dots, B_k$ are of the same form as the small matrices $A_1, \dots, A_k$ in the A_{n-1} case. Following the reasoning of the A_{n-1}-case, for each small matrix B_p, $1 \le p \le k$, the solid angle of the cone spanned by this matrix is given by

$$\alpha(B_p) = \frac{1}{l_p - l_{p-1}},$$

where we put $l_0 = 0$. Since $N(C^B(l_1, \dots, l_k), C^B)$ is the orthogonal direct product of the cones spanned by the rows of the individual matrices $B_1, \dots, B_{k+1}$, we arrive at

$$\alpha\big(N(C^B(l_1, \dots, l_k), C^B)\big) = \frac{\alpha(B_{k+1})}{l_1(l_2 - l_1) \cdot \ldots \cdot (l_k - l_{k-1})}.$$

In order to compute the angle of the normal face, we still need to determine the value of $\alpha(B_{k+1})$, where the $(n - l_k) \times (n - l_k)$-matrix B_{k+1} is given by

$$
\begin{bmatrix}
-1 & 1 & 0 & \dots & 0 & 0 \\
0 & -1 & 1 & \dots & 0 & 0 \\
 & & & \ddots & & \\
0 & 0 & 0 & \dots & -1 & 1 \\
0 & 0 & 0 & \dots & 0 & -1
\end{bmatrix}.
$$

We denote the rows of B_{k+1} by $w_1, \dots, w_{n-l_k}$. Using (2.1) we obtain

$$\begin{aligned}(\operatorname{pos}\{w_1,\dots,w_{n-l_k}\})^\circ &= \{x \in \mathbb{R}^{n-l_k} : \langle x, w_i\rangle \le 0,\ i = 1,\dots,n-l_k\}\\ &= \{x \in \mathbb{R}^{n-l_k} : x_1 \ge \dots \ge x_{n-l_k} \ge 0\}\\ &= \operatorname{pos}\{(1,0,\dots,0),(1,1,0,\dots,0),\dots,(1,\dots,1)\}.\end{aligned}$$

Thus, $x \in \operatorname{pos}\{w_1,\dots,w_{n-l_k}\}$ if and only if $\langle x, e_1+\dots+e_i\rangle \le 0$ for all $i = 1,\dots,n-l_k$, where we recall that $e_1,\dots,e_{n-l_k}$ denotes the standard Euclidean basis in $\mathbb{R}^{n-l_k}$. Now, take a $(n-l_k)$-dimensional standard Gaussian random vector $N = (N_1,\dots,N_{n-l_k})$. It follows that

$$\begin{aligned}\alpha(B_{k+1}) = \alpha(\operatorname{pos}\{w_1,\dots,w_{n-l_k}\}) &= \mathbb{P}(N \in \operatorname{pos}\{w_1,\dots,w_{n-l_k}\})\\ &= \mathbb{P}(\langle N, e_1+\dots+e_i\rangle \le 0\ \forall i = 1,\dots,n-l_k)\\ &= \mathbb{P}(N_1 \le 0, N_1+N_2 \le 0,\dots,N_1+\dots+N_{n-l_k} \le 0)\\ &= \binom{2(n-l_k)}{n-l_k}\frac{1}{2^{2(n-l_k)}}.\end{aligned}$$

In the last step, we applied the well-known formula of Sparre Andersen [5] which is valid in the case when $N_1,\dots,N_{n-l_k}$ are independent and standard normal distributed. This yields the external angle of $C^B(l_1,\dots,l_k)$

$$\alpha\big(N(C^B(l_1,\dots,l_k),C^B)\big) = \frac{\binom{2(n-l_k)}{n-l_k}}{l_1(l_2-l_1)\cdot\ldots\cdot(l_k-l_{k-1})2^{2(n-l_k)}}. \tag{3.5}$$

In the case $k = 0$, we want to compute the angle of $N(\{0\},C^B) = (C^B)^\circ$. We know that

$$(C^B)^\circ = \{x \in \mathbb{R}^n : x_1 \le 0, x_1+x_2 \le 0,\dots,x_1+\dots+x_n \le 0\}.$$

Thus, for an n-dimensional standard Gaussian random vector $N = (N_1,\dots,N_n)$, we have

$$\begin{aligned}\alpha(N(\{0\},C^B)) = \mathbb{P}(N \in (C^B)^\circ) &= \mathbb{P}(N_1 \le 0, N_1+N_2 \le 0,\dots,N_1+\dots+N_n \le 0)\\ &= \binom{2n}{n}\frac{1}{2^{2n}}\end{aligned} \tag{3.6}$$

again by the formula of Sparre Andersen [5].

Internal Angles. Now, we need to compute the internal angles for the faces of C^B, that is the angle $\alpha(C^B(l_1,\dots,l_k))$ for $1 \le l_1 < \dots < l_k \le n$. In order to do this, we consider the linear hull of $C^B(l_1,\dots,l_k)$ which forms a k-dimensional linear subspace. Recall that

$$\operatorname{lin} C^B(l_1,\dots,l_k) = \{x \in \mathbb{R}^n : x_1 = \dots = x_{l_1}, x_{l_1+1} = \dots = x_{l_2},\dots,x_{l_k+1} = \dots = x_n = 0\}.$$

Then, the following vectors $y_1,\dots,y_k$ form an orthonormal basis of $\operatorname{lin} C^B(l_1,\dots,l_k)$, where y_i is the normalized version of the vector with 1 in the entries $l_{i-1}+1,\dots,l_i$ and zeroes in the other entries, i.e.

$$y_1 = \frac{1}{\sqrt{l_1}}(\overbrace{1,\dots,1}^{1,\dots,l_1},0,\dots,0), y_2 = \frac{1}{\sqrt{l_2-l_1}}(0,\dots,0,\overbrace{1,\dots,1}^{l_1+1,\dots,l_2},0,\dots,0),\dots,$$

$$y_k = \frac{1}{\sqrt{l_k-l_{k-1}}}(0,\dots,0,\overbrace{1,\dots,1}^{l_{k-1}+1,\dots,l_k},0,\dots,0).$$

For independent and standard normal distributed random variables $\xi_1,\dots,\xi_k$, the random vector $N := \xi_1 y_1 + \dots + \xi_k y_k$ is k-dimensional standard normal distributed on the linear subspace $\operatorname{lin} C^B(l_1,\dots,l_k)$. Thus, we obtain for the angle of $C^B(l_1,\dots,l_k)$ the following formula:

$$\alpha(C^B(l_1,\dots,l_k)) = \mathbb{P}(N \in C^B(l_1,\dots,l_k)) = \mathbb{P}\Big(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2-l_1}} \geq \dots \geq \frac{\xi_k}{\sqrt{l_k-l_{k-1}}} \geq 0\Big). \tag{3.7}$$

Now, we are finally able to prove Theorem 3.2.

Proof of Theorem 3.2. The proof is more involved than that of Theorem 3.1. Let $1 \leq k \leq n$. Using the formulas (3.7) and (3.5) for the internal and external angles, we have

$$\begin{aligned}\upsilon_k(C^B) &= \sum_{1\leq l_1<\dots<l_k\leq n} \alpha(C^B(l_1,\dots,l_k))\alpha(N(C^B(l_1,\dots,l_k),C^B))\\ &= \sum_{1\leq l_1<\dots<l_k\leq n} \binom{2(n-l_k)}{n-l_k}\frac{\mathbb{P}\Big(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2-l_1}} \geq \dots \geq \frac{\xi_k}{\sqrt{l_k-l_{k-1}}} \geq 0\Big)}{l_1(l_2-l_1)\cdot\ldots\cdot(l_k-l_{k-1})2^{2(n-l_k)}}.\end{aligned}$$

Defining $i_1 = l_1, i_2 = l_2 - l_1, \dots, i_k = l_k - l_{k-1}$, we can change the summation of the above sum and obtain

$$\begin{aligned}&\sum_{\substack{i_1,\dots,i_k\in\mathbb{N}\\ i_1+\dots+i_k\leq n}} \binom{2(n-i_1-\dots-i_k)}{n-i_1-\dots-i_k}\frac{\mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_k}} \geq 0\Big)}{i_1 i_2\cdot\ldots\cdot i_k 2^{2(n-i_1-\dots-i_k)}}\\ &= \sum_{r=0}^{n-k}\sum_{\substack{i_1,\dots,i_k\in\mathbb{N}\\ i_1+\dots+i_k=n-r}} \binom{2r}{r}\frac{\mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_k}} \geq 0\Big)}{i_1 i_2\cdot\ldots\cdot i_k 2^{2r}}.\end{aligned} \tag{3.8}$$

Now, fix a number $r \in \{0,1,\dots,n-k\}$, a permutation $\pi \in \mathcal{S}_k$ and a vector of signs $\varepsilon = (\varepsilon_1,\dots,\varepsilon_k) \in \{\pm 1\}^k$. For each tuple $(i_1,\dots,i_k) \in \mathbb{N}^k$ satisfying $i_1+\dots+i_k = n-r$, the tuple $(i_{\pi(1)},\dots,i_{\pi(k)})$ also satisfies $i_{\pi(1)}+\dots+i_{\pi(k)} = n-r$. Thus, the inner sum in (3.8) does not change if we replace the tuple $(i_1,\dots,i_k)$ by $(i_{\pi(1)},\dots,i_{\pi(k)})$ inside the sum. Furthermore, the sum does not change if we additionally replace $\xi_1,\dots,\xi_k$ by $\varepsilon_1\xi_1,\dots,\varepsilon_k\xi_k$, respectively, since $\xi_1,\dots,\xi_k$ are independent and standard normal. Thus, we obtain

$$\begin{aligned}\upsilon_k(C^B) &= \sum_{r=0}^{n-k}\sum_{\substack{i_1,\dots,i_k\in\mathbb{N}\\ i_1+\dots+i_k=n-r}} \binom{2r}{r}\frac{\mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_k}} \geq 0\Big)}{i_1 i_2\cdot\ldots\cdot i_k 2^{2r}}\\ &= \sum_{r=0}^{n-k}\frac{1}{2^k k!}\sum_{(\varepsilon,\pi)\in\{\pm1\}^k\times\mathcal{S}_k}\sum_{\substack{i_1,\dots,i_k\in\mathbb{N}\\ i_1+\dots+i_k=n-r}} \binom{2r}{r}\frac{\mathbb{P}\Big(\frac{\varepsilon_1\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\varepsilon_2\xi_2}{\sqrt{i_{\pi(2)}}} \geq \dots \geq \frac{\varepsilon_k\xi_k}{\sqrt{i_{\pi(k)}}} \geq 0\Big)}{i_{\pi(1)} i_{\pi(2)}\cdot\ldots\cdot i_{\pi(k)} 2^{2r}}\\ &= \frac{1}{2^k k!}\sum_{r=0}^{n-k}\sum_{\substack{i_1,\dots,i_k\in\mathbb{N}\\ i_1+\dots+i_k=n-r}}\frac{\binom{2r}{r}}{i_1 i_2\cdot\ldots\cdot i_k 2^{2r}}\sum_{(\varepsilon,\pi)\in\{\pm1\}^k\times\mathcal{S}_k}\mathbb{P}\Big(\frac{\varepsilon_1\xi_1}{\sqrt{i_{\pi(1)}}} \geq \dots \geq \frac{\varepsilon_k\xi_k}{\sqrt{i_{\pi(k)}}} \geq 0\Big)\end{aligned}$$

$$= \frac{1}{2^k k!} \sum_{r=0}^{n-k} \sum_{\substack{i_1,\dots,i_k \in \mathbb{N} \\ i_1+\dots+i_k=n-r}} \frac{\binom{2r}{r}}{i_1 i_2 \cdot \ldots \cdot i_k 2^{2r}}.$$

In the last step, we used that for a given $r \in \{0,1,\dots,n-k\}$ and a composition $(i_1,\dots,i_k) \in \mathbb{N}^k$ satisfying $i_1+\dots+i_k = n-r$

$$\sum_{(\varepsilon,\pi)\in\{\pm1\}^k\times\mathcal{S}_k} \mathbb{P}\Big(\frac{\varepsilon_1\xi_{\pi(1)}}{\sqrt{i_{\pi(1)}}} \geq \dots \geq \frac{\varepsilon_k\xi_{\pi(k)}}{\sqrt{i_{\pi(k)}}} \geq 0\Big) = 1$$

and thus, also

$$\sum_{(\varepsilon,\pi)\in\{\pm1\}^k\times\mathcal{S}_k} \mathbb{P}\Big(\frac{\varepsilon_1\xi_1}{\sqrt{i_{\pi(1)}}} \geq \dots \geq \frac{\varepsilon_k\xi_k}{\sqrt{i_{\pi(k)}}} \geq 0\Big) = 1,$$

since $\xi_1,\dots,\xi_k$ are standard normal and independent and, in particular, symmetrically exchangeable. Using the representation (2.5) for the Stirling numbers of the first kind, we obtain

$$\upsilon_k(C^B) = \sum_{r=0}^{n-k} \frac{1}{2^k k!} \frac{\binom{2r}{r}}{2^{2r}} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{k!}{(n-r)!} = \sum_{r=0}^{n-k} 2^{-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{1}{(n-r)!}.$$

Note that this formula continues to hold the special case $k=0$. In order to observe this, note that since $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 1$, by convention, and $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 0$ for all $n > 0$, we have

$$\upsilon_0(C^B) = \alpha(N(\{0\}, C^B) = \binom{2n}{n} \frac{1}{2^{2n}} = \sum_{r=0}^{n} 2^{-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ 0 \end{bmatrix} \frac{1}{(n-r)!}$$

using (3.6). Thus, for $0 \leq k \leq n$, it is left to prove that

$$B(n,k) = \sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{n!}{(n-r)!} \tag{3.9}$$

holds true. In order to do so, we compare the generating functions of the sequence on the left-hand side and the sequence on the right-hand side. Recall from Proposition 2.3 that the generating function of the B-analogues of the Stirling numbers of the first kind is given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(n,k) \frac{t^n}{n!} y^k = (1-2t)^{-\frac{1}{2}(y+1)}.$$

Before we can evaluate the generating function of the right-hand side of (3.9), we consider the following sum:

$$a_k(t) := \sum_{n=k}^{\infty} \left(\sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{n!}{(n-r)!} \right) \frac{t^n}{n!}.$$

Then, we get

$$\begin{aligned} a_k(t) &= \sum_{n=k}^{\infty} \sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{t^n}{(n-r)!} \\ &= \sum_{r=0}^{\infty} \sum_{n=r+k}^{\infty} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{t^n}{(n-r)!} \end{aligned}$$

$$= \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \left(\sum_{n=r+k}^{\infty} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{(2t)^{n-r}}{(n-r)!} \right). \tag{3.10}$$

By shifting the index in the inner sum and using the generating function of the Stirling numbers of the first kind stated in (2.6), we can rewrite (3.10) to obtain

$$\begin{aligned} a_k(t) = \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \left(\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(2t)^n}{n!} \right) &= \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \frac{(-\log(1-2t))^k}{k!} \\ &= 2^{-k} \frac{(-\log(1-2t))^k}{k!} \sum_{r=0}^{\infty} \left(\frac{t}{2}\right)^r \binom{2r}{r} \\ &= 2^{-k} \frac{(-\log(1-2t))^k}{k!} \frac{1}{\sqrt{1-2t}} \end{aligned}$$

for $|t| < 1/2$. Thus, the generating function of the right-hand side of (3.9) is given by

$$\begin{aligned} \sum_{k=0}^{\infty} a_k(t) y^k &= \frac{1}{\sqrt{1-2t}} \sum_{k=0}^{\infty} 2^{-k} \frac{(-\log(1-2t))^k}{k!} y^k \\ &= \frac{1}{\sqrt{1-2t}} \sum_{k=0}^{\infty} \frac{\big(-\frac{y}{2}\log(1-2t)\big)^k}{k!} \\ &= \frac{1}{\sqrt{1-2t}} \exp\Big(-\frac{y}{2}\log(1-2t)\Big) \\ &= (1-2t)^{-\frac{1}{2}(y+1)}, \end{aligned}$$

which coincides with the generating function of the sequence $(B(n,k))_{n,k\geq 0}$. This completes the proof. □

3.3. **Type D_n.** At last, we can also compute the conical intrinsic volumes of the Weyl chambers of type D_n. They follow from the intrinsic volumes of the type B_n chambers and the additivity of the intrinsic volumes.

Theorem 3.3. *For $k = 0, 1, \ldots, n$ the k-th conical intrinsic volume of the Weyl chamber of type D_n is given by*

$$\upsilon_k(\mathcal{C}(D_n)) = \frac{D(n,k)}{2^{n-1} n!},$$

where the numbers $D(n,k)$ are given by (2.10).

Proof. Consider the Weyl chamber

$$\mathcal{C}(D_n) = \{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq |\beta_n|\}.$$

We have

$$\begin{aligned} \mathcal{C}(D_n) &= \big(\mathcal{C}(D_n) \cap \{\beta_n \geq 0\}\big) \cup \big(\mathcal{C}(D_n) \cap \{\beta_n \leq 0\}\big) \\ &= \{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq \beta_n \geq 0\} \cup \{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq -\beta_n \geq 0\}, \end{aligned}$$

where the first set on the right-hand side is the Weyl chamber of type B_n, while the second set is isometric to it. Their intersection

$$\{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq \beta_n \geq 0\} \cap \{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq -\beta_n \geq 0\}$$

$$= \{\beta_1 \geq \ldots \geq \beta_{n-1} \geq \beta_n = 0\}$$

is a Weyl chamber of type B_{n-1} if we identify $\mathbb{R}^{n-1}$ and $\mathbb{R}^{n-1} \times \{0\}$. The conical intrinsic volumes are additive functionals, see [15, Theorem 6.5.2], and thus, we obtain

$$\begin{aligned}
\upsilon_k(\mathcal{C}(D_n)) &= \upsilon_k(\{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq \beta_n \geq 0\}) \\
&\quad + \upsilon_k(\{\beta \in \mathbb{R}^n : \beta_1 \geq \ldots \geq \beta_{n-1} \geq -\beta_n \geq 0\}) - \upsilon_k(\{\beta_1 \geq \ldots \geq \beta_{n-1} \geq \beta_n = 0\}) \\
&= \frac{B(n,k)}{2^n n!} + \frac{B(n,k)}{2^n n!} - \frac{B(n-1,k)}{2^{n-1}(n-1)!} \\
&= \frac{B(n,k) - nB(n-1,k)}{2^{n-1}n!} \\
&= \frac{D(n,k)}{2^{n-1}n!}.
\end{aligned}$$

Here, we used the formula for the intrinsic volumes of the Weyl chambers of type B_n from Theorem 3.2, and in the last step the relation (2.11). □

References

[1] D. Amelunxen and P. Bürgisser. Intrinsic volumes of symmetric cones and applications in convex programming. *Math. Program.*, 149(1-2, Ser. A):105–130, 2015.

[2] D. Amelunxen and P. Bürgisser. Probabilistic analysis of the Grassmann condition number. *Found. Comput. Math.*, 15(1):3–51, 2015.

[3] D. Amelunxen and M. Lotz. Intrinsic volumes of polyhedral cones: A combinatorial perspective. *Discrete & Computational Geometry*, 58(2):371–409, jul 2017.

[4] D. Amelunxen, M. Lotz, M. B. McCoy, and J. A. Tropp. Living on the edge: phase transitions in convex programs with random data. *Information and Inference*, 3(3):224–294, jun 2014.

[5] E. S. Andersen. On the number of positive sums of random variables. *Scandinavian Actuarial Journal*, 1949(1):27–36, jan 1949.

[6] M. Bóna, editor. *Handbook of enumerative combinatorics.* Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2015.

[7] M. Drton and C. J. Klivans. A geometric interpretation of the characteristic polynomial of reflection arrangements. *Proc. Am. Math. Soc.*, 138(8):2873–2887, 2010.

[8] F. Gao. The mean of a maximum likelihood estimator associated with the Brownian bridge. *Electron. Comm. Probab.*, 8:1–5, 2003.

[9] F. Gao and R. A. Vitale. Intrinsic volumes of the brownian motion body. *Discrete & Computational Geometry*, 26(1):41–50, jan 2001.

[10] D. Hug and R. Schneider. Random conical tessellations. *Discrete & Computational Geometry*, 56(2):395–426, may 2016.

[11] Z. Kabluchko, V. Vysotsky, and D. Zaporozhets. Convex hulls of random walks, hyperplane arrangements, and Weyl chambers. *Geom. Funct. Anal.*, 27(4):880–918, 2017.

[12] C. J. Klivans and E. Swartz. Projection volumes of hyperplane arrangements. *Discrete Comput. Geom.*, 46(3):417–426, 2011.

[13] J. Pitman. *Combinatorial Stochastic Processes.* Lecture Notes in Mathematics. Springer-Verlag, 2006.

[14] R. Schneider. Combinatorial identities for polyhedral cones. *St. Petersburg Math. J.*, 29(1):209–221, 2018.
[15] R. Schneider and W. Weil. *Stochastic and Integral Geometry*. Springer Berlin Heidelberg, 2008.
[16] N. J. A. Sloane (editor). The On-Line Encyclopedia of Integer Sequences. https://oeis.org.
[17] E. Sparre Andersen. On the fluctuations of sums of random variables. *Math. Scand.*, 1:263–285, 1953.
[18] R. P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, pages 389–496. 2007.
[19] R. Suter. Two analogues of a classical sequence. *J. Integer Seq.*, 3(1):Article 00.1.8, 1 HTML document, 2000.

Institut für Mathematische Stochastik, Westfälische Wilhelms-Universität Münster, Orléans-Ring 10, 48149 Münster, Germany

Institut für Mathematische Stochastik, Westfälische Wilhelms-Universität Münster, Orléans-Ring 10, 48149 Münster, Germany