Transport of patterns by Burge transpose

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Abstract

We take the first steps in developing a theory of transport of patterns from Fishburn permutations to (modified) ascent sequences. Given a set of pattern avoiding Fishburn permutations, we provide an explicit construction for the basis of the corresponding set of modified ascent sequences. Our approach is in fact more general and can transport patterns between permutations and equivalence classes of so called Cayley permutations. This transport of patterns relies on a simple operation we call the Burge transpose. It operates on certain biwords called Burge words. Moreover, using mesh patterns on Cayley permutations, we present an alternative view of the transport of patterns as a Wilf-equivalence between subsets of Cayley permutations. We also highlight a connection with primitive ascent sequences.

Keywords: Fishburn permutation, Cayley permutation, Burge word, transpose, ascent sequence, pattern avoidance.

1 Introduction

In 2010 Bousquet-Mélou, Claesson, Dukes and Kitaev [7] introduced ascent sequences, which they used as an auxiliary set of objects that most transparently embodies the recursive structure that they discovered on (2+2)-free posets, Stoimenow's matchings and a set of pattern avoiding permutations, now called Fishburn permutations. All of these objects are enumerated by the Fishburn numbers, which is sequence A022493 in the OEIS [20]. This counting sequence has a beautiful generating function [20, 22]:

$$\sum_{n\geq 0}\prod_{k=1}^{n}\big(1-(1-x)^k\big) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + \cdots$$

Since then, ascent sequences have been studied in their own right. In particular, pattern avoiding ascent sequences have been quite thoroughly investigated [8, 10, 14, 16, 17]. The study of pattern avoidance on ascent sequences has proved itself to often

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be even more intricate than its analogue on permutations and a framework capable of producing general results is missing.

Recently, Gil and Weiner [15] studied pattern avoidance on Fishburn permutations. The main purpose of this work is to initiate the development of a theory of transport of patterns from Fishburn permutations to ascent sequences, and vice versa, aiming towards a more general understanding of pattern avoidance. Instead of ascent sequences, we use their modified version [7], whose recursive definition is given in Section 2.2. The main benefit is that modified ascent sequences as well as permutations are Cayley permutations. And Cayley permutations provide a natural setting for the transport of patterns. The necessary background on Cayley permutations and pattern avoidance is given in Section 2.1.

In Section 3 we introduce the Burge transpose of biwords. This operation provides a high-level description of a bijection ψ between modified ascent sequences and Fishburn permutations originally given by Bousquet-Mélou et al. [7]. In Section 4 we use the Burge transpose to define an equivalence relation on Cayley permutations and to equip its equivalence classes with a notion of pattern avoidance. The avoidance of a pattern on the quotient set is transported by Burge transposition to classical pattern avoidance on permutations, thus yielding a general result on the transport of patterns. This machinery can be specialized by suitably choosing representatives for the equivalence classes. The most striking example consists in a transport theorem for Fishburn permutations and modified ascent sequences: given a set of pattern avoiding Fishburn permutations, we describe an explicit construction for the basis of the corresponding set of modified ascent sequences. In a forthcoming paper, the same construction will be extensively used to derive a great number of structural and enumerative results on pattern avoiding (modified) ascent sequences by simply inspecting the corresponding Fishburn permutations. Two examples illustrating this approach are given at the end of the section.

In Section 6 we "lift" the mapping ψ^{-1}, whose domain is the set of Fishburn permutations, to a new mapping η whose domains is S, the set of all permutations. The map η encodes what we call the η-active sites of a permutation. In particular, η preserves the property of transporting patterns, thus generalizing the transport theorem for Fishburn permutations. We then characterize the image set $\eta(S)$ in terms of mesh patterns on Cayley permutations. This further allows us to characterize modified ascent sequences as pattern avoiding Cayley permutations, a noteworthy consequence of which is that the transport of patterns can be regarded as a theory of Wilf-equivalence on Cayley permutations. We close Section 6 by studying the set $\eta(S) \cap S$. This set can be described as the image under η of the set of permutations in which all sites are η-active, which in turn is shown to be in bijection with primitive ascent sequences.

In Section 7 we raise some natural questions, leaving two of them as open problems.

2 Preliminaries

2.1 Cayley permutations and pattern avoidance

A word consisting of positive integers that include at least one copy of each integer between one and its maximum value is called a *Cayley permutation* [2, 18]. We will denote by Cay_n the set of Cayley permutations on $[n] = \{1, \dots, n\}$. For instance, $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$ and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

Equivalently, a word $x_1x_2 \dots x_n$ belongs to Cay_n precisely when there is an endofunction $x : [n] \to [n]$ such that $\mathrm{Im}(x) = [k]$ for some $k \leq n$ and $x(i) = x_i$ for each i in $[n]$. We can also view x as encoding a ballot (ordered set partition) with blocks $B_1B_2 \dots B_k$ such that $i \in B_{x(i)}$. Thus, the cardinality of Cay_n is the n-th Fubini number, which is sequence A000670 in the OEIS [20].

A bijective endofunction $\pi : [n] \to [n]$ is called a *permutation* and n is said to be the length of π. We shall sometimes write permutations in so called one-line notation and thus identify π with its list of images $\pi(1)\pi(2) \cdots \pi(n)$. We will denote by id_n the identity permutation, $\mathrm{id}(i) = i$, in S_n. In fact, we shall often just write id (without the subscript) and let n be inferred by context. Denote by S_n the set of permutations of length n and by $S = \cup_{n\geq 0} S_n$ the set of permutations of any finite length. Note that $S \subseteq \mathrm{Cay}$.

Given two Cayley permutations u and v, we say that v is a *pattern* of u if u contains a subsequence $u(i_1)u(i_2) \cdots u(i_k)$, with $i_1 \leq i_2 \leq \cdots \leq i_k$, which is order isomorphic to v, that is, $u(i_s) < u(i_t)$ if and only if $v(s) < v(t)$ and $u(i_s) = u(i_t)$ if and only if $v(s) = v(t)$. The subsequence $u(i_1)u(i_2) \cdots u(i_k)$ is then called an *occurrence* of v in u. Otherwise, u *avoids* v. Denote by $\mathrm{Cay}(v)$ the set of Cayley permutations that avoid v and by $\mathrm{Cay}_n(v)$ the set $\mathrm{Cay}(v) \cap \mathrm{Cay}_n$ of Cayley permutations of length n avoiding v. For example, $S = \mathrm{Cay}(11)$ is the set of permutations. If B is a set of patterns, $\mathrm{Cay}(B)$ denotes the set of Cayley permutations avoiding every pattern in B and $\mathrm{Cay}_n(B)$ denotes $\mathrm{Cay}_n \cap \mathrm{Cay}(B)$. We use analogous notations for subsets of Cay. For instance, $\hat{A}(212, 312)$ denotes the set of modified ascent sequences (defined in Section 2.2) avoiding the two patterns 212 and 312. The containment relation is a partial order on S and downsets in this poset are called *permutation classes*. Similarly, the containment relation is a partial order on Cay and downsets in this poset are called *Cayley permutation classes*. The *basis* of a (Cayley) permutation class is the minimal set of (Cayley) permutations it avoids. For instance, the basis for S in Cay is $\{11\}$. For a more detailed introduction to permutation patterns we refer the reader to Bevan's note "Permutation patterns: basic definitions and notations" [4].

The set S of permutations can be equipped with more general notions of patterns [6, 7, 9, 11]. A *bivincular pattern* [7] of length k is a triple (σ, X, Y), where X and Y are subsets of $\{0, 1, \dots, k\}$ and $\sigma \in S_k$. An occurrence of (σ, X, Y) in a permutation $\pi \in S_n$ is then an occurrence $\pi(i_1) \cdots \pi(i_k)$ of σ (in the classical sense) such that:

- $i_{\ell+1} = i_\ell + 1$, for each $\ell \in X$;
- $j_{\ell+1} = j_\ell + 1$, for each $\ell \in Y$,

where $\{\pi(i_1), \ldots, \pi(i_k)\} = \{j_1, \ldots, j_k\}$, with $j_1 < \cdots < j_k$; by convention, $i_0 = j_0 = 0$ and $i_{k+1} = j_{k+1} = n + 1$. The set X identifies constraints of adjacency on the positions of the elements π, while the set Y, symmetrically, identifies constraints on their values. An example of a bivincular pattern is depicted in Figure 1.

By allowing more general constraints on positions and values we arrive at mesh patterns. A *mesh pattern* [6] is a pair (σ, R), where $\sigma \in S_k$ is a permutation (classical pattern) and $R \subseteq [0, k] \times [0, k]$ is a set of pairs of integers. The pairs in R identify the lower left corners of unit squares in the plot of π which specify forbidden regions. An occurrence of the mesh pattern (σ, R) in the permutation π is an occurrence of the classical pattern σ such that no other points of the permutation occur in the forbidden regions specified by R.

Two subsets of Cay are *equinumerous* if they contain the same number of Cayley permutations of each length. Equivalently, if they have the same generating function. Two sets of (generalized) patterns B_1 and B_2 are *Wilf-equivalent* if $S(B_1)$ and $S(B_2)$ are equinumerous. We extend this notion to Cayley permutations by saying that B_1 and B_2 are Wilf-equivalent (over Cay) if $\mathrm{Cay}(B_1)$ and $\mathrm{Cay}(B_2)$ are equinumerous.

2.2 Ascent sequences

Let $x : [n] \to [n]$ be an endofunction. We call $i \in [n-1]$ an *ascent* of x if $x(i) < x(i+1)$. Let $\mathrm{asc}(x)$ denote the number of ascents of x. Then x is an *ascent sequence* of length n if $x(1) = 1$ and $x(i+1) \leq 2 + \mathrm{asc}\big(x \circ \mathrm{id}_{i,n}\big)$ for each $i \in [n-1]$, where $\mathrm{id}_{i,n} : [i] \to [n]$ is the inclusion map. Let A_n be the set of ascent sequences of length n. For instance, $A_3 = \{111, 112, 121, 122, 123\}$. Note that some ascent sequences are not Cayley permutations, the smallest example of which is 12124. Note also that we depart slightly from the original definition of ascent sequences [7] in that our sequences are one-based rather then zero-based. The reason for this is that we want to bring all the families of sequences considered in this paper under one umbrella, namely that of endofunctions on $[n]$.

We shall now define the set of *modified ascent sequences* [7], denoted $\hat{A}_n$. This set has a recursive structure that is similar to, but more complicated than, that of A_n. The definition goes as follows. There is exactly one modified ascent sequence of length zero, namely the empty word. There is also exactly one modified ascent sequence of unit length, namely the single letter word 1. Suppose $n \geq 2$. Every $x \in \hat{A}_n$ is of one of two forms depending on whether the last letter forms an ascent with the penultimate letter:

- $x = \upsilon a$ and $1 \leq a \leq b$, or
- $x = \tilde{\upsilon} a$ and $b < a \leq 2 + \mathrm{asc}(\upsilon)$,

where $\upsilon \in \hat{A}_{n-1}$, the last letter of υ is b, and $\tilde{\upsilon}$ is obtained from υ by increasing each entry $c \geq a$ by one. Note that in the latter case a is the the only occurrence of the integer a in the resulting sequence. It is easy to see that $\max(x) = 1 + \mathrm{asc}(x)$ for any modified ascent sequence x and, consequently, $\hat{A}_n \subseteq \mathrm{Cay}_n$. To see that $|A_n| = |\hat{A}_n|$ we will give a bijection $x \mapsto \hat{x}$ from A_n to $\hat{A}_n$. Given an ascent sequence x, let

$$M(x, j) = x', \text{ where } x'(i) = x(i) + \begin{cases} 1 & \text{if } i < j \text{ and } x(i) \geq x(j+1), \\ 0 & \text{otherwise,} \end{cases}$$

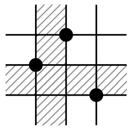
$\mathfrak{f}$ = 

Figure 1: Bivincular pattern $\mathfrak{f}$ characterizing Fishburn permutations

and extend the definition of M to multiple indices j_1, j_2, $\ldots$, j_k by

$$M(x, j_1, j_2, \ldots, j_k) = M\big(M(x, j_1, \ldots, j_{k-1}), j_k\big).$$

Then $\hat{x} = M\big(x, \mathrm{Asc}(x)\big)$, where $\mathrm{Asc}(x) = \big(i : x(i) < x(i+1)\big)$ denotes the vector of ascents of x. For example, if $x = 121242232$, then $\mathrm{Asc}(x) = (1, 3, 4, 7)$ and we get:

$$\begin{aligned} x &= 121242232 \\ M(x,1) &= \underline{12}1242232 \\ M(x,1,3) &= 1\mathit{3}\underline{12}42232 \\ M(x,1,3,4) &= 131\underline{24}2232 \\ M(x,1,3,4,7) &= 1\mathit{4}12\mathit{5}2\underline{23}2 = \hat{x} \end{aligned}$$

The construction described above can easily be inverted and thus the mapping $x \mapsto \hat{x}$ is a bijection. Indeed, the set of modified ascent sequences $\hat{A}_n$ was originally defined as the image of A_n under the $x \mapsto \hat{x}$ mapping.

2.3 Fishburn permutations

Define the bivincular pattern $\mathfrak{f} = (231, \{1\}, \{1\})$, as in Figure 1. Let $F = S(\mathfrak{f})$. We call these the *Fishburn permutations*. Bousquet-Mélou et al. [7] gave a length-preserving bijection between ascent sequences and Fishburn permutations. More precisely, ascent sequences encode the so called *active sites* of the Fishburn permutations. The term active site comes from the generating tree approach to enumeration. Each vertex in such a tree corresponds to a combinatorial object and the path from the root to a vertex encodes the choices made in the construction of the object. Regarding Fishburn permutations, let us construct an element of F_{n+1} by starting from an element of F_n and inserting a new maximum in some position. The avoidance of the pattern $\mathfrak{f}$ makes some of the positions forbidden, while the others are the active sites. More precisely, let $\pi \in F_n$ be a Fishburn permutation. For $i \in [n]$, define $J(i)$ as the index such that $\pi(J(i)) = \pi(i) - 1$. The position between $\pi(i)$ and $\pi(i+1)$ is active if and only if $J(i) < i$. Otherwise, the insertion of $n+1$ immediately after $\pi(i)$ would result in an occurrence $\pi(i), n+1, \pi(J(i))$ of $\mathfrak{f}$. Since the position after $\pi(j) = 1$ is always active, we assume $\pi(0) = 0$. Similarly, the position before $\pi(1)$ is always considered to be active. Now, the empty ascent sequence corresponds to the empty permutation. The ascent sequence corresponding to a nonempty Fishburn permutation $\pi \in F$ is constructed as follows. Start from the permutation 1 and the sequence 1. Record the position in which you insert the new maximum, step by step, until you get π. To illustrate this map consider the permutation $\pi = 61832547$. It is obtained by the following insertions, where the subscripts indicate the labels of the active sites, while positions between consecutive elements that have no subscript are forbidden sites.

$$
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "x \mapsto \hat{x}"'] & F \\
\hat{A} \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

Figure 2: How the bijections $x \mapsto \hat{x}$, φ and ψ are related

$$
\begin{aligned}
{}_01_1 \;&\xmapsto{\;x_2 = 2\;}\; {}_11_22_3 \\
&\xmapsto{\;x_3 = 2\;}\; {}_11_23\ 2_2 \\
&\xmapsto{\;x_4 = 3\;}\; {}_11_23\ 2_34_4 \\
&\xmapsto{\;x_5 = 3\;}\; {}_11_23\ 2_35\ 4_4 \\
&\xmapsto{\;x_6 = 1\;}\; {}_16\ 1_23\ 2_35\ 4_4 \\
&\xmapsto{\;x_7 = 4\;}\; {}_16\ 1_23\ 2_35\ 4_47_5 \\
&\xmapsto{\;x_8 = 2\;}\; \ 6\ 1\ 8\ 3\ 2\ 5\ 4\ 7.
\end{aligned}
$$

Therefore the ascent sequence corresponding to π is $x = 12233142$. This procedure can also be viewed as constructing π from a given ascent sequence by successive insertions of a new maximum in the active site specified by the ascent sequence. Throughout this paper we will denote this mapping from ascent sequence to Fishburn permutations by φ, so that $\varphi(x) = \pi$. For a proof that $\varphi : A \to F$ is a bijection the interested reader is again referred to Bousquet-Mélou et al. [7].

Next we recall (from [7]) the construction of a map $\psi : \hat{A} \to F$ such that $\psi(\hat{x}) = \varphi(x)$ for each ascent sequence x. It will play a central role in transporting patterns from ascent sequences to Fishburn permutations. As we will see, ψ is much easier to handle than φ. The relation between the bijections $x \mapsto \hat{x}$, φ and ψ is illustrated by the commutative diagram in Figure 2.

Let $\hat{x}$ be a modified ascent sequence. Write the integers 1 through n below it, and sort the pairs $\binom{\hat{x}(i)}{i}$ in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. The resulting bottom row is the permutation $\psi(\hat{x})$. For example, with $\hat{x} = 141252232$, the modified sequence of $x = 121242232$, we have

$$
\begin{pmatrix} \hat{x} \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} 1\,4\,1\,2\,5\,2\,2\,3\,2 \\ 1\,2\,3\,4\,5\,6\,7\,8\,9 \end{pmatrix} \longmapsto \begin{pmatrix} 1\,1\,2\,2\,2\,2\,3\,4\,5 \\ 3\,1\,9\,7\,6\,4\,8\,2\,5 \end{pmatrix} = \begin{pmatrix} \upsilon(\pi) \\ \pi \end{pmatrix}
$$

To reverse this process, annotate a given Fishburn permutation π with its active sites as in $\pi = {}_131_29764_38_42_55_6$. Write k above all entries $\pi(j)$ that lie between active sites k and $k+1$. In the example, this forms the word $\upsilon(\pi)$ above π. Then sort the pairs $\binom{k}{\pi(j)}$ in ascending order with respect to the bottom entry. This defines ψ^{-1}, the inverse of the map ψ.

It turns out that it is more natural to place the identity permutation above $\hat{x}$, rather than below it. Then

$$\binom{\upsilon(\pi)}{\pi}^T = \binom{\mathrm{id}}{\hat{x}}$$

is a special case of transposing matrices in a sense that we describe in the next section.

3 The Burge transpose

Let M_n be the set of matrices with nonnegative integer entries whose every row and column has at least one nonzero entry and are such that the sum of all entries is equal to n. For instance, M_2 consists of the following five matrices:

$$\begin{pmatrix}2\end{pmatrix}, \begin{pmatrix}1 & 1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

With each matrix $A = (a_{ij})$ in M_n we associate a biword in which any column $\binom{i}{j}$ appears a_{ij} times and the columns are sorted in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. The biwords corresponding to the five matrices above are

$$\binom{1\,1}{1\,1}, \binom{1\,1}{2\,1}, \binom{1\,2}{1\,1}, \binom{1\,2}{1\,2}, \binom{1\,2}{2\,1}.$$

Note that if i appears in the bottom row of such a biword, then each k such that $1 \leq k < i$ also appears in the bottom row. This follows from the requirement that each column of the corresponding matrix has at least one nonzero entry. In other words, the bottom row is a Cayley permutation. Similarly, the top row is a Cayley permutation. In fact, it is a weakly increasing Cayley permutation.

Let I_n be the subset of Cay_n consisting of the weakly increasing Cayley permutations:

$$I_n = \{u \in \mathrm{Cay}_n : u(1) \leq u(2) \leq \cdots \leq u(n)\}.$$

To ease notation we will often write biwords as pairs. As an example, the first two biwords in the list corresponding to matrices in M_2 would be written $(11, 11)$ and $(11, 21)$. In general, the set of biwords corresponding to matrices in M_n is

$$\mathrm{Bur}_n = \{(u, v) \in I_n \times \mathrm{Cay}_n : D(u) \subseteq D(v)\},$$

where $D(v) = \{i : v(i) \geq v(i+1)\}$ is the set of weak descents of v. We shall call the elements of Bur_n *Burge words.* This terminology is due to Alexandersson and Uhlin [1]. The connection to Burge is with his variant of the RSK correspondence [5]. Since u is weakly increasing we have $D(u) = \{i : u(i) = u(i+1)\}$. In particular,

$$|\mathrm{Bur}_n| = \sum_{v \in \mathrm{Cay}_n} 2^{\mathrm{des}(v)},$$

where $\mathrm{des}(v) = |D(v)|$ is the number of weak descents in v. This is sequence A120733 in the OEIS [20].

The simple operation of transposing a matrix in M_n turns out to be surprisingly useful. Assume that $A = (a_{ij}) \in M_n$ and that w is its corresponding biword in Bur_n.

Let w^T denote the biword corresponding to the transpose $A^T = (a_{ji})$ of A. It is easy to compute w^T without taking the detour via the matrix A. Turn each column of w upside down and then sort the columns as previously described. In particular, if π is a permutation, then

$$(\mathrm{id}, \pi)^T = (\mathrm{id}, \pi^{-1}).$$

Also, if $\pi = \psi(x)$ is the Fishburn permutation corresponding to the modified ascent sequence x and $\upsilon(\pi)$ is as described in the previous section. Then

$$\big(\upsilon(\pi), \pi\big)^T = \big(\,\mathrm{id}, x\,\big).$$

Let $\mathcal{F}_n = \{(\upsilon(\pi), \pi) : \pi \in F_n\}$ and $\hat{\mathcal{A}}_n = \{(\mathrm{id}, x) : x \in \hat{A}_n\}$. Then the correspondence between Fishburn permutations and modified ascent sequences is the identity

$$\mathcal{F}_n^T = \hat{\mathcal{A}}_n,$$

where $\mathcal{F}_n^T = \{w^T : w \in \mathcal{F}_n\}$ is the image of $\mathcal{F}_n$ under T.

It is clear that Bur_n is closed under transpose. In fact, the transpose gives an alternative characterization of the set Bur_n.

Lemma 3.1. *Let $w = (u, v) \in I_n \times \mathrm{Cay}_n$. Then $D(u) \subseteq D(v)$ if and only if $(w^T)^T = w$. Moreover, T is an involution on* Bur_n.

Proof. By definition of T, the biword w^T is a Burge word. Therefore $(I_n \times \mathrm{Cay}_n)^T \subseteq \mathrm{Bur}_n$. If $D(u) \subseteq D(v)$, then both w and $(w^T)^T$ are Burge words, and since they share the same set of columns, we must have $w = (w^T)^T$. Conversely, suppose that $(w^T)^T = w$. Then $w = z^T$, for $z = w^T$, and so w is a Burge word, or, equivalently, $D(u) \subseteq D(v)$, as desired. That T is an involution on Bur_n is immediate. □

It is well known that the n-th Eulerian polynomial evaluated at 2 equals the n-th Fubini number. That is,

$$|\mathrm{Cay}_n| = \sum_{\pi \in S_n} 2^{\mathrm{des}(\pi)} \tag{1}$$

The following proof is taken from Stanley [21]. To each pair (π, E), with $\pi \in S_n$ and $E \subseteq D(\pi)$, we bijectively associate a ballot of $[n]$: Draw a vertical bar between $\pi(i)$ and $\pi(i+1)$ if i is an ascent or $i \in E$. Thus, if $\pi = 319764825$ and $E = \{1, 5\} \subseteq D(\pi) = \{1, 3, 4, 5, 7, 8\}$ we get the ballot 3|1|976|4|82|5.

We shall reformulate this proof in terms of the transpose of Burge words. First a definition. The *direct sum* $u \oplus v$ of two Cayley permutations u and v is the concatenation uv', where v' is obtained from v by adding $\max(u)$ to each of its elements. For instance, $12 \oplus 1112 \oplus 11 \oplus 1 = 123334556$. We further extend the direct sum to sets U and V of Cayley permutations:

$$U \oplus V = \{u \oplus v : u \in U, v \in V\}.$$

Let us now return to the proof of Equation 1. Let π be a permutation of $[n]$. A *descending run* of π is a maximal sequence of consecutive descending letters $\pi(i) > \pi(i+1) > \cdots > \pi(i+d-1)$. Let $\pi = B_1 B_2 \cdots B_t$ be the decomposition of π into descending runs and let $\ell(i) = |B_i|$ be the length of the i-th descending run.

The descending runs of the example permutation $\pi = 319764825$ are 31, 9764, 82 and 5. The lengths of those runs are 2, 4, 2 and 1. The next step is to pick a weakly increasing Cayley permutation that is a direct sum of sequences of the same lengths as the descending runs. That is, we will pick u from $I_2 \oplus I_4 \oplus I_2 \oplus I_1$. Since $|I_k| = 2^{k-1}$ there are $2 \cdot 8 \cdot 2 \cdot 1 = 32$ possible choices for u. Say we pick $u = 12 \oplus 1112 \oplus 11 \oplus 1 = 123334556$. Then

$$\binom{u}{\pi}^T = \binom{1\,2\,3\,3\,3\,4\,5\,5\,6}{3\,1\,9\,7\,6\,4\,8\,2\,5}^T = \binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{2\,5\,1\,4\,6\,3\,3\,5\,3} = \binom{\mathrm{id}}{v}$$

and the resulting Cayley permutation is $v = 251463353$, which encodes the same ballot, $\{3\}\{1\}\{9,7,6\}\{4\}\{8,2\}\{5\}$, as in the previous example.

For $\pi \in S_n$, let

$$I(\pi) = I_{\ell(1)} \oplus \cdots \oplus I_{\ell(t)},$$

where t is the number of descending runs of π, or, equivalently,

$$I(\pi) = \{u \in I_n : D(u) \subseteq D(\pi)\}. \tag{2}$$

Define the set $B(\pi) \subseteq \mathrm{Cay}_n$ by

$$\big(I(\pi) \times \{\pi\}\big)^T = \{\mathrm{id}\} \times B(\pi).$$

We call $B(\pi)$ the *Fishburn basis* of π. The reason will become evident later. In particular,

$$|B(\pi)| = |I(\pi)| = 2^{\mathrm{des}(\pi)}. \tag{3}$$

Alternatively, let the underlying permutation of a ballot be obtained by sorting elements within blocks decreasingly and then removing the curly brackets. Thus, the underlying permutation of $\{3\}\{1\}\{9,7,6\}\{4\}\{8,2\}\{5\}$ is 319764825. This defines a natural surjection from ballots to permutations and $B(\pi)$ is exactly the collection of encodings of ballots whose underlying permutation is π. In particular,

$$\bigcup_{\pi \in S_n} B(\pi) = \mathrm{Cay}_n$$

in which the union is disjoint. Equation 1 follows.

4 The transport theorem

Consider the map $\Gamma : \mathrm{Bur}_n \to \mathrm{Cay}_n$ defined by

$$\binom{u}{v}^T = \binom{y}{\Gamma(u,v)},$$

for any $w = (u,v) \in \mathrm{Bur}_n$. Let us write $\mathrm{sort}(v)$ for the word obtained by sorting v in weakly increasing order. Then $y = \mathrm{sort}(v)$ and, since T is an involution, $u = \mathrm{sort}(\Gamma(u,v))$.

Definition 4.1. Let $E \subseteq \mathrm{Cay}$. A *Burge labeling* on E is a map $\lambda : E \to I$ such that $(\lambda(x), x)$ is a Burge word for each $x \in E$. Equivalently, $D(\lambda(x)) \subseteq D(x)$.

Let λ be a Burge labeling on E. Then λ induces a map $\Gamma_\lambda : E \to \mathrm{Cay}$ by

$$\Gamma_\lambda(x) = \Gamma(\lambda(x), x).$$

If λ is injective, then Γ_λ is also injective. Indeed suppose that $\Gamma_\lambda(x) = \Gamma_\lambda(y)$. Then $\lambda(x) = \mathrm{sort}(\Gamma_\lambda(x)) = \mathrm{sort}(\Gamma_\lambda(y)) = \lambda(y)$ and thus $x = y$, if λ is injective.

This construction becomes particularly meaningful for specific labelings. Let ι : $\mathrm{Cay} \to I$ be defined by $\iota(x) = \mathrm{id}_n$, for each $x \in \mathrm{Cay}_n$. Note that ι is a Burge labeling on Cay, since $D(\iota(x)) = \emptyset$. From now on, let $\gamma = \Gamma_\iota$.

Lemma 4.2. *We have* $\mathrm{sort}(x) = x \circ \gamma(x)$ *for each Cayley permutation* x.

Proof. We have

$$\big(\mathrm{id}_n, x\big)^T = \big(\mathrm{sort}(x), \gamma(x)\big) = \big(\mathrm{sort}(x), \mathrm{id}_n \circ \gamma(x)\big)$$

and thus $\mathrm{sort}(x) = x \circ \gamma(x)$ as claimed. □

Remark 4.3. If $\pi \in S_n$, then $\mathrm{id}_n = \mathrm{sort}(\pi) = \pi \circ \gamma(\pi)$ by Lemma 4.2. That is, $\gamma(\pi) = \pi^{-1}$. In this sense, $\gamma : \mathrm{Cay} \to S$ generalizes the permutation inverse to Cay.

Remark 4.4. Recall that, for any modified ascent sequence x, we have

$$\big(\mathrm{id}, x\big)^T = \big(\mathrm{sort}(x), \psi(x)\big),$$

where $\psi : \hat{A} \to F$ is the bijection described in Section 2.3. Thus, restricting ι to $\hat{A}$ gives the map $\Gamma_{\iota_{|\hat{A}}} = \psi$. That is, $\gamma_{|_{\hat{A}}} = \psi$ and in this sense γ generalizes ψ to Cay. On the other hand, consider the map $\upsilon : F \to I$ introduced in Section 2.3. It is easy to see that υ is a Burge labeling on F and $\Gamma_\upsilon : F \to \hat{A}$ is equal to ψ^{-1}, the inverse map of ψ.

Next we use the the map γ to define an equivalence relation $\sim$ on Cay: let $x \sim y$ if and only if $\gamma(x) = \gamma(y)$. Denote by $[x]$ the equivalence class

$$[x] = \{y \in \mathrm{Cay} : x \sim y\}$$

of x, and denote by [Cay] the quotient set

$$[\mathrm{Cay}] = \{[x] : x \in \mathrm{Cay}\}.$$

Since $\sim$ is the equivalence relation induced by γ, there is a unique injective map $\tilde{\gamma}$ such that the diagram

$$\begin{tikzcd} \mathrm{Cay} \arrow[r, "\gamma"] \arrow[d, "x \mapsto {[x]}"'] & S \\ {[\mathrm{Cay}]} \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

commutes. Furthermore, since γ is surjective, $\tilde{\gamma}$ is surjective too. Indeed, for any permutation π, we have $\tilde{\gamma}([\pi^{-1}]) = \gamma(\pi^{-1}) = \pi$. Thus $\tilde{\gamma}$ is a bijection and the quotient set [Cay] is equinumerous with S, the set of permutations. By slight abuse of notation we will write γ for $\tilde{\gamma}$ as well. That is, we have two functions $\gamma : \mathrm{Cay} \to S$ and $\gamma : [\mathrm{Cay}] \to S$, and it should be clear from the context which one is referred to.

Below we show that the relation $\sim$ does not depend on our choice of ι as Burge labeling in the definition of $\gamma = \Gamma_\iota$.

Lemma 4.5. *If $x, y \in \mathrm{Cay}_n$ and $x \sim y$, then $\Gamma(u, x) = \Gamma(u, y)$ for each $u \in I_n$.*

Proof. Let $\pi = \gamma(x) = \gamma(y)$. Note that $\pi \in S_n$. Let $u \in I_n$. By definition of Γ we have $(u, x)^T = (\mathrm{sort}(x), \Gamma(u, x))$. Moreover, by Lemma 4.2, $\mathrm{sort}(x) = x \circ \pi$ and hence

$$(u, x)^T = \big(x \circ \pi, \Gamma(u, x)\big).$$

It follows that $\Gamma(u, x) = u \circ \pi$. Similarly, $\Gamma(u, y) = u \circ \pi$, concluding the proof. □

The equivalence class of x is none other than the Fishburn basis of $\pi = \gamma(x)$:

Lemma 4.6. *Let $x \in \mathrm{Cay}$. Then $[x] = B(\gamma(x))$. Moreover, for each permutation π, we have $B(\pi) = [\pi^{-1}]$.*

Proof. Let $\pi = \gamma(x) \in S_n$. We will start by showing the inclusion $B(\pi) \subseteq [x]$. Let $y \in B(\pi)$. By definition of Fishburn basis we have

$$(\mathrm{id}, y)^T = (\mathrm{sort}(y), \pi)$$

and $\mathrm{sort}(y) \in I(\pi)$. On the other hand, by definition of γ we have

$$(\mathrm{id}, y)^T = (\mathrm{sort}(y), \gamma(y)).$$

Thus $\gamma(y) = \pi = \gamma(x)$ and $y \in [x]$. Conversely, let $y \in [x]$; that is, $\gamma(y) = \pi$. Then

$$(\mathrm{id}, y)^T = (\mathrm{sort}(y), \gamma(y)) = (\mathrm{sort}(y), \pi).$$

We need to show that $\mathrm{sort}(y) \in I(\pi)$. Since $(\mathrm{sort}(y), \pi) \in \mathrm{Bur}_n$ we have $D(\mathrm{sort}(y)) \subseteq D(\pi)$, which in turn is equivalent to $\mathrm{sort}(y) \in I(\pi)$ by Equation 2. Thus $B(\pi) = [x]$ as claimed. Finally, $\gamma(\pi^{-1}) = \Gamma(\mathrm{id}, \pi^{-1}) = \pi$, and therefore $\pi^{-1} \in B(\pi)$. □

The pattern containment relation on Cayley permutations can be extended to Burge words as follows. For (u', v') in Bur_k and (u, v) in Bur_n, let $(u', v') \leq (u, v)$ if there is an increasing injection $\alpha : [k] \to [n]$ such that $u \circ \alpha$ and $v \circ \alpha$ are order isomorphic to u' and v', respectively. As an important special case, $(\mathrm{id}_k, v') \leq (\mathrm{id}_n, v)$ if and only if $v' \leq v$.

Lemma 4.7. *Let $x \in \mathrm{Cay}_n$ and $y \in \mathrm{Cay}_k$.*

1. *If $x \geq y$, then $(\mathrm{id}, x)^T \geq (\mathrm{id}, y)^T$.*
2. *If $\gamma(x) \geq \gamma(y)$, then there exists $y' \in [y]$ such that $x \geq y'$.*

Proof. Let $x(i_1)\cdots x(i_k)$ be an occurrence of y in x. We have

$$\begin{pmatrix}\mathrm{id}_n\\ x\end{pmatrix}^T = \begin{pmatrix}\cdots & i_1 & \cdots & i_k & \cdots\\ \cdots & x(i_1) & \cdots & x(i_k) & \cdots\end{pmatrix}^T = \begin{pmatrix}\mathrm{sort}(x)\\ \gamma(x)\end{pmatrix}.$$

Let $\alpha : [k] \to [n]$ be the increasing injection defined by $\alpha(1) = i_1$, $\alpha(2) = i_2$, etc. We wish to show that $\mathrm{sort}(x) \circ \alpha$ and $\gamma(x) \circ \alpha$ are order isomorphic to $\mathrm{sort}(y)$ and $\gamma(y)$, respectively. In other words, the elements $i_1, \ldots, i_k$ are mapped to an occurrence of $\gamma(y)$ in $\gamma(x)$ under transposition; and the elements $x(i_1), \ldots, x(i_k)$ to an occurrence of $\mathrm{sort}(y)$. The relative order of any pair of columns is not affected by the remaining columns when T is applied. In particular, since $i_1 < \cdots < i_k$ and $x(i_1)\cdots x(i_k)$ is an occurrence of y, the relative order of the columns $\big(i_1, x(i_1)\big), \ldots, \big(i_k, x(i_k)\big)$ in $(\mathrm{id}_n, x)^T$ is determined by $(\mathrm{id}_k, y)^T$. More formally,

$$\big(x(i_s), i_s\big) \text{ precedes } \big(x(i_t), i_t\big) \text{ in } (\mathrm{id}_n, x)^T$$

if and only if

$$\big(y(s), s\big) \text{ precedes } \big(y(t), t\big) \text{ in } (\mathrm{id}_k, y)^T.$$

Since $(\mathrm{id}_k, y)^T = (\mathrm{sort}(y), \gamma(y))$, the first statement follows.

For the second statement, let $\pi = \gamma(x)$ and $\sigma = \gamma(y)$. By hypothesis, π contains an occurrence $\pi(i_1)\cdots\pi(i_k)$ of σ. Let $u = \mathrm{sort}(x)$. We have

$$\begin{pmatrix}\mathrm{id}_n\\ x\end{pmatrix}^T = \begin{pmatrix}u\\ \pi\end{pmatrix} = \begin{pmatrix}\cdots & u(i_1) & \cdots & u(i_k) & \cdots\\ \cdots & \pi(i_1) & \cdots & \pi(i_k) & \cdots\end{pmatrix}.$$

We shall prove that there is a Cayley permutation $y' \in [y]$ such that the elements $u(i_1), \ldots, u(i_k)$ map to an occurrence of y' in x under transpose. Since $(u, \pi) = (\mathrm{id}_n, x)^T$ and T is an involution on Bur_n we have

$$(u, \pi)^T = (\mathrm{id}_n, x).$$

Now, let $u' \in I_k$ be the unique weakly increasing Cayley permutation such that $u(i_1)\cdots u(i_k)$ is an occurrence of u' in u. Then, as in the previous case, the relative order of the columns $\big(u(i_1), \pi(i_1)\big), \ldots, \big(u(i_k), \pi(i_k)\big)$ in $(u, \pi)^T$ is determined by $(u', \sigma)^T$. Let y' be defined by $(u', \sigma)^T = (\mathrm{id}_k, y')$. It now only remains to show that $y' \in [y]$. That is, we need to show that $\gamma(y') = \gamma(y)$. Since $\sigma = \gamma(y)$ and T is an involution, we have

$$\big(\mathrm{sort}(y'), \gamma(y')\big) = (\mathrm{id}_k, y')^T = (u', \sigma) = (u', \gamma(y)).$$

and thus $\gamma(y') = \gamma(y)$. This completes the proof. ∎

Let us now define Burge versions of the sets Cay and S:

$$\mathcal{C}ay = \{(\mathrm{id}, x) : x \in \mathrm{Cay}\} \quad \text{and} \quad \mathcal{S} = \{(u, \pi) \in \mathrm{Bur} : \pi \in S\}.$$

Consider the equivalence relation on $\mathcal{C}ay$ induced by γ. That is, $(\mathrm{id}, x) \sim (\mathrm{id}, y)$ if $\gamma(x) = \gamma(y)$. Denote the equivalence class of (id, x) by $[(\mathrm{id}, x)]_{\mathcal{C}}$ and define $[\mathcal{C}ay]$ as the quotient set $\{[(\mathrm{id}, x)]_{\mathcal{C}} : (\mathrm{id}, x) \in \mathcal{C}ay\}$. Define an equivalence relation on $\mathcal{S}$ by $(u, \pi) \sim (v, \sigma)$ if $\pi = \sigma$. Denote the equivalence class of (u, π) by $[(u, \pi)]_{\mathcal{S}}$ and define $[\mathcal{S}]$ as the quotient set $\{[(u, \pi)]_{\mathcal{S}} : (u, \pi) \in \mathcal{S}\}$.

Lemma 4.8. *For any biword $w \in \mathcal{C}ay$ we have $[w]_{\mathcal{C}}^{T} = [w^T]_{\mathcal{S}}$.*

Proof. Let $x \in \mathrm{Cay}_n$. By definition we have

$$\begin{aligned}[(\mathrm{id}, x)]_{\mathcal{C}}^{T} &= \{(\mathrm{id}, x')^T : x' \in [x]\} \\ &= \{(\mathrm{sort}(x'), \gamma(x)) : x' \in [x]\}\end{aligned}$$

and

$$\begin{aligned}\big[(\mathrm{id}, x)^T\big]_{\mathcal{S}} &= \big[(\mathrm{sort}(x), \gamma(x))\big]_{\mathcal{S}} \\ &= \{(u, \gamma(x)) : u \in I_n, D(u) \subseteq D(\gamma(x))\}\end{aligned}$$

It thus suffices to show that

$$\{(\mathrm{sort}(x'), \gamma(x)) : x' \in [x]\} = \{(u, \gamma(x)) : u \in I_n, D(u) \subseteq D(\gamma(x))\}.$$

Since $(\mathrm{sort}(x'), \gamma(x)) = (\mathrm{id}, x')^T$ is a Burge word we have $D(\mathrm{sort}(x')) \subseteq D(\gamma(x))$ and hence the left-hand side is a subset of the right-hand side. For the reverse inclusion, let $u \in I_n$ be such that $D(u) \subseteq D(\gamma(x))$, and let x' be defined by $(u, \gamma(x))^T = (\mathrm{id}, x')$. Then $(u, \gamma(x))^T = (\mathrm{id}, x')^T = (\mathrm{sort}(x'), \gamma(x'))$ which gives $u = \mathrm{sort}(x')$ and $\gamma(x) = \gamma(x')$ (i.e. $x' \in [x]$). This concludes the proof. $\square$

We extend the notion of pattern containment to [Cay] by $[x] \geq [y]$ if $x' \geq y'$ for some $x' \in [x]$ and $y' \in [y]$. Following the same template, pattern containment on $[\mathcal{C}ay]$ and $[\mathcal{S}]$ is defined by $[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}}$ if $x' \geq y'$ for some $x' \in [x]$ and $y' \in [y]$; and $[(u, \pi)]_{\mathcal{S}} \geq [(v, \sigma)]_{\mathcal{S}}$ if $(u', \pi) \geq (v', \sigma)$ for some $(u', \pi) \in [(u, \pi)]_{\mathcal{S}}$ and $(v', \sigma) \in [(v, \sigma)]_{\mathcal{S}}$.

Lemma 4.9. *Let $x, y \in \mathrm{Cay}$. The following two statements are equivalent:*

1. *$[x] \geq [y]$.*
2. *For each $x' \in [x]$, there exists $y' \in [y]$ such that $x' \geq y'$.*

Proof. Suppose that $[x] \geq [y]$. That is, there are two Cayley permutations $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$. By Lemma 4.7, we have $\gamma(x') \geq \gamma(y')$. Let $\bar{x} \in [x]$. Then $\gamma(\bar{x}) = \gamma(x') \geq \gamma(y')$. Thus, again by Lemma 4.7, there exists $\bar{y} \in [y'] = [y]$ such that $\bar{x} \geq \bar{y}$, as desired. The other implication is trivial. $\square$

Proposition 4.10. *The containment relation is a partial order on* [Cay] *and* $[\mathcal{C}ay]$.

Proof. Reflexivity is trivial. To show transitivity, suppose that $[x] \geq [y]$ and $[y] \geq [z]$. Then there are $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$. Further, since $[y] \geq [z]$ and $y' \in [y]$ there is $z' \in [z]$ such that $y' \geq z'$ by Lemma 4.9. Thus $x' \geq z'$ and $[x] \geq [z]$. It remains to show antisymmetry. If $[x] \geq [y]$ and $[y] \geq [x]$, then, as in the proof of transitivity, there are three elements $x' \in [x]$, $y' \in [y]$ and $x'' \in [x]$ such that $x' \geq y' \geq x''$. But then $x' = x''$, since Cayley permutations in the same equivalence class have the same length. Thus $x' = y' = x''$ and $[x] = [y]$. Note that $(\mathrm{id}, x) \sim (\mathrm{id}, y)$ if and only if $x \sim y$ and $[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}}$ if and only if $[x] \geq [y]$. Therefore pattern containment is a partial order on $[\mathcal{C}ay]$ as well. $\square$

As before we allow ourselves to apply the Burge transpose to both Burge words and sets of such words. With these definitions we can now formulate the transport theorem on Burge words. As a slogan one might express it as “Burge transpose commutes with avoids”.

Theorem 4.11 (The transport theorem on Burge words)**.** *Let* $x, y \in \mathrm{Cay}$*. Then*

$$[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}} \iff [(\mathrm{id}, x)^T]_{\mathcal{S}} \geq [(\mathrm{id}, y)^T]_{\mathcal{S}}$$

or, equivalently,

$$\big([\mathcal{C}ay][(\mathrm{id}, y)]_{\mathcal{C}}\big)^T = \mathcal{S}[(\mathrm{id}, y)^T]_{\mathcal{S}}.$$

Proof. Suppose that $[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}}$. Then there are two Cayley permutations $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$. By Lemma 4.7, we have $(\mathrm{id}, x')^T \geq (\mathrm{id}, y')^T$. Since $(\mathrm{id}, x')^T \in [(\mathrm{id}, x)^T]_{\mathcal{S}}$ and $(\mathrm{id}, y')^T \in [(\mathrm{id}, y)^T]_{\mathcal{S}}$ we have $[(\mathrm{id}, x)^T]_{\mathcal{S}} \geq [(\mathrm{id}, y)^T]_{\mathcal{S}}$. Conversely, suppose that $[(\mathrm{id}, x)^T]_{\mathcal{S}} \geq [(\mathrm{id}, y)^T]_{\mathcal{S}}$; that is, there are two Cayley permutations $x' \in [x]$ and $y' \in [y]$ such that $(\mathrm{sort}(x'), \gamma(x)) \geq (\mathrm{sort}(y'), \gamma(y))$. In particular, $\gamma(x) \geq \gamma(y)$. By Lemma 4.7 there is a Cayley permutation $y'' \in [y]$ such that $x \geq y''$, which in turn is equivalent to $(\mathrm{id}, x) \geq (\mathrm{id}, y'')$. This implies $[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}}$, completing the proof. □

Next we use Theorem 4.11 to derive a transport theorem between permutations and equivalence classes of Cayley permutations. As previously noted, given $x, y \in \mathrm{Cay}$, we have $[(\mathrm{id}, x)]_{\mathcal{C}} \geq [(\mathrm{id}, y)]_{\mathcal{C}}$ if and only if $[x] \geq [y]$. This can be expressed by saying that pattern involvement on $[\mathcal{C}ay]$ depends solely on the second component of the biwords. The same happens for $[\mathcal{S}]$, as showed in the next lemma.

Lemma 4.12. *For* $(u, \pi), (v, \sigma) \in \mathcal{S}$ *we have* $[(u, \pi)]_{\mathcal{S}} \geq [(v, \sigma)]_{\mathcal{S}} \iff \pi \geq \sigma$*.*

Proof. Suppose that $[(u, \pi)]_{\mathcal{S}} \geq [(v, \sigma)]_{\mathcal{S}}$. Then $(u', \pi) \geq (v', \sigma)$, for some $(u', \pi) \in [(u, \pi)]_{\mathcal{S}}$ and $(v', \sigma) \in [(v, \sigma)]_{\mathcal{S}}$. Thus $\pi \geq \sigma$. Conversely, suppose that $\pi \geq \sigma$. Let $\pi = \gamma(x)$ and $\sigma = \gamma(y)$, for some $x, y \in \mathrm{Cay}$. By the first part of Lemma 4.7, there exists $y' \in [y]$ such that $x \geq y'$. Equivalently, $(\mathrm{id}, x) \geq (\mathrm{id}, y')$. Then, by the second part of the same lemma, we have $(\mathrm{sort}(x), \gamma(x)) \geq (\mathrm{sort}(y'), \gamma(y))$ and hence $[(\mathrm{sort}(x), \gamma(x))]_{\mathcal{S}} \geq [(\mathrm{sort}(y), \gamma(y))]_{\mathcal{S}}$. □

We have seen that pattern containment is a partial order on [Cay] and $[\mathcal{C}ay]$. Before moving on to the promised transport theorem, let us note that it follows immediately from Lemma 4.12 that the containment relation on $[\mathcal{S}]$ is a partial order as well.

Theorem 4.13 (The transport theorem)**.** *Let* $x, y \in \mathrm{Cay}$*. Then*

$$[x] \geq [y] \iff \gamma(x) \geq \gamma(y)$$

or, equivalently,

$$\gamma\big([\mathrm{Cay}][y]\big) = S\big(\gamma(y)\big).$$

Proof. Using Theorem 4.11 and Lemma 4.12 we have

$$\begin{aligned} [x] \geq [y] &\iff [(\mathrm{id}, x)]_{\mathfrak{C}} \geq [(\mathrm{id}, y)]_{\mathfrak{C}} \\ &\iff [(\mathrm{id}, x)^T]_{\mathcal{S}} \geq [(\mathrm{id}, y)^T]_{\mathcal{S}} \\ &\iff [(\mathrm{sort}(x), \gamma(x))]_{\mathcal{S}} \geq [(\mathrm{sort}(y), \gamma(y))]_{\mathcal{S}} \iff \gamma(x) \geq \gamma(y). \end{aligned}$$ □

If σ is a permutation, then—by Lemma 4.6—we can choose σ^{-1} as representative for the Fishburn basis of σ and thus we have the following corollary.

Corollary 4.14. *If σ is a permutation, then $\gamma(S(\sigma)) = [\mathrm{Cay}][\sigma^{-1}]$.*

A remarkable consequence is that the sets $S(\sigma)$ and $[\mathrm{Cay}][\sigma^{-1}]$ are equinumerous. In fact, we can say a bit more. By Lemma 4.6 and Equation 3 we have $|[x]| = 2^{\mathrm{des}(\gamma(x))}$, which leads to the following result relating the Eulerian polynomial on $S_n(\sigma)$ to a polynomial recording the distribution of (the logarithm of) sizes of equivalence classes in $[\mathrm{Cay}_n][\sigma^{-1}]$. The special case $t = 2$ can be seen as a generalization of Equation 1.

Corollary 4.15. *For any natural number n and permutation σ,*

$$\sum_{\pi \in S_n(\sigma)} t^{\mathrm{des}(\pi)} = \sum_{[x] \in [\mathrm{Cay}_n][\sigma^{-1}]} t^{\log |[x]|},$$

in which the logarithm is with respect to the base 2.

5 Transport of patterns from F to $\hat{A}$

Theorem 4.13 can be specialized by choosing a representative in each equivalence class of [Cay]. Among the resulting examples, the most significant one is that of transport of patterns between Fishburn permutations and modified ascent sequences. Consider the bijection $\psi = \gamma_{|_{\hat{A}}}$ of Remark 4.4. Let σ and π be Fishburn permutations. Note that $\psi^{-1}(\sigma) \in [\pi^{-1}]$. In other words, for any Fishburn permutation σ, the map ψ^{-1} picks exactly one representative in the equivalence class $[\sigma^{-1}]$. By Theorem 4.13

$$\pi \geq \sigma \iff [\psi^{-1}(\pi)] \geq [\psi^{-1}(\sigma)] = [\sigma^{-1}].$$

By Lemma 4.9 we therefore have $\pi \geq \sigma$ if and only if $\psi^{-1}(\pi) \geq \sigma'$ for some $\sigma' \in [\sigma^{-1}]$. Since $\psi : \hat{A} \to F$ is bijective and $[\sigma^{-1}] = B(\sigma)$ is the Fishburn basis of σ we obtain the following transport theorem.

Theorem 5.1 (Transport of patterns from F to $\hat{A}$)**.** *For any permutation σ and Cayley permutation y we have*

$$F(\sigma) = \gamma\big(\hat{A}[\sigma^{-1}]\big) \quad \text{and} \quad \gamma\big(\hat{A}[y]\big) = F\big(\gamma(y)\big)$$

In other words, the set $F(\sigma)$ of Fishburn permutations avoiding σ is mapped via the bijection ψ^{-1} to the set $\hat{A}(B(\sigma))$ of modified ascent sequences avoiding all patterns in the Fishburn basis $B(\sigma)$.

Corollary 5.2. *For any permutation σ we have $|F_n(\sigma)| = |\hat{A}_n(B(\sigma))|$.*

Recall that a constructive procedure for determining the Fishburn basis $B(\sigma)$ has been described at the end of Section 3.

Theorem 5.1 can be easily generalized to Fishburn permutations avoiding a set of patterns Σ:

$$F(\Sigma) = \gamma\big(\hat{A}[\Sigma^{-1}]\big), \text{ where } [\Sigma^{-1}] = \bigcup_{\sigma \in \Sigma} [\sigma^{-1}].$$

In a forthcoming paper, we will show many examples where the framework described in this section is fruitful in the sense that it leads to structural and enumerative results for sets of pattern avoiding modified ascent sequences by analyzing the corresponding Fishburn permutations. Two simple examples are illustrated below.

Example. Since id_k has no descents, $B(\mathrm{id}_k) = \{\mathrm{id}_k\}$ and

$$F(\mathrm{id}_k) = \gamma(\hat{A}(\mathrm{id}_k)).$$

On the other hand, for the decreasing permutation id_k^r we have $B(\mathrm{id}_k^r) = I_k$ and

$$F(\mathrm{id}_k^r) = \gamma(\hat{A}(I_k)).$$

Example. Since 231 is the classical pattern underlying the bivincular pattern $\mathfrak{f}$, we have $F(231) = S(231)$. The Fishburn basis of 231 is $B(231) = \{212, 312\}$. Thus

$$F(231) = \gamma(\hat{A}(212, 312))$$

and consequently $\hat{A}(212, 312)$ is equinumerous with $S(231)$. It is well known that $|S_n(231)|$ is the n-th Catalan number (A000108 in the OEIS [20]).

6 Picking a representative for each equivalence class

In Theorem 5.1 we exploited the maps $\psi : \hat{A} \to F$ and its inverse ψ^{-1} to transport patterns from F to $\hat{A}$. It seems natural to push this approach further by "lifting" ψ^{-1} to a map whose domain is S, the set of all permutations, thus extending the reach of Theorem 5.1. In effect, we will define a map, called η, that picks a representative for each equivalence class in [Cay]. The set of representatives will be called X and the lifted map will be $\eta : S \to X$. We now detail this construction.

Remark 4.4 shows that $\psi^{-1} = \Gamma_\upsilon$ is the map induced by the Burge labeling υ of Fishburn permutations described in Section 2.3. Let π be a Fishburn permutation. Recall that $\upsilon(\pi)$ is obtained by

1. annotating π with its active sites with respect to the Fishburn pattern $\mathfrak{f}$;
2. writing k above all entries $\pi(j)$ that lie between active sites k and $k + 1$.

Due to the avoidance of $\mathfrak{f}$, the site between $\pi(i)$ and $\pi(i + 1)$ is active if and only if $J(i) < i$, where $\pi(J(i)) = \pi(i) - 1$. In addition, the sites before $\pi(1)$ and after $\pi(j) = 1$ are always considered active. From now on, we call these sites $\mathfrak{f}$-*active*.

We wish to lift the map $\psi^{-1} : F \to \hat{A}$ to a map η, with S as its domain, by extending the labeling υ to a labeling $\tilde{\upsilon}$ on S. The lifted map η will then be $\eta = \Gamma_{\tilde{\upsilon}}$.

Let π be a permutation. The site between $\pi(i)$ and $\pi(i + 1)$ is η-*active* if $J(i) < i$ or $\pi(i) < \pi(i + 1)$. In addition, the sites before $\pi(1)$ and after $\pi(j) = 1$ are always considered η-active. The labeling $\tilde{\upsilon} : S_n \to I_n$ is defined by

1. annotating π with its η-active sites;
2. writing k above all entries $\pi(j)$ that lie between η-active sites k and $k + 1$.

Now, $\tilde{\upsilon}$ is a Burge labeling on S. Indeed the site between $\pi(i)$ and $\pi(i+1)$ is η-active if $\pi(i) < \pi(i+1)$, therefore $D(\tilde{\upsilon}(\pi)) \subseteq D(\pi)$. Next we prove that $\tilde{\upsilon} = \upsilon$ on Fishburn permutations.

n	$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
1	1	1	✓

n	$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
2	12	12	✓
	21	11	✓

n	$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
3	123	123	✓
	132	122	✓
	213	112	✓
	231	312	
	312	121	✓
	321	111	✓

n	$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
4	1234	1234	✓
	1243	1233	✓
	1324	1223	✓
	1342	1423	

n	$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
4	1423	1232	✓
	1432	1222	✓
	2134	1123	✓
	2143	1122	✓
	2314	3124	
	2341	4123	
	2413	2132	
	2431	3122	
	3124	1213	✓
	3142	1312	✓
	3214	1112	✓
	3241	3112	
	3412	3412	
	3421	3312	
	4123	1231	✓
	4132	1221	✓
	4213	1121	✓
	4231	3121	
	4312	1211	✓
	4321	1111	✓

Table 1: Permutations and corresponding members of X

Lemma 6.1. *If π is a Fishburn permutation, then $\upsilon(\pi) = \tilde{\upsilon}(\pi)$.*

Proof. We will show that each site of a Fishburn permutation π is $\mathfrak{f}$-active if and only if it is η-active. The sites before $\pi(1)$ and after $\pi(n)$ are both $\mathfrak{f}$-active and η-active by definition. Consider the site between $\pi(i)$ and $\pi(i+1)$, for $1 \leq i < n$. If the site is $\mathfrak{f}$-active, then $J(i) < i$ and thus it is also η-active. Conversely, suppose that the site is η-active. If $J(i) < i$, then it is also $\mathfrak{f}$-active. Otherwise, if $J(i) > i$, we must have $\pi(i) < \pi(i+1)$. But then $\pi(i)\pi(i+1)\pi(J(i))$ is an occurrence of $\mathfrak{f}$ in π, which is impossible. □

Since $\tilde{\upsilon}$ is a Burge labeling of S and the restriction of $\tilde{\upsilon}$ to F coincides with υ, the map $\eta = \Gamma_{\tilde{\upsilon}} : S \to \mathrm{Cay}$ lifts the map $\Gamma_\upsilon : F \to \hat{A}$. Moreover, since $\tilde{\upsilon}$ is injective, η is also injective, as shown below Definition 4.1. In other words, η picks one representative in the equivalence class $[\pi^{-1}]$, for each permutation π. If π is a Fishburn permutations, η chooses the same element as ψ^{-1}. Let

$$X = \eta(S).$$

Note that $\hat{A} \subseteq X$ by Lemma 6.1. In Table 1 we list permutations and members of X of length one through four. We also indicate which ones are Fishburn permutations and modified ascent sequences, respectively.

We defined η as a the function $\Gamma_{\tilde{\upsilon}} : S \to \mathrm{Cay}$. We then defined its range to be X.

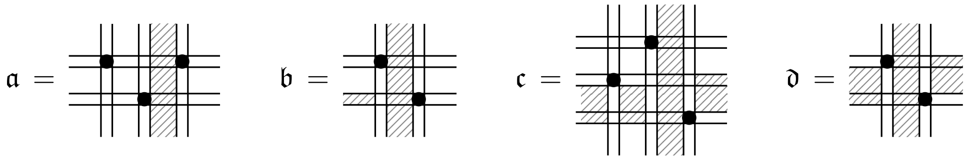


Figure 3: Mesh patterns such that $\hat{A} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b})$ and $X = \mathrm{Cay}(\mathfrak{a}, \mathfrak{c}, \mathfrak{d})$

From now on we will consider η as a function $\eta : S \to X$. It is clearly a bijection[1] and we have the following transport theorem.

Theorem 6.2 (Transport of patterns from S to X)**.** *For any permutation σ and Cayley permutation y we have*

$$S(\sigma) = \gamma\big(X[\sigma^{-1}]\big) \quad \textit{and} \quad \gamma\big(X[y]\big) = S\big(\gamma(y)\big).$$

In other words, $S(\sigma)$ is mapped via the bijection η to $X(B(\sigma))$.

As a direct consequence, $S(\sigma)$ and $X(B(\sigma))$ are equinumerous subsets of Cay.

Example. For each natural number n, we have

$$\begin{aligned} |S_n(1324)| &= |X_n(1223, 1324)|; \\ |S_n(4231)| &= |X_n(2121, 3121, 3231, 4231)|. \end{aligned}$$

The rest of this section is devoted to describing the set X. Mesh patterns on Cayley permutations were recently introduced by Cerbai [3]. They are defined like mesh patterns on permutations, but with additional regions to account for the possibility of having repeated elements. Instead of giving a formal definition, we refer the reader to [3] and Figure 3. From now on, let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ the mesh patterns depicted in Figure 3.

Lemma 6.3. *Let $x \in \hat{A}_n$ be a modified ascent sequence. An element $x(i) = k > 1$ is the leftmost occurrence of the integer k in x if and only if $x(i-1) < x(i)$.*

Proof. We proceed by induction on the length of x, using the recursive definition of $\hat{A}$ given in Section 2.2. If $n = 0$ or $n = 1$, then there is nothing to prove. Suppose $n \geq 2$ and let $x \in \hat{A}$. Let $a = x(n)$. We have either

- $x = va$ and $1 \leq a \leq b$, or
- $x = \tilde{v}a$ and $b < a \leq 2 + \mathrm{asc}(v)$,

where $v \in \hat{A}_{n-1}$, the last letter of v is b and $\tilde{v}$ is obtained from v by increasing each entry $c \geq a$ by one. By the inductive hypothesis, the thesis holds for v. Note that it holds for $\tilde{v}$ as well, since increasing by one each element greater than or equal to a certain value preserves the desired property. Now, if $1 \leq a \leq b$, then v already contains an occurrence of a (since v is a Cayley permutation) and therefore $x(n)$ is not the leftmost occurrence of a in x. Finally, if $b < a \leq 2 + \mathrm{asc}(v)$, then $x(n)$ is the only (and thus leftmost) occurrence of a in x. $\square$

[1] Is there some easier way (than using this bijection) to see that $|X_n| = n!$?

In particular, Lemma 6.3 tells us that the set of ascent tops of a modified ascent sequence x together with the first element, $x(1) = 1$, forms a permutation of length $\max(x)$. The converse of Lemma 6.3 is also true. To be precise, let $\mathrm{top}(x) = \{(1,1)\} \cup \{(i, x(i)) : 1 < i \leq n, x(i-1) < x(i)\}$ be the set of ascent tops and their indices—including the first element— and let $\mathrm{nub}(x) = \{(\min x^{-1}(j), j) : 1 \leq j \leq \max(x)\}$ be the set of first occurrences and their indices. Let x be a Cayley permutation. Then $x \in \hat{A}$ if and only if $\mathrm{top}(x) = \mathrm{nub}(x)$. This can be equivalently expressed in terms of avoidance of the two mesh patterns $\mathfrak{a}$ and $\mathfrak{b}$.

Theorem 6.4. *We have* $\hat{A} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b})$*, and hence the two sets* $\{\mathfrak{a}, \mathfrak{b}\}$ *and* $\{11, \mathfrak{f}\}$ *are Wilf-equivalent.*

Proof. Let $x \in \mathrm{Cay}_n$ be a Cayley permutation. We start by showing that if x contains $\mathfrak{a}$ or $\mathfrak{b}$, then x is not a modified ascent sequence. Suppose that $x(i)x(j)x(j+1)$ is an occurrence of $\mathfrak{a}$ in x. Then $x(j+1)$ is an ascent top and $x(j+1) = x(i)$ with $i < j+1$. Thus $x \notin \hat{A}_n$ by Lemma 6.3. Suppose that $x(i)x(i+1)$ is an occurrence of $\mathfrak{b}$ and let $k = x(i+1)$. Then $x(i+1)$ is the leftmost occurrence of k in x, but $x(i+1)$ is not an ascent top. Again, $x \notin \hat{A}_n$ by Lemma 6.3.

Conversely, suppose that x avoids both $\mathfrak{a}$ and $\mathfrak{b}$. We shall use the recursive definition of $\hat{A}$ to prove that x is a modified ascent sequence. Let $v = x(1) \cdots x(n-1)$ and let $a = x(n)$. Note that v avoids $\mathfrak{a}$ and $\mathfrak{b}$, but v is not necessarily a Cayley permutation. We distinguish the following three cases.

- If $x(n-1) > x(n)$, then $x(n)$ is not the leftmost occurrence of a in x (since x avoids $\mathfrak{b}$). Thus v is a Cayley permutation: it contains all the integers from 1 to $\max(v) = \max(x)$. By the inductive hypothesis, v is a modified ascent sequence. Since $x = va$, with $1 \leq a \leq x(n-1)$, we have that x is also a modified ascent sequence.
- If $x(n-1) = x(n)$, then v is again a Cayley permutation and we can proceed as in the previous case.
- If $x(n-1) < x(n)$, then $x(n)$ must be the only occurrence of a in x (since x avoids $\mathfrak{a}$). Because x is a Cayley permutation, the string w obtained from v by decreasing each entry $c > a$ by one must also be a Cayley permutation (that still avoids $\mathfrak{a}$ and $\mathfrak{b}$). By the inductive hypothesis, w is a modified ascent sequence and $x(n) \leq \max(w) + 1 = \mathrm{asc}(w) + 2$. Therefore x is a modified sequence (since $x = \tilde{w}x(n)$ with $x(n-1) < x(n) \leq \mathrm{asc}(w) + 2$).

Finally, $\hat{A}$ and the set of Fishburn permutations, $F = \mathrm{Cay}(11, \mathfrak{f})$, are equinumerous. Therefore the two sets $\{\mathfrak{a}, \mathfrak{b}\}$ and $\{11, \mathfrak{f}\}$ are Wilf-equivalent. ◻

Lemma 6.5. *We have* $X = \big\{x \in \mathrm{Cay} : (\tilde{v} \circ \gamma)(x) = \mathrm{sort}(x)\big\}$.

Proof. For any Cayley permutation x we have $(\mathrm{id}, x)^T = (\mathrm{sort}(x), \gamma(x))$. For any permutation π we have $(\tilde{v}(\pi), \pi)^T = (\mathrm{sort}(\pi), \eta(\pi)) = (\mathrm{id}, \eta(\pi))$. Thus

$$\begin{aligned}
x \in X \iff & x = \eta(\pi) && \text{for some } \pi \in S \\
\iff & (\tilde{v}(\pi), \pi)^T = (\mathrm{id}, x) && \text{for some } \pi \in S \\
\iff & (\tilde{v}(\pi), \pi) = (\mathrm{sort}(x), \gamma(x)) && \text{for some } \pi \in S \\
\iff & \tilde{v}(\gamma(x)) = \mathrm{sort}(x).
\end{aligned}$$

◻

Theorem 6.6. *We have* $X = \mathrm{Cay}(\mathfrak{a}, \mathfrak{c}, \mathfrak{d})$*, and hence the set* $\{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$ *is Wilf-equivalent to the pattern* 11.

Proof. Let $x \in \mathrm{Cay}_n$ be a Cayley permutation. We start by showing that if x contains $\mathfrak{a}$, $\mathfrak{c}$ or $\mathfrak{d}$, then $x \notin X$. Let $\pi = \gamma(x)$. By Lemma 6.5, it suffices to show that $\tilde{v}(\pi) \neq \mathrm{sort}(x)$. To ease notation, let $v = \tilde{v}(\pi)$.

Suppose that $x(i)x(j)x(j+1)$ is an occurrence of the pattern $\mathfrak{a}$ in x. Then $x(i) = x(j+1) > x(j)$ and

$$\begin{aligned}\binom{\mathrm{id}}{x}^T &= \begin{pmatrix} \cdots & i & \cdots & j & j+1 & \cdots \\ \cdots & x(i) & \cdots & x(j) & x(j+1) & \cdots \end{pmatrix}^T \\ &= \begin{pmatrix} \cdots & x(j) & \cdots & x(j+1) & \cdots & x(i) & \cdots \\ \cdots & j & \cdots & j+1 & \cdots & i & \cdots \end{pmatrix} = \binom{\mathrm{sort}(x)}{\pi}.\end{aligned}$$

For $\ell \in [n]$, let $K(\ell)$ be the index of the column $(x(\ell), \ell)$ in $(\mathrm{id}, x)^T$. In particular, $\mathrm{sort}(x)(K(\ell)) = x(\ell)$. Note that $K(j) < K(j+1) < K(i)$. In particular, the site in π immediately after $K(j+1)$ is η-active. Therefore $v(K(i)) > v(K(j+1))$, whereas $\mathrm{sort}(x)(K(i)) = x(i) = x(j+1) = \mathrm{sort}(x)(K(j+1))$, and hence $\mathrm{sort}(x) \neq v$.

Next, suppose that $x(i)x(j)x(j+1)$ is an occurrence of the pattern $\mathfrak{c}$ in x. Then $x(i) = x(j+1) + 1 < x(j)$ and

$$\begin{aligned}\binom{\mathrm{id}}{x}^T &= \begin{pmatrix} \cdots & i & \cdots & j & j+1 & \cdots \\ \cdots & x(i) & \cdots & x(j) & x(j+1) & \cdots \end{pmatrix}^T \\ &= \begin{pmatrix} \cdots & x(j+1) & x(K(t)) & \cdots & x(i) & \cdots & x(j) & \cdots \\ \cdots & j+1 & t & \cdots & i & \cdots & j & \cdots \end{pmatrix} = \binom{\mathrm{sort}(x)}{\pi}.\end{aligned}$$

Note that $K(j+1) < K(i) < K(j)$. Now, if $K(i) = K(j+1) + 1$, then $j+1 > i$ is a descent in π and, since $K(j) > K(j+1)$, the site in π immediately after $K(j+1)$ is not η-active. Therefore $v(K(i)) = v(K(j+1))$, whereas $\mathrm{sort}(x)(K(i)) = x(i) > x(j+1) = \mathrm{sort}(x)(K(j+1))$, and hence $\mathrm{sort}(x) \neq v$. Otherwise, consider the column $(x(t), t)$ immediately after the column $(x(j+1), j+1)$ in $(\mathrm{sort}(x), \pi)$. In other words, suppose that $K(t) = K(j+1)+1$. Since $x(i) = x(j+1)+1$, either $x(t) = x(j+1)$ or $x(t) = x(j+1)+1$. Suppose that $x(t) = x(j+1)$. We shall prove by contradiction that $v(K(t)) \neq v(K(j+1))$, and thus $\mathrm{sort}(x) \neq v$. If $v(K(t)) = v(K(j+1))$, then the site between $K(j+1)$ and $K(t)$ in π is not η-active. Therefore $j+1 > t$ is a descent. But then $x(t) = x(j+1)$ would precede $x(j+1)$ in x, which contradicts $x(i)x(j)x(j+1)$ being an occurrence of $\mathfrak{c}$ (since $x(t)$ would be placed in a forbidden region). Finally, suppose that $x(t) = x(j+1) + 1$. We wish to show that $v(K(t)) = v(K(j+1))$. By contradiction, suppose that $v(K(j+1)) < v(K(t))$; that is, the site between $K(j+1)$ and $K(t)$ is η-active. Since $K(j) > K(j+1)$, we have that $j+1 < t$ is an ascent. But then $x(t) = x(j+1) + 1 \leq x(i)$, which contradicts $x(i)x(j)x(j+1)$ being an occurrence of $\mathfrak{c}$ (again $x(t)$ would be placed in a forbidden region).

The pattern $\mathfrak{d}$ can be treated similarly, so we leave it to the reader.

Conversely, suppose that x avoids $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. Let $\pi = \gamma(x)$. We wish to prove that $x = \eta(\pi)$ or, equivalently, $\mathrm{sort}(x) = \tilde{v}(\pi)$. Due to the great amount of technical details, we just sketch the proof. To prove the contrapositive statement, suppose that $\mathrm{sort}(x) \neq \tilde{v}(\pi)$. There are two possibilities for $\mathrm{sort}(x)$ to be different from the η-labeling of π. Either $\mathrm{sort}(x)$ labels two consecutive elements $\pi(i)$ with k and $\pi(i+1)$

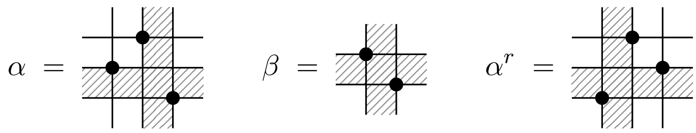


Figure 4: Bivincular patterns such that $\eta(S^0) = S(\alpha, \beta)$

with $k+1$, but i is not η-active. Or $\mathrm{sort}(x)$ labels $\pi(i)$ and $\pi(i+1)$ with the same integer k, but the site i is η-active. In the first case, $J(i) > i$ and $\pi(i) > \pi(i+1)$. If $J(i) = i+1$, then the labels k of $\pi(i)$ and $k+1$ of $\pi(i+1)$ necessarily result in an occurrence of $\mathfrak{d}$ in x. Similarly, if $J(i) > i+1$, then the labels of $\pi(i)$, $\pi(i+1)$ and $\pi(J(i))$ result in an occurrence of $\mathfrak{c}$ in x. Analogously, if $\pi(i)$ and $\pi(i+1)$ are labeled with the same integer k, but the site i is η-active, then it is possible to show that x contains an occurrence of $\mathfrak{a}$. This completes the proof. □

Theorems 6.4 and 6.6 characterize $\hat{A}$ and X as pattern avoiding Cayley permutations. As a result, we can interpret the transports of patterns described in Theorems 5.1 and 6.2 as Wilf-equivalences.

Corollary 6.7. *Let σ be a permutation.*

1. *The two sets $\{11, \mathfrak{f}, \sigma\}$ and $\{\mathfrak{a}, \mathfrak{b}\} \cup B(\sigma)$ are Wilf-equivalent. That is,*

$$|\mathrm{Cay}_n(11, \mathfrak{f}, \sigma)| \;=\; |\mathrm{Cay}_n(\mathfrak{a}, \mathfrak{b}, B(\sigma))|.$$

2. *The two sets $\{11, \sigma\}$ and $\{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\} \cup B(\sigma)$ are Wilf-equivalent. That is,*

$$|\mathrm{Cay}_n(11, \sigma)| \;=\; |\mathrm{Cay}_n(\mathfrak{a}, \mathfrak{c}, \mathfrak{d}, B(\sigma))|.$$

6.1 Permutations with no η-inactive sites

Let S^0 denote the set of permutations with no η-inactive sites. Note that if $\pi \in S^0$, then $\tilde{v}(\pi) = \mathrm{id}$, and so $\eta(\pi)$ contains no repeated letters. Indeed, $\eta(\pi) = \pi^{-1}$. Thus $\eta(S^0) = (S^0)^{-1} = X \cap S$. When restricting to permutations we can considerably simplify the mesh patterns $\mathfrak{a}$, $\mathfrak{c}$ and $\mathfrak{d}$ that characterize X: since the underlying pattern of $\mathfrak{a}$ is not a permutation we can remove it; the pattern $\mathfrak{c}$ is equivalent to the bivincular pattern $\alpha = (231, \{2\}, \{1\})$; and the pattern $\mathfrak{d}$ is equivalent to the bivincular pattern $\beta = (21, \{1\}, \{1\})$. Thus

$$\eta(S^0) = S(\alpha, \beta).$$

The patterns α and β are depicted in Figure 4.

We wish to construct a bijection between $\eta(S^0)$ and the set of ascent sequences with no flat steps (consecutive equal entries). An ascent sequence with no flat steps is said to be *primitive*. Primitive ascent sequences were enumerated by Dukes et al. [13]. Dukes and Parviainen [12] proved that primitive ascent sequences are in bijection with upper triangular matrices with non-negative entries such that all rows and columns contain at least one nonzero entry. The pattern α is closely related to the Fishburn pattern $\mathfrak{f}$. Let $\alpha^r = (132, \{1\}, \{1\})$, the reverse of α (see Figure 4). Recall from

Section 2.3 the step-wise procedure that associates each Fishburn permutation π with an ascent sequence through the construction of π from 1 by inserting a new maximum, at each step, and recording its position. Parviainen [19] observed that an alternative description of A can be obtained by performing the same construction on $S(\alpha^r)$ instead of F. The avoidance of α^r gives rise to an analogous notion of α^r-active site and the resulting bijection $\psi' : \hat{A} \to S(\alpha^r)$ can be computed using the Burge transpose by replacing $\mathfrak{f}$-active sites with α^r-active sites.

Lemma 6.8. *Let x be a modified ascent sequence and let $\pi = \psi'(x)$. Then π contains an occurrence of β^r if and only if x contains a flat step.*

Proof. Suppose that $\pi(i)\pi(i+1)$ is an occurrence of β^r in π, or, equivalently, that $\pi(i+1) = \pi(i) + 1$. Note that the site between $\pi(i)$ and $\pi(i+1)$ is not α^r-active, since inserting a new maximum $n+1$ in this position would create an occurrence $\pi(i), n+1, \pi(i+1)$ of α^r. Therefore the labels of $\pi(i)$ and $\pi(i+1)$ are equal. Since $\pi(i+1) = \pi(i) + 1$, this results in a flat step $x(\pi(i))\, x(\pi(i+1))$ in x. Conversely, suppose that $x(i)x(i+1)$ is a flat step in x. Then, by definition of Burge transpose, the elements $i+1$ and i are in consecutive positions in π, and $i+1$ precedes i. Thus $i+1, i$ is an occurrence of β, as desired. □

As a consequence of the proof of Lemma 6.8, ψ' is a bijection between the set of modified ascent sequences with no flat steps and $S(\alpha^r, \beta^r)$. Moreover, $\pi \mapsto \pi^r$ is a bijection between $S(\alpha^r, \beta^r)$ and $S(\alpha, \beta) = \eta(S^0)$. Finally, since flat steps are preserved when mapping a modified ascent sequence to its corresponding ascent sequence, we obtain by composition the desired bijection between $\eta(S^0)$ and the set of primitive ascent sequences. We close this section by stating this as a theorem.

Theorem 6.9. *There is one-to-one correspondence between permutations with no η-inactive sites and the set of primitive ascent sequences.*

7 Future directions

In this paper we have laid the theoretical foundations for the development of a theory of transport of patterns from Fishburn permutations to ascent sequences, and more generally between S and [Cay], leaving most applications for future work. Given a set of pattern avoiding Fishburn permutations, we have provided a construction for the basis of the corresponding set of modified ascent sequences. Using the bijection $\hat{A} \to A$, this result can be interpreted in terms of (plain) ascent sequences. Nevertheless, a more direct construction for a basis would be of interest.

Open Problem 7.1. *Given a permutation σ, determine a set of Cayley permutation $C(\sigma)$ such that*

$$\varphi^{-1}(F(\sigma)) = A(C(\sigma)).$$

To find analogous sets in the other direction also remains open problem.

Open Problem 7.2. *Given a Cayley permutation x, determine a set $B'(x)$ such that*

$$\psi(\hat{A}(x)) = F(B'(x)),$$

and a set $C'(x)$ such that

$$\varphi(A(x)) = F(C'(x)).$$

Understanding how the avoidance of a pattern on ascent sequences affects the corresponding set of modified ascent sequences, and vice versa, seems to be necessary if we want to answer these questions. In other words, we would like to describe the set of sequences obtained by modifying $A(x)$ in terms of avoidance of patterns, as well as the set obtained by applying the inverse construction to $\hat{A}(x)$.

Our work suggests that a natural setting for the transport of patterns is the set of Cayley permutations. Indeed, we showed how a transport theorem often can be regarded as an example of Wilf-equivalence over Cayley permutations. On the other hand, not all the ascent sequences are Cayley permutations. This raises at least two more questions. First, is there an analogue of the Burge transpose that allows us to incorporate (plain) ascent sequences in the same framework? Secondly, what natural superset Y do ascent sequences belong to? Ideally, since we would like to transport patterns between S and Y, the set Y should be equinumerous with the set of permutations. A reasonable guess could then be the set of inversion sequences, which properly contains A, but this remains to be investigated.

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