MONODROMY GROUPS OF CERTAIN KLOOSTERMAN AND HYPERGEOMETRIC SHEAVES

NICHOLAS M. KATZ AND PHAM HUU TIEP

Abstract. A certain "condition (**S**)" on reductive algebraic groups was introduced in [GT2], in which a slightly stronger condition (**S+**) was shown to have very strong consequences. We show that a wide class of Kloosterman and hypergeometric sheaves satisfy (**S+**). For this class of sheaves, we determine possible structure of their monodromy groups.

Contents

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Introduction

Given a prime p, it was conjectured by Abhyankar [Abh] and proven by Raynaud [Ray] (see also [Pop]) that any finite group G which is generated by its Sylow p-subgroups occurs as a quotient of the fundamental group of the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$. The analogous result for the multiplicative group $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$, also conjectured by Abhyankar and proven by Harbater [Har] is that any finite group G which, modulo the subgroup $\mathbf{O}^{p'}(G)$ generated by its Sylow p-subgroups, is cyclic, occurs as a quotient of the fundamental group of $\mathbb{G}_m/\overline{\mathbb{F}_p}$. In the ideal world, given such a finite group G, and a complex representation V of G, we would be able, for any prime $\ell \neq p$, to choose an embedding of $\mathbb{C}$ into $\overline{\mathbb{Q}_\ell}$, and to write down an explicit $\overline{\mathbb{Q}_\ell}$-local system on either $\mathbb{A}^1/\overline{\mathbb{F}_p}$ or on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ whose geometric monodromy group is G, in the given representation.

In some earlier papers, we have been able to do this for some particular pairs (G, V). When we were able to do this on $\mathbb{A}^1$, it was through one-parameter families of "simple to remember" exponential sums, often but not always rigid local systems on $\mathbb{A}^1$. When we have been able to do this on $\mathbb{G}_m$, it was through explicit irreducible hypergeometric sheaves.

Here we reverse this point of view, and investigate what possible (G, V) can hypergeometric sheaves give rise to? The first part of the paper is devoted to showing that for a wide class of hypergeometric sheaves $\mathcal{H}$, their geometric monodromy groups $G_{\rm geom}$ (which need not be finite) in their given representations satisfy a certain condition (**S**+) (which is a slightly strengthening of condition (**S**) introduced in [GT2], and roughly speaking, corresponds to Aschbacher's class $\mathcal{S}$ of maximal subgroups of classical groups [Asch]), see Theorems 1.7, 1.9, 1.11, and 1.12. When this condition holds, it imposes strong restrictions on the pair $(G_{\rm geom}, \mathcal{H})$. If G is infinite, then the identity component $G^\circ_{\rm geom}$ of $G_{\rm geom}$ is a simple algebraic group, still acting irreducibly. If G is finite, then either G is almost quasisimple (that is, $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for some non-abelian simple group S), or G is an "extraspecial normalizer", in particular, the dimension of the representation is a prime power r^n and there is an extraspecial r-group E in G of order r^{1+2n} acting irreducibly.

In this paper, we consider only geometrically irreducible hypergeometric sheaves, i.e., those on which $G_{\rm geom}$ acts irreducibly. One also knows that if $G_{\rm geom}$ is finite, then a generator of local monodromy at 0 is an element of G which has all distinct eigenvalues in the given representation (a "simple spectrum" element). And by Abhyankar, if $G_{\rm geom}$ is finite, then $G/\mathbf{O}^{p'}(G)$ is cyclic.

Let us say that a triple (G, V, g) satisfies *the Abhyankar condition at* p if G is a finite group such that $G/\mathbf{O}^{p'}(G)$ is cyclic, V a faithful, irreducible, finite-dimensional complex representation of G, and $g \in G$ an element of order coprime to p that has simple spectrum on V. So a natural question is which triples (G, V, g), with G a finite group, almost quasisimple or an extraspecial normalizer, that satisfy the Abhyankar condition at p, occur "hypergeometrically", that is, as $(G_{\rm geom}, \mathcal{H}, g)$ for a hypergeometric sheaf $\mathcal{H}$ and a generator $g \in G_{\rm geom}$ of local monodromy around 0 on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ (and V realizes the action of $G = G_{\rm geom}$ on $\mathcal{H}$).

Grosso modo, our main results essentially classify all such triples (G, V, g) that can arise from hypergeometric sheaves, and also determine the structure of geometric monodromy groups of hypergeometric sheaves that satisfy the condition (**S**+).

More precisely, in Theorems 6.2, 6.4, and 6.6 we classify all pairs (G, V), where G is a finite almost quasisimple group and V a faithful, irreducible, finite-dimensional complex representation of G such that some element g has simple spectrum on V. Next, in Theorem 7.4 we show that if such a group G occurs as G_{geom} for a hypergeometric sheaf in characteristic p and in addition G is a finite group of Lie type in characteristic r, then $p = r$ unless $\dim(V)$ is small. Theorem 7.5 gives an analogous result in the case G is an extraspecial normalizer. With these results in hand, we complete the classification of triples (G, V, g) that satisfy the Abhyankar condition at p, with G being almost quasisimple or an extraspecial normalizer, in §8. Further constraints for a finite group G to occur as G_{geom} of a hypergeometric sheaf are established in §§4, 5, 9. With an explicit, finite, list of exceptions, all the almost quasisimple triples (G, V, g), that satisfy the Abhyankar condition at p and in addition these extra constraints, are then shown (modulo a central subgroup) to occur hypergeometrically; the respective hypergeometric sheaves $\mathcal{H}$ are explicitly constructed in a series of companion papers [KRL], [KRLT1]–[KRLT4], [KT1]–[KT3], [KT5]–[KT8].

The hypergeometric sheaves satisfying $(\mathbf{S}+)$, but with infinite geometric monodromy groups, will be studied in a sequel to this paper.

1. The basic $(\mathbf{S}+)$ setting

1A. Conditions (S) and (S+). We work over an algebraically closed field $\mathbb{C}$ of characteristic zero, which we will take to be $\overline{\mathbb{Q}_\ell}$ for some prime ℓ in the rest of this paper. Given a nonzero finite-dimensional $\mathbb{C}$-vector space V and a Zariski closed subgroup $G \leq \mathrm{GL}(V)$, recall from [GT2, 2.1] that G (or more precisely the pair (G, V)) is said *to satisfy condition* $(\mathbf{S})$ if each of the following four conditions is satisfied.

(i) The G-module V is irreducible.
(ii) The G-module V is primitive.
(iii) The G-module V is tensor indecomposable.
(iv) The G-module V is not tensor induced.

Lemma 1.1. *Suppose* $1 \neq G \leq \mathrm{GL}(V)$ *is a Zariski closed, irreducible subgroup. Then the following statements holds.*

(i) *If G satisfies* $(\mathbf{S})$, $\dim(V) > 1$, *and* $\mathbf{Z}(G)$ *is finite, then we have three possibilities:*
  (a) *The identity component G° is a simple algebraic group, and $V|_{G^\circ}$ is irreducible.*
  (b) *G is finite, and almost quasisimple, i.e. there is a finite non-abelian simple group S such that $S \lhd G/\mathbf{Z}(G) < \mathrm{Aut}(S)$.*
  (c) *G is finite and it is an "extraspecial normalizer" (in characteristic r), that is,* $\dim(V) = r^n$ *for a prime r, and G contains a normal r-subgroup $R = \mathbf{Z}(R)E$, where E is an extraspecial r-group E of order r^{1+2n} acting irreducibly on V, and either $R = E$ or $\mathbf{Z}(R) \cong C_4$.*

(ii) $\mathbf{Z}(G)$ *is finite if and only if* $\det(G)$ *is finite.*

Proof. (i) The proof of [GT2, Prop. 2.8] (taking $H = G$) shows that one of (a)–(c) holds.

(ii) By Schur's lemma, $\mathbf{Z}(G)$ consists of scalar matrices, hence the finiteness of $\det(G)$ implies $|\mathbf{Z}(G)| < \infty$. Suppose now that $|\mathbf{Z}(G)| < \infty$. Note that the unipotent radical of G° has nonzero fixed points on V [Hum, 17.5], hence the irreducibility of $V \neq 0$ implies that G° is reductive, and so $G^\circ = T[G^\circ, G^\circ]$ with $T := \mathbf{Z}(G^\circ)^\circ$ and $[G^\circ, G^\circ] \leq \mathrm{SL}(V)$. As G/G° is finite, it suffices to show that $T \leq \mathrm{SL}(V)$.

We may assume the torus T has dimension $d \geq 1$, and let $\lambda_1, \ldots, \lambda_n$ denote the distinct weights of T acting on V. The irreducibility of V over $G \rhd T$ implies that G/G° acts transitively on $\{\lambda_1, \ldots, \lambda_m\}$; in particular, all these weights occur on V with the same multiplicity $e \geq 1$. Let $A < \mathrm{GL}_d(\mathbb{Z})$ denote the finite subgroup induced by the action of G/G° on the character group

$X(T) \cong \mathbb{Z}^d$, and let $W := X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. As A is finite, we can find an A-invariant Euclidean scalar product $(\cdot,\cdot)$ on W. Note that

$$W = [W,A] \oplus W^A, \tag{1.1.1}$$

where $[W,A] := \langle a(v) - v \mid a \in A, v \in W\rangle_{\mathbb{Q}}$ and $W^A := \{v \in W \mid a(v) = v,\ \forall a \in A\}$. [Indeed, for any $a \in A$, $v \in W$, and $w \in W^A$ we have

$$(a(v) - v, w) = (a(v), w) - (v, w) = (a(v), a(w)) - (v, w) = 0,$$

showing $[W,A] \perp W^A$. Also we have

$$|A| \cdot v = \sum_{a \in A} (v - a(v)) + \sum_{a \in A} a(v),$$

ensuring $W = [W,A] + W^A$.]

Choose a basis $\alpha_1, \ldots, \alpha_l \in X(T)$ of $[W,A]$ (over $\mathbb{Q}$). Consider any $g \in G$ and the element $a \in A$ induced by the conjugation action of g on T. Since $X(T)$ has finite rank d, we can find an integer $N_a > 0$ such that $N_a(a(\beta) - \beta) \in \langle \alpha_1, \ldots, \alpha_l\rangle_{\mathbb{Z}}$ for all $\beta \in X(T)$. As A is finite, taking $N := \mathrm{lcm}(N_a \mid a \in A)$, we have that

$$N(a(\beta) - \beta) \in \langle \alpha_1, \ldots, \alpha_l\rangle_{\mathbb{Z}}, \text{ for all } a \in A \text{ and } \beta \in X(T). \tag{1.1.2}$$

Now, if $l \leq d-1$, then $T_1 := \big(\bigcap_{j=1}^{l} \mathrm{Ker}(\alpha_j)\big)^\circ$ has dimension ≥ 1. On the other hand, by (1.1.2), for any $t \in T_1$ and any $\beta \in X(T)$, $g \in G$, we have

$$\beta(gt^N g^{-1} t^{-N}) = \beta((gtg^{-1})^N)/\beta(t^N) = \big(a(\beta)(t)\big)^N/\beta(t)^N = \big(N(a(\beta) - \beta)\big)(t) = 1$$

if g induces $a \in A$. Thus $gt^N g^{-1} = t^N$ for all $g \in G$, and so $t^N \in \mathbf{Z}(G)$ for all $t \in T_1$, a contradiction since $|\mathbf{Z}(G)| < \infty$ and $\dim T_1 \geq 1$. It follows that $l = d$, and so $W^A = 0$ by (1.1.1).

Recall that $\lambda_1, \ldots, \lambda_m$ is an A-orbit in W. Hence $\sum_{i=1}^m \lambda_i \in W^A$, and so $\sum_{i=1}^m \lambda_i = 0$. Finally, for any $t \in T$, note that

$$\det(t|_V) = \big(\prod_{i=1}^m \lambda_i(t)\big)^e = \big(e\sum_{i=1}^m \lambda_i\big)(t) = 1,$$

i.e. $T \leq \mathrm{SL}(V)$, as stated. □

Definition 1.2. A pair (G,V) is said *to satisfy the condition* $(\mathbf{S}+)$, if it satisfies $(\mathbf{S})$ and, in addition, $|\mathbf{Z}(G)|$ is finite (equivalently, $\det(G)$ is finite).

The following lemma is immediate from the definitions.

Lemma 1.3. *Given a Zariski closed subgroup $G \subset \mathrm{GL}(V)$ and a Zariski closed subgroup $H \leq G$, suppose that $(H, V|_H)$ satisfies $(\mathbf{S})$. Then (G,V) satisfies $(\mathbf{S})$. If in addition $\mathbf{Z}(G)$ is finite, then (G,V) satisfies $(\mathbf{S}+)$.*

Let us also recall the following lemma from [GT2, Lemma 2.5].

Lemma 1.4. *Given a Zariski closed subgroup $G \subset \mathrm{GL}(V)$ and a Zariski closed normal subgroup $H \lhd G$, suppose that (G,V) satisfies the first three conditions defining $(\mathbf{S})$, i.e., suppose that G is irreducible, primitive, and tensor indecomposable. Then either $H \leq \mathbf{Z}(G)$ or $V|_H$ is irreducible.*

Definition 1.5. More generally, if Γ is any group given with a finite-dimensional representation $\Phi : \Gamma \to \mathrm{GL}(V)$, then we say (Γ, V) *satisfies* $(\mathbf{S}+)$, if $(\Phi(\Gamma), V)$ satisfies the three conditions of $(\mathbf{S})$ and, in addition, $\det(\Phi(\Gamma))$ is finite.

Lemma 1.6. *Let Γ be a group, $\mathbb{C}$ an algebraically closed field of characteristic zero, $n \in \mathbb{Z}_{\geq 1}$, $\Phi : \Gamma \to \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$ a representation of Γ, and $G \leq \mathrm{GL}(V)$ the Zariski closure of $\Phi(\Gamma)$. Then (Γ, V) satisfies $(\mathbf{S+})$ if and only if (G, V) satisfies $(\mathbf{S+})$. This equivalence holds separately for each of the four conditions defining $(\mathbf{S+})$.*

Proof. If V is G-reducible, it is a fortiori Γ-reducible. Conversely, if $\Phi(\Gamma)$ stabilizes a proper subspace $U \neq 0$ of V, then, since the stabilizer of U in $\mathrm{GL}(V)$ is closed, G also stabilizes U and so is reducible on V. If V is G-imprimitive, any system of imprimitivity for G remains one for Γ. Conversely, if $\Phi(\Gamma)$ stabilizes an imprimitive decomposition $V = \oplus_{i=1}^m V_i$ of V, then, since the stabilizer of this decomposition in $\mathrm{GL}(V)$ is closed, G also stabilizes the decomposition and so is imprimitive on V. If V is tensor decomposable as a G-module, then a fortiori it is tensor decomposable for Γ. Conversely, if $\Phi(\Gamma)$ stabilizes a tensor decomposition $V = A \otimes B$ with $\dim A, \dim B > 1$, we use the fact that the image of the"Kronecker product" map $\mathrm{GL}(A) \times \mathrm{GL}(B) \to \mathrm{GL}(A \otimes B)$, namely the stabilizer $\mathrm{GL}(A) \otimes \mathrm{GL}(B)$, is closed, cf. [Hum, 7.4, Prop. B]. Therefore G also stabilizes the decomposition and so is tensor decomposable on V. The same argument shows that V is tensor induced for G if and only if it is tensor induced for Γ. Indeed if V is $V_1^{\otimes n}$ with $\dim(V_1) > 1$ and $n > 1$, use the fact that the image in $\mathrm{GL}(V_1^{\otimes n})$ of the wreath product $\mathrm{GL}(V_1) \wr \mathsf{S}_n$ is closed to see that $\Phi(\Gamma)$ lands in this image if and only if G does. If $\det(G)$ fails to be finite, then $\det(\Phi(\Gamma))$ is infinite, by the Zariski density of $\Phi(\Gamma)$ in G. If $\det(G)$ is finite, then a fortiori $\det(\Phi(\Gamma))$ is finite. □

1B. Statements of theorems of type (S+) for Kloosterman and hypergeometric sheaves. We work in characteristic p, and use $\overline{\mathbb{Q}_\ell}$-coefficients for a chosen prime $\ell \neq p$. We fix a nontrivial additive character ψ of $\mathbb{F}_p$, with values in $\mu_p(\overline{\mathbb{Q}_\ell})$. We will consider Kloosterman and hypergeometric sheaves on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ as representations of $\pi_1 := \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$, and prove that, under various hypotheses, they satisfy $(\mathbf{S+})$ as representations of π_1. As noted in Lemma 1.6, this is equivalent to their satisfying $(\mathbf{S+})$ as representations of their geometric monodromy groups.

On $\mathbb{G}_m/\overline{\mathbb{F}_p}$, we consider a Kloosterman sheaf

$$\mathcal{K}l := \mathcal{K}l_\psi(\chi_1, \ldots, \chi_D)$$

of rank $D \geq 2$, defined by an unordered list of D not necessarily distinct multiplicative characters of some finite subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$.

One knows that $\mathcal{K}l$ is absolutely irreducible, cf. [Ka-GKM, 4.1.2]. One also knows, by a result of Pink [Ka-MG, Lemmas 11 and 12] that $\mathcal{K}l$ is primitive so long as it is not Kummer induced. Recall that $\mathcal{K}l$ is Kummer induced if and only if there exists a nontrivial multiplicative character ρ such that the unordered list of the χ_i is equal to the unordered list of the $\rho\chi_i$. Thus primitivity (or imprimitivity) of $\mathcal{K}l$ is immediately visible.

Theorem 1.7. *Let $\mathcal{K}l$ be a Kloosterman sheaf of rank $D \geq 2$ in characteristic p which is primitive. Suppose that D is not 4. If $p = 2$, suppose also that $D \neq 8$. Then $\mathcal{K}l$ satisfies $(\mathbf{S+})$.*

Remark 1.8. We exclude $D = 4$ because in any odd characteristic p, there are Kloosterman sheaves of rank $D = 4$ which are 2-tensor induced, cf. [Ka-CC, Theorem 6.3].

We next consider a hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > m \geq 0$, thus

$$\mathcal{H} = \mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m).$$

Here the χ_i and ρ_j are (possibly trivial) multiplicative characters of some finite subfield $\mathbb{F}_q^\times$, with the proviso that no χ_i is any ρ_j. [The case $m = 0$ is precisely the $\mathcal{K}l$ case.] One knows [Ka-ESDE, 8.4.2, (1)] that such an $\mathcal{H}$ is lisse on $\mathbb{G}_m$, geometrically irreducible. Its local monodromy at 0 is tame, a successive extension of the χ_i. It is of finite order if and only if the χ_i are pairwise distinct, in which case that local monodromy is their direct sum $\oplus_i \chi_i$, cf. [Ka-ESDE, 8.4.2, (5)]. Its local monodromy at ∞ is the direct sum of a tame part which is a successive extension of the ρ_j, with a totally wild representation Wild_{D-m} of rank $D-m$ and Swan conductor one, i.e. it has all ∞-breaks $1/(D-m)$. It is of finite order if and only the ρ_j are pairwise distinct, in which case that local monodromy is the direct sum of $\oplus_j \rho_j$ with Wild_{D-m}. We denote by $W := D-m$ the dimension of the wild part Wild.

In the case of a hypergeometric sheaf $\mathcal{H}$ with $m > 0$, primitivity is less easy to determine at first glance, because there is also the possibility of Belyi induction, cf. [KRLT3, Proposition 1.2]. It is known that an $\mathcal{H}$ of type $(D, 1)$ is primitive unless D is a power of p, cf. [KRLT3, Cor 1.3]. It is also known [KRLT3, Proposition 1.4] that an $\mathcal{H}$ of type (D, m), with $D > m \geq 2$ and D a power of p, is primitive.

Theorem 1.9. *Let $\mathcal{H}$ be a hypergeometric sheaf of type (D, m) with $D > m > 0$, with $D \geq 4$. Suppose that $\mathcal{H}$ is primitive, $p \nmid D$, and $W > D/2$. If p is odd and $D = 8$, suppose $W > 6$. If $p \neq 3$, suppose that either $D \neq 9$, or that both $D = 9$ and $W > 6$. Then $\mathcal{H}$ satisfies* $(\mathbf{S+})$.

Remark 1.10. In the case $D = 4$, the condition $W > D/2$ is sharp. In any odd characteristic p, there are hypergeometric sheaves of type $(4, 2)$ which are 2-tensor induced, cf. [Ka-CC, Theorem 6.5]. There are also hypergeometric sheaves of type $(4, 2)$ which are tensor decomposable, cf. [Ka-CC, Theorem 5.3].

Here is a slight variant, which visibly implies the above Theorem 1.9.

Theorem 1.11. *Let $\mathcal{H}$ be a hypergeometric of type (D, m) with $D > m > 0$, with $D \geq 4$. Suppose that $\mathcal{H}$ is primitive. Suppose that $D > 4$ is prime to p. Denote by p_0 the least prime divisor of D. Suppose that either*

(i) *$D = p_0$, or*
(ii) *$D = p_0^2$ and $W > 2p_0$, or*
(iii) *D is neither p_0 nor p_0^2, and $W > D/p_0$, or*
(iv) *$D = 4$ and $W = 3$.*
(v) *$D = 8$ and $W > 6$.*

Then $\mathcal{H}$ satisfies $(\mathbf{S+})$.

In the case when p divides D, we need stronger hypotheses to show that $(\mathbf{S+})$ holds.

Theorem 1.12. *Let $\mathcal{H}$ be a hypergeometric of type (D, m) with $D > m > 0$, with $D > 4$. Suppose that $\mathcal{H}$ is primitive. Suppose that $p|D$, and $W > (2/3)(D-1)$. If $p = 2$, suppose $D \neq 8$. If $p = 3$, suppose (D, m) is not $(9, 1)$. Then $\mathcal{H}$ satisfies* $(\mathbf{S+})$.

2. Tensor indecomposability

In this section, we will prove the tensor indecomposability for the Kloosterman and hypergeometric sheaves of Theorems 1.7, 1.9, 1.11, 1.12. We being with a general statement on "linearization".

Let k be an algebraically closed field of characteristic $p > 0$, and U/k an affine curve which smooth and connected, X/k the complete nonsingular model of U/k, and ∞ a k-point of $X \setminus U$. Denote by $\pi_1(U)$ the fundamental group of U (with respect to some geometric point as base point), and denote by $I(\infty) \subset \pi_1(U)$ a choice of inertia group at ∞. Fix a choice of a prime ℓ.

Proposition 2.1. *Suppose we are given a finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space V with on which π_1 acts continuously, by a representation ρ. Suppose further that we are given an expression of the vector space V as a tensor product $V = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ of $n \geq 2$ vector spaces A_i, each of dimension ≥ 2, such that the image of $\rho(\pi_1(U))$ lands in the subgroup*

$$\mathrm{GL}(A_1) \otimes \mathrm{GL}(A_2) \otimes \cdots \otimes \mathrm{GL}(A_n) < \mathrm{GL}(V).$$

[This is the subgroup of those automorphisms of V which have (non-unique !) expressions as n-fold tensor products of automorphisms of the A_i.] *Then we have the following results.*

(i) *There exists a lifting of ρ to a homomorphism*

$$\tilde{\rho} : \pi_1(U) \to \mathrm{GL}(A_1) \times \mathrm{GL}(A_2) \times \cdots \times \mathrm{GL}(A_n).$$

(ii) *Suppose that for $i = 1$ to $n-1$, $\dim(A_i)$ is prime to p. Suppose that in the representation ρ, all the ∞-slopes are $\leq r$ for some real number $r \geq 0$, i.e., for each real $x > r$, the upper numbering subgroup $I(\infty)^{(x)}$ acts trivially on V. Then $\tilde{\rho}$ can be chosen so that each A_i (viewed as a representation of $\pi_1(U)$ by applying $\tilde{\rho}$ and then projecting onto the A_i factor) has all its ∞-slopes are $\leq r$.*

Proof. To prove the first assertion, we argue as follows. In an expression of an element of $\otimes_{i=1}^n \mathrm{GL}(A_i)$ as $\otimes_{i=1}^n \alpha_i$, we are free to multiply each α_i by an invertible scalar λ_i, so long as $\prod_i \lambda_1 = 1$. Doing this, we can move the first $n-1$ of the α_i into $\mathrm{SL}(A_i)$. In other words, we have an equality of groups

$$(\otimes_{i=1}^{n-1} \mathrm{SL}(A_i)) \otimes \mathrm{GL}(A_n) = \otimes_{i=1}^n \mathrm{GL}(A_i)$$

inside $\mathrm{GL}(V)$. So we have a short exact sequence

$$1 \to \prod_{i=1}^{n-1} \mu_{\dim(A_i)} \to (\prod_{i=1}^{n-1} \mathrm{SL}(A_i)) \times \mathrm{GL}(A_n) \to \otimes_{i=1}^n \mathrm{GL}(A_i) \to 1,$$

the first map sending $(\zeta_1, \cdots, \zeta_{n-1})$ to $(\zeta_1, \cdots, \zeta_{n-1}, 1/\prod_{i=1}^{n-1} \zeta_i)$. Now use the fact that $\pi_1(U)$ has cohomological dimension ≤ 1, to lift ρ.

If the first $n-1$ factors A_i have dimensions prime to p, then the group $\prod_{i=1}^{n-1} \mu_{\dim(A_i)}$ has order prime to p. If a given $I(\infty)^{(x)}$ with $x > r$ dies under ρ, then its image under $\tilde{\rho}$ lands in $\prod_{i=1}^{n-1} \mu_{\dim(A_i)}$. But $I(\infty)^{(x)}$ with $x > r$ is a pro-p group, so must die in the prime to p group $\prod_{i=1}^{n-1} \mu_{\dim(A_i)}$. Thus $I(\infty)^{(x)}$ with $x > r$ dies under $\tilde{\rho}$. In other words, each A_i has all its $I(\infty)$-slopes $\leq r$. $\square$

Lemma 2.2. *Let $\mathcal{K}l$ be a Kloosterman sheaf of rank $D \geq 2$ in characteristic p. Then $\mathcal{K}l$ is tensor indecomposable.*

Proof. If D is a prime number, there is nothing to prove. If D is not prime, suppose that $D = AB$ with A, B both ≥ 2. Suppose that the image of $\pi_1 := \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$ lies in $\mathrm{GL}(A) \otimes \mathrm{GL}(B)$. In view of Proposition 2.1, there exist local systems $\mathcal{A}$ and $\mathcal{B}$ on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, of ranks A and B respectively, such that we have an isomorphism $\mathcal{K}l \cong \mathcal{A} \otimes \mathcal{B}$ as representations of π_1. We argue by contradiction.

Consider first the "easy" case, in which p^2 does not divide D. Then p does not divide at least one of A or B. The largest ∞-slope of $\mathcal{K}l$ is $1/D$. In view of part (ii) of Proposition 2.1, we may choose the local systems $\mathcal{A}$ and $\mathcal{B}$ so that each of them has largest ∞-slope $\leq 1/D$. Then their Swan conductors at ∞ satisfy

$$\mathsf{Swan}_\infty(\mathcal{A}) \leq A/D < 1, \mathsf{Swan}_\infty(\mathcal{B}) \leq B/D < 1.$$

But Swan conductors are nonnegative integers, so we have $\mathsf{Swan}_\infty(\mathcal{A}) = \mathsf{Swan}_\infty(\mathcal{B}) = 0$, i.e., both $\mathcal{A}$ and $\mathcal{B}$ are tame at ∞. But then $\mathcal{K}l \cong \mathcal{A} \otimes \mathcal{B}$ is tame at ∞, contradiction.

Suppose now that $\mathcal{K}l \cong \mathcal{A} \otimes \mathcal{B}$, but both A and B are divisible by p. In this case, we use the argument of Šuch, cf. [Such, Prop. 12.1, second paragraph]. We have

$$\begin{aligned} \mathrm{End}(\mathcal{K}l) &\cong \mathrm{End}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{End}(\mathcal{A}) \otimes \mathrm{End}(\mathcal{B}) = (\mathbb{1} \oplus \mathrm{End}^0(\mathcal{A})) \otimes (\mathbb{1} \oplus \mathrm{End}^0(\mathcal{B})) \\ &= \mathbb{1} \oplus \mathrm{End}^0(\mathcal{A})) \oplus \mathrm{End}^0(\mathcal{B})) \oplus \mathrm{End}^0(\mathcal{A})) \otimes \mathrm{End}^0(\mathcal{B})). \end{aligned}$$

In particular, each of $\mathrm{End}^0(\mathcal{A})), \mathrm{End}^0(\mathcal{B}))$ is a direct factor of $\mathrm{End}(\mathcal{K}l)$. To fix ideas, assume $A \le B$. Then $A^2 \le D$, and hence $\mathrm{End}^0(\mathcal{A}))$ has rank $\le D-1$. The largest ∞-slope of $\mathcal{K}l$ is $1/D$, as is the largest slope of its dual (itself another Kloosterman sheaf of the same rank D). There $\mathrm{End}(\mathcal{K}l)$ has all ∞-slopes $\le 1/D$. Therefore $\mathrm{End}^0(\mathcal{A}))$ has $\mathsf{Swan}_\infty \le (D-1)/D < 1$. Just as above, this forces $\mathrm{End}^0(\mathcal{A}))$ to be tame at ∞. Hence also $\mathrm{End}(\mathcal{A}))$ (being the sum of $\mathrm{End}^0(\mathcal{A}))$ and $\mathbb{1}$) is tame at ∞. Thus the wild inertia group $P(\infty)$ acts trivially on $\mathrm{End}(\mathcal{A}))$, and hence acts by a scalar character on $\mathcal{A}$. Observe that $\mathcal{A}$ is $I(\infty)$-irreducible, simply because $\mathcal{A}\otimes\mathcal{B}$ is $I(\infty)$-irreducible. Recalling that $p|A$, write A as n_0q with n_0 prime to p and with q a positive power of p. From [Ka-GKM, 1.14], we know that the restriction of $\mathcal{A}$ to $P(\infty)$ is the sum of n_0 pairwise distinct irreducible representations of $P(\infty)$, each of dimension q. This contradicts having $P(\infty)$ act on $\mathcal{A}$ by a scalar character. ☐

Lemma 2.3. *Let $\mathcal{H}$ be a hypergeometric sheaf of type (D, m) with $D > m > 0$ in characteristic p. Then $\mathcal{H}$ is tensor indecomposable under each of the following hypotheses.*

(i) $D \ne 4$.
(ii) $D = 4$, *p odd, and* $(D, m) \ne (4, 2)$.
(iii) $D = 4$, $p = 2$, *and* $(D, m) \ne (4, 1)$.

Proof. This is proven in [KRLT3, Cor. 10.3], in the stronger form that under the stated hypotheses, the $I(\infty)$ representation of $\mathcal{H}$ is tensor indecomposable. ☐

3. Tensor induced sheaves

3A. Dealing with tensor induction: First steps. Given (G, V) as in the first section, and an integer $n \ge 2$, we say that (G, V) is *n-tensor induced* if $D := \dim(V)$ is an n^{th} power $D = D_0^n$ with $D_0 \ge 2$ and there exists a tensor factorization of V as $V = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ with each $\dim(A_i) = D_0$, such that $G \le (\otimes_{i=1}^n \mathrm{GL}(A_i)) \rtimes \mathsf{S}_n$, with the symmetric group S_n acting by permuting the factors.

One says that (G, V) is *not tensor induced* if it is not n-tensor induced for any $n \ge 2$.

We have the following obvious but useful lemma.

Lemma 3.1. *Given (G, V) whose dimension $D := \dim(V) \ge 2$ not a power (i.e., not an n^{th} power for any $n \ge 2$), then (G, V) is not tensor induced.*

To deal with the case when D is a power, we begin with the following lemma.

Lemma 3.2. *Let $\mathcal{F}$ be either a Kloosterman sheaf $\mathcal{K}l$ of rank $D \ge 4$ or a hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > m > 0$ and $D \ge 4$. Suppose $\mathcal{F}$ is n-tensor induced for a given $n \ge 2$. Consider the composite homomorphism*

$$\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \to (\otimes_{i=1}^n \mathrm{GL}(A_i)) \rtimes \mathsf{S}_n \to \mathsf{S}_n,$$

obtained by projecting onto the last factor. Suppose we are in either of the following four situations.

(i) *$\mathcal{F}$ is a Kloosterman sheaf of rank $D \ge 4$.*
(ii) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D \ne 4$. Denote by p_0 the least prime dividing D, and suppose we have the inequality $W > D/p_0^2$.*
(iii) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type $(4, 1)$ and p is odd.*
(iv) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type $(4, 2)$ and $p = 2$.*

Then this composite homomorphism factors through the tame quotient $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})^{\text{tame at } 0,\infty}$, and its image is an n-cycle in S_n. Moreover, n is prime to p.

Proof. Via the deleted permutation representation, we have $\mathsf{S}_n \subset O(n-1)$. View the composite homomorphism as an $n-1$ dimensional representation of π_1. It is tame at 0, and its largest ∞

slope is $\leq 1/W$. We first show that this homomorphism is tame at ∞. For this, via the inequality $\mathsf{Swan}_\infty \leq (n-1)/W$, it suffices to show that $W > n-1$.

In the Kloosterman case, $W = D$ and $D = D_0^n$ with $D_0 \geq 2$ and $n \geq 2$. So we must show in this case that $D_0^n > n-1$ for $D_0 \geq 2$ and $n \geq 2$. Hence we are done, since $2^n > n-1$ for all $n \in \mathbb{Z}_{\geq 0}$.

In the two hypergeometric cases with $D = 4$, the only possible n is $n = 2$. In both of these cases, we have $W > 1$.

In the hypergeometric case with $D \neq 4$, we are given $W > D/p_0^2 = D_0^n/p_0^2$, so it suffices to show that $D_0^n/p_0^2 \geq n-1$ for $n \geq 2$. Because $p_0|D$ and $D = D_0^n$, p_0 must divide D_0. Thus $D_0^n/p_0^2 \geq p_0^{n-2}$, and it suffices to show that $p_0^{n-2} \geq n-1$. Again for given $n \geq 2$, it suffices to show that $2^{n-2} \geq n-1$, which holds for all $n \geq 2$.

The tame quotient $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})^{\text{tame at } 0,\infty}$ is the pro-cyclic group $\prod_{\ell \neq p} \mathbb{Z}_\ell(1)$, of pro-order prime to p. So its image in S_n is a cyclic group of order prime to p. But this image must be transitive, otherwise our $\mathcal{K}l$ would be tensor decomposed (never) or our $\mathcal{H}$ would be tensor decomposed (not under the $D \neq 4$ and (D,m) not (an even power of $p, 1$) hypothesis). Thus the image is (the cyclic group generated by) an n-cycle. Because the tame quotient is pro-cyclic or pro-order prime to p, and cyclic quotient has order prime to p. Thus n is prime to p. $\square$

Corollary 3.3. *Let $\mathcal{F}$ be either a Kloosterman sheaf or a hypergeometric sheaf which satisfies one of the hypotheses of Lemma 3.2 above, if $\mathcal{F}$ is n-tensor induced for a given $n \geq 2$ (n necessarily prime to p), then we have a tensor decomposition of the Kummer pullback $[n]^\star\mathcal{F}$,*

$$[n]^\star\mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$$

with local systems $\mathcal{A}_i$ each of rank $D_0 \geq 2$. Moreover, if D is prime to p, then we can choose this tensor decomposition so that each $\mathcal{A}_i$ has all ∞ slopes $\leq n/W$.

Proof. In view of Lemma 3.2, after this Kummer pullback, π_1 lands in $\otimes_{i=1}^n \mathrm{GL}(A_i)$. Then apply the linearization Proposition 2.1. The largest ∞ slope of $[n]^\star\mathcal{F}$ is n/W, so in the case when D is prime to p, we apply part (ii) of Proposition 2.1. $\square$

3B. Tensor induction: the case when $p \nmid D$.

Proposition 3.4. *Let $\mathcal{F}$ be either a Kloosterman sheaf $\mathcal{K}l$ of rank $D > 4$ or a hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $D > m > 0$ and $D \geq 4$. Suppose further we are in one of the following three situations.*

(i) *$\mathcal{F}$ is a Kloosteman sheaf of rank $D \geq 4$ and D is prime to p..*
(ii) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $D \neq 4$ and D prime to p. Denote by p_0 the least prime dividing D, and suppose we have the inequality $W > D/p_0$. If $D = p_0^2$ (possible only if $p_0 > 2$, given that $D > 4$), suppose in addition that $W > 2p_0$. If $D = 8$, suppose in addition that $W > 6$.*
(iii) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type $(4,1)$ and $p \neq 2$.*

Then $\mathcal{F}$ is not tensor induced.

Proof. We treat first the case of a hypergeometric sheaf $\mathcal{H}$ of type $(4,1)$ in characteristic $p \neq 2$. We must show that $\mathcal{H}$ is not 2 tensor induced. If it were, then the $I(\infty)$ of $[2]^\star\mathcal{H}$ would be tensor decomposed. But its slopes are $2/3$ repeated 3 times, and 0. Thus the $I(\infty)$ of $[2]^\star\mathcal{H}$ is the sum of a one-dimensional tame part and a single wild irreducible of dimension 3, hence is not tensor decomposable, cf. [KRLT3, Cor. 10.4 (ii)].

The idea is to show that in the other cases, each $\mathcal{A}_i$ is tame at ∞. [For this, it suffices to show that its Swan conductor is < 1.] This tameness forces $[n]^\star\mathcal{F}$ to be tame at ∞, which is nonsense.

We begin with the Kloosterman case. If we are n-tensor induced, then $D = D_0^n$, each A_i has rank D_0 and all ∞ slopes $\le n/W = n/D = n/D_0^n$. It suffices to show that each $\mathcal{A}_i$ has $Swan_\infty < 1$. This Swan conductor is $\le D_0(n/D_0^n)$, so it suffices to show that

$$n < D_0^{n-1}$$

when $n \ge 2$ and $D_0 \ge 2$, except in the case $(n = 2, D_0 = 2)$, which is ruled out by the $D > 4$ hypothesis. For $n = 2$, the worst remaining case is $D_0 = 3$, and indeed $3 > 2$. For $n \ge 3$, the worst case is $2^{n-1} > n$, which indeed holds.

In the hypergeometric case, we again have $D_0(n/W)$ as an upper bound for the Swan conductor of any $\mathcal{A}_i$. We have $W > D/p_0$, so we wish to show $nD_0 < W$, which is implied by

$$nD_0 \le D_0^n/p_0, \text{i.e., } D_0^{n-1} \ge np_0.$$

Because p_0 divides $D = D_0^n$, p_0 divides D_0, so we write $D_0 = n_0p_0$ for some integer $n_0 \ge 1$. It suffices to show

$$n_0^{n-1}p_0^{n-2} \ge n.$$

This last equality is visibly false for $n = 2$ if $n_0 = 1$, i.e, if we are dealing with the case $D = p_0^2$. But in that case we assumed that $W > 2p_p$, and with this estimate we do have $nD_0 < W$ in the $n = 2$ case with $D = p_0^2$.

Suppose now that $n = 3$. Then we need $3D_0 \le D_0^3/p_0$, i.e., we need $D_0^2 \ge 3p_0$, i.e., $n_0^2p_0 \ge 3$. This is fine so long as $p_0 \ge 3$ or $n_0 > 1$. In the case $D_0 = p_0 = 2$, the desired inequality for $n = 3$ is $3.2 < W$, which is precisely what we assumed in the $D = 8$ case.

Finally, for $n \ge 4$, where we need $nD_0 \le D_0^n/p_0$, this is implied by $p_0^{n-2} \ge n$, which for $n \ge 4$ already holds for the worst case $p_0 = 2$. ☐

Remark 3.5. In [Ka-ESDE, 10.6.9 and 10.9.1], there are examples of hypergeometric sheaves of type $(9, 3)$ which are 2-tensor induced. In [Ka-ESDE, 10.8.1], there are examples of hypergeometric sheaves of type $(8, 2)$ which are 3-tensor induced.

3C. Tensor induction: the case when $p|D$.

Proposition 3.6. *Let $\mathcal{F}$ be either a Kloosterman sheaf $\mathcal{K}l$ of rank $D > 4$ or a hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > m > 0$ and $D \ge 4$. Suppose further we are in one of the following three situations.*

(i) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type $(4, 2)$ in characteristic $p = 2$.*
(ii) *$\mathcal{F}$ is a Kloosteman sheaf of rank $D > 4$ and $p|D$. If $p = 2$, suppose also that $D \ne 8$.*
(iii) *$\mathcal{F}$ is a hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > 4$ and $p|D$. Suppose that $W > (2/3)(D - 1)$. If $p = 2$, suppose $D \ne 8$. If $p = 3$, suppose (D, m) is not $(9, 1)$.*

Then $\mathcal{F}$ is not tensor induced.

Proof. We first treat case (i), a hypergeometric sheaf $\mathcal{H}$ of type $(4, 2)$ in characteristic $p = 2$. It could only possibly be n-tensor induced for $n = 2$, but this is impossible as $p \nmid n$, cf. Lemma 3.2.

We next treat the Kloosterman case. If $\mathcal{K}l$ is n-tensor induced for a given $n \ge 2$, then n is prime to p, $D = D_0^n$ and we have a tensor decomposition

$$[n]^\star\mathcal{K}l = \mathcal{A}_1 \otimes \mathcal{A}_2 \cdots \otimes \mathcal{A}_n,$$

with each $\mathcal{A}_i$ of rank $D_0 \ge 2$. We use the argument of Šuch, cf. [Such, Prop. 12.1, second paragraph], which we already used in the proof of Lemma 2.2. Exactly as there, each $\mathrm{End}^0(\mathcal{A}_i)$ is a direct factor of $\mathrm{End}([n]^\star\mathcal{K}l)$, hence has all ∞ slopes $\le n/D = n/D_0^n$.

If $n = 2$, then each $\mathcal{A}_i$ has

$$\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) \le (D_0^2 - 1)(2/D_0^2) < 2.$$

Thus $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i))$, which is equal to $\mathsf{Swan}_\infty(\mathrm{End}(\mathcal{A}_i))$, is either 0 or 1. If it is 1 for at least one of $\mathcal{A}_1$ or $\mathcal{A}_2$, that End is the direct sum of a nonzero tame part (from scalar endomorphisms) and an $I(\infty)$-irreducible part with Swan conductor 1, this latter part being totally wild. Its expression as an End violates its tensor indecomposability, cf. [KRLT3, Cor. 10.3], because the rank of End, here $D_0^2 = D$, is not 4.

If both the $\mathrm{End}^0(\mathcal{A}_i)$ are tame at ∞, then $P(\infty)$ acts by a scalar character on each of $\mathcal{A}_1$ and $\mathcal{A}_2$, and hence $P(\infty)$ acts by a scalar character on $[2]^\star(\mathcal{K}l)$. The $I(\infty)$ representation of $\mathcal{K}l$ is irreducible of Swan conductor 1. Its rank D is divisible by p, so we write $D = n_0 q$ with $n_0 \geq 1$ prime to p and with q a positive power of p. Then the $P(\infty)$ representation of $\mathcal{K}l$ is the direct sum of n_0 inequivalent irreducible $P(\infty)$-representations, each of dimension q. The $P(\infty)$ representation does not change under Kummer pullback, so $P(\infty)$ representation of $[2]^\star(\mathcal{K}l)$ is the direct sum of n_0 inequivalent irreducible $P(\infty)$-representations, each of dimension q. Therefore $P(\infty)$ does not act as scalars on any of these q dimensional $P(\infty)$-irreducibles.

Suppose now that $\mathcal{K}l$ is n-tensor induced for a given $n \geq 3$. Then each $\mathcal{A}_i$ has

$$\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) \leq (D_0^2 - 1)(n/D_0^n) \leq (n/D_0^{n-2})\frac{D_0^2 - 1}{D_0^2} < n/D_0^{n-2}.$$

For $n = 3$, we cannot have $D_0 = 2$ unless $p = 2$, but we have ruled out $D = 8$ when $p = 2$. So for $n = 3$, we have $D_0 \geq 3$, and so each $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) < 1$ in the $n = 3$ case. For $n \geq 4$, we have $n/D_0^{n-2} \leq 1$, as one sees already from the worst case $D_0 = 2$, where it amounts to the inequality $n \leq 2^{n-2}$ for $n \geq 4$.

We now turn to the hypergeometric case. Because $D \neq 4$, $\mathcal{H}$ is tensor indecomposable. So if $\mathcal{H}$ is n-tensor induced for some $n \geq 2$ (necessarily prime to p), we have $D = D_0^n$ and a tensor decomposition

$$[n]^\star\mathcal{H} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n,$$

with each $\mathcal{A}_i$ of rank $D_0 \geq 2$.

We first consider the case $n \geq 3$. By the Šuch argument, each $\mathrm{End}^0(\mathcal{A}_i)$ is a direct factor of $\mathrm{End}([n]^\star\mathcal{H})$, hence has all ∞ slopes $\leq n/W$. We claim that each $\mathrm{End}^0(\mathcal{A}_i)$ is tame at ∞. If so, we reach a contradiction as follows. Each $\mathrm{End}(\mathcal{A}_i)$ is then tame at ∞, so $P(\infty)$ acts on each $\mathcal{A}_i$ by a scalar character, and hence $P(\infty)$ acts on $[n]^\star\mathcal{H}$ by a scalar character. Because $[n]^\star\mathcal{H}$ has a tame part of rank $m > 0$, this scalar character must be trivial. This in turn implies that $[n]^\star\mathcal{H}$ is tame at ∞, contradiction.

To show that each $\mathrm{End}^0(\mathcal{A}_i)$ is tame at ∞, it suffices to show that its Swan conductor is < 1. Using the estimate $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i))$ is $\leq (D_0^2 - 1)(n/W)$, it suffices to show that

$$n(D_0^2 - 1) < W.$$

By hypothesis, $W > (2/3)(D - 1)$. So for $D \neq 27$ it suffices to show that

$$n(D_0^2 - 1) \leq (2/3)(D - 1) = 2/3)(D_0^n - 1),$$

so long D is neither 8 when $p = 2$ nor 27 when $p = 3$. For $n = 3$, we need

$$3(D_0^2 - 1) \leq (2/3)(D_0^3 - 1) \text{ for } p \geq 3.$$

This inequality holds for $D_0 \geq 5$, which for p odd rules out $D_0 = 3$. But this $p = 3, n = 3, D_0 = 3$ case does not arise, because in characteristic p, here 3, we can only be n-tensor induced when n is prime to p.. For $p = 2$, it rules out $D_0 = 2$, the excluded $D = 8$ case.

But we must still deal with the case $n = 3, p = 2, D_0 = 4$. Here the estimate for $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)$

$$\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i) < \frac{3(4^2 - 1)}{(2/3)(4^3 - 1)} = 15/14 < 2.$$

So each $\mathcal{A}_i$ has $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i) = \mathsf{Swan}_\infty(\mathrm{End}(\mathcal{A}_i)$ either 0 or 1. The $\mathsf{Swan}_\infty(\mathrm{End}(\mathcal{A}_i) = 1$ is impossible, because it violates tensor indecomposability, cf. [KRLT3, Cor. 10.3], because the rank of End, here $D_0^2 = 16$, is not 4. Thus each $\mathrm{End}^0(\mathcal{A}_i)$ is tame at ∞ in this case as well.

For $n \geq 4$, we need

$$n(D_0^2 - 1) \leq (2/3)(D_0^n - 1) \text{ for } p \geq 3.$$

This holds for $D_0 \geq 3$ for all $n \geq 4$, and for $D_0 \geq 2$ for all $n \geq 5$. The case $n = 4, D_0 = 2$ is excluded because when $p = 2$, n-tensor induction is only possible when n is odd.

It remains to treat the case $n = 2$. In this case, p must be odd. Thus $D = D_0^2$,

$$[2]^\star\mathcal{H} = \mathcal{A}_1 \otimes \mathcal{A}_2,$$

with each $\mathcal{A}_i$ of rank D_0. We first claim that $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i))$ is $0, 1$, or 2, i.e. that

$$\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) < 3.$$

This Swan conductor is $\leq (D_0^2 - 1)(2/W) = 2(D-1)$, so we must show $2(D-1) < 3W$, which is precisely our hypothesis.

If both $\mathcal{A}_i$ have End^0, and hence End, tame at ∞, then just as in the $n \geq 3$ case above, we reach a contradiction.

If one of the $\mathcal{A}_i$ has $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) = 1$, then the rank D of this $\mathrm{End}(\mathcal{A}_i)$ must be 4. But $D > 4$ in the hypergeometric case (ii) we are considering. [Alternatively, $D = 4$ and $p|D$ forces $p = 2$, in which case n-tensor induction for $n = 2$ is impossible.]

If one of the $\mathcal{A}_i$ has $\mathsf{Swan}_\infty(\mathrm{End}^0(\mathcal{A}_i)) = 2$, we argue as follows. Either $\mathrm{End}(\mathcal{A}_i)$ is the sum of a nonzero tame part and a single $I(\infty)$-irreducible whose Swan conductor is 2, or $\mathrm{End}(\mathcal{A}_i)$ is the sum of a nonzero tame part and of two $I(\infty)$-irreducibles, each of Swan conductor 1. In the first case, we again (by [KRLT3, Cor. 10.3]) then the rank D of this $\mathrm{End}(\mathcal{A}_i)$ must be 4, an excluded case.

It remains now to analyze the case when each of the $\mathrm{End}(\mathcal{A}_i)$, $i = 1, 2$, is the sum of a nonzero tame part and of two $I(\infty)$-irreducibles, each of Swan conductor 1. We first show that in this case, the rank D of $\mathrm{End}(\mathcal{A}_i)$ must be q^2, for q some positive power of p. We show this in the next lemma.

Lemma 3.7. *Let $\mathcal{A}$ be an $I(\infty)$-representation of dimension $D_0 \geq 2$ with $p|D_0$, p odd, such that $\mathrm{End}(\mathcal{A}) := \mathcal{A} \otimes \mathcal{A}^\vee$ is the sum of a nonzero tame part and of two wild $I(\infty)$-irreducibles. If such an $\mathcal{A}$ exists, then it is an $I(\infty)$-irreducible of dimension q, for q some positive power of p.*

Proof. We first show that $\mathcal{A}$ is totally wild. It cannot be totally tame, otherwise its End would be tame. It cannot contain both a nonzero tame part T and two wild two $I(\infty)$-irreducibles W_1 and W_2, for then its End contains the four totally wild components $T \otimes W_1^\vee, T \otimes W_2^\vee, T^\vee \otimes W_1, T^\vee \otimes W_2$, which each themselves contain at least one wild $I(\infty)$-irreducible.

If it is of the form $T + W$ with T a nonzero tame part and W a wild $I(\infty)$-irreducible,then its End contains $T \otimes W^\vee, T^\vee \otimes W, W \otimes W^\vee$. If this End contains only two wild $I(\infty)$-irreducibles, then T is one-dimensional and $W \otimes W^\vee$ is totally tame. But if $W \otimes W^\vee$ is totally tame, then W is one-dimensional, cf. [KRLT3, Lemma 10.2]). Thus our $\mathcal{A}$, if not totally wild, has dimension $D_0 = 2$. But as $p|D_0$, and p is odd, this cannot happen.

Thus $\mathcal{A}$ is totally wild. We next show that it is $I(\infty)$-irreducible. If $\mathcal{A}$ contains two wild irreducibles W_1 and W_2, at least one of which has dimension > 1, we reach a contradiction as follows. Then its End contains the four terms $W_1 \otimes W_2^\vee, W_2 \otimes W_1^\vee, W_1 \otimes W_1^\vee, W_2 \otimes W_2^\vee$. Neither of ths two cross terms, nor whichever of $W_i \otimes W_i^\vee$ has dimension > 1, can be totally tame, again by [KRLT3, Lemma 10.2]).

To finish the proof that $\mathcal{A}$ is $I(\infty)$-irreducible, we must rule out the case when $\mathcal{A}$ contains only wild irreducibles of dimension one. In this case, $\mathcal{A}$ contains at least 3 such (because $D_0 > 2$). Partition them according to the equivalence relation $W_1 \equiv W_2$ if and only if $W_1 \otimes W_2^\vee$ is tame.

Then $\mathcal{A}$ is the sum of terms $T_i \otimes W_i$, with T_i tame of dimension $d_i \geq 1$, W_i wild of dimension one, and $W_i \otimes W_j^\vee$ is wild whenever $i \neq j$. Then its End contains precisely $\sum_{i \neq j} d_i d_j$ wild summands (namely the $T_i \otimes T_j^\vee \otimes W_i \otimes W_j^\vee$), and $\sum_i d_i = D_0 \geq 3$. There must be more than one such summand, otherwise the End is tame. If there are at least three such summands, then $\mathcal{A}$ contains $W_1 + W_2 + W_3$, with $W_i \otimes W_j^\vee$ wild for $i \neq j$. In this case, the End contains the six wild summands $W_i \otimes W_j^\vee$ with $i, j \in [1, 3]$ and $i \neq j$. So $\mathcal{A}$ must be $T_1 \otimes W_1 + T_2 \otimes W_2$ with $d_1 + d_2 = D_0 \geq 3$. Interchanging the two indices if necessary, we may assume $d_1 \geq 2$ (and $d_1 \geq 1$). Then the End contains at least $2d_1 \geq 4$ wild summands, contradiction.

Thus $\mathcal{A}$ is $I(\infty)$-irreducible. We write its dimension D_0 as $n_0 q$, with $n_0 \geq 1$ and q a strictly positive power of p. Then $\mathcal{A}$ is the Kummer direct image

$$\mathcal{A} = [n_0]_\star \mathcal{B}$$

for $\mathcal{B}$ a q dimensional $I(\infty)$-irreducible. We know further that $\mathcal{B}$ is $P(\infty)$-irreducible, and that $\mathcal{B}$ is, as $P(\infty)$-representation, the direct sum of n_0 pairwise inequivalent irreducibles. Indeed, under the multiplicative translation action of μ_{n_0}, the n_0 multiplicative translates $\{\mathrm{MT}_\zeta \mathcal{B}\}_{\zeta \in \mu_{n_0}}$ are pairwise inequivalent $P(\infty)$-irreducibles.

We next claim that we have a direct sum decomposition

$$\mathrm{End}([n_0]_\star \mathcal{B}) = \bigoplus_{\zeta \in \mu_{n_0}} [n_0]_\star (\mathcal{B} \otimes \mathrm{MT}_\zeta \mathcal{B}^\vee).$$

To see this, we argue as follows. Denote by $I(n_0) \lhd I(\infty)$ the open subgroup of index n_0. For any $I(n_0)$-representation V, the character of its direct image $[n_0]_\star V$ (i.e.the group theoretic induction of V from $I(n_0)$ to $I(\infty)$) is supported in $I(n_0)$ (simply because $I(n_0) \lhd I(\infty)$ is a normal subgroup). Then $\mathrm{End}(([n_0]_\star V) = ([n_0]_\star V) \otimes ([n_0]_\star V^\vee)$ has its character supported in $I(n_0)$. Therefore the character of $\mathrm{End}([n_0]_\star V)$ is determined by its pullback to $I(n_0)$.

We now apply this with V taken to be $\mathcal{B}$. Because $\mathcal{B}$ is $I(n_0)$-irreducible, its induction $[n_0]_\star \mathcal{B}$ and its $\mathrm{End}([n_0]_\star \mathcal{B})$ are both $I(\infty)$-semisimple, so determined by their characters, and hence by the characters of their pullbacks $[n_0]^\star$. We have

$$[n_0]^\star [n_0]_\star \mathcal{B} = \bigoplus_{\zeta \in \mu_{n_0}} \mathrm{MT}_\zeta \mathcal{B},\ [n_0]^\star [n_0]_\star \mathcal{B}^\vee = \bigoplus_{\zeta \in \mu_{n_0}} \mathrm{MT}_\zeta \mathcal{B}^\vee.$$

Thus

$$[n_0]^\star \mathrm{End}([n_0]_\star \mathcal{B}) = \bigoplus_{(\zeta_1, \zeta_2) \in \mu_{n_0} \times \mu_{n_0}} (\mathrm{MT}_{\zeta_1} \mathcal{B}) \otimes (\mathrm{MT}_{\zeta_2} \mathcal{B}^\vee) = \bigoplus_{\zeta_2 \in \mu_{n_0}} \bigoplus_{\zeta_1 \in \mu_{n_0}} \mathrm{MT}_{\zeta_1} (\mathcal{B} \otimes (\mathrm{MT}_{\zeta_2} \mathcal{B}^\vee)),$$

which is the pullback to $I(n_0)$ of the character of

$$\bigoplus_{\zeta_2 \in \mu_{n_0}} [n_0]_\star (\mathcal{A} \otimes \mathrm{MT}_{\zeta_2} \mathcal{B}^\vee).$$

With this formula at hand, we continue as follows. Because the various $\mathrm{MT}_\zeta \mathcal{B}$ are pairwise inequivalent irreducible $P(\infty)$-representations, we have

$$\mathcal{B} \otimes \mathcal{B}^\vee = \mathbb{1} + \text{totally wild},$$

and for each $\zeta \neq 1$,

$$\mathcal{B} \otimes \mathrm{MT}_\zeta \mathcal{B}^\vee = \text{totally wild}.$$

Now $V \mapsto [n_0]_\star V$ preserves being totally wild, so we find

$$\begin{aligned}\mathrm{End}(\mathcal{A}) = \mathrm{End}([n_0]_\star\mathcal{B}) &= [n_0]_\star\mathbb{1} + [n_0]_\star\mathrm{End}^0(\mathcal{B}) \oplus_{\zeta\neq 1\in\mu_{n_0}} [n_0]_\star(\mathcal{B}\otimes \mathrm{MT}_\zeta\mathcal{B}^\vee)\\ &= [n_0]_\star\mathbb{1} + \text{the sum of } n_0 \text{ totally wild summands.}\end{aligned}$$

In order for there to be precisely two irreducible wild summands in $\mathrm{End}(\mathcal{A})$, we must have $n_0 \le 2$.

If $n_0 = 2$, then both $[2]_\star(\mathrm{End}^0(\mathcal{B}))$ and $[2]_\star(\mathcal{B}\otimes \mathrm{MT}_{-1}\mathcal{B}^\vee)$ must be irreducible. In particular, $\mathrm{End}^0(\mathcal{B})$ must be irreducible, i.e. we must have $\mathrm{End}(\mathcal{B}) = \mathbb{1} +$ irreducible. This is possible only when $\mathcal{B}$ has rank 2, cf. [KRLT3, Cor. 10.4]. Then $\mathcal{A} = [2]_\star\mathcal{B}$ has rank $D_0 = 4$. As $p|D_0$, p must be 2, an excluded case.

If $n_0 = 1$, then $\mathcal{A}$ is an $I(\infty)$-irreducible of dimension q. $\square$

Returning to our situation

$$[2]^\star\mathcal{H} = \mathcal{A}_1\otimes\mathcal{A}_2,$$

we now know that $D = q^2, D_0 = q$, and that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are $I(\infty)$-irreducibles. Having dimension q, they are each $P(\infty)$-irreducible. Because $\mathcal{H}$ has type (D,m) with $m > 0$, $[2]^\star\mathcal{H}$ has an $I(\infty)$-tame part of dimension $m > 0$. At the expense of tensoring $\mathcal{H}$ with a tame character, we may assume further that among the "bottom" characters in $\mathcal{H}$ is $\mathbb{1}$. Then $\mathcal{A}_1\otimes\mathcal{A}_2$ as $I(\infty)$-representation contains $\mathbb{1}$. The projection of $\mathcal{A}_1\otimes\mathcal{A}_2$ onto $\mathbb{1}$ is then a nonzero $I(\infty)$-linear map $\mathcal{A}_2 \to \mathcal{A}_1^\vee$, which must be an $I(\infty)$-isomorphism because source and target are $I(\infty)$-irreducible. Thus $\mathcal{A}_1\otimes\mathcal{A}_2$ is $I(\infty)$-isomorphic to $\mathrm{End}(\mathcal{A}_1)$. Because $\mathcal{A}_1$ is $P(\infty)$-irreducible, the space of $P(\infty)$-invariants in $\mathrm{End}(\mathcal{A}_1)$ is one dimensional. But this space of $P(\infty)$-invariants is precisely the tame part of $\mathcal{A}_1\otimes\mathcal{A}_2 = [2]^\star\mathcal{H}$. Therefore $m = 1$, and $\mathcal{H}$ has type $(D,m) = (q^2,1)$.

In the next section, we will deal with this $(q^2,1)$ case.

3D. Completion of the proof of Proposition 3.6.

In this subsection, q is a positive power p^a of the odd prime p, and $\mathcal{H}$ is a hypergeometric of type $(q^2,1)$ whose "bottom" character is $\mathbb{1}$. The $I(\infty)$-representation of $\mathcal{H}$ is the direct sum $\mathcal{W}+\mathbb{1}$, with $\mathcal{W}$ totally wild of rank q^2-1 and Swan conductor one. Because q^2-1 is prime to p, we know [Ka-GKM, 1.14] that $\mathcal{W}$ is the Kummer direct image $[q^2-1]_\star(\mathcal{L})$ for some rank one $\mathcal{L}$ of Swan conductor one. Furthermore, the restriction of $\mathcal{W}$ to $P(\infty)$ is the direct sum of q^2-1 pairwise distinct characters of $P(\infty)$ which are cyclically permuted [Ka-GKM, 1.14(3)] by $I(\infty)/P(\infty)$, acting through its μ_{q^2-1} quotient.

We denote by

$$J_1 := \text{ the image of } I(\infty) \text{ acting on } \mathcal{W}+\mathbb{1}.$$

Because our $\mathcal{H}$ began life on $\mathbb{G}_m$ over a finite extension of $\mathbb{F}_p$, we know [Ka-GKM, 1.11 (3)] that J_1 is finite, with a normal Sylow p-subgroup P_1 such that J_1/P_1 is cyclic of p'-order $m(q^2-1)$ for some $m\in\mathbb{Z}_{\ge 1}$. Moreover, any element of J_1 of order $m(q^2-1)$ induces, by conjugation, an automorphism of P_1 of order q^2-1. [Indeed this action cyclically permutes q^2-1 distinct characters of P_1 on $\mathcal{W}$.]

Our concern is with the Kummer pullback $[2]^\star\mathcal{H}$, whose $I(\infty)$ representation is $[2]^\star\mathcal{W}+\mathbb{1}$. We readily decompose

$$\begin{gathered}[2]^\star\mathcal{W} = [2]^\star[q^2-1]_\star\mathcal{L} = [2]^\star[2]_\star[(q^2-1)/2]_\star\mathcal{L} =\\ = [2]^\star[2]^\star\mathcal{X} = \mathcal{X} + [-1]_\star\mathcal{X}, \text{ for } \mathcal{X} := [(q^2-1)/2]_\star\mathcal{L}.\end{gathered}$$

$\mathcal{X}$ is itself irreducible of Swan conductor one. One knows that for any irreducible $I(\infty)$-representation $\mathcal{X}$ of Swan conductor one, $\mathcal{X}$ and its multiplicative translate $[-1]_\star\mathcal{X}$ are inequivalent. [If they were isomorphic, $\mathcal{X}$ would descend through [2], i.e. would be of the form $[2]^\star\mathcal{Y}$, which would force its Swan conductor to be even, cf. [Ka-ESDE, proof of 3.7.6] for the $\mathcal{D}$-module analogue.]

Thus the $I(\infty)$ representation of $[2]^\star\mathcal{H}$ is the sum of three distinct irreducibles:

$$[2]^\star\mathcal{H} \cong \mathcal{X} + [-1]_\star\mathcal{X} + \mathbb{1}.$$

We denote by

$$J_2 \lhd J_1$$

the subgroup of index 2 which is the image of $I(\infty)$ acting on $[2]^\star\mathcal{H}$. As $p>2$, J_2 has the **same** P_1 as its normal Sylow p-subgroup P_2, and the quotient J_2/P_2 is cyclic of p'-order $m(q^2-1)/2$. Moreover, any element of J_2 of order $m(q^2-1)/2$ induces, by conjugation, an automorphism of P_2 of order $(q^2-1)/2$.

Suppose now that $\mathcal{H}$ is 2-tensor induced. The $I(\infty)$ representation (indeed the π_1-representation, but we do not know how to use this much stronger information) of $[2]^\star\mathcal{H}$ lands in $\mathrm{GL}(A_1)\otimes\mathrm{GL}(A_2)$, with each A_i of dimension q. We wrote $\mathrm{GL}(A_1)\otimes\mathrm{GL}(A_2)$ as $\mathrm{SL}(A_1)\otimes\mathrm{GL}(A_2)$. This allowed us to lift the action of $I(\infty)$ (indeed of π_1) on $[2]^\star\mathcal{H}$ to a map

$$I(\infty)\to\mathrm{SL}(A_1)\times\mathrm{GL}(A_2).$$

The image of this map we denote J_3. This group J_3 is finite, and maps $J_3 \twoheadrightarrow J_2$ with kernel the intersection of J_3 with the subgroup μ_q, embedded as $(\lambda,1/\lambda)\in\mathrm{SL}(A_1)\times\mathrm{GL}(A_2)$. Thus the kernel is a cyclic group of order dividing q, so lies in the Sylow p-subgroup P_3 of J_3. Thus J_3 has a normal Sylow p-subgroup P_3, and J_3/P_3 is cyclic of p'-order $m(q^2-1)/2$, in fact the same order as J_2/P_2. Thus we get J_3-representations $\mathcal{A}_1\times\mathcal{A}_2$ such that

$$[2]^\star\mathcal{H}\cong\mathcal{A}_1\otimes\mathcal{A}_2,$$

and we showed that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $I(\infty)$ (and hence J_3) duals of each other.

Next, we define

$$J_4 < \mathrm{SL}(A_1)$$

to be the image of $J_3<\mathrm{SL}(A_1)\times\mathrm{GL}(A_2)$ by the first projection. We denote by $P_4 \lhd J_4$ its normal Sylow p-subgroup. The image of J_4 in $\mathrm{End}(\mathcal{A}_1)\cong[2]^\star\mathcal{H}$ is J_2. The kernel K of the surjection $J_4 \twoheadrightarrow J_2$ is the intersection of J_4 with the scalars μ_q of $\mathrm{SL}(A_1)$, so lies in P_4. Therefore P_4 maps onto P_2 with kernel K, and J_4/P_4 maps isomorphically to J_2/P_2. Any element $x\in J_4$ of order $m(q^2-1)/2$ induces, by conjugation, an automorphism φ_x of P_4 of order $(q^2-1)/2$. [Indeed, this is already the case in the quotient situation (J_2,P_2), hence φ_x has order divisible by $(q^2-1)/2$. It also follows that $\varphi_x^{(q^2-1)/2}$ acts trivially on $P_2=P_4/K$ and on the central p-subgroup K, and so the order of $\varphi_x^{(q^2-1)/2}$ is a p-power. As x is a p'-element, we conclude that $\varphi_x^{(q^2-1)/2}=1$.]

We will apply the next lemmas with

$$J := J_4 < \mathrm{SL}(\mathcal{A}_1), V := \mathcal{A}_1.$$

We know that J is a finite group with a normal p-Sylow subgroup P (thus $\mathbf{O}_p(J)=P$), that J/P is cyclic of prime to p order divisible by $(q^2-1)/2$ and that any element of J of order $m(q^2-1)/2$ induces, by conjugation, an automorphism of P of order $(q^2-1)/2$. We have a faithful irreducible q-dimensional representation V of J, and we know that $\mathrm{End}(V)$ is the sum of three distinct irreducible submodules. Note that J is solvable, and furthermore has cyclic Sylow 2-subgroups if $2\nmid q$. Hence the subsequent Lemmas 3.8–3.10 apply to J.

Lemma 3.8. *Let V be a faithful irreducible $\mathbb{C}J$-module of dimension $d\geq 3$ and $d\neq 4$, where J is a finite solvable group, which has abelian Sylow 2-subgroups if $2\nmid d$. Suppose that* $\mathrm{End}(V)$ *is a sum of three irreducible submodules, but the J-module V does not satisfy condition* ($\mathbf{S}$). *Then the J-module V is tensor indecomposable, not tensor induced, and every imprimitivity decomposition for V has the form $V=\oplus_{i=1}^d V_i$, with* $\dim V_i=1$ *and J permuting $\{V_1,\dots,V_d\}$ primitively and 2-homogeneously.*

Proof. Let χ denote the character afforded by the $\mathbb{C}J$-module V. By assumption, $\chi\overline{\chi} = 1_J + \alpha_1 + \alpha_2$, where $\alpha_i \in \mathrm{Irr}(J)$. First we note that $\alpha_1 \neq \alpha_2$. Indeed, by Burnside's theorem (3.15) of [Is], $\chi(g) = 0$ for some $g \in J$. Hence, in the case $\alpha_1 = \alpha_2$ we would have that $\alpha_1(g) = -1/2$, which is not an algebraic integer, a contradiction. Next, the irreducibility of χ implies that $\alpha_i \neq 1_J$. It follows that

$$3 = M_4(J,V) = [\chi\overline{\chi}, \chi\overline{\chi}]_J = [\chi^2, \chi^2]_J.$$

On the other hand, $V^{\otimes 2} = \mathsf{S}^2(V) \oplus \wedge^2(V)$. So we conclude that either $\mathsf{S}^2(V)$ is irreducible, or $\wedge^2(V)$ is irreducible.

If $\mathsf{S}^2(V)$ is irreducible, then the J-module V satisfies ($\mathbf{S}$) by [GT2, Lemma 2.1]. Hence $\wedge^2(V)$ is irreducible. Assume in addition that V is imprimitive: J stabilizes a decomposition $V = \oplus_{i=1}^t V_i$ with $t > 1$ and $\dim V_i = d/t$. The proof of [GT2, Lemma 2.4] then shows that $t = d$ and J acts on the set $\{V_1, \ldots, V_d\}$ 2-homogeneously.

In the same proof, it was shown that V is tensor indecomposable, and furthermore, if it is tensor induced, then any tensor induced decomposition has the form $V = V_1 \otimes V_2$, with $J_0 := \mathrm{Stab}_J(V_1, V_2)$ of index 2 in J, and $\mathsf{S}^2(V_i)$ and $\wedge^2(V_i)$ being irreducible over J_0 for $i = 1, 2$. In the latter case, $\dim V_i = \sqrt{d} \geq 3$ by hypothesis. Furthermore, V_i is irreducible, and $\mathsf{S}^2(V_i) \not\cong \wedge^2(V_i)$ by dimension consideration. It follows that $M_4(J_0, V) = 2$. Certainly, J_0 is solvable as so is J. If furthermore $2 \nmid \dim V_i$, then $2 \nmid d$ and so Sylow 2-subgroups of J are abelian, whence so are Sylow 2-subgroups of J_0. Thus [KRLT3, Theorem 2.3] applies to the J_0-module V_i and yields $M_4(J_0, V_i) \geq 3$, a contradiction. $\square$

Next we will analyze the situations arising in Lemma 3.8, under the assumption that $d = \dim V = q = p^a$, where p is a prime and $a \geq 1$. We will fix a *primitive prime divisor* $\ell = \mathrm{ppd}(p, 2a)$ of $p^{2a} - 1$, that is, a prime divisor of $p^{2a} - 1$ that does not divide $\prod_{i=1}^{2a-1}(p^i - 1)$, when it exists. Such a prime always exists, unless either $(p, a) = (2, 3)$, or $a = 1$ and p is a Mersenne prime, see [Zs].

Lemma 3.9. *In the situation of Lemma 3.8, assume that $d = p^a \geq 3$ for a prime p and that the conjugation by some element $h \in J$ induces an automorphism of $\mathbf{O}_p(J)$ of order $(p^{2a}-1)/\gcd(2, p-1)$. Then J acts primitively on V.*

Proof. Assume the contrary. By Lemma 3.8, the action of J on $\{V_1, \ldots, V_d\}$ induces a solvable, primitive subgroup H of S_d. Since H is solvable, it possesses an abelian minimal normal subgroup N. By the O'Nan-Scott theorem, see e.g. [LPS], H is a subgroup of the affine group $\mathrm{AGL}(U) = \mathrm{AGL}_a(p)$ in its action on the points of $U = \mathbb{F}_p^a$ (with N acting via translations). Let $B \lhd J$ consist of all elements that fact trivially on $\{V_1, \ldots, V_d\}$, so that $H = J/B$. Then B is contained in a maximal torus of $\mathrm{GL}(V)$ and so is abelian.

First we consider the case $\ell = \mathrm{ppd}(p, 2a)$ exists. Then ℓ does not divide $|\mathrm{GL}_a(p)|$. It follows that any ℓ-element $g \in J$ has trivial image in H, that is, $g \in B$ and so $g \in \mathbf{O}_\ell(B) \lhd J$ (since B is abelian). For any $x \in \mathbf{O}_p(J)$ we then have $[g, x] \in \mathbf{O}_p(J) \cap \mathbf{O}_\ell(B) = 1$. We have shown that $[g, \mathbf{O}_p(J)] = 1$, and so $\mathbf{O}_p(J)$ is centralized by $\mathbf{O}^{\ell'}(J)$. Thus the action of J on $\mathbf{O}_p(J)$ induces a subgroup of $\mathrm{Aut}(\mathbf{O}_p(J))$ of order coprime to ℓ, a contradiction.

Now we may assume that ℓ does not exist. Assume furthermore that $a = 1$, but $p \geq 3$ is a Mersenne prime. Now we can find a 2-element $g \in \langle h \rangle$ such that the conjugation by g induces an automorphism φ_g of order 4 of $\mathbf{O}_p(J)$. Since the 2-part of $|H|$ divides $|\mathrm{GL}_1(p)| = p-1$, g^2 has trivial image in H, and so $g^2 \in \mathbf{O}_2(B) \lhd J$. For any $x \in \mathbf{O}_p(J)$ we then have $[g^2, x] \in \mathbf{O}_p(J) \cap \mathbf{O}_2(B) = 1$. We have shown that $[g^2, \mathbf{O}_p(J)] = 1$, contrary to $|\varphi_g| = 4$.

It remains to consider the case $p^a = 8$. In this case we can find a 3-element $g \in \langle h \rangle$ such that the conjugation by g induces an automorphism φ_g of order 9 of $\mathbf{O}_p(J)$. Since the 3-part of $|H|$ divides

$|\mathrm{GL}_3(2)| = 168$, g^3 has trivial image in H, and so $g^3 \in \mathbf{O}_3(B) \lhd J$. For any $x \in \mathbf{O}_p(J)$ we then have $[g^3, x] \in \mathbf{O}_p(J) \cap \mathbf{O}_3(B) = 1$. Thus $[g^3, \mathbf{O}_p(J)] = 1$, again contradicting the equality $|\varphi_g| = 9$. □

Now we complete the analysis of the situations arising in Lemma 3.8:

Lemma 3.10. *Let J be a finite solvable group, and let V be a faithful irreducible $\mathbb{C}J$-module of dimension $d = p^a \geq 5$ for some prime p. Assume in addition that J has abelian Sylow 2-subgroups if $2 \nmid d$, and that the conjugation by some element $h \in J$ induces an automorphism of $\mathbf{O}_p(J)$ of order $(p^{2a} - 1)/\gcd(2, p-1)$. Then $\mathrm{End}(V)$ cannot be a sum of three irreducible J-submodules.*

Proof. (i) Assume the contrary: $\mathrm{End}(V)$ a sum of three irreducible J-submodules. By Lemmas 3.8 and 3.9, the J-module V satisfies condition ($\mathbf{S}$). As explained in §1, the proof of [GT2, Proposition 3.8] shows that J contains a normal p-subgroup Q, where $Q = \mathbf{Z}(Q)E$ for some extraspecial p-group E of order p^{1+2a} acting irreducibly on V; furthermore, either $Q = E$ or $|\mathbf{Z}(Q)| = 4$. Let A denote the subgroup of $\mathrm{Aut}(Q)$ induced by the conjugation action of J.

(ii) Observe that $\mathbf{C}_J(\mathbf{O}_p(J)) = \mathbf{C}_J(Q) = \mathbf{Z}(J)$. (Indeed, $\mathbf{Z}(J) \leq \mathbf{C}_J(\mathbf{O}_p(J)) \leq \mathbf{C}_J(Q)$ as $Q \lhd \mathbf{O}_p(J)$. As $E \leq Q$ is irreducible on V, $\mathbf{C}_J(Q) = \mathbf{Z}(J)$ by Schur's lemma.) From the equality $\mathbf{C}_J(\mathbf{O}_p(J)) = \mathbf{C}_J(Q)$, we see that the image of J in $\mathrm{Aut}(\mathbf{O}_p(J))$, namely $J/\mathbf{C}_J(\mathbf{O}_p(J))$, maps isomorphically to $A \cong J/\mathbf{C}_J(Q)$.

Hence, by hypothesis, $A \leq \mathrm{Aut}(Q)$ contains a cyclic p'-subgroup C of order $(p^{2a}-1)/\gcd(2, p-1)$. In fact, A acts trivially on $\mathbf{Z}(Q)$, so A is contained in $\mathrm{Aut}_0(Q)$, the subgroup of all automorphisms of Q that act trivially on $\mathbf{Z}(Q)$. Next, if $Q = E$, then $E/\mathbf{Z}(E) \lhd \mathrm{Aut}_0(Q) \leq (E/\mathbf{Z}(E)) \cdot \mathrm{Sp}_{2a}(p)$. The same also holds in the case $Q > E$, see [Gri, §1]. As $p \nmid |C|$, C injects into $\mathrm{Sp}(U) \cong \mathrm{Sp}_{2a}(p)$, and we can view C as a subgroup of $\mathrm{Sp}(U)$, with $U := E/\mathbf{Z}(E) \cong \mathbb{F}_p^{2a}$.

(iii) Here we consider the case $\ell = \mathrm{ppd}(p, 2a)$ exists. As ℓ divides $|C|$, $C < \mathrm{Sp}(U)$ acts irreducibly on U. As explained in part (a) of the proof of [BNRT, Theorem 5], $|C|$ divides $p^a + 1$. However, $|C| = (p^{2a} - 1)/\gcd(2, p-1)$, so we obtain $p^a - 1 \leq \gcd(2, p-1)$, and so $d = p^a \leq 3$, which is excluded.

Next we consider the case $a = 1$ and $p \geq 5$ a Mersenne prime. Then C is a cyclic subgroup of $\mathrm{SL}_2(p)$. Any such subgroup has order $\leq p + 1 < (p^2 - 1)/2$, contrary to the assumptions.

Finally, the case $p^a = 8$ is excluded since $\mathrm{Sp}_6(2)$ does not contain any cyclic subgroup of order $2^6 - 1$, see [Atlas]. □

With this Lemma 3.10, we have completed the proof of Proposition 3.6. □

Remark 3.11. Here we construct two examples related to the situations in Lemma 3.8. First we give an example of an imprimitive $\mathbb{C}J$-module of dimension $d = p^a$ that satisfies the conditions of Lemma 3.8, but does not satisfy ($\mathbf{S}$), for any odd prime power $p^a \geq 3$. Consider the d-dimensional vector space $V = \mathbb{C}^d$ with basis $\{e_v \mid v \in \mathbb{F}_q\}$. Let $H := \mathrm{AGL}_1(q)$ act on this basis as follows: the normal p-subgroup Q_1 of order q acts via translations $e_v \mapsto e_{u+v}$, $u \in \mathbb{F}_q$, and the complement $C := \{c_\lambda \mid \lambda \in \mathbb{F}_q^\times\}$ acts via $e_v \mapsto e_{\lambda v}$. Also consider the unique elementary abelian subgroup Q of order p^q of $\mathrm{GL}(V)$ that acts diagonally in the given basis. Then $J := Q \rtimes H = (Q \rtimes Q_1) \rtimes C$ acts imprimitively on V and has $M_4(J, V) = 3$. (Indeed, $\wedge^2(V)$ is irreducible and $\mathsf{S}^2(V)$ is the sum of two irreducible submodules. The equality $M_4(J, V) = 3$ then follows from the fact that c_{-1} acts as 1 on $e_v \otimes e_{-v} + e_{-v} \otimes e_v$ but as -1 on $e_v \otimes e_{-v} - e_{-v} \otimes e_v$ for any $0 \neq v \in V$.)

Next, let $d = p = 3$ and consider the faithful irreducible representation of the extraspecial 3-group $P = 3_+^{1+2}$ on $V = \mathbb{C}^3$. It is well known that this representation extends to $P \rtimes \mathrm{SL}_2(3)$. Now we can take $J = P \rtimes C_4$ inside $\mathrm{GL}(V)$ and observe that $M_4(J, V) = 3$.

Remark 3.12. If we drop the hypothesis that $p|D_0$, there is a three dimensional $\mathcal{A}$ in any characteristic $p \neq 3$ whose End consists of a nonzero tame part and two wild irreducibles. Start with

$\mathcal{L}_{\psi(x)}$ and form its Kummer direct image $[3]_\star\mathcal{L}_{\psi(x)}$. This is $I(\infty)$ irreducible. We first show that

$$\mathsf{Swan}_\infty(\mathrm{End}([3]_\star\mathcal{L}_{\psi(x)}) = 2.$$

In fact, for any $n \geq 1$ prime to p, we will have

$$\mathsf{Swan}_\infty(\mathrm{End}([n]_\star\mathcal{L}_{\psi(x)})) = n - 1.$$

To see this, use the fact that for n prime to p, and any $I(\infty)$ representation V, we have

$$\mathsf{Swan}_\infty([n]^\star V) = n\mathsf{Swan}_\infty(V).$$

Applied to $\mathrm{End}([n]_\star\mathcal{L}_{\psi(x)})$, this gives

$$\mathsf{Swan}_\infty(\mathrm{End}([n]_\star\mathcal{L}_{\psi(x)})) = (1/n)\mathsf{Swan}_\infty([n]^\star\mathrm{End}([n]_\star\mathcal{L}_{\psi(x)})) =$$
$$= (1/n)\mathsf{Swan}_\infty(\mathrm{End}([n]^\star[n]_\star\mathcal{L}_{\psi(x)})).$$

But $[n]^\star[n]_\star\mathcal{L}_{\psi(x)}$ is the direct sum $\oplus_{\zeta\in\mu_n}\mathcal{L}_{\psi(\zeta x)}$, whose End is

$$\oplus_{(\zeta_1,\zeta_2)\in\mu_n\times\mu_n}\mathcal{L}_{\psi((\zeta_1-\zeta_2)x)},$$

whose Swan conductor is visibly $n(n-1)$.

In fact, $\mathrm{End}([n]_\star\mathcal{L}_{\psi(x)})$ is the direct sum of the tame piece $[n]_\star(\mathbb{1})$ with the direct sum of the $n-1$ irreducible wild summands

$$\oplus_{\zeta\neq 1,\zeta\in\mu_n}[n]_\star(\mathcal{L}_{\psi((1-\zeta)x}),$$

each of which has Swan conductor one. To see this, we repeat the argument in Lemma 3.7. Denote by $I(n)$ the unique subgroup of $I(\infty)$ of index n. Because $I(n) \lhd I(\infty)$ is a normal subgroup, the induction $[n]_\star\mathcal{L}_{\psi(x)}$ has its character supported in $I(n)$. Hence also $\mathrm{End}([n]^\star[n]_\star\mathcal{L}_{\psi(x)})$ has its character supported in $I(n)$. Similarly, each term $[n]_\star\mathcal{L}_{\psi((1-\zeta)x)}$ has its character supported in $I(n)$. So it suffices to check that the two sides of the asserted identity have, after $[n]^\star$, the same character, which is visibly the case.

3E. Interesting special cases. For an integer $N \geq 2$ prime to p, we have the local system $\mathcal{F}_N$ on $\mathbb{A}^1$ of rank $D = N-1$ in characteristic p attached to the family of exponential sums

$$t \mapsto -\sum_x \psi(x^N + tx).$$

One knows that $\mathcal{F}_N$ is the Kummer pullback

$$\mathcal{F}_N \cong [N]^\star\mathcal{K}l(\psi; \text{all nontrivial } \chi \text{ with } \chi^N = \mathbb{1}).$$

This $\mathcal{K}l$ is visibly primitive (i.e., not Kummer induced), and thus satisfies $(\mathbf{S+})$ so long as $D := N-1$ is not 4 (or 8, if $p = 2$). We expect that $\mathcal{F}_5$ satisfies $(\mathbf{S+})$, but we do not know how to prove it. We also note that $\mathcal{F}_N$ itself is primitive when D is not a power of p, but is imprimitive (indeed its G_{geom} is a finite p-group) when D is any power of p.

For an integer $D \geq 2$ prime to p, and a nontrivial multiplicative character χ, we have the local system $\mathcal{G}_D$ on $\mathbb{A}^1$ of rank D in characteristic p attached to the family of exponential sums

$$t \mapsto -\sum_x \psi(x^D + tx)\chi(x).$$

One knows that $\mathcal{F}_N$ is the Kummer pullback

$$\mathcal{G}_D \cong [D]^\star\mathcal{H}(\psi; \text{all } \chi \text{ with } \chi^D = \mathbb{1}; \rho),$$

for any ρ with $\rho^D = \chi$. One knows [KRLT1, Lemma 1.1] that $\mathcal{G}_D$ is primitive for any D prime to p and any nontrivial χ, so a fortiori $\mathcal{H}$ is primitive as well. Then by Theorem 2.3, applied in the $W = D-1$ case, $\mathcal{H}$ satisfies $(\mathbf{S+})$ whenever D is prime to p.

3F. Another (S+) result. In this section, we work in a fixed characteristic p, and we denote by q a positive power of p.

Theorem 3.13. *Let $\mathcal{H}$ be a hypergeometric sheaf in characteristic p of type $(D, D-q)$, with $D > q = p^a$. Equivalently, $\mathcal{H}$ is of type (D,m) with $D > m > 0$ and wild part of dimension $W = q = p^a$. If D is a power, let n be the largest integer such that D is an n^{th} power, and suppose we have the inequality $W \geq n$. Then $\mathcal{H}$ has* $(\mathbf{S+})$.

Proof. We first show that such an $\mathcal{H}$ is primitive. To see that it cannot be Kummer induced, let $d \geq 2$ be the (necessarily prime to p) degree of the Kummer induction. Then $d|D$ and $d|m$, and hence $d|q$, a contradiction since d is prime to p.

We next show that $\mathcal{H}$ is not Belyi induced. Looking at the three types of Belyi induction in [KRLT3, Prop. 1.2], we see that whenever there are fewer "downstairs" characters than "upstairs" characters, the difference, i.e. W, is always of the form

$$d_0 p^r - d_0 = d_0(p^r - 1),$$

for some d_0 prime to p and some positive power p^r of p. This difference is prime to p, so cannot be q.

We next observe that $\mathcal{H}$ is tensor indecomposable, since already its $I(\infty)$ representation is tensor indecomposable, cf. [KRLT3, Prop. 10.1 or Thm. 2.1]. This applies when $D \neq 4$.

It also applies in the two cases $D = 4$ and either $p = 2$ and $m \geq 2$ or p odd and $m \neq 2$. Let us see that this is good enough for us. We are working in characteristic p, with $D = 4 > q$. So either we are in characteristic $p = 2$ and $W = 2$, which has $m = 2$, an allowed case, or we are in characteristic $p = 3$, $W = 3$, and $m = 1$, another allowed case.

If D is not a power, then $\mathcal{H}$ cannot be tensor induced, and we are done. If D is a power, recall that n is the largest integer such that D is an n^{th} power, in which case we assume $W \geq n$. Suppose that D is an r^{th} power, and that $\mathcal{H}$ is r-tensor induced. Then $W \geq r$, and, as explained in the first paragraph of the proof of Lemma 3.2, the composite map from π_1 to S_r is tame at both 0 and ∞, so its image is (the cyclic group generated by) an r-cycle, and r is prime to p. So we would find that the Kummer pullback $[r]^\star\mathcal{H}$ is tensor decomposable (of a very specific shape, but this will not matter). The key point is that the wild part Wild of the $I(\infty)$ representation of $\mathcal{H}$, having dimension q is irreducible on $P(\infty)$. Therefore all of its Kummer pullbacks, e.g. $[r]^\star\mathsf{Wild}$, are irreducible on $P(\infty)$, and hence a fortiori on $I(\infty)$. So we have only to apply [KRLT3, Prop. 10.1 or Thm. 2.1] to know that the $I(\infty)$ representation of $[r]^\star\mathcal{H}$ is tensor indecomposable. ☐

4. General results on G_{geom}

In this section, we consider a ($\overline{\mathbb{Q}_\ell}$-adic) hypergeometric sheaf,

$$\mathcal{H} := \mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m)$$

of type (D,m) with $D > m \geq 0$, defined over some finite subfield of $\overline{\mathbb{F}_p}$, $p \neq \ell$, and write $W := D - m$. The $I(\infty)$ representation on $\mathcal{H}$ is then the direct sum of a tame part of rank m and a totally wild part of rank W, all of whose ∞-breaks are $1/W$. Let us denote by

$$J := \text{the image of } I(\infty) \text{ on } \mathcal{H}. \tag{4.0.1}$$

One knows that J is a finite group if and only if the ρ_j are all distinct, and that $\mathcal{H}$ is geometrically irreducible if and only if none of χ_i is among the ρ_j.

Theorem 4.1. *Let $\mathcal{H}$ be an irreducible $\overline{\mathbb{Q}_\ell}$-hypergeometric sheaf on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, with $p \neq \ell$, and of type (D,m) with $W := D - m \geq 2$. Denote by G_0 the Zariski closure inside the geometric monodromy group G_{geom} of the normal subgroup generated by all G_{geom}-conjugates of the image of $I(0)$. Then*

$G_0 = G_{\rm geom}$. *In particular, if* $G_{\rm geom}$ *is finite then it is generated by all* $G_{\rm geom}$*-conjugates of the image of* $I(0)$*, and* $G_{\rm geom} = \mathbf{O}^p(G_{\rm geom})$.

Proof. Let $K := G_{\rm geom}/G_0$. Because $\mathcal{H}$ is geometrically irreducible, $G_{\rm geom}$ has a faithful irreducible representation, and hence is reductive. Therefore its quotient K is reductive.

Suppose K is nontrivial. Then it has at least one nontrivial irreducible representation, say ρ. View ρ as a representation of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$. So viewed, ρ is trivial on $I(0)$, so may be viewed as a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_\rho$ on the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$ which is irreducible and nontrivial. Therefore $H^i_c(\mathbb{A}^1/\overline{\mathbb{F}_p}, \mathcal{F}_\rho) = 0$ for $i \neq 1$. By the Euler-Poincare formula [Ka-GKM, 2.3.1],

$$\chi_c(\mathbb{A}^1/\overline{\mathbb{F}_p}, \mathcal{F}_\rho) = {\rm rank}(\mathcal{F}_\rho) - \mathsf{Swan}_\infty(\mathcal{F}_\rho),$$

and hence

$$h^1_c(\mathbb{A}^1/\overline{\mathbb{F}_p}, \mathcal{F}_\rho) = \mathsf{Swan}_\infty(\mathcal{F}_\rho) - {\rm rank}(\mathcal{F}_\rho).$$

As $h^1_c \geq 0$, we find that

$$\mathsf{Swan}_\infty(\mathcal{F}_\rho) \geq {\rm rank}(\mathcal{F}_\rho).$$

On the other hand, the upper numbering subgroup $I(\infty)^{1/W+\epsilon}$ dies in $G_{\rm geom}$ for all $\epsilon > 0$. So a fortiori, it dies in K, and hence all ∞-slopes of $\mathcal{F}_\rho$ are $\leq 1/W$. Hence

$$\mathsf{Swan}_\infty(\mathcal{F}_\rho) \leq (1/W){\rm rank}(\mathcal{F}_\rho) \leq (1/2){\rm rank}(\mathcal{F}_\rho),$$

a contradiction.

In the case $G_{\rm geom}$ is finite, G_0 is the normal closure of the image of $I(0)$, and is contained in the subgroup $\mathbf{O}^p(G_{\rm geom})$ generated by all p'-elements of $G_{\rm geom}$, whence the statements follow. $\square$

If we replace 0 by ∞ in Theorem 4.1, we get the following result.

Lemma 4.2. *Let* $\mathcal{H}$ *be an irreducible* $\overline{\mathbb{Q}_\ell}$*-hypergeometric sheaf on* $\mathbb{G}_m/\overline{\mathbb{F}_p}$*. Denote by* G_∞ *the Zariski closure inside the geometric monodromy group* $G_{\rm geom}$ *of the normal subgroup generated by all* $G_{\rm geom}$*-conjugates of the image* J *of* $I(\infty)$*. Then* $G_\infty = G_{\rm geom}$.

Proof. In this case, representations of the quotient $G_{\rm geom}/G_\infty$ correspond to lisse sheaves on $\mathbb{P}^1 \setminus \{0\}$ which are tame at 0, and any such is trivial. $\square$

Here is another companion result to Theorem 4.1.

Theorem 4.3. *Let* $\mathcal{H}$ *be an irreducible* $\overline{\mathbb{Q}_\ell}$*-hypergeometric sheaf on* $\mathbb{G}_m/\overline{\mathbb{F}_p}$ *definable on* $\mathbb{G}_m/\mathbb{F}_q$ *for some finite extension* $\mathbb{F}_q/\mathbb{F}_p$*, with* $p \neq \ell$*, and of type* (D, m) *with* $D > m$*. Denote by* $G_{P(\infty)}$ *the Zariski closure inside the geometric monodromy group* $G_{\rm geom}$ *of the normal subgroup generated by all* $G_{\rm geom}$*-conjugates of the image of the wild inertia group* $P(\infty)$*. Then* $G_{\rm geom}/G_{P(\infty)}$ *is a finite cyclic group of order prime to* p*.*

Proof. Let $K := G_{\rm geom}/G_{P(\infty)}$. Because $\mathcal{H}$ is definable on $\mathbb{G}_m/\mathbb{F}_q$, one knows [Ka-ESDE, 8.4.2 (4)] it is pure (of weight $D + m - 1$). It is geometrically, and hence arithmetically irreducible; therefore by [De-Weil II, 1.3.9] $G_{\rm geom}$ is a semisimple group (in the sense that its identity component $G^0_{\rm geom}$ is semisimple). Therefore the quotient K is semisimple. Let V be an irreducible representation of K. Then V is given by a geometrically irreducible lisse sheaf $\mathcal{F}$ on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ which is tame at both 0 and ∞. As a representation of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$, it factors through the quotient $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})^{\text{tame at }0,\infty}$, which is the pro-cyclic group $\prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ of pro-order prime to p. Any irreducible representation of this group is one-dimensional. Therefore $\mathcal{F}$ is lisse of rank one, and tame at 0 and ∞. Because K is semisimple, it admits a faithful finite dimensional representation, which is necessarity a direct sum of rank one sheaves $\mathcal{F}$ as above. Therefore K embeds into a finite product of groups ${\rm GL}_1(\overline{\mathbb{Q}_\ell})$. Thus K is abelian, and therefore (being semisimple) is finite. But the image of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$ in K is

Zariski dense (this already being true for its image in $G_{\rm geom}$). Therefore K is a finite quotient of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})^{\text{tame at }0,\infty}$, hence is cyclic of order prime to p. □

Proposition 4.4. *Let $\mathcal{H}$ be an (irreducible) hypergeometric sheaf of type (D,m) in characteristic p, with $D>m$ and with finite geometric monodromy group $G=G_{\rm geom}$. Then the following statements hold for the image Q of $P(\infty)$ in G:*

(i) *If $\mathcal{H}$ is not Kloosterman, i.e. if $m>0$, then $Q\cap \mathbf{Z}(G)=1$.*
(ii) *Suppose $\mathcal{H}$ is Kloosterman and $D>1$. Then $Q\not\leq \mathbf{Z}(G)$. If $p\nmid D$, then $Q\cap\mathbf{Z}(G)=1$. If $p|D$ then either $Q\cap\mathbf{Z}(G)=1$ or $Q\cap\mathbf{Z}(G)\cong C_p$.*
(iii) *If $D>1$, then $1\neq Q/(Q\cap\mathbf{Z}(G))\hookrightarrow G/\mathbf{Z}(G)$ and p divides $|G/\mathbf{Z}(G)|$.*
(iv) *If $D-m\geq 2$, the determinant of G is a p'-group. If moreover $p\nmid D$, then $\mathbf{Z}(G)$ is a p'-group.*
(v) *Suppose $p=2$. Then the trace of any element $g\in G$ on $\mathcal{H}$ is 2-rational* (i.e. lies in a cyclotomic field $\mathbb{Q}(\zeta_N)$ for some odd integer N)*; in particular, the 2-part of $|\mathbf{Z}(G)|$ is at most 2.*

Proof. (a) For (i), note that if $g\in Q\cap\mathbf{Z}(G)$, then g acts as a scalar on $\mathcal{H}$ and trivially on the (nonzero) tame part, hence $g=1$.

Suppose now that $\mathcal{H}$ is Kloosterman. If $Q\leq\mathbf{Z}(G)$, then Q acts as scalars on $\mathcal{H}$. Hence, the Q-module $\mathcal{H}$ is a direct sum of D copies of a 1-dimensional module. But the wild part, which is $\mathcal{H}$ in this case, is a direct sum of pairwise non-isomorphic simple Q-modules. So $D=1$.

Next assume that $\mathcal{H}$ is Kloosterman and $p\nmid D$. Then by [KRLT1, proof of Lemma 1.2], we know that Q can be identified with the additive group of the field $\mathbb{F}_{p^a}$, where $\mathbb{F}_{p^a}=\mathbb{F}_p(\zeta_D)$ and $\zeta_D\in\overline{\mathbb{F}_p}^\times$ has order D. Moreover, a generator of the tame quotient acts via conjugation on Q as multiplication by $\zeta_D\neq 1$. Hence, if $g\in Q\cap\mathbf{Z}(G)$, then $g\zeta_D=g$ and $g=0$ in Q viewed as $\mathbb{F}_p(\zeta_D)$, as stated.

(b) Now we consider the case $\mathcal{H}$ is Kloosterman and $p|D$. Recall [Ka-ESDE, 8.6.3] that the action of $I(\infty)$ on a Kloosterman $\mathcal{H}$ is uniquely determined by the rank D, up to tensoring with a one-dimensional representation and multiplicative translation.

Consider first the case of Kloosterman sheaf $\mathcal{H}$ of rank $q=p^f$. To analyze the Q-action, we may, by [Ka-ESDE, 8.6.3], assume that our $\mathcal{H}$ is

$$\mathcal{H}:=\mathcal{K}l(\text{all nontrivial characters of order dividing } q+1).$$

The action of Q does not change if we replace this $\mathcal{H}$ by its (prime to p) Kummer pullback

$$\mathcal{F}:=[q+1]^\star\mathcal{H},$$

which is the local system on $\mathbb{A}^1/\mathbb{F}_{q^2}$ whose trace function at $t\in k$, k a finite extension of $\mathbb{F}_{q^2}$, is

$$\mathcal{F}:t\mapsto -\sum_{x\in k}\psi_k(x^{q+1}+tx).$$

By a result of Pink [KT1, Corollary 20.3], the geometric monodromy group $G_{{\rm geom},\mathcal{F}}$ of $\mathcal{F}$ is a finite p-group. Using [Ka-LGE, Prop. 1.4.2], we see that $G_{{\rm geom},\mathcal{F}}$ is precisely Q. This allows us to apply to Q the known results about $G_{{\rm geom},\mathcal{F}}$, due to Pink and Sawin.

By the result [KT1, Corollary 20.2] of Pink, the image of Q on $\mathrm{End}(\mathcal{H})$ is the additive group $\mathbb{W}_q:=\{t\in\mathbb{F}_{q^4}\mid t+t^{q^2}=0\}$. By the irreducibility of the action of Q on $\mathcal{H}$, this tells us that $Q/\mathbf{Z}(Q)\cong\mathbb{W}_q$. To compute the order of $\mathbf{Z}(Q)$, it suffices to compute the order of Q. In the first part of the proof of Sawin's p-odd result [KT1, top of page 841], valid in any characteristic, he writes down an explicit description of the action of Q which shows that its order is pq^2. Therefore $\mathbf{Z}(Q)\cong C_p$. Because the Q-action is irreducible and faithful, $\mathbf{Z}(Q)\cong C_p$ acts by scalars, and faithfully. But any element of Q that acts by a scalar lies in $\mathbf{Z}(G)$. Thus $\mathbf{Z}(Q)\leq\mathbf{Z}(G)$. Conversely, any element of $Q\cap\mathbf{Z}(G)$ acts as a scalar, so (by the irrreducibility of the Q action) lies in $\mathbf{Z}(Q)$. So in this rank q case, we have $Q\cap\mathbf{Z}(G)\cong C_p$.

Now we consider the case when our Kloosterman sheaf $\mathcal{H}$ has rank dq with d prime to p. As $\mathbf{Z}(G)$ acts as scalars on $\mathcal{H}$, $Q \cap \mathbf{Z}(G) = \langle g \rangle$ is cyclic. We also know that $\mathcal{H}$ is a direct sum of d pairwise non-isomorphic simple Q-modules, Wild_i of dimension q, $1 \leq i \leq d$. As Q maps to the image Q_i of $P(\infty)$ on Wild_i, it follows from the preceding rank q result that g^p acts trivially on every Wild_i, and hence that g^p acts trivially on $\mathcal{H} \cong \oplus_{i=1}^d \mathsf{Wild}_i$. By the faithfulness of the action of Q on $\mathcal{H}$, $g^p = 1$. Therefore either $Q \cap \mathbf{Z}(G)$ is trivial, or $Q \cap \mathbf{Z}(G) \cong C_p$.

(c) For (iii), we note that $\mathcal{H}$ is not tame at ∞, hence $Q \neq 1$. It follows from (i) and (ii) that $Q \not\leq \mathbf{Z}(G)$, and so $1 \neq Q/(Q \cap \mathbf{Z}(G)) \hookrightarrow G/\mathbf{Z}(G)$. In particular, p divides $|G/\mathbf{Z}(G)|$.

(d) Now we establish (iv). By [Ka-ESDE, 8.11.6] $\det(G)$ is equal to the product of the D upstairs characters of $\mathcal{H}$, whence it is a p'-group. In particular, for any $g \in G_{\mathrm{geom}}$, $\det(g)$ is a p'-root of unity. If $z \in \mathbf{Z}(G)$ acts as the scalar $\alpha \in \mathbb{C}^\times$, then $\det(z) = \alpha^D$ is a p'-root of unity. So if $p \nmid D$, then α is a p'-root of unity, and hence z has p'-order. Thus $\mathbf{Z}(G)$ is a p'-group.

Finally, for (v) we note that any additive character of a finite field of characteristic 2 takes only integer values ± 1. Sp for any finite extension $k/\mathbb{F}_2$, and any multiplicative character χ of $k^\times$, the Gauss sum $\mathsf{Gauss}(\psi_k, \chi)$ lies in the field $\mathbb{Q}(\chi)$, which is $\mathbb{Q}(\zeta_N)$ for some odd integer N. View our $\mathcal{H}$ on $\mathbb{G}_m/\mathbb{F}_q$ for a finite extension $\mathbb{F}_q/\mathbb{F}_2$ such that all the "upstairs" characters χ_i and all the "downstairs" characters ρ_j of $\mathcal{H}$ are characters of $F_q^\times$, and define

$$\Lambda := \prod_i \chi_i, \quad A := \Lambda((-1)^{D-1}) q^{D(D-1)/2} \prod_{i,j} (-\mathsf{Gauss}(\psi_{\mathbb{F}_q}, \chi_i/\rho_j)).$$

Notice that A lies in the field $\mathbb{Q}(\zeta_{q-1})$, itself $\mathbb{Q}(\zeta_N)$ for some odd integer N. According to the arithmetic determinant formula [Ka-ESDE, 8.12.2], we have

$$\det(\mathcal{H}) \cong \begin{cases} \mathcal{L}_\Lambda \otimes A^{\deg/\mathbb{F}_q}, & \text{if } D - m \geq 2, \\ \mathcal{L}_\psi \otimes \mathcal{L}_\Lambda \otimes A^{\deg/\mathbb{F}_q}, & \text{if } D - m = 1. \end{cases}$$

The group G_{geom} of $\mathcal{H}$ does not change if we make an extension of the ground field, so we may consider $\mathcal{H}$ viewed on $\mathbb{G}_m/\mathbb{F}_{q^D}$. Relative to this ground field, we have

$$\det(\mathcal{H}) \cong \begin{cases} \mathcal{L}_\Lambda \otimes (A^D)^{\deg/\mathbb{F}_{q^D}}, & \text{if } D - m \geq 2, \\ \mathcal{L}_{\overline{\psi}} \otimes \mathcal{L}_\Lambda \otimes (A^D)^{\deg/\mathbb{F}_{q^D}}, & \text{if } D - m = 1. \end{cases}$$

The key point is that over this ground field, $\mathcal{H} \otimes A^{-\deg/\mathbb{F}_{q^D}}$ has finite arithmetic determinant, but all Frobenius traces still lie in $\mathbb{Q}(\zeta_{q-1})$. Because the determinant of $\mathcal{H} \otimes A^{-\deg/\mathbb{F}_{q^D}}$ is of finite order, and its G_{arith} normalizes the irreducible subgroup G_{geom}, G_{arith} itself is finite. The trace of any element $g \in G$ (indeed of any element g in G_{arith}, being the trace of some Frobenius, is 2-rational. In particular, if a 2-element $g \in \mathbf{Z}(G)$ acts on $\mathcal{H}$ as scalar $\alpha \in \mathbb{C}^\times$, then α is both a root of unity in some $\mathbb{Q}(\zeta_N)$ with N odd, and a 2-power root of unity, and so $\alpha = \pm 1$. $\square$

Theorem 4.5. *In the above situation of* (4.0.1), *suppose that J is a finite group. Let $\mathbb{F}$ be an algebraically closed field of characteristic $r = 0$ or $r \neq p$, and let*

$$\Lambda : J \to \mathrm{GL}_d(\mathbb{F})$$

be an $\mathbb{F}J$-representation of dimension $d \geq 1$. If $d < W$, then Λ is tame, and the image $\Lambda(J)$ is a finite cyclic group of order prime to p. If in addition Λ is irreducible, then $d = 1$.

Proof. Let

$$\rho_{\mathcal{H}} : I(\infty) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$$

be the representation of $I(\infty)$ on $\mathcal{H}$. The highest ∞-break of ρ is $1/W$, meaning precisely that the upper numbering subgroup $I(\infty)^{(1/W+)}$ of $I(\infty)$ lies in the kernel of ρ. The composite representation

$$\Lambda \circ \rho_{\mathcal{H}} : I(\infty) \twoheadrightarrow J \to \mathrm{GL}_d(\mathbb{F})$$

then has $I(\infty)^{(1/W+)}$ in its kernel, and hence has highest slope $\leq 1/W$. The Swan conductor of $\Lambda \circ \rho_{\mathcal{H}}$ then satisfies

$$\mathsf{Swan}(\Lambda \circ \rho_{\mathcal{H}}) \leq \text{rank} \times \text{highest slope} \leq d/W < 1,$$

and hence and hence by [Ka-GKM, 1.9] $\mathsf{Swan}(\Lambda \circ \rho_{\mathcal{H}}) = 0$. Thus $\Lambda \circ \rho_{\mathcal{H}}$ is tame, i.e., is a representation of the tame quotient $I(\infty)/P(\infty)$, which is abelian and pro-cyclic, of pro-order prime to p. Therefore the image $\Lambda(J)$ is a finite cyclic group of order prime to p. If in addition, Λ is irreducible, then $d = 1$, simply because $I(\infty)/P(\infty)$ is abelian. □

We now give a global version of this result.

Theorem 4.6. *Consider a ($\overline{\mathbb{Q}_\ell}$-adic) hypergeometric sheaf,*

$$\mathcal{H} := \mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m)$$

of type (D, m) with $D > m \geq 0$, defined over a finite subfield of $\overline{\mathbb{F}_p}$. Suppose that $W := D - m \geq 2$ and that $\mathcal{H}$ has finite geometric monodromy group G_{geom}. Suppose further that we are given a finite group Γ together with a surjective homomorphism

$$\phi : \Gamma \twoheadrightarrow G_{\mathrm{geom}}$$

whose kernel $\mathrm{Ker}(\phi)$ *is an abelian group of order prime to p. Let $\mathbb{F}$ be an algebraically closed field of characteristic $r = 0$ or $r \neq p$, and let*

$$\Lambda : \Gamma \to \mathrm{GL}_d(\mathbb{F})$$

be an $\mathbb{F}\Gamma$-representation of dimension $d \geq 1$. If $d < W$, then Λ is tame, and the image $\Lambda(\Gamma)$ is a finite cyclic group of order prime to p. If in addition Λ is irreducible, then $d = 1$.

Proof. Let us write $\pi_1^{\mathrm{geom}} := \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})$, and denote by

$$\rho_{\mathcal{H}} : \pi_1^{\mathrm{geom}} \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$$

the representation which "is" $\mathcal{H}$. By definition, $G_{\mathrm{geom}} = \rho_{\mathcal{H}}(\pi_1^{\mathrm{geom}})$, and we view $\rho_{\mathcal{H}}$ as a homomorphism

$$\rho_{\mathcal{H}} : \pi_1^{\mathrm{geom}} \twoheadrightarrow G_{\mathrm{geom}}.$$

From the short exact sequence

$$1 \to \mathrm{Ker}(\phi) \to \Gamma \to G_{\mathrm{geom}} \to 1,$$

we see that the obstruction to lifting $\rho_{\mathcal{H}}$ to a homomorphism

$$\tilde{\rho} : \pi_1^{\mathrm{geom}} \to \Gamma$$

lies in the group $H^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathrm{Ker}(\phi)) = 0$, the vanishing because open curves have cohomological dimension ≤ 1, cf. [SGA4t3, Cor. 2.7, Exp. IX and Thm. 5.1, Exp. X]. Let us choose such a lifting

$$\tilde{\rho} : \pi_1^{\mathrm{geom}} \to \Gamma.$$

The composite map

$$\pi_1^{\mathrm{geom}} \to \Gamma \twoheadrightarrow G_{\mathrm{geom}}$$

is tame at 0, i.e., trivial on $P(0)$, and has highest ∞-break $1/W$, i.e., trivial on $I(\infty)^{(1/W+)}$. Because $\mathrm{Ker}(\phi)$ has order prime to p, the map $\tilde{\rho}$ itself is trivial on the p-groups $P(0)$ and $I(\infty)^{(1/W+)}$. Therefore $\tilde{\rho}$ is tame at 0 and has highest ∞-break $\leq 1/W$. If we now compose $\tilde{\rho}$ with Λ, we find that $\Lambda \circ \tilde{\rho}$ is tame at 0 and has $\mathsf{Swan}_\infty \le d/W$. Hence if $d < W$, then $\Lambda \circ \tilde{\rho}$ has $\mathsf{Swan}_\infty = 0$, hence is tame at both 0 and ∞. But $\pi_1^{\text{geom,tame at } 0,\infty}$ is a pro-cyclic group of pro-order prime to p. Hence if $d < W$, then the image $\Lambda(\Gamma)$ is a finite cyclic group of order prime to p. In particular, if Λ is irreducible and $d < W$, then $d = 1$, simply because $\Lambda(\Gamma)$ is abelian. $\square$

Corollary 4.7. *In the situation of the theorem above, the group Γ has no faithful $\mathbb{F}\Gamma$-representation of dimension $d < W$. In particular, $G_{\rm geom}$ itself has no faithful $\mathbb{F}\Gamma$-representation of dimension $d < W$.*

Proof. By Theorem 4.6, any such representation Λ has image an abelian group. But Γ is not abelian, indeed its quotient $G_{\rm geom}$ is not abelian, as it has an irreducble $\overline{\mathbb{Q}_\ell}$ representation of dimension $n \ge W \ge 2$, namely the one coming from $\mathcal{H}$. $\square$

Here is another application of Theorem 4.6.

Theorem 4.8. *Let $\mathcal{H}$ be a hypergeometric sheaf of type (D, m) with $D > m \ge 0$ in characteristic p, and let G be the geometric monodromy group of $\mathcal{H}$. Suppose that*

(a) *G is a finite almost quasisimple group: $S \lhd G/\mathbf{Z}(G) \le \mathrm{Aut}(S)$ for some finite non-abelian simple group S;*
(b) *For some normal subgroup R of $G/\mathbf{Z}(G)$ containing S, R admits either a d-dimensional linear representation $\Phi : R \to \mathrm{GL}_d(\mathbb{F})$, or an e-dimensional projective representation $\Psi : R \to \mathrm{PGL}_e(\mathbb{F})$, over an algebraically closed field $\mathbb{F}$ of characteristic $\ne p$ and nontrivial over S.*

Then for the dimension $W = D - m$ of the wild part of $I(\infty)$ on $\mathcal{H}$ we have

$$W \le d \cdot [G/\mathbf{Z}(G) : R] \le d \cdot |\mathrm{Out}(S)|,$$

respectively

$$W \le (e^2 - 1) \cdot [G/\mathbf{Z}(G) : R] \le (e^2 - 1) \cdot |\mathrm{Out}(S)|.$$

Proof. In the case Ψ is given, we note that Ψ is faithful. [Indeed, $\mathrm{Ker}(\Psi) \lhd R$ does not contain S, and so intersects S trivially by simplicity of S. Because both S and $\mathrm{Ker}(\Psi)$ are normal in R, the commutator $[S, \mathrm{Ker}(\Psi)] \subset S \cap \mathrm{Ker}(\Psi) = 1$. Thus $\mathrm{Ker}(\Psi) \le \mathbf{C}_R(S) \le \mathbf{C}_{\mathrm{Aut}(S)}(S) = 1$. Hence R is embedded in $\mathrm{PGL}(U)$, where $U = \mathbb{F}^d$. Composing this embedding with the faithful action of $\mathrm{PGL}(U)$ on $\mathrm{End}(U)$/scalars, we obtain a faithful action of R on a module of dimension $\le e^2 - 1$. Thus it suffices to prove the bound $W \le d \cdot [G/\mathbf{Z}(G) : R]$ in the case $\Phi : R \to \mathrm{GL}(V)$ is given.

So assume the contrary: $\Phi : R \to \mathrm{GL}(V)$ is faithful with $\dim(V) = d$, but

$$W > d \cdot [G/\mathbf{Z}(G) : R]. \tag{4.8.1}$$

Let $\tilde{V}$ denote the $\bar{G}$-module $\mathrm{Ind}_R^{\bar{G}}(V)$ for $\bar{G} := G/\mathbf{Z}(G)$. Note that $\bar{G}$ acts faithfully on $\tilde{V}$. Indeed, let $K \lhd \bar{G}$ denote the kernel of the action of $\bar{G}$ on $\tilde{V}$. By the construction of V as the induced representation, the R-module $\tilde{V}$ contains V as a submodule. But S acts faithfully on V, hence $S \cap K = 1$. As $S \lhd \bar{G}$, it follows that $[S, K] = 1$, and so

$$K \le \mathbf{C}_{\bar{G}}(S) \le \mathbf{C}_{\mathrm{Aut}(S)}(S) = 1.$$

We also note that

$$\dim(\tilde{V}) = [\bar{G} : R] \cdot \dim(V) \le d \cdot [\bar{G} : R] < W$$

by (4.8.1); in particular, $D \ge W \ge 2$.

Now view $\tilde{V}$ as a representation of G, of dimension $< W$. By Theorem 4.6, applied with its Γ taken to be G, this representation is tame at both 0 and ∞. Thus the image Q in G of $P(\infty)$ acts trivially on $\tilde{V}$. But $G/\mathbf{Z}(G)$ acts faithfully on $\tilde{V}$. Therefore Q lands in $\mathbf{Z}(G)$, contradicting Proposition 4.4(iii). $\square$

5. Hypergeometricity results

In this section, we consider the question of when a $\overline{\mathbb{Q}_\ell}$-local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, $\ell \neq p$, is given by a hypergeometric sheaf.

Theorem 5.1. *Let G be a finite group, and $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$ a surjective homomorphism. Suppose we are given two irreducible representations*

$$\Phi_i : G \to \mathrm{GL}_{D_i}(\overline{\mathbb{Q}_\ell}), \quad i = 1, 2.$$

Let $\mathcal{H}_i$ the the local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ which realizes Φ_i; i.e., $\mathcal{H}_i$ is the local system given by the composite

$$\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \xrightarrow{\phi} G \xrightarrow{\Phi_i} \mathrm{GL}_{D_i}(\overline{\mathbb{Q}_\ell}), \quad i = 1, 2,$$

Suppose that there exists an integer a such that for every g in $\phi(P(0)) \cup \phi(P(\infty))$, we have

$$(**_p) \qquad \mathrm{Trace}(\Phi_2(g)) = a + \mathrm{Trace}(\Phi_1(g)).$$

Then the following conditions are equivalent.

(i) *$\mathcal{H}_1$ is hypergeometric, of type (D_1, m_1) with $D_1 > m_1$.*
(ii) *$\mathcal{H}_2$ is hypergeometric, of type (D_2, m_2) with $D_2 > m_2$; moreover $(D_2, m_2) = (D_1+a, m_1+a)$.*

Proof. By symmetry, it suffices to show that (i) implies (ii). Because $D_1 > m_1$, $\mathcal{H}_1$ is tame at 0. Apply $(**_p)$ to the image $P_G(0) = \phi(P(0))$ of the wild inertia group $P(0)$ at 0. For any $\gamma \in P_G(0)$, we have

$$\mathrm{Trace}(\Phi_1(\gamma)) = D_1,$$

simply because $\mathcal{H}_1$ is tame at 0. Therefore we have

$$\mathrm{Trace}(\Phi_2(\gamma)) = D_1 + a$$

for every $\gamma \in P_G(0)$. Thus the $P_G(0)$-representation on $\mathcal{H}_2$ has the same trace at $D_1 + a$ copies of the trivial representation, and hence $P_G(0)$ acts trivially on $\mathcal{H}_2$, and $\mathcal{H}_2$ has rank $D_1 + a$. In particular, $\mathcal{H}_2$ is tame at 0.

We next consider the action of the image $P_G(\infty) = \phi(P(\infty))$ of the wild inertia group $P(\infty)$ at ∞. Because $\mathcal{H}_1$ is hypergeometric and tame at 0 (and lisse on $\mathbb{G}_m$), its $P_G(\infty)$-representation has Swan conductor $\mathsf{Swan}_\infty(\mathcal{H}_1) = 1$. [Recall that the Swan conductor of a representation of the inertia group $I(\infty)$ is defined completely in terms of its restriction to $P(\infty)$ and of the restriction to $P(\infty)$ of the upper numbering filtration on $I(\infty)$, cf [Ka-GKM, 1.7].] From the equality $(**_p)$ applied to elements of $P_G(\infty) < G$, we see that $\mathcal{H}_2$ as a $P(\infty)$-representation is isomorphic as a virtual representation to direct sum of $\mathcal{H}_1$ as a $P(\infty)$-representation and a copies of the trivial representation. As Swan conductors pass to virtual representations, and trivial representations have Swan conductor zero, it follows that $\mathsf{Swan}_\infty(\mathcal{H}_2) = 1$. By [Ka-ESDE, Theorem 8.5.3], it follows that $\mathcal{H}_2$, being irreducible, tame at 0 and with $\mathsf{Swan}_\infty(\mathcal{H}_2) = 1$, is hypergeometric, of type (D_2, m_2) with $D_2 > m_2$. We have already seen, from the $P_G(0)$ analysis, that $D_2 = D_1 + a$.

We will now show that $m_2 = m_1 + a$. Break the $P_G(\infty)$-representations of $\mathcal{H}_1$ and $\mathcal{H}_2$ into tame and totally wild parts, say

$$\mathcal{H}_1 = \mathsf{Wild}_1 + m_1\mathbb{1}, \quad \mathcal{H}_2 = \mathsf{Wild}_2 + m_2\mathbb{1}.$$

From the equality of traces on $P_G(\infty)$, we have an equality of virtual representations of $P_G(\infty)$,

$$\mathsf{Wild}_2 + m_2\mathbb{1} = \mathsf{Wild}_1 + m_1\mathbb{1} + a\mathbb{1},$$

which we rewrite as

$$\mathsf{Wild}_2 - \mathsf{Wild}_1 = (m_1 + a - m_2)\mathbb{1}.$$

If, for example, $m_1 + a - m_2 \geq 0$, we get an isomorphism of representations

$$\mathsf{Wild}_2 = \mathsf{Wild}_1 + (m_1 + a - m_2)\mathbb{1}.$$

But Wild_2 is totally wild, hence it has no trivial components, and hence $m_1+a-m_2 = 0$. Similarly, if $m_1 + a - m_2 \leq 0$, then we get an isomorphism of representations

$$\mathsf{Wild}_1 = \mathsf{Wild}_2 + (m_2 - a - m_1)\mathbb{1},$$

and again infer that $m_1 + a - m_2 = 0$. $\square$

Some particularly useful consequences of Theorem 5.1 are the following:

Corollary 5.2. *Let G be a finite group, and $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$ a surjective homomorphism. Suppose $G = Z \times H$ for a p'-subgroup $Z \leq \mathbf{Z}(G)$ and $H \leq G$. Denote by $\pi : G \twoheadrightarrow H$ the projection and $\iota : H \to G$ the inclusion. Suppose we are given an irreducible representation $\Phi : G \to \mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$, and let $\mathcal{H}$ and $\mathcal{H}'$ be the local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ given by*

$$\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \xrightarrow{\phi} G \xrightarrow{\Phi} \mathrm{GL}_D(\overline{\mathbb{Q}_\ell}) \text{ and } \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \xrightarrow{\phi} G \xrightarrow{\pi} H \xrightarrow{\iota} G \xrightarrow{\Phi} \mathrm{GL}_D(\overline{\mathbb{Q}_\ell}),$$

respectively. Then $\mathcal{H}$ is hypergeometric, of type (D, m) with $D > m$, if and only if $\mathcal{H}'$ is hypergeometric, of type (D, m) with $D > m$.

Proof. Note that $(\iota \circ \pi)(h) = h$ for all $h \in H$, and, furthermore, any p-element $g \in G$ is contained in H as $p \nmid |Z|$. Moreover, $(\Phi \circ \iota \circ \pi)(G) = \Phi(H)$ is irreducible since $Z \leq \mathbf{Z}(G)$. Now for any p-element $g \in G$ we have $(\Phi \circ \iota \circ \pi)(g) = \Phi(g)$. Hence $(**_p)$ holds with $a = 0$, and the statement follows from Theorem 5.1. $\square$

Corollary 5.3. *Let G be a finite group, and $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$ a surjective homomorphism. Suppose we are given an irreducible representation $\Phi : G \to \mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$ and a tame representation $\Lambda : G \to \mathrm{GL}_1(\overline{\mathbb{Q}_\ell})$ of odd order such that $\Phi^* \cong \Phi \otimes \Lambda$. Then there exists a tame representation $\Theta : G \to \mathrm{GL}_1(\overline{\mathbb{Q}_\ell})$ such that $\Phi \otimes \Theta$ is self-dual. Let $\mathcal{H}$ and $\mathcal{H}'$ be the local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ given by*

$$\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \xrightarrow{\phi} G \xrightarrow{\Phi} \mathrm{GL}_D(\overline{\mathbb{Q}_\ell}) \text{ and } \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \xrightarrow{\phi} G \xrightarrow{\Phi\otimes\Theta} \mathrm{GL}_D(\overline{\mathbb{Q}_\ell}),$$

respectively. Then $\mathcal{H}$ is hypergeometric, of type (D, m) with $D > m$, if and only if $\mathcal{H}'$ is hypergeometric, of type (D, m) with $D > m$.

Proof. Let $N = 2m + 1$ denote the order of Λ, so that $\gcd(N, 2p) = 1$. Then

$$(\Phi \otimes \Lambda^{m+1})^* \cong \Phi^* \otimes \Lambda^{-m-1} \cong \Phi \otimes \Lambda^{-m} \cong \Phi \otimes \Lambda^{m+1},$$

i.e. we can take $\Theta = \Lambda^m$. Now, for any p-element $g \in G$, $\Theta(g) = 1$ as Θ is tame, whence $(\Phi \otimes \Theta)(g) = \Phi(g)$. Hence $(**_p)$ holds with $a = 0$, and the statement follows from Theorem 5.1. $\square$

In connection to the last statement, we prove the following useful fact:

Lemma 5.4. *Let $\mathcal{H}$ be a hypergeometric sheaf of type (D, m), where $D > m \geq 1$ and $D > 2$, with finite geometric monodromy group $G = G_{\mathrm{geom}}$. Let $\Phi : G \to \mathrm{GL}_D(\overline{\mathbb{Q}_\ell})$ denote the corresponding representation, and assume that, for the image Q of $P(\infty)$ in G, $(\Phi|_Q)^* \cong (\Phi|_Q) \otimes \Lambda$ for some 1-dimensional Q-representation Λ. Then Λ is trivial, unless $m = 1$, D is a power of p, and Q is elementary abelian of order D. In all cases, the Q-representation $\Phi|_Q$ is self-dual.*

Proof. Let φ denote the character of Φ. Write the dimension $w := D - m$ of Wild as tp^n with t prime to p and $n \geq 0$. Then one knows [Ka-GKM, 1.14] that $\varphi|_Q = \sum_{i=1}^{t} \theta_i + m \cdot 1_Q$, for t pairwise distinct nontrivial irreducible characters θ_i of Q, each of degree p^n, that are permuted transitively by J, the image of $I(\infty)$ in G. By assumption, there exists $\lambda \in \mathrm{Irr}(Q)$ such that $\overline{\varphi}|_Q = \varphi|_Q \cdot \lambda$, whence

$$\sum_{i=1}^{t} \overline{\theta}_i + m \cdot 1_Q = \overline{\varphi}|_Q = \varphi|_Q \cdot \lambda = \sum_{i=1}^{t} \theta_i \cdot \lambda|_Q + m\lambda.$$

Note that all the characters $\overline{\theta}_i$ are still irreducible and distinct. Hence, if $m \geq 2$, we must have that $\lambda = 1_Q$, and so $\Phi|_Q$ is self-dual.

Suppose now that $m = 1$ but $\lambda \neq 1_Q$. Then there exists some i such that $\lambda = \overline{\theta}_i$. Therefore θ_i has degree 1. Therefore $p^n = 1$, $t = w$, and every θ_j is a linear character of order p. Therefore Q is elementary abelian, and $\theta_j(1) = 1$ for all j. We now have that $\theta_i \cdot \overline{\sigma} = \sigma$ for $\sigma := \varphi|_Q$. Conjugating this equality by elements in J which acts transitively on $\{\theta_1, \ldots, \theta_t\}$, we see that

$$\theta_j \cdot \overline{\sigma} = \sigma \tag{5.4.1}$$

for all $1 \leq j \leq t$. It follows that

$$\sigma\overline{\sigma} = \big(1_Q + \sum_{j=1}^{t} \theta_j\big)\overline{\sigma} = \overline{\sigma} + \sum_{j=1}^{t} \theta_j \cdot \overline{\sigma} = \overline{\sigma} + t\sigma.$$

Taking complex conjugate and subtracting, we get $(t-1)(\overline{\sigma} - \sigma) = 0$. But note that $t = w = D - m = D - 1 > 1$ in this case, so $\overline{\sigma} = \sigma$. Next we show that $\sigma = \varphi|_Q$ is the regular character $\mathbf{reg}_Q$ and that $D = |Q|$ in this case. Indeed, consider any $g \in Q$ with $\sigma(g) \neq 0$. Then by (5.4.1) for the root of unity $z := \theta_j(g)$ we have $\sigma(g) = z\overline{\sigma}(g)$, for all j. It follows that $1 + tz = z(1 + t\bar{z}) = z + t$, and so $z = 1$ as $t > 1$. Thus $\sigma(g) = t + 1$ and $g \in \mathrm{Ker}(\Phi) = 1$. Thus $\sigma(x) = 0$ for all $1 \neq x \in Q$. Now

$$|Q| = |Q| \cdot [\sigma, 1_Q]_Q = \sum_{x \in Q} \sigma(x) = \sigma(1) = t + 1 = D.$$

As the $t + 1$ characters $1_Q, \theta_1, \ldots, \theta_t$ are all distinct, we conclude that $\sigma = \mathbf{reg}_Q$. □

6. Almost quasisimple groups containing elements with simple spectra

The goal of this section is to describe triples (G, V, g) subject to the following condition:

($\star$): G is an almost quasisimple finite group, with S the unique non-abelian composition factor, V a faithful irreducible $\mathbb{C}G$-module, and $g \in G$ has simple spectrum on V.

With G as in ($\star$), let $E(G)$ denote the *layer* of G, so that $E(G)$ is quasisimple and $S \cong E(G)/\mathbf{Z}(E(G))$. On the other hand, $G/\mathbf{Z}(G)$ is almost simple: $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$. We will frequently identify G with its image in $\mathrm{GL}(V)$. Let $\mathfrak{d}(S)$ denote the smallest degree of faithful projective irreducible complex representations of S, and let $\bar{\mathsf{o}}(g)$ denote the order of the element $g\mathbf{Z}(G)$ in $G/\mathbf{Z}(G)$. Adopting the notation of [GMPS], let $\mathrm{meo}(X)$ denote the largest order of elements in a finite group X. An element $g \in G \leq \mathrm{GL}(V)$ is called an ss-element, or an element with simple spectrum, if the multiplicity of any eigenvalue of g acting on V is 1. (Note that in ($\star$), we do *not* (yet) assume that $V|_{E(G)}$ is irreducible.)

We begin with a useful observation:

Lemma 6.1. *In the situation of ($\star$), we have*

$$\mathfrak{d}(S) \leq \dim(V) \leq \bar{\mathsf{o}}(g) \leq \mathrm{meo}(G/\mathbf{Z}(G)) \leq \mathrm{meo}(\mathrm{Aut}(S)).$$

Proof. For the first inequality, let U denote an irreducible summand of the $\mathbb{C}E(G)$-module V. Since G is almost quasisimple, $\mathbf{Z}(E(G)) \leq \mathbf{C}_G(S) = \mathbf{Z}(G)$. As the G-module V is faithful and irreducible, it follows that $\mathbf{Z}(E(G))$ acts faithfully (via scalars) on U, and so $E(G)$ is faithful on U. Thus U induces a faithful projective irreducible action of S, whence $n := \dim(V) \geq \dim(U) \geq \mathfrak{d}(S)$.

Next, let $\{\epsilon_1, \ldots, \epsilon_n\}$ denote the set of eigenvalues of g acting on V, and let $m := \bar{\mathsf{o}}(g)$. Then $g^m \in \mathbf{Z}(G)$ acts a scalar γ on V, hence $\epsilon_i^m = \gamma$ for all i. Since g has simple spectrum on V, we conclude that $n \leq m$, and the statement follows. $\square$

6A. Non-Lie-type groups. The goal of this subsection is to address the case where $S = \mathsf{A}_n$, the alternating group of degree $n \geq 7$, or one of the 26 sporadic simple groups. (We omit explicit results in the cases $S = \mathsf{A}_{5,6}$, since there are too many cases, all of which are tiny and can easily be looked up using [GAP].) For any partition $\lambda \vdash n$, let S^λ denote an irreducible $\mathbb{C}\mathsf{S}_n$-module labeled by λ. In particular, $S^{(n-1,1)}$ is just the *deleted permutation module* of S_n. We will also need to consider the so-called *basic spin modules* (acted on faithfully by the double cover $\hat{\mathsf{A}}_n$), see e.g. [KlT, §2].

The following result extends [GKT, Theorem 9.7]. (We note that the case $n = 6$ of [GKT, Theorem 9.7] inadvertently omitted a triple (G, V, g) with $G \cong \mathsf{A}_6$, $\dim(V) = 5 = |g|$.)

Theorem 6.2. *In the situation of* $(\star)$, *assume that* $S = \mathsf{A}_n$ *with* $n \geq 8$. *Then one of the following statements holds.*

(i) $E(G) = \mathsf{A}_n$ *and one of the following holds.*
  - (a) $\dim V = n-1$, $V|_{\mathsf{A}_n} \cong S^{(n-1,1)}|_{\mathsf{A}_n}$, *and, up to a scalar, g is either an n-cycle, or a disjoint product of a k-cycle and an $(n-k)$-cycle for some $1 \leq k \leq n-1$ coprime to n.*
  - (b) $n = 8$, $\dim V = 14$, *and, up to a scalar, g is an element of order* 15 *in* A_8.

(ii) $E(G) = \hat{\mathsf{A}}_n$ *and one of the following holds.*
  - (a) $n = 8$, $\dim V = 8$, $V|_{E(G)}$ *is a basic spin module, and* $\bar{\mathsf{o}}(g) = 10$, 12, *or* 15.
  - (b) $G/\mathbf{Z}(G) \cong \mathsf{A}_9$, $\dim V = 8$, $V|_{E(G)}$ *is a basic spin module, and* $\bar{\mathsf{o}}(g) = 9$, 10, 12, *or* 15.
  - (c) $G/\mathbf{Z}(G) \cong \mathsf{S}_9$, $\dim V = 16$, $V|_{E(G)}$ *is the sum of two basic spin modules, and* $\bar{\mathsf{o}}(g) = 20$.
  - (d) $G/\mathbf{Z}(G) \cong \mathsf{S}_{10}$, $\dim V = 16$, $V|_{E(G)}$ *is a basic spin module, and* $\bar{\mathsf{o}}(g) = 20$ *or* 30.
  - (e) $G/\mathbf{Z}(G) \cong \mathsf{A}_{11}$, $\dim V = 16$, $V|_{E(G)}$ *is a basic spin module, and* $\bar{\mathsf{o}}(g) = 20$.
  - (f) $G/\mathbf{Z}(G) \cong \mathsf{S}_{12}$, $\dim V = 32$, $V|_{E(G)}$ *is a basic spin module, and* $\bar{\mathsf{o}}(g) = 60$.

Proof. It is more convenient to work with a modified version H of G which may differ from G only by scalars and whose representation theory is better understood. If $G/\mathbf{Z}(G) = S$, we take $H = E(G)$.

Suppose $G/\mathbf{Z}(G) \cong \mathsf{S}_n$. Then there is an element $z \in G$ the conjugation by which induces the same automorphism of $E(G)$ as the one induced by the 2-cycle $(1,2)$. In particular, z^2 centralizes $E(G)$ and so $z^2 = \delta \cdot 1_V$ for some $\delta \in \mathbb{C}^\times$. In this case, taking $t := \delta^{-1/2}z$, we have that $t^2 = 1_V$ and choose $H := \langle E(G), t\rangle$. Our construction of H ensures that $\mathbf{Z}(\mathrm{GL}(V))G = \mathbf{Z}(\mathrm{GL}(V))H$; in particular, H is irreducible on V. If furthermore $E(G) = \mathsf{A}_n$, then since $|H| = 2|E(G)|$ and H induces the full $\mathrm{Aut}(S) \cong \mathsf{S}_n$, we have that $H \cong \mathsf{S}_n$. Consider the case $E(G) = \hat{\mathsf{A}}_n$. Then $\mathbf{Z}(H) = \mathbf{Z}(E(G)) < E(G) = [H,H]$ and $H/\mathbf{Z}(H) \cong G/\mathbf{C}_G(S) \cong \mathrm{Aut}(S) = \mathsf{S}_n$. Thus H is a central extension of S_n with kernel $\mathbf{Z}(H)$ of order 2 contained in $[H,H]$. By [Is, Corollary (11.20)], H is isomorphic to a universal cover of S_n, namely the one with order 2 inverse images of transpositions, usually denoted $\hat{\mathsf{S}}_n$ [KlT, §1].

From now on, we will replace G by H, so that $G \in \{\mathsf{A}_n, \mathsf{S}_n\}$ in the case $E(G) = \mathsf{A}_n$, and $G \in \{\hat{\mathsf{A}}_n, \hat{\mathsf{S}}_n\}$ in the case $E(G) = \hat{\mathsf{A}}_n$. We will let $\mathrm{cyc}(g)$ denote the number of disjoint cycles of the image of g in $G/\mathbf{Z}(G) \leq \mathsf{S}_n$.

(i) Here we assume that $E(G) \cong \mathsf{A}_n \cong S$, in particular, $\mathsf{A}_n \lhd G \leq \mathsf{S}_n$, and proceed by induction on $n \geq 8$. The cases $8 \leq n \leq 14$ can be checked directly using [Atlas] and [GAP], so we may assume $n \geq 15$. If furthermore $\dim(V) \leq n-1$, then by [Ra, Result 1] without loss we may assume that $V = S^{(n-1,1)}|_G$. In this case, if $\mathrm{cyc}(g) \geq 3$, then $\dim V^g \geq 2$, a contradiction. If $\mathrm{cyc}(g) = 2$: g is a product of disjoint k-cycle and $(n-k)$-cycle with $1 \leq k \leq n-1$ but $\gcd(k,n) > 1$, then

$\exp(2\pi i/\gcd(k,n))$ is an eigenvalue of g of multiplicity 2, again a contradiction. Thus we arrive at conclusion (i)(a).

We may now assume that $\dim(V) \geq n$. Assume furthermore that $\mathrm{cyc}(g) \leq 3$. Then $|g| \leq n^3/27$, whence $\dim(V) \leq n^3/27$ by Lemma 6.1. In particular, if W is an irreducible $\mathbb{C}\mathsf{S}_n$-module that contains V as a submodule upon restriction to G, then $\dim W \leq 2n^3/27 < n(n-1)(n-5)/6$. It follows from [Ra, Result 3] and the assumption $\dim(V) \geq n$ that $\dim W = \dim V$ and, up to tensoring with the sign representation, $V = S^{(n-2,1^2)}|_G$ or $S^{(n-2,2)}|_G$. Direct calculation shows that $\dim V^g \geq 2$ for all g with $\mathrm{cyc}(g) \leq 3$, a contradiction.

Thus we may assume that $s := \mathrm{cyc}(g) \geq 4$. Let $a_1 \geq a_2 \geq \ldots \geq a_s \geq 1$ denote the length of the disjoint cycles of g. Then we take m to be a_s if $2 \nmid a_s$, a_{s-1} if $2|a_s$ but $2 \nmid a_{s-1}$, and $a_{s-1}+a_s$ if $2|a_s, a_{s-1}$. Our choice of m ensures that (a conjugate of) g is contained in $\mathsf{A}_{n-m} \times \mathsf{A}_m$, with $n-m \geq 8$, and the A_{n-m}-component h of g has disjoint cycles of length a_1, a_2 (and possibly others) and $\mathrm{cyc}(h) \geq s-2$. Let $U_1 \otimes U_2$ be an irreducible summand of the module $V|_{\mathsf{A}_{n-m}\times\mathsf{A}_m}$ on which A_{n-m} acts nontrivially. Since $\mathrm{Spec}\,(g,V)$ is simple, $\mathrm{Spec}\,(h,U_1)$ is simple. By the induction hypothesis applied to U_1, $\mathrm{cyc}(h) \leq 2$, which implies $s=4$, $2|a_3,a_4$, and a_1,a_2 are coprime. Since $h \in \mathsf{A}_{n-m}$, we see that $2 \nmid a_1a_2$. Noting that $a_1+a_3+a_4 \geq 5+2+2 = 9$, we can now put g in $\mathsf{A}_{n-a_2} \times \mathsf{A}_{a_2}$ and repeat the above argument to get a contradiction, as the A_{n-a_2}-component h' of g now has $\mathrm{cyc}(h') = 3$.

(ii) Now we consider the case $E(G) \cong \hat{\mathsf{A}}_n$, in particular, $\hat{\mathsf{A}}_n \lhd G \leq \hat{\mathsf{S}}_n$. The cases $8 \leq n \leq 13$ can again be checked directly using [Atlas] and [GAP] (and they lead to examples (i)(a)–(f)), so we may assume $n \geq 14$. Note that

$$\dim(V) = \begin{cases} 2^{\lfloor (n-1)/2 \rfloor}, & G = \hat{\mathsf{S}}_n, \\ 2^{\lfloor (n-2)/2 \rfloor}, & G = \hat{\mathsf{A}}_n, \end{cases} \tag{6.2.1}$$

in particular, $\dim(V) \geq 2^{(n-3)/2}$. Now, if $n \geq 40$, then

$$\dim(V) \geq 2^{(n-3)/2} > e^{1.05314(n\ln n)^{1/2}} > \mathrm{meo}(\mathsf{S}_n) \geq \bar{\mathsf{o}}(g),$$

(where the second inequality follows from [Mas]), contradicting Lemma 6.1. For $20 \leq n \leq 39$, we can use the values of $\mathrm{meo}(\mathsf{S}_n)$ stored in the sequence A000793 of [Slo] to verify that

$$\dim(V) \geq 2^{\lfloor (n-2)/2 \rfloor} > \mathrm{meo}(\mathsf{S}_n) \geq \bar{\mathsf{o}}(g),$$

and again arrive at a contradiction. Using (6.2.1) and [GAP], we can verify that $\dim(V) > \mathrm{meo}(G)$ for $17 \leq n \leq 19$.

Now, the cases $G = \hat{\mathsf{S}}_n$ with $14 \leq n \leq 16$ can be checked using character tables available in [GAP]. We also have $\dim(V) \geq 128 > 105 = \mathrm{meo}(\mathsf{A}_{16})$ and $\dim(V) \geq 64 > 60 = \mathrm{meo}(\mathsf{A}_{14})$ when $n = 14, 16$. It remains to consider the case $G = \hat{\mathsf{A}}_{15}$. As $\bar{\mathsf{o}}(g) \geq \dim(V) \geq 64$, we must have that $\bar{\mathsf{o}}(g) = 105$ and that V is a basic spin module of $\hat{\mathsf{A}}_{15}$ of dimension 64 (as non-basic spin modules of $\hat{\mathsf{S}}_{15}$ have dimension ≥ 864, cf. [GAP]). Without loss, we may assume $|g| = 105$ and that $g = g_3g_5g_7$ lies in a central product $\hat{\mathsf{A}}_3 \circ \hat{\mathsf{A}}_5 \circ \hat{\mathsf{A}}_7$, with $g_j \in \hat{\mathsf{A}}_j$ has order j for $j = 3,5,7$. Note that g_j has $j-1$ distinct eigenvalues on basic spin modules of $\hat{\mathsf{A}}_j$ for $j = 3,5$, all different from 1. Furthermore, the restriction of V to any standard subgroup $\hat{\mathsf{A}}_{n'}$ of $\hat{\mathsf{A}}_n$ involves only basic spin modules of $\hat{\mathsf{A}}_{n'}$, see [KlT, Lemma 2.4]. It follows that g can have at most $2 \times 4 \times 7 = 56 < \dim(V)$ distinct eigenvalues on V, a contradiction. $\square$

Note that case (i)(b) of Theorem 6.2 does give rise to a hypergeometric sheaf in characteristic 2 with $G_{\mathrm{geom}} = \mathsf{A}_8 \cong \mathrm{GL}_4(2)$, see [KT5, Corollary 8.2]. Case (i)(a) is shown to occur in Theorem 9.3, whereas cases of dimension 16 or 32 of Theorem 6.2(ii) are ruled out in Lemma 9.1.

Next we record the following statement, which is useful in studying representations with irrational traces:

Lemma 6.3. *Let $\Phi : G \to \mathrm{GL}(V) \cong \mathrm{GL}_{n-1}(\mathbb{C})$ be a faithful irreducible representation of a finite almost quasisimple group G, which contains a normal subgroup $S \cong \mathsf{A}_n$ with $n \geq 7$. Suppose that*
(a) $V|_S \cong S^{(n-1,1)}|_S$, *and*
(b) $\mathbb{Q}(\varphi) \subseteq \mathbb{K}$ *for some number field $\mathbb{K}$, if φ denotes the character of Φ.*
Then $\mathbb{Q}(\varphi) \subseteq \mathbb{K}_0$, the subfield obtained by joining to $\mathbb{Q}$ all roots of unity that belong to $\mathbb{K}$. In fact, $\mathbb{Q}(\varphi)$ is some cyclotomic extension $\mathbb{Q}(\zeta_m)$ contained in $\mathbb{K}$, and $\mathrm{Tr}(\Phi(g))$ is an integer multiple of a root of unity for any $g \in G$.

Proof. (i) By Schur's lemma, $\mathbf{C}_G(S) = \mathbf{Z}(G)$ acts in Φ via scalars, and so by finiteness $\mathbf{Z}(G)$ is cyclic of order say k. Then $\varphi(x) \in \mathbb{Q}(\zeta_k) \subseteq \mathbb{K}_0$ for all $x \in \mathbf{Z}(G)$. Now if G induces only inner automorphisms of S, then $G = \mathbf{Z}(G)S = \mathbf{Z}(G) \times S$, and we are done since $\varphi(y) \in \mathbb{Z}$ for all $y \in S$; in this case, $\mathbb{Q}(\varphi) = \mathbb{Q}(\zeta_k)$. It is also clear that, for any $g \in G$, $\varphi(g) \in \mathbb{Z}\xi$ for some root of unity ξ.

(ii) It remains to consider the case G induces some outer automorphisms on S. As $n \geq 7$, it follows that $[G : \mathbf{Z}(G)S] = 2$, and we need to look at $\varphi(g)$ for all $g \in G \smallsetminus \mathbf{Z}(G)S$ with $\varphi(g) \neq 0$. Note that we can extend $\Phi|_S$ to S_n which without loss we also denote by Φ, and then $\mathrm{Tr}(\Phi(y)) \in \mathbb{Q}$ for all $y \in \mathsf{S}_n$. Given $g \in G \smallsetminus \mathbf{Z}(G)S$ with $\varphi(g) \neq 0$, we can find $h \in \mathsf{S}_n$ that induces the same action on S. It follows by Schur's lemma that $\Phi(g) = \xi\Phi(h)$ for some $\xi \in \mathbb{C}^\times$. Since both g and h have finite order, ξ is a root of unity. Also we have that $\mathbb{K}^\times \ni \varphi(g) = a\xi$ where $a := \mathrm{Tr}(\Phi(h)) \in \mathbb{Z}$. It follows that $\xi \in \mathbb{K}$, and so $\mathbb{K}_0$ contains ξ and $\varphi(g)$. We also note that $g^2, h^2 \in \mathbf{Z}(G)S$, and so $\Phi(g^2h^{-2}) = \xi^2 \cdot \mathrm{Id}$ belongs to $\Phi(\mathbf{Z}(G))$, whence $\xi^{2k} = 1$. Together with (i), we have shown that

$$\mathbb{Q}(\zeta_k) \subseteq \mathbb{Q}(\varphi) \subseteq \mathbb{K}_0 \cap \mathbb{Q}(\zeta_{2k}).$$

As $[\mathbb{Q}(\zeta_{2k} : \mathbb{Q}(\zeta_k)] \leq 2$, $\mathbb{Q}(\varphi)$ is either $\mathbb{Q}(\zeta_k)$ or $\mathbb{Q}(\zeta_{2k})$. □

Table 1 summarizes the classification of ss-elements in the non-generic cases of sporadic groups and A_7 and some small rank Lie-type groups, *under the additional condition that $V|_{E(G)}$ is irreducible*. For each V, we list all almost quasisimple groups G with common $E(G)$ that act on V, and we list the number of isomorphism classes of such representations in a given dimension, for a largest possible G up to scalars (if no number is given, it means the representation is unique up to equivalence in given dimension). For each representation, we list the names of conjugacy classes of ss-elements in a largest possible G, as listed in [GAP], and/or the total number of them. We also give a reference where a local system realizing the given representation is constructed. The indicator $\sharp$ signifies that we have a conjectured local system realizing the given representation, whereas (-) means that no hypergeometric sheaf with G as monodromy group can exist.

Theorem 6.4. *In the situation of $(\star)$, assume that S is one of* 26 *sporadic simple groups, or A_7, and that $V|_{E(G)}$ is irreducible. Then (S, G, V, g) are as listed in* Table 1.

Proof. We apply Lemma 6.1 to (G, V, g) to rule out 12 sporadic groups, listed in Table 2, because they all satisfy $\mathrm{meo}(\mathrm{Aut}(S)) < \mathfrak{d}(S)$. For the remaining 15 cases, we use [GAP] to find possible candidates for (G, V, g) (certainly, it suffices to search among representations of dimension at most $\mathrm{meo}(\mathrm{Aut}(S))$). □

Furthermore, we list in Table 3 certain hypergeometric sheaves

$$\mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m)$$

in characteristic p that are conjectured to produce G as geometric monodromy groups. All of them have been proved in [KRLT4] to have finite G_{geom}, and the cases marked with a reference to [KRLT4]

S	meo(Aut(S))	$\mathfrak{d}(S)$	G	dim(V)	ss-classes
A_7	12	4	$2\mathsf{A}_7$	4 (2 reps)	9 classes
			S_7	6 (2 reps)	$7A$, $6C$, $10A$, $12A$ (4 classes)
			$3\mathsf{A}_7$	6 (2 reps)	6 classes
			$6\mathsf{A}_7$	6 (4 reps)	15 classes
M_{11}	11	10	M_{11}	10 (3 reps) $\sharp$	$11AB$ (2 classes)
				11 $\sharp$	$11AB$ (2 classes)
M_{12}	12	10	$2\mathrm{M}_{12} \cdot 2$	10 (4 reps) (-)	11 classes
			M_{12}	11 (2 reps) (-)	$11AB$ (2 classes)
			$2\mathrm{M}_{12} \cdot 2$	12 (2 reps) (-)	$24AB$ (2 classes)
M_{22}	14	10	$2\mathrm{M}_{22} \cdot 2$	10 (4 reps) $\sharp$	10 classes
M_{23}	23	22	M_{23}	22 $\sharp$	$23AB$ (2 classes)
M_{24}	23	23	M_{24}	23 $\sharp$	$23AB$ (2 classes)
J_2	24	6	$2\mathrm{J}_2$	6 (2 reps) [KRL]	17 classes
			$2\mathrm{J}_2 \cdot 2$	14 (2 reps) $\sharp$	$28AB$, $24CDEF$ (6 classes)
J_3	34	18	$3\mathrm{J}_3$	18 (4 reps)	$19AB$, $57ABCD$ (6 classes)
HS	30	22	$\mathrm{HS} \cdot 2$	22 (2 reps) (-)	$30A$
McL	30	22	$\mathrm{McL} \cdot 2$	22 (2 reps) $\sharp$	$30A$, $22AB$ (3 classes)
Ru	29	28	2Ru	28	$29AB$, $58AB$ (4 classes)
Suz	40	12	6Suz	12 (2 reps) [KRLT3]	57 classes
Co_1	60	24	$2\mathrm{Co}_1$	24 [KRLT3]	17 classes
Co_2	30	23	Co_2	23 [KRLT2]	$23AB$, $30AB$ (4 classes)
Co_3	30	23	Co_3	23 [KRLT1]	$23AB$, $30A$ (3 classes)
$\mathrm{PSL}_3(4)$	21	6	$6S \cdot 2_1$	6 (4 reps)	many classes
			$4_1S \cdot 2_3$	8 (8 reps)	12 classes
			$2S \cdot 2_2$	10 (4 reps) $\sharp$	$14CDEF$ (4 classes)
$\mathrm{PSU}_4(3)$	28	6	$6_1S \cdot 2_2$	6 (4 reps) $\sharp$	many classes
$\mathrm{Sp}_6(2)$	15	7	$\mathrm{Sp}_6(2)$	7	$7A$, $8B$, $9A$, $12C$, $15A$
			$2\mathrm{Sp}_6(2)$	8 $\sharp$	8 classes
			$\mathrm{Sp}_6(2)$	15 (-)	$15A$
$\Omega_8^+(2)$	30	8	$2\Omega_8^+(2) \cdot 2$	8 $\sharp$	22 classes
${}^2B_2(8)$	15	14	${}^2B_2(8) \cdot 3$	14 (6 reps) $\sharp$	$15AB$ (2 classes)
$G_2(3)$	18	14	$G_2(3) \cdot 2$	14 (2 reps) $\sharp$	$14A$, $18ABC$ (4 classes)
$G_2(4)$	24	12	$2G_2(4) \cdot 2$	12 (2 reps) $\sharp$	20 classes

TABLE 1. Elements with simple spectra in non-generic cases

are proved therein to have the conjectured G as G_{geom}. For any natural number N, the notation Char_N denotes the set of all characters of order dividing N, $\mathsf{Char}_N^\times$ denotes the set of all characters of order exactly N, and ξ_N denotes a fixed character of order N. The last column indicates the conjectured image of $I(\infty)$.

6B. Finite groups of Lie type. In this subsection, we will deal with almost quasisimple groups G, where S is a finite simple group of Lie type. We will need the following well-known consequences of the Lang-Steinberg theorem:

S	$\mathrm{meo}(\mathrm{Aut}(S))$	$\mathfrak{d}(S)$	S	$\mathrm{meo}(\mathrm{Aut}(S))$	$\mathfrak{d}(S)$	S	$\mathrm{meo}(\mathrm{Aut}(S))$	$\mathfrak{d}(S)$
J_1	19	56	J_4	66	1333	He	42	51
Ly	62	2480	O′N	56	342	HN	60	133
Fi_{22}	42	78	Fi_{23}	60	782	Fi'_{24}	84	783
Th	39	248	BM	70	4371	M	119	196883

TABLE 2. Maximal element order and minimal degree for some sporadic groups

S	G	p	rank	$\chi_1, \ldots, \chi_D$	$\rho_1, \ldots, \rho_m$	Image of $I(\infty)$
M_{11}	S	3	10 (WS)	$\mathsf{Char}_{11}^{\times}$	Char_2	$3^2 : 8$
	S	3	10 [Lem. 9.5]	$\mathsf{Char}_{11}^{\times}$	ξ_8, ξ_8^3	$3^2 : 8$
	S	3	11 (WS)	Char_{11}	$\mathsf{Char}_4 \smallsetminus \{\mathbb{1}\}$	$3^2 : 8$
M_{22}	$2S$	2	10 [KRLT4]	$\mathsf{Char}_{11}^{\times}$	ξ_7, ξ_7^2, ξ_7^4	$2^3 : 7$
M_{23}	S	2	22	$\mathsf{Char}_{23}^{\times}$	$\mathsf{Char}_{15} \smallsetminus \mathsf{Char}_{15}^{\times}$	$2^4 : 15$
M_{24}	S	2	23 (WS)	Char_{23}	$\mathsf{Char}_{3}^{\times}$	$2^6 : 21$
McL	$S \cdot 2$	3	22 [KRLT4]	Char_{22}	$\mathsf{Char}_{5}^{\times}$	$3^{1+4} : 20$
	$S \cdot 2$	5	22 [KRLT4]	Char_{22}	$\mathsf{Char}_{3}^{\times}$	$5^{1+2} : 24$
J_2	$2S \cdot 2$	5	14 [KRLT4]	$\mathsf{Char}_{28} \smallsetminus \mathsf{Char}_{14}$	ξ_8, ξ_8^{-1}	$5^2 : 24$
J_3	$3S$	2	18 [KRLT4]	$\xi_3 \cdot \mathsf{Char}_{19}^{\times}$	$\mathbb{1}, \xi_5, \xi_5^{-1}$	$2^4 : 15$
Ru	$2S$	5	28 [KRLT4]	$\mathsf{Char}_{29}^{\times}$	$\xi_{12}, \xi_{12}^3, \xi_{12}^5, \xi_{12}^9$	$5^2 : 24$
$\mathrm{PSU}_4(3)$	$6_1 \cdot S$	3	6 [KRLT4]	$\mathsf{Char}_{7}^{\times}$	ξ_2	$3^4 : 10$
$\mathrm{Sp}_6(2)$	$2S$	7	8	$\mathsf{Char}_9 \smallsetminus \{\mathbb{1}\}$	Char_2	$7 : 6$
$\Omega_8^+(2)$	$2S \cdot 2$	3	8 [KRLT4]	$\mathsf{Char}_{20}^{\times}$	Char_2	$3^{1+2} : 8$
	$2S \cdot 2$	7	8 [KRLT4]	$\mathsf{Char}_{20}^{\times}$	Char_2	$7 : 6$
$\mathrm{PSL}_3(4)$	$2S \cdot 2_2$	3	10	$\mathsf{Char}_{14} \smallsetminus \{\mathbb{1}, \xi_7, \xi_7^2, \xi_7^4\}$	$\mathsf{Char}_{4}^{\times}$	$3^2 : 8$
$G_2(4)$	$2 \cdot S$	2	12 [KRLT4]	$\mathsf{Char}_{13}^{\times}$	$\mathsf{Char}_{3}^{\times}$	2-group : 15
$G_2(3)$	$S \cdot 2$	13	14 [KRLT4]	$\mathsf{Char}_{18} \smallsetminus \{\mathbb{1}, \xi_6, \xi_6^2, \xi_6^3\}$	$\mathsf{Char}_{4}^{\times}$	$13 : 12$
${}^2B_2(8)$	$S \cdot 3$	13	14 [KRLT4]	$\mathsf{Char}_{15} \smallsetminus \{\mathbb{1}\}$	ξ_{12}, ξ_{12}^5	$13 : 12$

TABLE 3. Hypergeometric sheaves in non-generic cases

Lemma 6.5. *Let $\mathcal{G}$ be a connected algebraic group over an algebraically closed field of characteristic $p > 0$ and let $\sigma : \mathcal{G} \to \mathcal{G}$ be a surjective morphism with finite $\mathcal{G}^\sigma := \{x \in \mathcal{G} \mid \sigma(x) = x\}$.*

(i) *Suppose the $\mathcal{G}$-conjugacy class of $g \in \mathcal{G}$ is σ-stable. Then some $\mathcal{G}$-conjugate of g is σ-fixed, in particular, $|g| \leq \mathrm{meo}(\mathcal{G}^\sigma)$.*

(ii) *Suppose that $[\mathcal{G}, \mathcal{G}]$ is simply connected and $g \in \mathcal{G}^\sigma$ is semisimple. Then, for any $t \in \mathcal{G}$ with $tgt^{-1} \in \mathcal{G}^\sigma$, tgt^{-1} is $\mathcal{G}^\sigma$-conjugate to g.*

Proof. (i) By assumption, $\sigma(g) = xgx^{-1}$ for some $x \in \mathcal{G}$. Since $\mathcal{G}$ is connected, the Lang map $y \mapsto y^{-1}\sigma(y)$ is surjective on $\mathcal{G}$. Hence $x = y^{-1}\sigma(y)$ for some $y \in \mathcal{G}$. Thus $\sigma(g) = \sigma(y^{-1})ygy^{-1}\sigma(y)$, whence $ygy^{-1} \in \mathcal{G}^\sigma$, and the statement follows.

(ii) By assumption, $t^{-1}\sigma(t) \in \mathbf{C}_{\mathcal{G}}(g)$. Since $\sigma(g) = g$ and $[\mathcal{G}, \mathcal{G}]$ is simply connected, by [C, Theorem 3.5.6] $\mathbf{C}_{\mathcal{G}}(g)$ is connected and σ-stable. By the Lang-Steinberg theorem applied to $\mathbf{C}_{\mathcal{G}}(g)$, $t^{-1}\sigma(t) = c^{-1}\sigma(c)$ for some $c \in \mathbf{C}_{\mathcal{G}}(g)$. Now $u := tc^{-1} \in \mathcal{G}^\sigma$ and $tgt^{-1} = tc^{-1}gct^{-1} = ugu^{-1}$ is $\mathcal{G}^\sigma$-conjugate to g. $\square$

Theorem 6.6. *In the situation of* ($\star$), *assume that* S *is a finite simple group of Lie type. Then one of the following statements holds.*

(i) $S \cong \mathrm{PSL}_2(q)$ *and* $\dim(V) \le \bar{\mathsf{o}}(g) \le q+1$.

(ii) $S = \mathrm{PSL}_n(q)$, $n \ge 3$, $E(G)$ *is a quotient of* $\mathrm{SL}_n(q)$, *and* $V|_{E(G)}$ *is one of* $q-1$ *Weil modules, of dimension* $(q^n-1)/(q-1)$ *or* $(q^n-q)/(q-1)$. *Moreover,* $\dim(V) \le \bar{\mathsf{o}}(g) \le (q^n-1)/(q-1)$.

(iii) $S = \mathrm{PSU}_n(q)$, $n \ge 3$, $E(G)$ *is a quotient of* $\mathrm{SU}_n(q)$, *and* $V|_{E(G)}$ *is one of* $q+1$ *Weil modules, of dimension* $(q^n-(-1)^n)/(q+1)$ *or* $(q^n+q(-1)^n)/(q+1)$.

(iv) $S = \mathrm{PSp}_{2n}(q)$, $n \ge 2$, $2 \nmid q$, $E(G)$ *is a quotient of* $\mathrm{Sp}_{2n}(q)$, *every irreducible constituent of* $V|_{E(G)}$ *is one of four Weil modules, of dimension* $d := (q^n \pm 1)/2$, *and* $\dim(V) = d$ *or* $2d$.

(v) *Non-generic cases:*

  (a) S *is one of the following groups:* $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_4(3)$, $\mathrm{Sp}_6(2)$, $\Omega_8^+(2)$, ${}^2B_2(8)$, $G_2(3)$, $G_2(4)$, $V|_{E(G)}$ *is simple, and the classification of* ss*-elements in* G *can be read off from Table I.*

  (b) $V|_{E(G)}$ *is the direct sum of two simple modules of equal dimension, and one of the following possibilities occurs.*

    (α) $E(G) = S = \mathrm{SU}_4(2)$, $G/\mathbf{Z}(G) = \mathrm{Aut}(S)$, *either* $\dim(V) = 8$ *and* $\bar{\mathsf{o}}(g) = 9, 10, 12$, *or* $\dim(V) = 10$ *and* $\bar{\mathsf{o}}(g) = 10, 12$.

    (β) $S = \mathrm{SU}_5(2)$, $G/\mathbf{Z}(G) = \mathrm{Aut}(S)$, $\dim(V) = 22$, *and* $\bar{\mathsf{o}}(g) = 24$.

Proof. By Lemma 6.1,

$$\mathrm{meo}(\mathrm{Aut}(S)) \ge \dim(V) \ge \mathfrak{d}(S). \tag{6.6.1}$$

We will use the upper bounds on $\mathrm{meo}(\mathrm{Aut}(S))$ available from [KSe] and [GMPS], on the one hand, and the (precise or lower) bounds on $\mathfrak{d}(S)$ as recorded in [T1, Table I], to show that most of the possibilities for S contradict (6.6.1). We will frequently use the obvious estimate

$$\mathrm{meo}(\mathrm{Aut}(S)) \le \mathrm{meo}(S) \cdot |\mathrm{Out}(S)|. \tag{6.6.2}$$

(A) First we consider exceptional groups of Lie type.

(A1) Assume $S = {}^2G_2(q)$, with $q = 3^{2a+1} \ge 27$. By [KSe, Table A.7], $\mathrm{meo}(S) \le q + \sqrt{3q} + 1$, hence $\mathrm{meo}(\mathrm{Aut}(S)) \le (q + \sqrt{3q} + 1)(2a+1)$ by (6.6.2). On the other hand, $\mathfrak{d}(S) = q^2 - q + 1$, contradicting (6.6.1).

Similarly, if $S = {}^2B_2(q)$ with $q = 2^{2a+1} \ge 128$, then $\mathrm{meo}(S) \le q + \sqrt{2q} + 1$, hence $\mathrm{meo}(\mathrm{Aut}(S)) \le (q + \sqrt{2q} + 1)(2a+1)$ by (6.6.2). On the other hand, $\mathfrak{d}(S) = (q-1)\sqrt{q/2}$, contradicting (6.6.1). The cases $S = {}^2B_2(q)$ with $q = 8, 32$ can be checked directly using [GAP].

Let $S = {}^2F_4(q)$, with $q = 2^{2a+1} \ge 8$. By [GMPS, Table 5],

$$\mathrm{meo}(\mathrm{Aut}(S)) \le 16(q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1)(2a+1).$$

This contradicts (6.6.1), since $\mathfrak{d}(S) = (q^3+1)(q^2-1)\sqrt{q/2}$. If $S = {}^2F_4(2)'$, then, according to [GAP], $\mathrm{meo}(\mathrm{Aut}(S)) = 20 < 27 = \mathfrak{d}(S)$.

(A2) Assume $S = {}^3D_4(q)$ with $q = p^f > 2$. We will show that

$$\mathrm{meo}(\mathrm{Aut}(S)) < \mathfrak{d}(S) = q(q^4 - q^2 + 1), \tag{6.6.3}$$

which contradicts (6.6.1). Indeed, if $p > 2$, then $\mathrm{meo}(S) = (q^3-1)(q+1)$ by [KSe, Table A.7]. On the other hand, if $p = 2$ then, by Propositions 2.1–2.3 of [DMi], the order of any element $s \in S$ is at most $(q^3-1)(q+1)$ if s is semisimple, and $\max(2(q^3+1), 8(q^2+q+1)) \le (q^3-1)(q+1)$ otherwise, and so $\mathrm{meo}(S) = (q^3-1)(q+1)$ again. Hence, if $q \ne 3, 4, 8$, then $\mathrm{meo}(\mathrm{Aut}(S)) \le 3f(q^3-1)(q+1)$ by (6.6.2), and so (6.6.3) holds.

Assume now that $q = 3, 4$, or 8, and view $S = \mathcal{G}^{\sigma^f \tau}$, where $\mathcal{G} = \mathrm{Spin}_8(\overline{\mathbb{F}_p})$, $\sigma : \mathcal{G} \to \mathcal{G}$ the standard Frobenius morphism induced by the map $x \mapsto x^p$ of $\overline{\mathbb{F}_p}$, and τ a triality automorphism of

$\mathcal{G}$ that commutes with σ. Then the restriction $\alpha := \sigma|_S$ induces an automorphism of order $3f$ of S, and $A := \mathrm{Aut}(S) = S \rtimes \langle \alpha \rangle$, cf. [GLS, Theorem 2.5.12]. Consider any element $g \in \mathrm{Aut}(S)$. If $S\langle g \rangle < A$, then

$$|g| \leq (3f/2)\mathrm{meo}(S) \leq 3f(q^3-1)(q+1)/2 < q(q^4-q^2+1).$$

In the remaining case, $S\langle g \rangle = A$. Note that $h := g^{3f} \in S$ is centralized by g, and so $[\mathbf{C}_A(h) : \mathbf{C}_S(h)] = [A : S]$. Hence $\#(h^A) = \#(h^S)$; in particular,

$$\sigma(h) = \alpha h \alpha^{-1} = tht^{-1}$$

for some $t \in S$. Thus the $\mathcal{G}$-conjugacy class of h is σ-stable, and so

$$|g| \leq 3f \cdot |h| \leq 3f \cdot \mathrm{meo}(\mathcal{G}^\sigma) = 3f \cdot \mathrm{meo}(\mathrm{Spin}_8^+(p))$$

by Lemma 6.5(i). Using [Atlas] one can check that $\mathrm{meo}(\mathrm{Spin}_8^+(3)) \leq 2 \cdot \mathrm{meo}(P\Omega_8^+(3)) = 40$ and $\mathrm{meo}(\mathrm{Spin}_8^+(2)) = 15$. Thus $|g| \leq 120$, respectively 210, 305, when $q = 3$, 4, and 8, respectively. It follows that $|g| < q(q^4-q^2+1)$, completing the proof of (6.6.3).

If $S = {}^3D_4(2)$, then $\mathrm{meo}(\mathrm{Aut}(S)) = 28$ and $\mathfrak{d}(S) = 26$ according to [GAP]. However, using character tables in [GAP], one can check that no ss-element exists.

(A3) Assume $S = G_2(q)$ with $q = p^f \geq 5$. If $p > 2$, then $\mathrm{meo}(S) = q^2+q+1$ by [KSe, Table A.7], and so $\mathrm{meo}(\mathrm{Aut}(S)) \leq f(q^2+q+1)$ if $p > 3$ and $\mathrm{meo}(\mathrm{Aut}(S)) \leq 2f(q^2+q+1)$ if $p = 3$. If $p = 2$ and $q \geq 8$, then using [EY] one can check that the order of any element $g \in S$ is at most q^2+q+1 if g is semisimple, and $2(q^2-1)$ otherwise, and so $\mathrm{meo}(\mathrm{Aut}(S)) < 2f(q^2+q+1)$. On the other hand, $\mathfrak{d}(S) \geq q^3-1$ if $p \neq 3$ and $\mathfrak{d}(S) = q^4+q^2+1$ if $p = 3$, see [T1, Table I], and we arrive at a contradiction when $q \geq 5$. The cases $q = 3, 4$ are handled directly using [GAP].

Let $S = F_4(q)$, with $q = p^f \geq 3$. Arguing as in the proof of [GMPS, Theorem 1.2], also using [KSe, Table A.7], we get $\mathrm{meo}(\mathrm{Aut}(S)) \leq 32fq(q^2-1)(q+1)$. This contradicts (6.6.1), since $\mathfrak{d}(S) \geq q^8-q^4+1$. If $S = F_4(2)$, then, according to [GAP], $\mathrm{meo}(\mathrm{Aut}(S)) = 40 < 52 = \mathfrak{d}(S)$.

Likewise, if $S = {}^2E_6(q)$ with $q = p^f \geq 3$, then arguing as in the proof of [GMPS, Theorem 1.2] and using [KSe, Table A.7], we get $\mathrm{meo}(\mathrm{Aut}(S)) \leq 32f(q^3-1)(q^2+1)(q+1)$. This contradicts (6.6.1), since $\mathfrak{d}(S) = q(q^4+1)(q^6-q^3+1)$. If $S = {}^2E_6(2)$, then, according to [GAP], $\mathrm{meo}(\mathrm{Aut}(S)) = 105 < 1938 = \mathfrak{d}(S)$. If $S = E_6(q)$ with $q = p^f \geq 3$, then the same arguments show that $\mathrm{meo}(\mathrm{Aut}(S)) \leq 32f(q^6-1)/(q-1)$. This contradicts (6.6.1), since $\mathfrak{d}(S) = q(q^4+1)(q^6+q^3+1)$. If $S = E_6(2)$, then $\mathrm{meo}(S) = 126$ according to [GAP], hence $\mathrm{meo}(\mathrm{Aut}(S)) \leq 252 < 2482 = \mathfrak{d}(S)$.

The same arguments apply to the last two exceptional types. If $S = E_7(q)$, then

$$\mathrm{meo}(\mathrm{Aut}(S)) \leq 32f(q+1)(q^2+1)(q^4+1) < q^{15}(q^2-1) < \mathfrak{d}(S).$$

If $S = E_8(q)$, then

$$\mathrm{meo}(\mathrm{Aut}(S)) \leq 32f(q+1)(q^2+q+1)(q^5-1) < q^{27}(q^2-1) < \mathfrak{d}(S).$$

(B) Now we analyze the simple classical groups.

(B1) Suppose $S = \mathrm{Sp}_{2n}(q)$ with $n \geq 2$ and $2|q$. Then $\mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n+1}/(q-1)$ by [GMPS, Theorem 2.16], whereas $\mathfrak{d}(S) = (q^n-1)(q^n-q)/2(q+1)$ by [T1, Table I], and this contradicts (6.6.1), unless $(n,q) = (3,2)$, $(2,4)$, $(2,2)$. The remaining exceptions are handled using [GAP]. Likewise, if $S = \Omega_{2n+1}(q)$ with $n \geq 3$, $2 \nmid q$, and $(n,q) \neq (3,3)$, then $\mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n+1}/(q-1)$ by [GMPS, Theorem 2.16], and $\mathfrak{d}(S) \geq (q^n-1)(q^n-q)/(q^2-1)$ by [T1, Table I], again contradicting (6.6.1). If $S = P\Omega^\epsilon_{2n}(q)$ with $n \geq 4$ and $(n,q,\epsilon) \neq (4,2,+)$, then $\mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n+1}/(q-1)$ by [GMPS, Theorem 2.16] and $\mathfrak{d}(S) \geq (q^n+1)(q^{n-1}-q)/(q^2-1)$ by [T1, Table I], contradicting (6.6.1). The cases $S = \Omega_7(3)$ and $\Omega_8^+(2)$ are handled using [GAP].

(B2) Assume now that $S = \mathrm{PSL}_n(q)$ with $n \geq 2$, $(n,q) \neq (3,4)$, $(4,3)$, and $q \geq 11$ if $n = 2$. Then by [GMPS, Theorem 2.16] and (6.6.1) we have

$$\dim(V) \leq \bar{\mathsf{o}}(g) \leq \mathrm{meo}(\mathrm{Aut}(S)) = (q^n - 1)/(q-1).$$

In particular, if $n = 2$ then we arrive at conclusion (i). If $n \geq 3$, then it follows from [TZ1, Theorem 3.1] that $E(G)$ is a quotient of $\mathrm{SL}_n(q)$ and that $V|_{E(G)}$ has an irreducible constituent U, which is a Weil module of dimension $(q^n - q)/(q-1)$ or $(q^n-1)/(q-1)$. In particular, $\dim(U) > \dim(V)/2$, and so $U = V|_{E(G)}$, and we arrive at conclusion (ii). The remaining cases are handled using [GAP].

(B3) Suppose $S = \mathrm{PSp}_{2n}(q)$ with $n \geq 2$, $2 \nmid q$, and $(n,q) \neq (2,3)$. Then by [GMPS, Theorem 2.16] and (6.6.1) we have

$$\dim(V) \leq \mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n+1}/(q-1).$$

It follows from [TZ1, Theorem 5.2] that $E(G)$ is a quotient of $\mathrm{Sp}_{2n}(q)$ and that $V|_{E(G)}$ has an irreducible constituent U, which is a Weil module of dimension $d = (q^n \pm 1)/2$. Now, if $q \geq 5$, then $q^{n+1}/(q-1) < 3(q^n-1)/2$, hence $\dim(V) = d$ or $2d$. Consider the case $q = 3$, for which $q^{n+1}/(q-1) < 4d$. Here, either $G = \mathbf{Z}(G)E(G)$, and so $\dim(V) = d$, or $[G : \mathbf{Z}(G)E(G)] = 2$, G induces a diagonal automorphism of $E(G)$ and fuses two irreducible Weil modules of $E(G)$ of dimension d, whence $\dim(V) = 2d$. Thus we arrive at conclusion (iv). The remaining case of $S = \mathrm{PSp}_4(3)$ is handled using [GAP].

(B4) Finally, we consider the case $S = \mathrm{PSU}_n(q)$ with $n \geq 3$. If $n = 3$ and $q \neq 3, 5$, then by [GMPS, Theorem 2.16] and (6.6.1) we have

$$\dim(V) \leq \mathrm{meo}(\mathrm{Aut}(S)) \leq q(q+1) < (q^2 - q + 1)(q-1)/\gcd(3, q+1). \tag{6.6.4}$$

If $n = 4$ and $q \geq 4$ and we have

$$\dim(V) \leq \mathrm{meo}(\mathrm{Aut}(S)) \leq q^3 + 1 < (q^2 - q + 1)(q^2+1)/2. \tag{6.6.5}$$

If $2|n \geq 6$ and $(n,q) \neq (6,2)$, then we have

$$\dim(V) \leq \mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n-1} + q^2 < (q^n - 1)(q^{n-1} - q)/(q+1)(q^2-1). \tag{6.6.6}$$

If $2 \nmid n \geq 5$ and $(n,q) \neq (5,2)$ and we have

$$\dim(V) \leq \mathrm{meo}(\mathrm{Aut}(S)) \leq q^{n-1} + q < (q^n + 1)(q^{n-1} - q^2)/(q+1)(q^2-1). \tag{6.6.7}$$

In all these cases, the upper bound on $\dim(V)$ obtained in (6.6.4)–(6.6.7) implies by [TZ1, Theorem 4.1] that $E(G)$ is a quotient of $\mathrm{SU}_n(q)$ and that $V|_{E(G)}$ has an irreducible constituent U, which is a Weil module of dimension $(q^n + q(-1)^n)/(q+1)$ or $(q^n - (-1)^n)/(q+1)$. In particular, $\dim(U) > \dim(V)/2$, and so $U = V|_{E(G)}$, and we arrive at conclusion (iii). The remaining cases $(n,q) = (3,3)$, $(3,5)$, $(4,3)$, $(5,2)$, and $(6,2)$ can be checked directly using [GAP]. □

7. The characteristic of hypergeometric sheaves

In this section, we assume $\mathcal{H} = \mathcal{H}yp_\psi(\chi_1, \ldots, \chi_D; \rho_1, \ldots, \rho_m)$ is geometrically irreducible (i.e, no χ_i is any ρ_j) ℓ-adic (Kloosterman or) hypergeometric sheaf of type (D,m), $D > m$, on $\mathbb{G}_m$ over a finite extension of $\mathbb{F}_p$ that admits a *finite* geometric monodromy group G_{geom}. In particular, the image of $I(0)$ on $\mathcal{H}$ is a finite cyclic group whose generator has D distinct eigenvalues $\zeta^{a_1}, \ldots, \zeta^{a_D}$, where $\zeta \in \overline{\mathbb{F}_p}^\times$ has order N, and $\chi_i = \chi^{a_i}$ for a fixed multiplicative character χ of order N and $1 \leq i \leq D$. We will show that, in most cases the characteristic p of the sheaf can be read off from the structure of G_{geom}.

7A. The Lie-type case. In this subsection, we will assume that $G = G_{\rm geom}$ is an *almost quasisimple group of Lie type*, that is, $S \le G/\mathbf{Z}(G) \le \mathrm{Aut}(S)$ for some finite simple group of Lie type, in some characteristic r which may a priori differ from p.

The principal result of this section is Theorem 7.4 stating that in the generic situation we in fact have $r = p$, that is, the characteristic of the sheaf and of the group S are equal.

In view of Theorem 6.6, we will first prove some auxiliary results concerning Weil representations of finite classical groups.

Lemma 7.1. *Let G be a finite classical group and φ be a complex irreducible character of G, such that at least one of the following conditions holds:*

(a) $G = \mathrm{SL}_2(q)$ *with* $q \ge 7$;
(b) $G = \mathrm{GL}_n(q)$ *with* $n \ge 3$, *and φ is one of the irreducible Weil characters* $\tau^i_{n,q}$, $0 \le i \le q-2$;
(c) $G = \mathrm{GU}_n(q)$ *with* $n \ge 3$ *and* $(n,q) \ne (3,2)$, *and φ is one of the irreducible Weil characters* $\zeta^i_{n,q}$, $0 \le i \le q$; *or*
(d) $G = \mathrm{Sp}_{2n}(q)$ *with* $n \ge 2$, $2 \nmid q$, *and φ is one of the four irreducible Weil characters* ξ_i, η_i, $i = 1, 2$.

Let $g \in G \smallsetminus \mathbf{Z}(G)$. Then $|\varphi(g)|/\varphi(1) < 2/3$ in the case of (d) *and $|\varphi(g)|/\varphi(1) \le 3/5$ in the other cases. Moreover, if $G = \mathrm{SL}_2(q)$ with $q \ge 25$ then*

$$|\varphi(g)|/\varphi(1) \le 1/(\sqrt{q}-1) \le 1/4.$$

Furthermore, if $G = \mathrm{Sp}_{2n}(q)$ and g is a p'-element, then $|\varphi(g)|/\varphi(1) \le (q^{n-1}+q)/(q^n-1)$.

Proof. In the case of (a), one can check using the well-known character tables of G, see e.g. [Do, §38], that $|\varphi(g)|/\varphi(1) \le 1/(\sqrt{q}-1) < 3/5$ when $q \ge 8$, and $|\varphi(g)|/\varphi(1) \le \sqrt{2}/3 < 3/5$ when $q = 7$. If $q \ge 25$, then $|\varphi(g)|/\varphi(1) \le 1/(\sqrt{q}-1) \le 1/4$.

In the remaining cases, we will consider G as a classical group with natural module V and let $e(g)$ denote the largest dimension of g-eigenspaces on $V \otimes \overline{\mathbb{F}_q}$. As $g \notin \mathbf{Z}(G)$, we have $e(g) \le \dim V - 1$.

Consider the case of (b) and view $G = \mathrm{GL}(V)$ with $V = \mathbb{F}_q^n$. If $q = 2$, then $\varphi(g) + 2 = \tau^0_{n,2}(g) + 2$ is the number of g-fixed vectors in V, whence $-2 \le \varphi(g) \le 2^{n-1} - 2$ and so $|\varphi(g)|/\varphi(1) < 1/2$. Assume $q \ge 3$, and let $\delta \in \overline{\mathbb{F}_q}^\times$ and $\tilde\delta \in \mathbb{C}^\times$ be of order $q-1$. By the character formula [T2, (1.1)],

$$\tau^i_{n,q}(g) = \frac{1}{q-1}\sum_{k=0}^{q-2} \tilde\delta^{ik} q^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g - \delta^k \cdot 1_V)} - \delta_{i,0}.$$

It is easy to see that $\bigl|\sum_{k=0}^{q-2} \tilde\delta^{ik} q^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g-\delta^k\cdot 1_V)}\bigr|$ is at most $q^{n-1} + 2q - 3$ if $e(g) = n-1$, and at most $q^{n-2}(q-1)$ otherwise. It follows that

$$\frac{|\varphi(g)|}{\varphi(1)} \le \frac{(q^{n-1}+2q-3)/(q-1)+1}{(q^n-q)/(q-1)} < 3/5.$$

In the case of (c), view $G = \mathrm{GU}(V)$ with $V = \mathbb{F}_{q^2}^n$. Let $\xi \in \overline{\mathbb{F}_q}^\times$ and $\tilde\xi \in \mathbb{C}^\times$ be of order $q+1$. By the character formula [TZ2, Lemma 4.1],

$$\zeta^i_{n,q}(g) = \frac{(-1)^n}{q+1}\sum_{k=0}^{q} \tilde\xi^{ik} (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g - \xi^k \cdot 1_V)}.$$

Again, it is easy to see that $\bigl|\sum_{k=0}^{q}\tilde{\xi}^{ik}q^{\dim_{\mathbb{F}_{q^2}}\mathrm{Ker}(g-\xi^k\cdot 1_V)}\bigr|$ is at most $q^{n-1}+2q-1$ if $e(g)=n-1$, $q^{n-2}+q^2+q-1$ if $e(g)=n-2$, and at most $q^{n-3}(q+1)$ otherwise. It follows that

$$\frac{|\varphi(g)|}{\varphi(1)} \le \frac{(q^{n-1}+2q-1)/(q+1)}{(q^n-q)/(q+1)} \le 3/5$$

unless $(n,q)=(4,2)$, $(5,2)$. In the cases $(n,q)=(4,2)$, $(5,2)$, the desired bound can be checked directly using [Atlas]. If $(n,q)=(3,16)$, then $|\varphi(g)|/\varphi(1)<1/14$.

Finally, we consider the case of (d), where $G=\mathrm{Sp}(V)$ and $V=\mathbb{F}_q^{2n}$. By the character formula for the Weil characters, see e.g. Theorem 2.1 and Lemma 3.1 of [GMT],

$$|\varphi(g)| \le (q^{\dim_{\mathbb{F}_q}\mathrm{Ker}(g-1_V)}+q^{\dim_{\mathbb{F}_q}\mathrm{Ker}(g+1_V)})/2 \le (q^{n-1/2}+q^{1/2})/2$$

as $e(g)\le 2n-1$. It follows that

$$\frac{|\varphi(g)|}{\varphi(1)} \le \frac{q^{n-1/2}+q^{1/2}}{q^n-1} \le 2/3$$

unless $(n,q)=(2,3)$. The remaining case $(n,q)=(2,3)$ can be checked directly using [Atlas]. If in addition g is a p'-element, then $\dim_{\mathbb{F}_q}\mathrm{Ker}(g\pm 1_V)\le 2n-2$, and so $|\varphi(g)|\le (q^{n-1}+q)/2$. □

Proposition 7.2. *Let G be a finite almost quasisimple group: $S \lhd G/\mathbf{Z}(G) \le \mathrm{Aut}(S)$ for some simple non-abelian group S. Suppose that at least one of the following conditions holds for (L,φ), where $L := G^{(\infty)}$ and $\varphi \in \mathrm{Irr}(G)$ is any faithful irreducible character:*

(a) *L is a quotient of $\mathrm{SL}_2(q)$ for some prime power $q\ge 7$ and $\varphi(1)\le q+1$;*
(b) *L is a quotient of $\mathrm{SL}_n(q)$ for some prime power q and some $n\ge 3$, and φ viewed as a character of $\mathrm{SL}_n(q)$ is one of the irreducible Weil characters $\tau_{n,q}^i$, $0\le i\le q-2$;*
(c) *L is a quotient of $\mathrm{SU}_n(q)$ for some prime power q and some $n\ge 3$, and φ viewed as a character of $\mathrm{SU}_n(q)$ is one of the irreducible Weil characters $\zeta_{n,q}^i$, $0\le i\le q$; or*
(d) *L is a quotient of $\mathrm{Sp}_{2n}(q)$ for some odd prime power q and some $n\ge 2$, and every irreducible constituent of $\varphi|_L$ viewed as a character of $\mathrm{Sp}_{2n}(q)$ is one of the four irreducible Weil characters ξ_i,η_i, $i=1,2$.*

Let $1\ne Q\le G$ be any subgroup and let $w(Q) := \varphi(1)-[\varphi|_Q,1_Q]_Q$ be the codimension of the fixed point subspace of Q in a $\mathbb{C}G$-representation Φ affording the character φ. Then

$$\frac{w(Q)}{\varphi(1)} \ge \begin{cases} (1/3)\cdot\bigl(1-1/|Q|\bigr)\ge 1/6, & \textit{in the case of (d)},\\ (1/10)\cdot\bigl(1-1/|Q|\bigr)\ge 1/20, & \textit{in the cases of (a)–(c)},\\ (3/16)\cdot\bigl(1-1/|Q|\bigr), & \textit{in the case of (a), with } q\ge 25,\\ 1/4-2/(5|Q|), & \textit{in the cases of (b), (c), with } q \textit{ prime},\\ 0.377-0.345/|Q|, & \textit{in the cases of (c), with } (n,q)=(6,3).\end{cases}$$

Proof. (i) The faithfulness of φ implies that any non-identity central element $z\in\mathbf{Z}(G)$ acts without nonzero fixed points in Φ. In particular, $w(Q)=\varphi(1)$ if $Q\cap\mathbf{Z}(G)\ne 1$. So in what follows we may assume that $Q\cap\mathbf{Z}(G)=1$. Suppose we can find an explicit constant $0<\alpha<1$ such that

$$|\varphi(g)|/\varphi(1)\le\alpha$$

for all $g\in Q\smallsetminus\mathbf{Z}(G)$. Then

$$\frac{[\varphi|_Q,1_Q]_Q}{\varphi(1)} = \left|\frac{1}{|Q|\cdot\varphi(1)}\sum_{g\in Q}\varphi(g)\right| \le \frac{1+\alpha(|Q|-1)}{|Q|} = \alpha+\frac{1-\alpha}{|Q|}, \tag{7.2.1}$$

and so

$$\frac{w(Q)}{\varphi(1)} \geq (1-\alpha)\big(1-\frac{1}{|Q|}\big). \tag{7.2.2}$$

(ii) Assume we are in the case of (d). First we consider the case where all irreducible constituents of $\varphi|_L$ are equal to a single irreducible Weil character, say θ (when considered as a character of $\mathrm{Sp}_{2n}(q)$). It is well known that, each such θ is stable under field automorphisms of $\mathrm{Sp}_{2n}(q)$ – in fact, it extend to a certain extension $\mathrm{Sp}_{2n}(q) \rtimes C_f \leq \mathrm{Sp}_{2nf}(p)$ that induces the full subgroup of outer field automorphisms of $\mathrm{Sp}_{2n}(q)$, where $q = p^f$ and p is prime – but θ is not stable under outer diagonal automorphisms. See e.g. [KT6, §6]. As $\mathbf{Z}(G)$ acts via scalars in Φ, we can extend θ to a character of $\mathbf{Z}(G)L$, which is still G-invariant. But $G/\mathbf{Z}(G)L$ embeds in the subgroup C_f of field automorphisms of L and so it is cyclic. Hence, by [Is, (6.17), (11.22)], θ extends to G and in fact $\varphi|_L = \theta$. Thus we may assume that Φ extends to $\Phi : \mathrm{Sp}_{2nf}(p) \to \mathrm{GL}(V)$ and that

$$\Phi(G) \leq \mathbf{N}_{\mathrm{GL}(V)}(\Phi(\mathrm{Sp}_{2n}(q))) \leq \Phi(\mathrm{Sp}_{2nf}(p))\mathbf{Z}(\mathrm{GL}(V)).$$

It follows that $\Phi(g)$ is a scalar multiple of $\Phi(h)$ for some non-central element $h \in \mathrm{Sp}_{2nf}(p)$. Applying case (d) of Lemma 7.1 to $\varphi(h)$, we obtain $|\varphi(g)| = |\varphi(h)| \leq (2/3)\varphi(1)$. Thus we can take $\alpha = 2/3$ in this case.

Assume now that the set of irreducible constituents of $\varphi|L$ is $\{\xi_1, \xi_2\}$ or $\{\eta_1, \eta_2\}$. By Clifford's theorem G permutes these two constituents transtively; let H denote the stabilizer of one of them, say θ_1. Then $|G/H| = 2$ and H fixes both θ_1 and the other constituent θ_2. Moreover, $\Phi|_H = \Phi_1 \oplus \Phi_2$, where all irreducible constituents of the character φ_i on restriction to L are equal to θ_i for $i = 1, 2$, and $\mathbf{Z}(G)$ acts the same in Φ_1 and Φ_2. The preceding analysis applied to φ_i shows that $|\varphi_i(g)|/\varphi_i(1) \leq 2/3$ and so $|\varphi(g)|/\varphi(1) \leq 2/3$ for all $g \in (Q \cap H) \smallsetminus \mathbf{Z}(G)$. On the other hand, if $g \in Q \smallsetminus H$, then g interchanges Φ_1 and Φ_2 and so $\varphi(g) = 0$. Thus we have $|\varphi(g)| \leq (2/3)\varphi(1)$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$, and can take $\alpha = 2/3$ as above.

(iii) In the remaining cases of (a)–(c), note that for any $g \in G \smallsetminus \mathbf{Z}(G)$, we can find $h \in L$ such that $[g, h] \leq L \smallsetminus \mathbf{Z}(G)$. [Indeed, suppose $[g, x] \in \mathbf{Z}(G)$ for all $x \in L$. Then for all $y \in L$ we have $[[x, y], g] = \big([y, g], x][[g, x], y]\big)^{-1} = 1$, and so g centralizes $[L, L] = L$. But this implies $g \in \mathbf{C}_G(L) = \mathbf{Z}(G)$.] By Lemma 7.1 (applied to each irreducible constituent of $\varphi|_L$), $|\varphi(h)| \leq (3/5)\varphi(1)$. Hence, by [GT3, Corollary 2.14] we have

$$|\varphi(g)| \leq (3/4)\varphi(1) + (1/4)|\varphi(h)| \leq (9/10)\varphi(1).$$

Thus we can take $\alpha = 9/10$ in these remaining cases. If L is a quotient of $\mathrm{SL}_2(q)$ with $q \geq 25$ in (a), then $|\varphi(h)| \leq \varphi(1)/4$ by Lemma 7.1, and so we can take $\alpha = 3/4 + 1/16 = 13/16$.

(iv) Finally, assume we are in the case of (b) or (c), and q is prime. Then by [GLS, Theorem 2.5.12], $\mathrm{Aut}(S) = \mathrm{PGL}^\epsilon_n(q) \rtimes \langle\tau\rangle$ if $S = \mathrm{PSL}^\epsilon_n(q)$, where $\epsilon = +$ in the GL-case and $\epsilon = -$ in the GU-case. In particular, at least half of the elements $g \in Q$ must induce only inner-diagonal automorphisms of S. Also, irreducible Weil representations of $L = \mathrm{SL}^\epsilon_n(q)$ extend to $\Phi : \mathrm{GL}^\epsilon_n(q) \to \mathrm{GL}(V)$. Hence, for any such element $g \in Q$, we may assume that $\Phi(g) = \gamma\Phi(g')$ for some $g' \in \mathrm{GL}^\epsilon_n(q)$ and a root of unity $\gamma \in \mathbb{C}^\times$, whence $|\varphi(g)| = |\varphi(g')| \leq (3/5)\varphi(1)$ by Lemma 7.1. It follows that

$$\frac{[\varphi|_Q, 1_Q]_Q}{\varphi(1)} = \Big|\frac{1}{|Q|\cdot\varphi(1)}\sum_{g\in Q}\varphi(g)\Big| \leq \frac{1 + (9/10)(|Q|/2) + (3/5)(|Q|/2-1)}{|Q|} = \frac{3}{4} + \frac{2/5}{|Q|},$$

whence $w(Q)/\varphi(1) \geq 1/4 - 2/(5|Q|)$.

Assume furthermore that we are in (c) and L is a quotient of $\mathrm{SU}_6(3)$. The proof of Lemma 7.1 for $\mathrm{SU}_6(3)$ and the character table of $\mathrm{SU}_5(3)$ in [GAP] show that $|\varphi(g')|/\varphi(1) < 0.345$ for the aforementioned element g'. Hence, replacing $3/5$ by 0.345 in the above estimate, we obtain $w(Q)/\varphi(1) \geq 0.377 - 0.345/|Q|$. □

Proposition 7.3. *Let $\mathcal{H}$ be an irreducible hypergeometric sheaf on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ of type (D, m) with $W := D - m > 0$ the dimension of the wild part* Wild *of the $I(\infty)$-representation. If $p \nmid W$, then we have the following results.*

(i) Wild *is the Kummer direct image $[W]_\star(\mathcal{L})$ of some linear character $\mathcal{L}$ of Swan conductor* 1.

(ii) Wild *as a $P(\infty)$ representation is the direct sum of the W multiplicative translates of $\mathcal{L}|_{P(\infty)}$ by μ_W.*

(iii) *Any element of $I(\infty)$ of pro-order prime to p which maps onto a generator of $I(\infty)/P(\infty)$ acts on the set of the W irreducible consituents of* $\mathsf{Wild}|_{P(\infty)}$ *through the quotient μ_W of $I(\infty)$, cyclically permuting these multiplicative translates of $\mathcal{L}|_{P(\infty)}$.*

(iv) *The image of $P(\infty)$ is isomorphic to the additive group of the finite field $\mathbb{F}_p(\mu_W)$.*

Proof. Statement (i) is proven in [Ka-GKM, 1.14 (2)]. Statements (ii) and (iii) result from (i), cf. [KRLT3, proof of 3.1]. For (iv), there is nothing to prove if $W = 1$. If $W \geq 2$, by [Ka-ESDE, 8.6.3], the $I(\infty)$-isomorphism class of Wild up to multiplicative translation depends only on $\det(\mathsf{Wild})$, a tame character which we can change as we like by tensoring Wild with a tame character. Such tensoring with a tame character does not alter the action of $P(\infty)$, and allows us to reduce to the case where $\det(\mathsf{Wild}) = \chi_2^{W-1}$, and then apply [KRLT3, 3.1]. □

Theorem 7.4. *Let $\mathcal{H}$ be an irreducible hypergeometric sheaf in characteristic p of rank D with finite geometric monodromy group $G = G_{\rm geom}$. Suppose that G is an almost quasisimple group of Lie type:*

$$S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$$

for some finite simple group S of Lie type in characteristic r. Then at least one of the following statements holds.

(i) *$p = r$, i.e. $\mathcal{H}$ and S have the same characteristic.*

(ii) *$D \leq 22$ and S is one of the following simple groups:* $\mathrm{PSL}_2(5,7,8,9,11,25)$, $\mathrm{SL}_{3,4}(2)$, $\mathrm{PSL}_3(3,4)$, $\mathrm{PSU}_{4,5,6}(2)$, $\mathrm{PSU}_{3,4}(3)$, $\mathrm{PSU}_3(4,5)$, $\mathrm{Sp}_6(2)$, $\mathrm{PSp}_{4,6}(3)$, $\mathrm{PSp}_4(5)$, $\Omega_8^+(2)$, ${}^2B_2(8)$, $G_2(3,4)$.

Proof. (a) As explained above, a generator g of the image $I(0)$ on $\mathcal{H}$ has simple spectrum on $\mathcal{H}$. Hence we can apply Theorem 6.6 to the faithful irreducible representation $\Phi : G \to \mathrm{GL}(\mathbb{C}^D)$ induced by the action of G on $\mathcal{H}$. Since the non-generic cases of Theorem 6.6(v) are already included in (ii), we may assume that the character φ of Φ and the subgroup $L = G^{(\infty)}$ fulfills the assumptions of Proposition 7.2. We can therefore apply Proposition 7.2 to the subgroup $Q = \mathbf{O}_p(J)$, the image of $P(\infty)$ on $\mathcal{H}$, where $J = QC$ is the image of $I(\infty)$ on $\mathcal{H}$, with C the cyclic tame quotient, and $W = w(Q)$ is the dimension of the wild part for $I(\infty)$ on $\mathcal{H}$.

First we note that if $|Q| = 2$, then Q has a unique nontrivial irreducible character (of degree 1), and so $W \leq 1$ and $D \leq 20$ by Proposition 7.2. Thus, by assuming $D \geq 21$, we may assume that $|Q| \geq 3$. In the rest of the proof we will assume that $|Q| \geq 3$ and that $r \neq p$, and work to bound $D = \mathrm{rank}(\mathcal{H})$.

(b) Here we consider the symplectic case: $S = \mathrm{PSp}_{2n}(q)$, where $n \geq 2$, $q = r^f$, and $r \neq 2$. By Proposition 7.2, $W \geq (2/9)D$, and $D \geq (q^n - 1)/2$. Since any irreducible Weil character is invariant under field automorphisms of $\mathrm{Sp}_{2n}(q)$, we have by Gallagher's theorem [Is, (6.17)] that D or $D/2$ is $(q^n \pm 1)/2$.

We can view $\mathrm{Sp}_{2n}(q)$ as $\mathrm{Sp}(V)$ with $V = \mathbb{F}_q^{2n}$, where V is endowed with a non-degenerate symplectic form $(\cdot,\cdot)$. We also consider the *conformal symplectic group*

$$\mathrm{CSp}(V) = \bigl\{X \in \mathrm{GL}(V) \mid \exists \kappa(X) \in \mathbb{F}_q^\times,\ (Xu, Xv) = \kappa(X)(u, v),\ \forall u, v \in V\bigr\}$$

which contains $\mathrm{Sp}(V)$ as a normal subgroup with cyclic quotient of order $q-1$. Next we consider the representation $\Lambda : \mathrm{GL}(V) \to \mathrm{GL}(\wedge^2(V))$, and its twisted restriction

$$\Lambda' : \mathrm{CSp}(V) \to \mathrm{GL}(\wedge^2(V)),\ \Lambda'(X) = \kappa(X)^{-1}\Lambda(X)$$

to $\mathrm{CSp}(V)$. It is straightforward to check that $\mathrm{Ker}(\Lambda') = \mathbf{Z}(\mathrm{CSp}(V))$, and Λ' induces a faithful action of S on the quotient of $\wedge^2(V)$ by the trivial submodule that extends to $\mathrm{PCSp}(V) := \mathrm{CSp}(V)/\mathbf{Z}(\mathrm{CSp}(V))$, which is the group of inner-diagonal automorphisms of S, cf. [GLS, Theorem 2.5.12].

Now we can embed $G/\mathbf{Z}(G)$ in $\mathrm{Aut}(S) \cong \mathrm{PCSp}(V) \rtimes C_f$ and set $R := G/\mathbf{Z}(G) \cap \mathrm{PCSp}(V)$. Applying Theorem 4.8 with $d := (2n+1)(n-1)$, we get

$$(q^n-1)/9 \le 2D/9 \le W \le d \cdot [G/\mathbf{Z}(G) : R] \le (2n+1)(n-1)f. \tag{7.4.1}$$

Now, if $n \ge 6$, then $q^{n-1} \ge 3^{n-1} > 6n^2$, whence

$$q^n > 6n^2 q \ge 18n^2 f,$$

contradicting (7.4.1). If $2 \le n \le 5$, then (7.4.1) leads us to one of the following possibilities.

(b1) $q = 3$, $2 \le n \le 5$, D or $D/2$ is $(q^n \pm 1)/2$. Assume $(n,q) = (5,3)$ or $(4,3)$. In this case, as $p \ne r = 3$, any $g \in Q \smallsetminus \mathbf{Z}(G)$ is a $3'$-element, and so $|\varphi(g)|/\varphi(1) \le 3/8$ by Lemma 7.1 if $g \in \mathbf{Z}(G)L$. If $g \in Q \smallsetminus \mathbf{Z}(G)L$, then g fuses the two irreducible Weil characters of any given degree $(q^n \pm 1)/2$, whence we must have that $D = q^n \pm 1$ and $\varphi(g) = 0$. Thus we have $|\varphi(g)|/\varphi(1) \le 3/8$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. Also, since $|Q| > 2$ and Q is not a 3-group, we have that $|Q| \ge 4$. The proof of Proposition 7.2 now implies that

$$W \ge D\big(1 - \frac{3}{8}\big)\big(1 - \frac{1}{|Q|}\big) \ge 15D/32,$$

whence (7.4.1) yields that $n = 4$, $D \in \{40, 41\}$, $G = \mathbf{Z}(G)L$, and

$$19 \le W \le 27,\ Q \cap \mathbf{Z}(G) = 1, \tag{7.4.2}$$

where the second conclusion follows from $W < D$ and Proposition 4.4(i). Assume in addition that $D = 40$. Then $L = \mathrm{Sp}_8(3)$, and it is easy to see that $G/\mathbf{O}_3(\mathbf{Z}(G))$ admits a faithful 8-dimensional representation Λ over $\overline{\mathbb{F}_3}$. Applying Theorem 4.5 to $\Lambda|_J$ and using (7.4.2), we obtain that $Q \le \mathbf{Z}(G)$, a contradiction. Now we consider the other possibility $D = 41$, for which $G = \mathbf{Z}(G) \times S$. By (7.4.2), $p \in \{2, 5, 7, 13, 41\}$ and Q embeds in a Sylow p-subgroup of $S = \mathrm{PSp}_8(3)$. Also, recall that the Q-module Wild is a sum of a pairwise non-isomorphic simple Q-modules of dimension p^c, where $W = ap^c$ and $p \nmid a$. If $p = 7$ or 13, then $Q \cap C_p$, whence $a \le p-1 \le 12$ and $p^c = 1$, yielding the contradiction $W \le 12$. If $p = 5$ or 13, then using the character table of S given in [GAP] one check that $|\varphi(g)| \le 4$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$, whence $W \ge (37/41)(1 - 1/|Q|)D > 29$, contradicting (7.4.2). Thus we must have $p = 2$. Again using the character table of S we get $|\varphi(g)| \le 15$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. Also, if $|Q| \le 16$, then Q has at most 16 irreducible characters of degree 1, 3 of degree 2, and none of degree > 2, whence $W \le 15$, contradicting (7.4.2). Thus $|Q| \ge 32$, and so $W \ge (26/41)(1 - 1/|Q|)D > 25$, i.e. $W = 27$ or 26. In the former case, we know that Q can be identified with the additive group of $\mathbb{F}_2(\tilde{\zeta}_{27}) \cong \mathbb{F}_{2^{18}}$, which is impossible since Q embeds in $\mathrm{PSp}_8(3)$. Consider the latter case $W = 26$, in which a generator g of the tame quotient of $I(\infty)$ permutes cyclically $a = 13$ simple Q-modules in Wild, and has a simple spectrum on the tame part of dimension 15, since $I(\infty)$ has finite image J. Thus the $2'$-element g has order divisible by 13 but larger than 15. Hence, we can write $g = zh$, where $z \in \mathbf{Z}(G)$ acts as a scalar on $\mathcal{H}$, and $h \in S$ is an element of order 39. Using the character table of S and letting $\zeta = \zeta_{39} \in \mathbb{C}^\times$, we have that the spectrum of h on $\mathcal{H}$ is the disjoint union $X \sqcup Y \sqcup Y \sqcup \{1, \zeta^{26}\}$ (with counting multiplicities), where $X = \mu_{13} = \langle \zeta^3 \rangle$, and $Y = \zeta^2 \mu_{13}$. On the other hand, because of the cyclic action of g (and h) on the

13 Q-summands in Wild, the spectrum of g (and h) on Wild must be the union of some μ_{13}-cosets, whereas the spectrum on the tame part is simple. Now, if the spectrum A of h on Wild is $Y \sqcup Y$, then the spectrum B of h on the tame part contains 1 twice. If A is $X \sqcup Y$, then B contains ζ^{26} twice, again a contradiction.

Next, we consider the case $n = 3$ and $D = q^n \pm 1$. Using the character table of $\mathrm{Sp}_6(3) \cdot 2$ [GAP], one can check that $|\varphi(g)| \leq D - 16$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. As $|Q| \geq 4$, it follows that $W \geq 16(1 - 1/4) = 12$. Since Q acts on the wild part of $\mathcal{H}$ with pairwise non-isomorphic simple summands, it follows that $|Q| \geq 9$, forcing $W \geq 16(8/9) > 14$, contradicting (7.4.1). We have shown that $n \leq 3$ and $D \leq 14$.

(b2) $(n, q) = (3, 5)$, $(2, 9)$, $(2, 5)$. In the case $(n, q) = (3, 5)$, the same arguments as in (b1) also apply, yielding $|\varphi(g)|/\varphi(1) \leq 15/62$, whence $W \geq (47/62)(1 - 1/3)D > 14$, contradicting (7.4.1). In the remaining cases (7.4.3) forces $D = (q^n \pm 1)/2$, and so G can only induce inner and field automorphisms of S. In the case $(n, q) = (2, 9)$, this implies that, modulo scalars, $\Phi(Q)$ is contained in the image of $\mathrm{Sp}_8(3)$ in a Weil representation, and so the arguments in (b1) again apply, yielding $W \geq 15D/32 > 18$, contradicting (7.4.1). Thus $(n, q) = (2, 5)$, and $D \leq 13$.

(c) Next we consider the linear case: $S = \mathrm{PSL}_n(q)$, where $n \geq 3$ and $q = r^f$. By Proposition 7.2, $W \geq D/15$, and $D \geq (q^n - q)/(q - 1)$. Recall [GLS, Theorem 2.5.12] that

$$\mathrm{Aut}(S) \cong \mathrm{PGL}_n(q) \rtimes C,$$

where C is an abelian group of order $2f$. Embedding $\bar{G} := G/\mathbf{Z}(G)$ in $\mathrm{Aut}(S)$, we let

$$R_1 := \bar{G} \cap \mathrm{PGL}_n(q) \lhd \bar{G},$$

so that $[\bar{G} : R_1]$ divides $|C| = 2f$. As noted in the proof of Theorem 4.8, $\mathrm{PGL}_n(q) \leq \mathrm{PGL}(V)$ acts faithfully and irreducibly on a subquotient $A(V)$ of $V \otimes V^*$ (of dimension $n^2 - 1$ if $r \nmid n$ and $n^2 - 2$ if $r|n$), where $V = \overline{\mathbb{F}_q}^n$, and moreover this action is extendible to $\mathrm{PGL}(V) \rtimes \langle\tau\rangle$, where τ is the transpose-inverse automorphism of $\mathrm{PGL}(V)$. Viewing R_1 inside $\mathrm{PGL}(V)$, we get a faithful irreducible action of R_1 which also extends to $R_1 \rtimes \langle\tau\rangle$.

If $[\bar{G} : R_1] \leq f$, we set $R := R_1$. If $\bar{G} : R_1] > f$, then $\bar{G}/R_1 \cong C = C_f \rtimes \langle\tau\rangle$. In this latter case, there is some element $\bar{x} \in \bar{G} \smallsetminus R_1$ such that $\bar{x}^2 \in R_1$ and $\bar{x}$ induces the automorphism τ on R_1. Then we set $R := \langle R_1, \bar{x}\rangle$ and obtain a faithful (at least on R_1) irreducible action on $A(V)$. Now we can apply Theorem 4.8 with $d := \dim(A(V)) \leq n^2 - 1$ to get

$$\frac{q^n - q}{15(q - 1)} \leq D/15 \leq W \leq (n^2 - 1) \cdot [\bar{G} : R] \leq (n^2 - 1)f. \tag{7.4.3}$$

Note that if $\gcd(n, q - 1) = 1$, then $\mathrm{PGL}_n(q) \cong \mathrm{SL}_n(q)$ (but the action of S on V does *not* extend to $S \rtimes \langle\tau\rangle$), and so we can apply Theorem 4.8 with $d := n$ to get

$$\frac{q^n - q}{15(q - 1)} \leq D/15 \leq W \leq d \cdot [\bar{G} : R_1] \leq 2nf. \tag{7.4.4}$$

Furthermore, if $q = r$ is prime, then by Proposition 7.2 we have $W > D/8.6$, hence the constants 15 in (7.4.3) and (7.4.4) can be replaced throughout by 8.6. Another observation is that, when $r = 2$ *and* f is a 2-power, since $p \neq r = 2$, $Q\mathbf{Z}(G)/\mathbf{Z}(G) \leq R_1$ for the p-group Q. Hence $|\varphi(g)|/\varphi(1) \leq 0.6$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$ by Lemma 7.1. Now the proof of Proposition 7.2 show that $W/D > 0.4(1 - 1/|Q|) \geq 4/15$, hence the constant 15 in (7.4.3) and (7.4.4) can be replaced by 15/4.

Now, if $n \geq 11$, or if $n \geq 7$ and $q \geq 3$, then $(q^{n-1} - 1)/(q - 1) > 7.5n^2$, whence

$$q^n - q > 7.5n^2q(q - 1) \geq 15n^2(q - 1)f,$$

contradicting (7.4.3). If $2 \leq n \leq 10$, then (7.4.3) and (7.4.4) imply that one of the following holds.

(c1) $q = 2$, $n \le 5$, and $D = 2^n - 2$. If $n = 5$, then since Q is not a 2-group, the character table of $\mathrm{SL}_5(2)$ [GAP] shows that $|\varphi(g)|/\varphi(1) \le 1/3$, whence $W \ge (2/3)(1 - 1/|Q|)D > 13$, contradicting (7.4.4). Thus $n \le 4$ and $D = 2^n - 2 \in \{6, 14\}$.

(c2) $q = 3$, $n = 4$, and $D = (3^n - 3)/2, (3^n - 1)/2$. Using the character tables of $L \cdot 2_1$, $L \cdot 2_2$, and $L \cdot 2_3$ given in [Atlas], one can check that $|\varphi(g)|/\varphi(1) \le 1/3$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$, whence $W \ge (2/3)(1 - 1/|Q|)D > 18$, contradicting (7.4.3).

(c3) $n = 3$, $3 \le q \le 9$, $D = q(q+1)$ or $q^2 + q + 1$. Suppose $q = 9$, whence $S = \mathrm{SL}_3(9)$ and $\mathrm{Aut}(S) = S \rtimes C_2^2$, or $q = 8$, whence $S = \mathrm{SL}_3(8)$ and $\mathrm{Aut}(S) = S \rtimes C_3$. The character tables of all groups between S and $\mathrm{Aut}(S)$ are known in [GAP], from which we can check that $|\varphi(g)|/\varphi(1) \le 1/7$ for all $g \in Q \smallsetminus \mathbf{Z}(Q)$. Hence the proof of Proposition 7.2 implies that $W/D \ge 6/7(1 - 1/|Q|) > 1/2$. Thus we can use 2 instead of the constant 15 in (7.4.4), which now leads to a contradiction.

Next suppose that $q = 7$, whence $D = 56$ or 57. Note that when $D = 57$, $L = \mathrm{SL}_3(7)$ has its center inverted by the transpose-inverse automorphism of S, hence, up to scalars, $\Phi(G)$ is contained in the image of $\mathrm{GL}_3(7)$ in a Weil representation of degree 57. If $D = 56$, then either $p = 2$ and, up to scalars, $\Phi(Q)$ is contained the image of $\mathrm{PSL}_3(7) \rtimes C_2$, or $p \ne 2$ and, up to scalars, $\Phi(Q)$ is contained the image of $\mathrm{PGL}_3(7)$, in a representation of degree 56. Hence, using the proof of Lemma 7.1 for $\mathrm{GL}_3(7)$ and the character table of $\mathrm{PSL}_3(7) \cdot C_2$ in [GAP], we can check that $|\varphi(g)|/\varphi(1) \le 11/56$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. It follows that $W \ge (45/56)(1 - 1/3)D \ge 30 > 8$, contradicting (7.4.3).

If $q = 5$, then the character table of $\mathrm{SL}_3(5) \cdot 2$ [GAP] shows that $|\varphi(g)|/\varphi(1) \le 7/31$, whence $W \ge (24/31)(1 - 1/|Q|)D > 15$, contradicting (7.4.4). Hence $q \le 4$ and $D \le 21$.

(d) Now we handle the unitary case: $S = \mathrm{PSU}_n(q)$, where $n \ge 3$ and $q = r^f$. By Proposition 7.2, $W \ge D/15$, and $D \ge (q^n - q)/(q+1)$. Here we have

$$\mathrm{Aut}(S) \cong \mathrm{PGU}_n(q) \rtimes C,$$

where $C \cong C_{2f}$, by [GLS, Theorem 2.5.12]. Embedding $\bar{G} := G/\mathbf{Z}(G)$ in $\mathrm{Aut}(S)$, we let

$$R_1 := \bar{G} \cap \mathrm{PGU}_n(q) \lhd \bar{G},$$

so that $[\bar{G} : R_1]$ divides $2f$. As in (c), $\mathrm{PGU}_n(q) \le \mathrm{PGL}(V)$ acts faithfully and irreducibly on a subquotient $A(V)$ of $V \otimes V^*$ (of dimension $n^2 - 1$ if $r \nmid n$ and $n^2 - 2$ if $r|n$), where $V = \overline{\mathbb{F}_q}^n$, and this action is extendible to $\mathrm{PGL}(V) \rtimes \langle \tau \rangle$, where τ is the transpose-inverse automorphism of $\mathrm{PGL}(V)$. Viewing R_1 inside $\mathrm{PGL}(V)$, we get a faithful irreducible action of R_1 which also extends to $R_1 \rtimes \langle \tau \rangle$. Note that τ can be identified with an involution in the subgroup C_{2f} of $\mathrm{Aut}(S)$.

If $[\bar{G} : R_1] \le f$, we set $R := R_1$. If $\bar{G} : R_1] > f$, then $\bar{G}/R_1 \cong C$. In this latter case, there is some element $\bar{x} \in \bar{G} \smallsetminus R_1$ such that $\bar{x}^2 \in R_1$ and $\bar{x}$ induces the automorphism τ on R_1. Then we set $R := \langle R_1, \bar{x} \rangle$ and obtain a faithful (at least on R_1) irreducible action on $A(V)$. Now we can apply Theorem 4.8 with $d := \dim(A(V)) \le n^2 - 1$ to get

$$\frac{q^n - q}{15(q+1)} \le D/15 \le W \le (n^2 - 1) \cdot [\bar{G} : R] \le (n^2 - 1)f. \tag{7.4.5}$$

Note that if $\gcd(n, q+1) = 1$, then $\mathrm{PGU}_n(q) \cong \mathrm{SU}_n(q)$ (but the action of S on V does *not* extend to $S \rtimes \langle \tau \rangle$), and so we can apply Theorem 4.8 with $d := n$ to get

$$\frac{q^n - q}{15(q+1)} \le D/15 \le W \le d \cdot [\bar{G} : R_1] \le 2nf. \tag{7.4.6}$$

Furthermore, if $q = r$ is prime, then by Proposition 7.2 we have $W > D/8.6$, hence the constants 15 in (7.4.5) and (7.4.6) can be replaced throughout by 8.6. If, on the other hand, $r = 2$ and f is a 2-power, then as in (c) the constant 15 in (7.4.3) and (7.4.4) can be replaced by 15/4.

Now, if $n \geq 13$, or if $n \geq 8$ and $q \geq 3$, then $(q^{n-1}-1)/(q+1) > 7.5n^2$, whence

$$q^n - q > 7.5n^2 q(q+1) \geq 15n^2(q+1)f,$$

contradicting (7.4.5). If $2 \leq n \leq 12$, then (7.4.5) and (7.4.6) imply that one of the following holds.

(d1) $q = 2$ and $n \leq 9$ but $n \neq 8$. Assume we are in the case $(n,q) = (9,2)$, so that $D = 170$ or 171. As mentioned above, since $r = 2$ and $f = 1$, we have the bound $W/D \geq (4/15)D$ and so $W \geq 46$. Now if the p-abelian group Q is non-abelian, then $|Q| \geq p^3 \geq 27$. If Q is abelian, then since the wild part of $\mathcal{H}$ is a sum on non-isomorphic irreducible Q-modules, we get $|Q| \geq w+1 \geq 47$. Thus in either case $|Q| \geq 27$. Since $Q\mathbf{Z}(G)/\mathbf{Z}(G) \leq R_1$, the proof of Lemma 7.1 shows that $|\varphi(g)|/\varphi(1) < 0.508$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. Arguing as in the proof of Proposition 7.2, we obtain

$$W/D > (1-0.508)(1-1/27) > 1/2.12.$$

Thus we can use 2.12 instead of the constant 15 in (7.4.5), which now leads to a contradiction.

Next suppose that $n = 7$. Since Q is not a 2-group and $|\mathrm{Out}(S)| = 2$, $Q\mathbf{Z}(G) \leq \mathbf{Z}(G)L$. Using the character table of $\mathrm{SU}_7(2)$ [GAP] one can check that $|\varphi(g)|/\varphi(1) < 0.51$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$, whence $W > 0.49(1-1/3)D > 13$. This in turn implies that $|Q| \geq 5$, and so $W > 0.49(1-1/5)D > 16$, contradicting (7.4.6). Hence, in fact we have $n \leq 6$, and so $D = (2^n+2(-1)^n)/3, (2^n-(-1)^n)/3 \leq 22$.

(d2) $q = 3$ and $n \leq 6$. As $|Q| > 2$ and Q is not a 3-group, we have $|Q| \geq 4$. If moreover $n = 5$ or 6, then by Proposition 7.2, $W > D/4$, hence we can use (7.4.6), respectively (7.4.5), with 8.6 replaced by 4, yielding a contradiction ruling out this case. Hence $3 \leq n \leq 4$, and $D = (3^n + 3(-1)^n)/4, (3^n-(-1)^n)/4 \in \{6, 7, 20, 21\}$.

(d3) $n = 4$ and $q = 4, 5$. Suppose $q = 5$. Then the proof of Lemma 7.1 shows that $|\varphi(g')|/\varphi(1) < 1/4$ for all $g' \in \mathrm{GU}_4(5) \smallsetminus \mathbf{Z}(\mathrm{GU}_4(5))$. Arguing as in part (iii) of the proof of Proposition 7.2 we get $|\varphi(g)|/\varphi(1) \leq 3/4 + 1/16 = 13/16$ for all $g \in G \smallsetminus \mathbf{Z}(G)$. Now, arguing as in part (iv) of the proof of Proposition 7.2 we obtain $W/D \geq 15/32 - 3/4|Q| > 1/5$. Thus we can use 5 instead of the constant 15 in (7.4.5), which now leads to a contradiction. Next suppose that $q = 4$. As $\mathrm{Out}(S) = C_4$ and Q is not a 2-group, $Q/\mathbf{Z}(G) \leq S$. Hence, using the character table of $\mathrm{SU}_4(4)$ [GAP], one can check that $|\varphi(g)|/\varphi(1) < 1/2$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$, and so $W/D > (1/2)(1-1/|Q|) > 1/3$. Thus we can use 3 instead of the constant 15 in (7.4.6), which again leads to a contradiction.

(d4) $n = 3$, $4 \leq q \leq 9$, and $D = q(q-1)$ or $q^2 - q + 1$. Suppose $q = 9$, whence $S = \mathrm{SU}_3(9)$ and $\mathrm{Aut}(S) = S \rtimes C_4$. The character tables of all groups between S and $\mathrm{Aut}(S)$ are known in [GAP], from which we can check that $|\varphi(g)|/\varphi(1) < 1/7$ for all $g \in Q \smallsetminus \mathbf{Z}(Q)$. Hence the proof of Proposition 7.2 implies that $W/D \geq 6/7(1-1/|Q|) > 1/2$. Thus we can use 2 instead of the constant 15 in (7.4.6), which now leads to a contradiction.

Next suppose that $q = 8$, whence $\mathrm{Out}(S) = C_3 \times \mathsf{S}_3$. As Q is not a 2-group, for any $g \in Q \smallsetminus \mathbf{Z}(G)$ the coset $g\mathbf{Z}(G)$ belongs to one of the three almost simple groups $S \cdot 3_1$, $S \cdot 3_2$, and $S \cdot 3_3$ listed in [Atlas]. Using the character tables of covers of these groups given in [Atlas], we can check that $|\varphi(g)|/\varphi(1) \leq 1/7$, whence $W \geq (6/7)(1-1/|Q|)D > 8f$, contradicting (7.4.5).

If $q = 7$, then using [Atlas] one can check that $|\varphi(g)|/\varphi(1) \leq 7/43$, whence $W \geq (36/43)(1 - 1/|Q|)D > 8f$, again contradicting (7.4.5). Hence $q \leq 5$ and $D \leq 21$.

(v) Finally, we consider the case $S = \mathrm{PSL}_2(q)$ with $q = r^f$, whence $D \leq q+1$. First we analyze the cases with $D \geq 25$; in particular, $q \geq 25$. By Proposition 7.2, $W \geq D/8$, and $D \geq (q-1)/\gcd(2, q-1)$. We also note in this case that $|Q| \geq 5$ (because if $|Q| \leq 4$, then Q is abelian and has at most 3 nontrivial irreducible characters, all of degree 1, when $W \leq 3$ and so $D \leq 24$). Lemma 7.1 and Proposition 7.2 now imply that $W/D \geq 3/20$ (with equality possibly only when

$q = 25$). Arguing as in (c) using $R = \bar{G} \cap \mathrm{PGL}_2(q)$, instead of (7.4.3) and (7.4.4) we now have

$$q - 1 \leq \begin{cases} 2D \leq 40f, & r > 2, \\ D \leq 40f/3, & r = 2. \end{cases} \tag{7.4.7}$$

This can happen only when $q \leq 3^4$.

We will now analyze the remaining cases $q \leq 3^4$ further, following the proof of Proposition 7.2 (and using (7.4.7) only when $D \geq 25$). If $q = 2^6$, then the character table of $\mathrm{SL}_2(64)$ [GAP] shows that $|\varphi(g')|/\varphi(1) \leq 2/63$ for all $g' \in L \smallsetminus \mathbf{Z}(G)$, whence $|\varphi(g)|/\varphi(1) \leq 3/4 + (1/4)(2/63) = 191/252$. Thus $W \geq (61/252)(1 - 1/5)D > 12 = 2f$, a contradiction. If $q = 2^5$, then the character tables of $\mathrm{SL}_2(32)$ and $\mathrm{SL}_3(32) \cdot C_5$ in [GAP] shows that $|\varphi(g)|/\varphi(1) \leq 2/31$ for all $g \in G \smallsetminus \mathbf{Z}(G)$. Hence $W \geq (29/31)(1 - 1/5)D > 23 > 2f$, again a contradiction.

If $q = 3^4$, then since $D < 80$ by (7.4.7) (note that the equality in (7.4.7) when $r > 2$ can occur only when $q = 25$), we must have $D = (q \pm 1)/2$. In particular, G can only induce inner and fields automorphisms of S. Thus, modulo scalars, $\Phi(G)$ is contained in the image of $\mathrm{Sp}_8(3)$ in a Weil representation of degree D. Arguing as in (b1), we obtain $W > 15D/32 > 18 > 3f$, a contradiction.

If $q = 7^2$, then since $D \leq 40$, we must have $D = (q \pm 1)/2$, whence again G can only induce inner and fields automorphisms of S. Thus, modulo scalars, $\Phi(G)$ is contained in the image of $\mathrm{Sp}_4(9)$ in a Weil representation of degree D. Arguing as in (b1), we obtain $W > 15D/32 > 11 > 3f$, again a contradiction.

If $q = 3^3$ or $q = r \geq 13$, then since Q is not an r-group, we must have that $Q\mathbf{Z}(G)/\mathbf{Z}(G) \leq \mathrm{PGL}_2(q)$. The character tables of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$ [DM, Ch. 15] show that $|\varphi(g)|/\varphi(1) \leq 2/(q-1) \leq 1/6$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. Hence, $W \geq (5/6)(1 - 1/|Q|)D \geq 6(5/6)(2/3) > 3$, contradicting (7.4.3) when $q = r$. When $q = 3^3$, the same bound but using $D \geq 13$ yields $W > 7$, and so $|Q| \geq 8$. Using the same bound again, we get $W \geq (5/6)(7/8)13 > 9 = 3f$, again contradicting (7.4.3).

If $q = 5^2$, then using the character tables in [Atlas] we can check that $|\varphi(g)|/\varphi(1) \leq 5/13$ for all $g \in Q \smallsetminus \mathbf{Z}(G)$. By (7.4.3) we now have $6 \geq W > (8/13)(2/3)D$, whence $D = (q \pm 1)/2 \leq 13$. Thus we have shown that $D \leq 13$, and either $q = 25$ or $q \leq 11$. □

7B. The extraspecial case. Next we determine the characteristic of hypergeometric sheaves $\mathcal{H}$ whose geometric monodromy groups are in the extraspecial case (iii) of [GT2, Proposition 2.8].

Theorem 7.5. *Let $\mathcal{H}$ be a hypergeometric sheaf in characteristic p, of type (D, m) with $D > m$. Suppose that $D = r^n > 1$ for some prime r and that the geometric monodromy group $G = G_{\mathrm{geom}}$ of $\mathcal{H}$ contains a normal r-subgroup R, such that $R = \mathbf{Z}(R)E$ for an extraspecial r-group E of order r^{1+2n} that acts irreducibly on $\mathcal{H}$, and either $R = E$ or $\mathbf{Z}(R) \cong C_4$. Then either $p = r$, or $D \in \{2, 3, 4, 5, 8, 9\}$.*

Proof. Assume $p \neq r$, and let $J = QC$ denote the image of $I(\infty)$ on $\mathcal{H}$, with $Q = \mathbf{O}_p(J)$ being the image of $P(\infty)$ and C the image of the tame quotient. Also let Φ denote the representation of G on $\mathcal{H}$, and φ denote the character of Φ. As in the proof of Theorem 7.4, first we show that the dimension $W = \varphi(1) - [\varphi|_Q, 1_Q]_Q$ of the wild part of $\mathcal{H}$ satisfies

$$W \geq D/3. \tag{7.5.1}$$

Note that $D > m$ implies that $Q \neq 1$. Moreover, $W = D$ if $Q \cap \mathbf{Z}(G) \neq 1$. Hence we may assume that $Q \cap \mathbf{Z}(G) = 1$, and consider any $g \in Q \smallsetminus \mathbf{Z}(G)$. We will use the well-known fact (see e.g. [Wi, Theorem 1]) that the group $\mathrm{Out}_1(R)$ of all outer automorphisms of R that act trivially on $\mathbf{Z}(R)$ is contained in $\mathrm{Sp}_{2n}(r)$, and so (identifying the groups in consideration with their images on $\mathcal{H}$)

$$G \leq \mathbf{N}_{\mathrm{GL}(\mathcal{H})}(R) \leq \mathbf{Z}(\mathrm{GL}(\mathcal{H}))R \cdot \mathrm{Sp}_{2n}(r).$$

Since $p \neq r$ and $g \notin \mathbf{Z}(G)$, g projects onto a nontrivial *semisimple* element $\bar{g}$ of $\mathrm{Sp}_{2n}(r)$. In particular, if we view $\mathrm{Sp}_{2n}(r)$ as $\mathrm{Sp}(U)$ with $U := \mathbb{F}_r^{2n}$, then $\dim \mathrm{Ker}_{\mathbb{F}_r}(\bar{g} - 1_U) \leq 2n - 2$. Applying [GT1, Lemma 2.4], we obtain

$$|\varphi(g)| \leq r^{n-1} = \varphi(1)/r. \tag{7.5.2}$$

Now, using (7.2.2), we see that $W/D \geq (1 - 1/3)(1 - 1/2) = 1/3$ if $r \geq 3$. If $r = 2$, then $|Q| \geq 3$ (as $Q \neq 1$ is a p-group), and so $W/D \geq (1/2)(1 - 1/3) = 1/3$ again.

(ii) Recall that the conjugation action of G on R induces a homomorphism $\Psi : G \to \mathrm{Aut}(R)$, with $\mathrm{Ker}(\Psi) = \mathbf{C}_G(R) = \mathbf{Z}(G)$. Composing with the projection $\mathrm{Aut}(R) \twoheadrightarrow \mathrm{Out}(R)$ (with kernel $R/\mathbf{Z}(R)$), we obtain a homomorphism $\Lambda : G \to \mathrm{Out}_1(R) \leq \mathrm{Sp}(U)$, with $\mathrm{Ker}(\Lambda) = \mathbf{Z}(G)R$. Suppose now that $2n < W$. Then, by Theorem 4.5, $\Lambda|_J$ is tame, i.e. $Q \leq \mathbf{Z}(G)R$. As Q is a p-group and $p \neq r$, it follows that $Q \leq \mathbf{Z}(G)$. But this is impossible when $D > 1$ by Proposition 4.4.

Together with (7.5.1), we have shown that

$$2n \geq W \geq D/3 = r^n/3. \tag{7.5.3}$$

This is possible only when $D \in \{2, 3, 4, 5, 8, 9, 16\}$. Assume now that $D = 16$, so that $r = 2$ and $p > 2$. Then $W \in \{6, 7, 8\}$ by (7.5.3). We now show that $W = 8$. First, as $W \geq 6$, we must have by Proposition 7.3 that $|Q| \geq 7$, whence (7.5.2) and (7.2.2) imply $W \geq 7$. This in turn implies by Proposition 7.3 that $Q| \geq 9$, whence (7.5.2) and (7.2.2) ensure that $W \geq 8$, i.e. $W = 8$ by (7.5.3). Suppose that $\dim \mathrm{Ker}_{\mathbb{F}_r}(\bar{h} - 1_U) \leq 2n - 3$ for all $1 \neq h \in Q$. Then instead of (7.5.2) we now have $|\varphi(h)| \leq \varphi(1)/4$, and so (7.2.2) implies $W > 10$, a contradiction. Thus Q contains an element $g \neq 1$ with $|\varphi(g)| = \varphi(1)/2$, hence necessarily $\dim \mathrm{Ker}_{\mathbb{F}_r}(\bar{g} - 1_U) = 2n - 2$. As $\bar{g}$ is a $2'$-element in $\mathrm{Sp}(U) = \mathrm{Sp}_8(2)$, by [GT1, Lemma 2.4] this can happen only when $\bar{g} \in \mathrm{O}_2^-(2) < \mathrm{Sp}_8(2)$ is an element of order 3, whence $p = 3$, and g has eigenvalues $\lambda\zeta_3$ and $\lambda\zeta_3^2$, both with multiplicity 8, for some root of unity $\lambda \in \mathbb{C}^\times$. On the other hand, g acts trivially on the tame part of dimension $D - W = 8$, so we may assume $\lambda = \zeta_3^2$, whence $\varphi(g) + \varphi(g^{-1}) = 8$. Let $x \geq 1$ be the number of pairs (h, h^{-1}) of elements $h \in Q$ with $|\varphi(h)| = \varphi(1)/2$, and let $y \geq 0$ by the number of remaining pairs of nontrivial elements in Q (for which we have $|\varphi(h)| \leq \varphi(1)/4$). Then

$$8(1 + 2x + 2y) = W \cdot |Q| = [\varphi|_Q, 1_Q]_Q \cdot |Q| \leq 16 + 8x + 8y,$$

whence $(x, y) = (1, 0)$. Thus $|Q| = 3$, and this contradicts Proposition 7.3 since $W = 8$. □

8. Elements with simple spectra in finite groups of Lie type

In this section, we continue the classification of triples (G, V, g) satisfying the condition $(\star)$ introduced at the beginning of §6 in the *generic* situation, that is, when $\dim V \geq 23$ and $S = \mathrm{soc}(G/\mathbf{Z}(G))$ is a simple group of Lie type in characteristic p. The non-generic cases, that is where either $\dim V \leq 22$ or S is an alternating group, have already been dealt with in §6. Furthermore, because of the main application to hypergeometric sheaves, by Theorem 7.4 and using the assumption $\dim V \geq 23$, we will assume in some, explicitly described, cases that g is a semisimple element. The ss-elements g will be classified modulo scalars, that is, inside $G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$, and we let $\bar{g}$ denote the coset $g\mathbf{Z}(G)$ as an element of $G/\mathbf{Z}(G)$.

First we start with the linear case:

Theorem 8.1. *In the situation of $(\star)$, suppose that $S = \mathrm{PSL}_n(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2)$, $(3, 3)$, $(3, 4)$, so that case* (ii) *of Theorem 6.6 holds. Then $\bar{\mathsf{o}}(g) = (q^n - 1)/(q - 1)$, $\bar{g} \in \mathrm{PGL}_n(q)$, and $\bar{g}$ generates the unique, up to $\mathrm{PGL}_n(q)$-conjugacy, maximal torus of order $(q^n - 1)/(q - 1)$ of $\mathrm{PGL}_n(q)$.*

Proof. (i) The cases $(n,q) = (4,2)$, $(4,4)$, $(5,2)$, $(6,2)$, $(7,2)$ can be checked directly using [GAP], so we will assume none of these cases occurs. By Theorem 6.6(ii),

$$(q^n-1)/(q-1) \geq \bar{\mathsf{o}}(g) \geq \dim(V) \geq (q^n-q)/(q-1). \tag{8.1.1}$$

Recall [GLS, Theorem 2.5.12] that $\mathrm{Aut}(S) = Y \rtimes A$, where $Y := \mathrm{PGL}_n(q)$ and $A = \langle \phi, \tau \rangle \cong C_f \times C_2$, with ϕ the field automorphism induced by the Frobenius map $x \mapsto x^p$, $q = p^f$, and τ the transpose-inverse automorphism. We will follow the analysis in the proof of [GMPS, Theorem 2.16] to show that $\bar{g} \in Y$.

First suppose that $\bar{g} \in \langle Y, \phi \rangle \smallsetminus Y$. Then, as shown on [GMPS, p. 7679], there is a divisor $e > 1$ of f such that

$$\bar{\mathsf{o}}(g) \leq e \cdot \mathrm{meo}\big(\mathrm{PGL}_n(q^{1/e})\big) \leq e\frac{r^n-1}{r-1},$$

where $r := q^{1/e} \geq 2$. Note that, since $n \geq 3$ we have

$$r^{(n-1)e-n} \geq \begin{cases} e, & \text{if } e \geq 2, \text{ with equality only when } (n,r,e) = (3,2,2), \\ 2e, & \text{if } e \geq 3, \end{cases} \tag{8.1.2}$$

in particular, $q^{n-1} \geq er^n$. Hence

$$\frac{q^n-1}{q-1} - e\frac{r^n-1}{r-1} > q^{n-1} + q^{n-2} - er^n \geq 2,$$

and so $\bar{\mathsf{o}}(g) < (q^n-1)/(q-1) - 2$, violating (8.1.1).

Next suppose that $\bar{g} = y\psi\tau$, where $y \in Y$ and $\psi \in \langle \phi \rangle$ has order $e|f$. If $2|e$, then, as shown on [GMPS, p. 7680], $\bar{\mathsf{o}}(g) \leq e \cdot \mathrm{meo}\big(\mathrm{PGU}_n(q^{1/e})\big)$. By [GMPS, Lemma 2.15], $\mathrm{meo}\big(\mathrm{PGU}_n(q^{1/e})\big) < \mathrm{meo}\big(\mathrm{PGL}_n(q^{1/e})\big)$, so as above we have $\bar{\mathsf{o}}(g) < (q^n-q)/(q-1)$, contradicting (8.1.1). On the other hand, if $2 \nmid e \geq 3$, then

$$\bar{\mathsf{o}}(g) \leq 2e\frac{r^n-1}{r-1} < \frac{q^n-1}{q-1} - 2,$$

where we use (8.1.2) for $r = q^{1/e}$, and this contradicts (8.1.1).

It remains to consider the case $e = \mathsf{o}(\psi) = 1$. As shown on [GMPS, p. 7680], we have one of the following cases:

- $2|n$ and $\bar{\mathsf{o}}(g) \leq 2q^{n/2+1}/(q-1) < (q^n-q)/(q-1)$, since $(n,q) \neq (4,2)$;
- $n = 3$ and $\bar{\mathsf{o}}(g) \leq \max(8, 2q+2) < (q^3-q)/(q-1)$, since $(n,q) \neq (3,2)$;
- $n \geq 4$ and $\bar{\mathsf{o}}(g) \leq 2p^{\lceil \log_p(2k+1) \rceil} q^{(n-2k+1)/2}$ for some $1 \leq k \leq (n-1)/2$. Since $(n,q) \neq (4,2)$, $(5,2)$, we again have $\bar{\mathsf{o}}(g) < (q^n-q)/(q-1)$.

(ii) We have shown that $\bar{g} \in \mathrm{PGL}_n(q)$. The cases $(n,q) = (3,3)$ or $(3,7)$ can be checked directly using [GAP], so assume we are not in these cases. Consider an inverse image h of $\bar{g}$ in $\mathrm{GL}_n(q) = \mathrm{GL}(V)$ with $V = \mathbb{F}_q^n$, and suppose first that h is not semisimple. Then p divides $\mathsf{o}(h)$ and $\bar{\mathsf{o}}(g)$, and so $\bar{\mathsf{o}}(g) = (q^n-q)/(q-1)$ by (8.1.1). Note that h centralizes its unipotent part $u \neq 1$. If u is regular unipotent, then $\mathsf{o}(h)$ divides $|\mathbf{C}_{\mathrm{GL}_n(q)}(u)| = q^{n-1}(q-1)$, a contradiction. If, on the opposite, u is a transvection, then $\mathsf{o}(h)$ divides $|\mathbf{C}_{\mathrm{GL}_n(q)}(u)| = q^{2n-3}(q-1)^2 \cdot |\mathrm{GL}_{n-2}(q)|$, a contradiction when $n = 3$ since $(n,q) \neq (3,3)$, $(3,7)$. In particular, we may assume now that $n \geq 4$. Our assumptions on (n,q) imply that there exists a primitive prime divisor $\ell = \ell(p,(n-1)f)$ of $p^{(n-1)f} - 1 = q^{n-1} - 1$ by [Zs]. Since ℓ divides $\bar{\mathsf{o}}(g)$ but not $|\mathrm{GL}_{n-2}(q)|$, this rules out the case u is a transvection. Thus u is neither regular nor a transvection, whence the u-fixed point subspace U on V has dimension $2 \leq m \leq n-2$. Now h fixes U, so it belongs to $\mathrm{Stab}_{\mathrm{GL}(V)}(U)$, which is a p-group extended by $\mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$, and so has order coprime to ℓ, again a contradiction.

We have shown that h is semisimple, and so $\bar{\mathsf{o}}(g) = (q^n-1)/(q-1)$ by (8.1.1). Our assumptions on (n,q) imply that there exists a primitive prime divisor $\ell_1 = \ell(p,nf)$ of $p^{nf} - 1 = q^n - 1$

by [Zs]. Let h_1 denote the ℓ_1-part of h. The structure of centralizers of semisimple elements in $\mathrm{GL}_n(q)$ is well known, in particular, the choice of ℓ_1 implies that $\mathbf{C}_{\mathrm{GL}_n(q)}(h_1) \cong \mathrm{GL}_1(q^n)$, and this maximal torus is unique in $\mathrm{GL}_n(q)$ up to conjugacy. It is now clear that $h \in \mathrm{GL}_1(q^n)$, and, since $\bar{\mathsf{o}}(g) = (q^n-1)/(q-1)$, $\bar{g}$ generates $\mathrm{GL}_1(q^n)$ modulo $\mathbf{Z}(\mathrm{GL}_n(q))$. □

Next we consider the symplectic case:

Theorem 8.2. *In the situation of* $(\star)$, *suppose that* $S = \mathrm{PSp}_{2n}(q)$ *with* $n \geq 2$, $2 \nmid q = p^f$, *and* $(n,q) \neq (2,3)$, *so that case* (iv) *of Theorem 6.6 holds. Then* $\bar{g} \in \mathrm{PCSp}_{2n}(q)$. *Assume furthermore that* $\bar{g}$ *is a* p'*-element. Then one of the following cases occurs.*

(i) $V|_{E(G)}$ *is an irreducible Weil module,* $\bar{g} \in \mathrm{PSp}_{2n}(q)$, *and one of the following statements hold.*
 (α) $\bar{\mathsf{o}}(g) = (q^n \pm 1)/2$, *and* $\bar{g}$ *generates a unique, up to* $\mathrm{PSp}_{2n}(q)$*-conjugacy, cyclic maximal torus* $T_\pm < \mathrm{PSL}_2(q^n)$ *of order* $(q^n \pm 1)/2$ *in* $\mathrm{PSp}_{2n}(q)$.
 (β) $n = a + b$ *with* $a,b \in \mathbb{Z}_{\geq 1}$, $2e|a$ *for* $e := \gcd(a,b)$, $\bar{\mathsf{o}}(g) = (q^a+1)(q^b+1)/2$, *and* $\bar{g}$ *generates a unique, up to* $\mathrm{PSp}_{2n}(q)$*-conjugacy, cyclic maximal torus* $T_{a,b} < (\mathrm{Sp}_{2a}(q) \times \mathrm{Sp}_{2b}(q))/\mathbf{Z}(\mathrm{Sp}_{2n}(q))$ *of order* $(q^a+1)(q^b+1)/2$ *in* $\mathrm{PSp}_{2n}(q)$.

(ii) $V|_{E(G)}$ *is reducible,* $\dim(V) = q^n \pm 1$, $\bar{g} \notin \mathrm{PSp}_{2n}(q)$, *and its square* $\bar{g}^2$ *fulfills the conclusions of* (i).

Proof. (A) By Theorem 6.6(iv) and [GMPS, Theorem 2.16],

$$q^{n+1}/(q-1) \geq \bar{\mathsf{o}}(g) \geq \dim(V) \geq (q^n-1)/2. \tag{8.2.1}$$

Recall [GLS, Theorem 2.5.12] that $\mathrm{Aut}(S) = Y \rtimes \langle \phi \rangle$, where $Y := \mathrm{PCSp}_{2n}(q)$ and ϕ is the field automorphism induced by the Frobenius map $x \mapsto x^p$. Now suppose that $\bar{g} \notin Y$. Then, as shown on [GMPS, p. 7679], there is a divisor $e > 1$ of f such that

$$\bar{\mathsf{o}}(g) \leq e \cdot \mathrm{meo}\big(\mathrm{PCSp}_{2n}(q^{1/e})\big) \leq er^{n+1}/(r-1),$$

where $r := q^{1/e} \geq 3$. By (8.1.2) applied to $(n+1, r, e)$, we have that

$$q^n = r^{ne} \geq (e+1)r^{n+1} > er^{n+1} + 1,$$

and so $\bar{\mathsf{o}}(g) \leq er^{n+1}/(r-1) \leq er^{n+1}/2 < (q^n-1)/2$, violating (8.2.1). Thus we have shown that $\bar{g} \in \mathrm{PCSp}_{2n}(q)$.

(B) From now on we will assume that $\bar{g}$ is a p'-element. First we consider the case $V|_{E(G)}$ is irreducible, and so it is a Weil module of dimension $d = (q^n \pm 1)/2$. Since the outer diagonal automorphism of $E(G)$ fuses the two irreducible Weil modules of dimension d but ϕ stabilizes each of them, $\bar{g} \in Y \cap \langle S, \phi \rangle = S = \mathrm{PSp}_{2n}(q)$. View $S = \mathrm{PSp}(W)$ with $W = \mathbb{F}_q^{2n}$, and let $h \in \mathrm{Sp}(W)$ be a (semisimple) inverse image of $\bar{g}$.

(B1) Here we consider the case where the $\langle h \rangle$-module W cannot be decomposed as an orthogonal sum of h-invariant nonzero non-degenerate subspaces, and, for further use, we also allow $n = 1$ and $(n,q) = (2,3)$ here. In this case, by [Hup, Satz 2], either $\mathsf{o}(h)|(q^n+1)$ and W is an irreducible $\mathbb{F}_q\langle h \rangle$-module, or $\mathsf{o}(h)|(q^n-1)$ and $W = W_1 \oplus W_2$ with W_i an irreducible $\mathbb{F}_q\langle h \rangle$-module, also being a totally isotropic subspace of W. Set $\epsilon = -$, respectively $\epsilon = +$, in these two cases. Then, up to $\mathrm{Sp}(W)$-conjugacy, there is a unique cyclic maximal torus $\hat{T}_\epsilon = \langle h_\epsilon \rangle \cong C_{q^n-\epsilon}$, which can be chosen to be inside a standard subgroup $\mathrm{SL}_2(q^n)$ of $\mathrm{Sp}(W)$ and to contain h. Note that $\bar{\mathsf{o}}(h_\epsilon) = (q^n-\epsilon)/2$; on the other hand, $\bar{\mathsf{o}}(g) \geq (q^n-1)/2$ by (8.2.1). Hence, if $\bar{\mathsf{o}}(g) > (q^n-1)/2$, we must have that $\epsilon = -$, $\bar{\mathsf{o}}(g) = (q^n+1)/2$, and $\langle \bar{g} \rangle = \hat{T}_-/\mathbf{Z}(\mathrm{Sp}(W)) =: T_-$. Otherwise we have $\bar{\mathsf{o}}(g) = (q^n-1)/2$. If moreover $(n,q) \neq (1,3)$, then $(q^n+1)/4 < (q^n-1)/2$, and so $\epsilon = +$, $\bar{\mathsf{o}}(g) = (q^n-1)/2$, and $\langle \bar{g} \rangle = \hat{T}_+/\mathbf{Z}(\mathrm{Sp}(W)) =: T_+$. In the remaining case, we have $(n,q) = (1,3)$ and $g \in \mathbf{Z}(G)$. In particular, we have arrived at conclusion (α).

(B2) Now we may assume that $W = \oplus_{i=1}^{k} W_i$ is an orthogonal sum of minimal h-invariant nonzero non-degenerate subspaces W_i for some $k \geq 2$. Correspondingly, we can write

$$h = \mathrm{diag}(h_1, h_2, \ldots, h_k) \in H := \mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2) \times \ldots \times \mathrm{Sp}(W_k),$$

with $h_i \in \mathrm{Sp}(W_i)$, $\dim W_i = 2n_i$ and $\sum_{i=1}^{k} n_i = n$.

By the analysis in (B1), $\bar{\mathsf{o}}(h_i) \leq (q^{n_i}+1)/2$ for all i. Suppose for instance that $\bar{\mathsf{o}}(h_1) \leq (q^{n_1}-1)/2$. Recall that, a total Weil module of $\mathrm{Sp}_{2n}(q)$ has character $\omega_n = \xi_n + \eta_n$, with $\xi_n(1) = (q^n+1)/2$ and $\eta_n(1) = (q^n-1)/2$; in particular, the character of V considered as $\mathrm{Sp}(W)$-module is either ξ_n or η_n. Using the branching rule [TZ2, Proposition 2.2(iv)], we see that at least one irreducible constituent of $V|_{\mathrm{Sp}(W_1)}$ affords the character ξ_{n_1} of degree $(q^{n_1}+1)/2$. It follows that at least one irreducible constituent V_0 of $V|_H$ affords the character

$$\xi_{n_1} \boxtimes \alpha_2 \boxtimes \ldots \boxtimes \alpha_k$$

for some irreducible (Weil) characters α_i of $\mathrm{Sp}(W_i)$, $2 \leq i \leq k$. Since $\xi_{n_1}(1) > \bar{\mathsf{o}}(h_1)$, $\mathrm{Spec}\,(h_1, \xi_{n_1})$ is *not* simple, whence the same holds for $\mathrm{Spec}\,(h, V_0)$ and $\mathrm{Spec}\,(g, V)$, a contradiction.

We have shown that $\bar{\mathsf{o}}(h_i) = (q^{n_i}+1)/2$ for all i. The analysis in (B1) shows that $\mathsf{o}(h_i^{(q^{n_i}+1)/2}) \leq 2$. In particular, $h^{2M} = 1$ and $\bar{\mathsf{o}}(g) \leq 2M$, where

$$M := \mathrm{lcm}\big((q^{n_1}+1)/2, (q^{n_2}+1)/2, \ldots, (q^{n_k}+1)/2\big). \tag{8.2.2}$$

Suppose that $k \geq 3$. If $q \geq 5$, or if $q = 3$ but $k \geq 4$, then

$$M \leq \frac{q^n}{2^k} \cdot \prod_{i=1}^{k} \big(1 + \frac{1}{q^{n_i}}\big) \leq \frac{q^n}{2^k} \cdot \big(1 + \frac{1}{q}\big)^k < \frac{q^n}{4} \cdot \big(1 - \frac{1}{q^n}\big) = \frac{q^n - 1}{4},$$

and so $\bar{\mathsf{o}}(g) \leq 2M < (q^n-1)/2$, contradicting (8.2.1). If $q = k = 3$ but $n \geq 5$, then

$$M \leq \frac{q^n}{2^3} \cdot \prod_{i=1}^{3} \big(1 + \frac{1}{q^{n_i}}\big) \leq \frac{q^n}{8} \cdot \big(1 + \frac{1}{3}\big)^2 \cdot \big(1 + \frac{1}{9}\big) < \frac{q^n}{4} \cdot \big(1 - \frac{1}{q^n}\big) = \frac{q^n - 1}{4},$$

again leading to the same contradiction. If $q = k = n = 3$, then $2M = 4 < (q^3-1)/2$ by (8.2.2), and if $q = k = 3$ but $n = 4$, then $2M = 20 < (q^4-1)/2$, again contradicting (8.2.1).

We have shown that $k = 2$. Now we have $n = a + b$ with $a := n_1$ and $b := n_2$. Let $e := \gcd(a, b)$, and consider the case both a/e and b/e are odd. Then by (8.2.2) we have

$$\bar{\mathsf{o}}(g) \leq 2M \leq (q^a+1)(q^b+1)/(q^e+1) \leq (q^a+1)(q^b+1)/4 < (q^n-1)/2,$$

unless $(n, q) = (2, 3)$ which is ruled out by assumption. Thus, renaming a and b if necessary, we have that $2e|a$ (and so $2 \nmid (b/e)$ necessarily). In this case, $(q^a-1)/2$ is divisible by $(q^{2e}-1)/2$, and one can check that $\gcd((q^a+1)/2, (q^b+1)/2)) = 1$. Now, $\bar{\mathsf{o}}(h_1) = (q^a+1)/2$ and $\bar{\mathsf{o}}(h_2) = (q^b+1)/2$, hence $\bar{\mathsf{o}}(g) = \bar{\mathsf{o}}(h)$ is divisible by $M = (q^a+1)(q^b+1)/4 > (q^n+1)/4$. Together with (8.2.1) and $h^{2M} = 1$, this implies that $\bar{\mathsf{o}}(g) = 2M$. Also, we have shown in (B1) that h_i generates a cyclic maximal torus (of order $(q^{n_i}+1)/2$) in $\mathrm{PSp}(W_i)$ for $i = 1, 2$, and this torus is unique in $\mathrm{PSp}(W_i)$ up to conjugacy. Hence, $\bar{g}$ generates a cyclic maximal torus $T_{a,b}$ (of order $2M = (q^a+1)(q^b+1)/2$) in $\mathrm{PSp}(W)$. Note that such a torus T is unique in $\mathrm{PSp}(W)$ up to conjugacy. [Indeed, applying the above analysis to an inverse image $h' \in \mathrm{Sp}(W)$ of a generator of T we see that case (B1) does not occur for h', since $\bar{\mathsf{o}}(h') = |T| > (q^n+1)/2$. Next, the analysis in (B2) using (8.2.2) shows that the $\langle h' \rangle$-module W decomposes as the orthogonal sum $W_1' \oplus W_2'$ of two minimal h'-invariant non-degenerate subspaces of dimension $2c$ and $2d$, with $1 \leq c \leq d$ and $c + d = n$. Now, using $(q^c-1)(q^d-1) < (q^c-1)(q^d+1) < q^n$ and $(q^c+1)(q^d-1) \equiv -1 \not\equiv (q^a+1)(q^b+1) (\mathrm{mod}\ p)$ but $\bar{\mathsf{o}}(h') = \bar{\mathsf{o}}(h) = (q^a+1)(q^b+1)/2$, we must have that $\bar{\mathsf{o}}(h') = (q^c+1)(q^d+1)/2$ and $\{c, d\} = \{a, b\}$, and thus T is conjugate to $T_{a,b}$.] We have arrived at conclusion (β).

(C) Now we consider the case $V|_{E(G)}$ is reducible, whence $\dim(V) = q^n \pm 1$ and $V|_{E(G)} = A \oplus B$ is the sum of two irreducible Weil modules A, B of dimension $d = (q^n \pm 1)/2$ by Theorem 6.6(iv). If moreover $A \cong B$, then G cannot induce an outer diagonal automorphism of $E(G)$, and so $G/\mathbf{Z}(G) \leq \langle S, \phi \rangle$. In particular, $G/\mathbf{Z}(G)E(G)$ is cyclic. On the other hand, the $E(G)$-module A extends to a simple $\mathbf{Z}(G)E(G)$-module $\tilde{A}$ which is G-stable. It follows from Gallagher's theorem [Is, (6.17)] that $\dim(V) = \dim(A)$, a contradiction. Thus $A \not\cong B$, but A and B are fused by any element $t \in Y \smallsetminus S$: $B \cong A^t$.

Suppose $\bar{g} \in S$. Then, up to scalar, g acts on V as some p'-element $h \in \mathrm{Sp}(V)$ and stabilizes each of A and B. As g has simple spectrum on V, the same holds for the actions of h on A and on B. Next, viewing $\mathrm{Sp}_{2n}(q) = \mathcal{G}^\sigma$ for a Frobenius endomorphism $\sigma : \mathcal{G} \to \mathcal{G}$ of the simply connected algebraic group $\mathcal{G} = \mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$, we have that $Y = (\mathcal{G}/\mathbf{Z}(\mathcal{G}))^\sigma$. In particular, we can take $t \in \mathcal{G}$ and also have that $tht^{-1} \in \mathcal{G}^\sigma$ (since $S \lhd Y$). By Lemma 6.5(ii), $tht^{-1} = uhu^{-1}$ for some $u \in \mathcal{G}^\sigma$. It follows that

$$\mathrm{Spec}\,(h, B) = \mathrm{Spec}\,(h, A^t) = \mathrm{Spec}\,(tht^{-1}, A) = \mathrm{Spec}\,(uhu^{-1}, A) = \mathrm{Spec}\,(h, A),$$

and this contradicts the simple spectrum of g on $V = A \oplus B$.

We have shown that $\bar{g} \in Y \smallsetminus S$, whence g interchanges A and B and g^2 stabilizes each of A and B. In this case, $\mathrm{Spec}\,(g, V) = \{\pm\sqrt{\alpha} \mid \alpha \in \mathrm{Spec}\,(g^2, A)\}$ is simple if and only if $\mathrm{Spec}\,(g^2, A)$ is simple. It follows that $\bar{g}^2$ fulfills the conclusions of (i). $\square$

Next, we treat the unitary case:

Theorem 8.3. *In the situation of* $(\star)$*, suppose that* $S = \mathrm{PSU}_n(q)$ *with* $n \geq 3$*,* $q = p^f$*, and* $(n, q) \neq (3, 2)$*,* $(3, 3)$*,* $(3, 4)$*,* $(4, 2)$*,* $(4, 3)$*,* $(5, 2)$*,* $(6, 2)$*, so that case* (iii) *of Theorem 6.6 holds. Then* $\bar{g} \in \mathrm{PGU}_n(q)$*. Assume furthermore that* $\bar{g}$ *is a* p'*-element. Then* $G/\mathbf{Z}(G) \rhd \mathrm{PGU}_n(q)$ *and one of the following cases occurs.*

(i) $\mathsf{\bar{o}}(g) = (q^n - (-1)^n)/(q + 1)$*, and* $\bar{g}$ *generates a unique, up to* $\mathrm{PGU}_n(q)$*-conjugacy, cyclic maximal torus of order* $(q^n - (-1)^n)/(q + 1)$ *in* $\mathrm{PGU}_n(q)$*. Moreover, if* $2|n$ *then* $\dim(V) = (q^n - 1)/(q + 1)$*.*

(ii) $2 \nmid n$*,* $\mathsf{\bar{o}}(g) = q^{n-1} - 1$*, and* $\bar{g}$ *generates a unique, up to* $\mathrm{PGU}_n(q)$*-conjugacy, cyclic maximal torus* $T_{n-1,1}$ *of order* $q^{n-1} - 1$ *in* $\mathrm{PGU}_n(q)$*. Moreover,* $\dim(V) = (q^n - q)/(q + 1)$*.*

(iii) $2|n = a + b$ *with* $2 \nmid a, b \in \mathbb{Z}_{\geq 1}$*,* $\gcd(a, b) = 1$*,* $\mathsf{\bar{o}}(g) = (q^a + 1)(q^b + 1)/(q + 1)$*, and* $\bar{g}$ *generates a unique, up to* $\mathrm{PGU}_n(q)$*-conjugacy, cyclic maximal torus* $T_{a,b} < (\mathrm{GU}_a(q) \times \mathrm{GU}_b(q))/\mathbf{Z}(\mathrm{GU}_n(q))$ *of order* $(q^a + 1)(q^b + 1)/(q + 1)$ *in* $\mathrm{PGU}_n(q)$*.*

Proof. (A) The cases $(n, q) = (4, 4)$, $(4, 5)$ can be checked directly using [GAP], so we will assume $(n, q) \neq (4, 4)$, $(4, 5)$. By Theorem 6.6(iii) and [GMPS, Theorem 2.16],

$$q^{n-1} + q^{\min(2,n-2)} \geq \mathsf{\bar{o}}(g) \geq \dim(V) \geq (q^n - q)/(q + 1). \tag{8.3.1}$$

Recall [GLS, Theorem 2.5.12] that $\mathrm{Aut}(S) = Y \rtimes \langle \phi \rangle$, where $Y := \mathrm{PGU}_n(q)$ and ϕ is an outer automorphism of order $2f$. Now suppose that $\bar{g} \notin Y$, and write $\bar{g} = x\psi$ with $x \in Y$ and $\psi \in \langle \phi \rangle$ of order $1 < e|2f$. Suppose first that $2 \nmid e$. Then, as shown on [GMPS, p. 7679],

$$\mathsf{\bar{o}}(g) \leq e \cdot \mathrm{meo}\big(\mathrm{PGU}_n(q^{1/e})\big) \leq e\big(r^{n-1} + r^{\min(2,n-2)}\big) < (8/9)(q^{n-1} - 1) \leq (q^n - q)/(q + 1),$$

where $r := q^{1/e} \geq 2$, provided $(n, r) \neq (5, 2)$; and this contradicts (8.3.1). We also achieve a contradiction in the case $(n, r) = (5, 2)$ using $\mathrm{meo}(\mathrm{PGU}_5(2)) = 24$. Next we consider the case $2|e \geq 4$. Then

$$\mathsf{\bar{o}}(g) \leq e \cdot \mathrm{meo}\big(\mathrm{PGL}_n(q^{2/e})\big) \leq e(r^n - 1)/(r - 1) < (q^n - q)/(q + 1),$$

with $r := q^{2/e}$, $(n,r) \neq (3,2)$, and $(n,q) \neq (4,4)$, again contradicting (8.3.1). If $(n,r) = (3,2)$, then $e \geq 6$ as $(n,q) \neq (3,4)$, and we also achieve a contradiction using $\mathrm{meo}(\mathrm{PGL}_3(2)) = 8$.

It remains to consider the case $e = \mathsf{o}(\psi) = 2$. As shown on [GMPS, p. 7680], we have one of the following cases:

- $2|n$ and $\bar{\mathsf{o}}(g) \leq 2q^{n/2+1}/(q-1) < (q^n - q)/(q+1)$, since $(n,q) \neq (4,2)$, $(4,3)$, and $(6,2)$;
- $n = 3$ and $\bar{\mathsf{o}}(g) \leq \max(8, 2q+2) < (q^3 - q)/(q+1)$, since $(n,q) \neq (3,2)$, $(3,3)$;
- $n \geq 4$ and $\bar{\mathsf{o}}(g) \leq 2p^{\lceil \log_p(2k+1) \rceil} q^{(n-2k+1)/2}$ for some $1 \leq k \leq (n-1)/2$. Since $(n,q) \neq (4,2)$, $(4,3)$, $(4,4)$, $(4,5)$, and $(5,2)$, we again have $\bar{\mathsf{o}}(g) < (q^n - q)/(q+1)$.

Thus we have shown that $\bar{g} \in \mathrm{PGU}_n(q)$. Note that the same conclusion holds in the cases $(n,q) = (3,4)$, $(4,2)$, $(4,4)$, and $(6,2)$, since in these cases $p = 2$ and f is a 2-power.

(B) From now on we will assume that $\bar{g}$ is a p'-element. View $S = \mathrm{PSU}(W)$ with $W = \mathbb{F}_{q^2}^n$, and let $h \in \mathrm{GU}(W)$ be a (semisimple) inverse image of $\bar{g}$.

(B1) Here we consider the case where the $\langle h \rangle$-module W cannot be decomposed as an orthogonal sum of h-invariant nonzero non-degenerate subspaces, and, for further use, we assume only that $n \geq 2$. In this case, by [Hup, Satz 2], either $2 \nmid n$, $\mathsf{o}(h)|(q^n + 1)$ and W is an irreducible $\mathbb{F}_{q^2}\langle h \rangle$-module, or $2|n$, $\mathsf{o}(h)|(q^n - 1)$ and $W = W_1 \oplus W_2$ with W_i an irreducible $\mathbb{F}_{q^2}\langle h \rangle$-module, also being a totally isotropic subspace of W. Furthermore, up to $\mathrm{GU}(W)$-conjugacy, there is a unique cyclic maximal torus $\hat{T} = \langle t \rangle \cong C_{q^n - (-1)^n}$, which can be chosen to contain h. Note that $\bar{\mathsf{o}}(t) = (q^n - (-1)^n)/(q+1)$; on the other hand, $\bar{\mathsf{o}}(g) > (q^n - (-1)^n)/2(q+1)$ by (8.3.1). Hence, we must have that $\bar{\mathsf{o}}(g) = (q^n - (-1)^n)/(q+1)$, and $\langle \bar{g} \rangle = \hat{T}/\mathbf{Z}(\mathrm{GU}(W))$. In particular, we have arrived at conclusion (i) (with the value of $\dim V$ following from (8.3.1) when $2|n$).

(B2) Now we may assume that $W = \oplus_{i=1}^k W_i$ is an orthogonal sum of minimal h-invariant nonzero non-degenerate subspaces W_i for some $k \geq 2$. Correspondingly, we can write

$$h = \mathrm{diag}(h_1, h_2, \ldots, h_k) \in H := \mathrm{GU}(W_1) \times \mathrm{GU}(W_2) \times \ldots \times \mathrm{GU}(W_k),$$

with $h_i \in \mathrm{GU}(W_i)$, $\dim W_i = n_i$ and $\sum_{i=1}^k n_i = n$.

By the analysis in (B1), $\mathsf{o}(h_i^{(q^{n_i} - (-1)^{n_i})/(q+1)})|(q+1)$. In particular, $h^{(q+1)M} = 1$ and $\bar{\mathsf{o}}(g)|(q+1)M$, where

$$M := \mathrm{lcm}\big((q^{n_1} - (-1)^{n_1})/(q+1), (q^{n_2} - (-1)^{n_2})/(q+1), \ldots, (q^{n_k} - (-1)^{n_k})/(q+1)\big). \tag{8.3.2}$$

Also note that for any $c, d \in \mathbb{Z}_{\geq 1}$,

$$\frac{q^a - (-1)^a}{q+1} \cdot \frac{q^b - (-1)^b}{q+1} \leq \frac{q^{a+b-1} - (-1)^{a+b-1}}{q+1}. \tag{8.3.3}$$

Suppose that $n \geq 4$ and $k \geq 3$. Applying (8.3.3) repeatedly, we obtain

$$(q+1)M \leq (q+1)\prod_{i=1}^k \frac{q^{n_i} - (-1)^{n_i}}{q+1} \leq q^{n-k+1} - (-1)^{n-k+1} \leq q^{n-2} - (-1)^{n-2} < \frac{q^n - q}{q+1},$$

and so $\bar{\mathsf{o}}(g) < (q^n - q)/(q+1)$, contradicting (8.3.1). If $n = k = 3$, then $q > 2$ and $n_i = 1$ for all i, and so $(q+1)M = q+1 < (q^3 - q)/(q+1)$ by (8.3.2), again contradicting (8.2.1).

We have shown that $k = 2$. Now we have $n = a + b$ with $a := n_1$ and $b := n_2$. If $a, b \geq 2$, then we note that $(q^a - (-1)^a)(q^b - (-1)^b) < 2(q^{a+b} - q)$. Hence, if $\gcd(q^a - (-1)^a, q^b - (-1)^b) \geq 2(q+1)$, then

$$\bar{\mathsf{o}}(g) \leq (q+1)M \leq (q+1)\frac{(q^a - (-1)^a)(q^b - (-1)^b)}{2(q+1)^2} < \frac{q^n - q}{q+1},$$

contradicting (8.3.1). Since $\gcd(q^a-(-1)^a, q^b-(-1)^b) = q^e-(-1)^e$ for $e := \gcd(a,b)$, we must therefore have that $e=1$, or $e=q=2$. In the latter case we also have

$$\bar{\mathsf{o}}(g) \le (q+1)M = (q+1)\frac{(q^a-(-1)^a)(q^b-(-1)^b)}{(q+1)^2} = \frac{(q^a-1)(q^b-1)}{q+1} < \frac{q^n-q}{q+1},$$

again a contradiction. Thus a and b are coprime.

Consider the case $a,b \ge 2$, but $2|b$. As $n \ge 3$, all irreducible Weil characters of $\mathrm{SU}_n(q)$ extend to $\mathrm{GU}_n(q)$, see e.g. [TZ2, §4], so we may extend V to $\mathrm{GU}(W)$. Using the branching rule [KT4, (2.0.3)], we see that at least one irreducible constituent of $V|_{\mathrm{SU}(W_2)}$ affords the Weil character $\zeta^0_{b,q}$ of degree $(q^b+q)/(q+1)$, and the same holds for the restriction to $\mathrm{GU}(W_2)$. Thus at least one irreducible constituent V_0 of V to $H = \mathrm{GU}(W_1) \times \mathrm{GU}(W_2)$ affords the character $\beta_1 \boxtimes \beta_2$ for some irreducible (Weil) characters β_i of $\mathrm{GU}(W_i)$, $i=1,2$, and $\beta_2(1) = (q^b+q)/(q+1) > \bar{\mathsf{o}}(h_2)$. It follows that $\mathrm{Spec}\,(h_2,\beta_2)$ is *not* simple, whence the same holds for $\mathrm{Spec}\,(h,V_0)$ and $\mathrm{Spec}\,(g,V)$, a contradiction.

Thus either $2 \nmid n$ and $(a,b) = (n-1,1)$, or $2|n$ and $\gcd(a,b)=1$. In the former case, h is contained in a maximal torus $C_{q^{n-1}-1} \times C_{q+1} < \mathrm{GU}_{n-1}(q) \times \mathrm{GU}_1(q)$ which projects onto a cyclic maximal torus $T_{n-1,1} \cong C_{q^{n-1}-1}$ of $\mathrm{PGU}_n(q)$. Since

$$\bar{\mathsf{o}}(h) = \bar{\mathsf{o}}(g) \ge (q^n-q)/(q+1) > (q^{n-1}-1)/2,$$

we must have that $\langle \bar{g} \rangle = T_{n-1,1}$. In fact, multiplying h by a suitable central element of $\mathrm{GU}(W)$, we may assume that $h_2 = 1_{W_2}$. Now, if $\dim V = (q^n+1)/(q+1)$, then again using the branching rule, we see that at least one irreducible constituent V_1 of $V|_{\mathrm{GU}(W_1)}$ has degree $(q^{n-1}+q)/(q+1)$. On the other hand, h_1 has order $(q^{n-1}-1)/(q+1)$ *modulo* $\mathbf{Z}(\mathrm{GU}(W_1))$. It follows that $\mathrm{Spec}\,(h_1,V_1)$ is not simple, whence so is $\mathrm{Spec}\,(g,V)$ by the above argument. Hence $\dim(V) = (q^n-q)/(q+1)$, and we arrive at conclusion (ii).

In the latter case, h is contained in a maximal torus $C_{q^a+1} \times C_{q^b+1} < \mathrm{GU}_a(q) \times \mathrm{GU}_b(q)$ which again projects onto a maximal torus $T_{a,b}$ of $\mathrm{PGU}_n(q)$; moreover, $T_{a,b}$ is cyclic of order $(q^a+1)(q^b+1)/(q+1)$, since $\gcd(q^a+1, q^b+1) = q+1$. Since

$$\bar{\mathsf{o}}(h) = \bar{\mathsf{o}}(g) \ge (q^n-q)/(q+1) > (q^a+1)(q^b+1)/2(q+1),$$

we must have that $\langle \bar{g} \rangle = T_{a,b}$, and so we arrive at conclusion (iii).

In both cases of (ii) and (iii), the uniqueness of cyclic maximal tori $T_{a,b}$ of order $(q^a-(-1)^a)/(q^b-(-1)^b)/(q+1)$ follows from the well-known order formula and classification of maximal tori in $\mathrm{GU}_n(q)$ (or from repeating the analysis in (B1) and (B2) for an inverse image $h' \in \mathrm{GU}(W)$ of a generator of such a torus). Finally, since $\bar{g}$ generates a maximal torus of $\mathrm{PGU}_n(q)$, we have $G/\mathbf{Z}(G) \rhd \mathrm{PGU}_n(q)$. $\square$

Corollary 8.4. *In the situation of* $(\star)$*, assume we are in one of the cases considered in Theorem 8.1, respectively Theorem 8.2(i), Theorem 8.3. Suppose* $(\star)$ *gives rise to a hypergeometric sheaf* $\mathcal{H}$ *of type* (D,m) *with* $D-m \ge 2$*, with* $G = G_{\mathrm{geom}}$*,* g *a generator of the image of* $I(0)$ *in* G*, and* V *realizes the action of* G *on* $\mathcal{H}$*. Then* $G/\mathbf{Z}(G) \cong \mathrm{PGL}_n(q)$*, respectively* $\mathrm{PSp}_{2n}(q)$*,* $\mathrm{PGU}_n(q)$*.*

Proof. Since G is almost quasisimple, $G^{(\infty)}$ is a quasisimple cover of $S = \mathrm{PSL}_n(q)$, respectively $\mathrm{PSp}_{2n}(q)$, $\mathrm{PSU}_n(q)$, and $S \lhd G/\mathbf{Z}(G) \le \mathrm{Aut}(S)$. By Theorem 8.1, respectively Theorem 8.2(i), Theorem 8.3, $H/\mathbf{Z}(G)$, with $H := \langle G^{(\infty)}, \mathbf{Z}(G), g\rangle$, is the normal subgroup $\mathrm{PGL}_n(q)$, respectively $\mathrm{PSp}_{2n}(q)$, $\mathrm{PGU}_n(q)$, of $\mathrm{Aut}(S)$; in particular, $H \lhd G$. As H contains the normal closure of the image $\langle g \rangle$ of $I(0)$ in G, it follows by Theorem 4.1 that $G = H$, whence the statement follows. $\square$

Finally, we treat the extraspecial normalizers:

Theorem 8.5. *Let p be a prime. Let G be a finite irreducible subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_{p^n}(\mathbb{C})$ that satisfies* $(\mathbf{S}+)$ *and is an extraspecial normalizer, so that $G \rhd R = \mathbf{Z}(R)E$ for some some extraspecial p-group E of order p^{1+2n} that acts irreducibly on V, and furthermore either $R = E$ or $\mathbf{Z}(R) \cong C_4$, as in* [GT2, Proposition 2.8(iii)]. *Suppose that a p'-element $g \in G$ has simple spectrum on V and that $p^n \geq 11$. Then the following statements hold.*

(i) *Suppose $p > 2$. Then $\exp(R) = p$, $\bar{\mathsf{o}}(g) = p^n + 1$, and the coset $g\mathbf{Z}(G)R$ as an element of $G/\mathbf{Z}(G)R \hookrightarrow \mathrm{Sp}_{2n}(p)$ generates a cyclic maximal torus C_{p^n+1} of $\mathrm{Sp}_{2n}(p)$.*

(ii) *Suppose $p = 2$. Then one can find integers $a_1 > a_2 > \ldots > a_t \geq 1$ such that $n = \sum_{i=1}^{t} a_i$, $\gcd(2^{a_i}+1, 2^{a_j}+1) = 1$ if $i \neq j$, $\bar{\mathsf{o}}(g) = \prod_{i=1}^{t}(2^{a_i}+1)$, and the coset $g\mathbf{Z}(G)R$ as an element of $G/\mathbf{Z}(G)R \hookrightarrow \mathrm{Sp}_{2n}(2)$ generates a cyclic maximal torus $C_{2^{a_1}+1} \times \ldots \times C_{2^{a_t}+1}$ of $\mathrm{Sp}_{2n}(2)$.*

Proof. By Schur's lemma, the irreducibility of R on V implies that $\mathbf{C}_G(R) = \mathbf{Z}(G) < \mathbf{Z}(\mathrm{GL}(V))$, and so $G/\mathbf{Z}(G)$ embeds in the group $\mathrm{Aut}_1(R)$ of all automorphisms of R that act trivially on $\mathbf{Z}(R)$, and $G/\mathbf{Z}(G)R \hookrightarrow \mathrm{Out}_1(R) = \mathrm{Aut}_1(R)/(R/\mathbf{Z}(R)) \hookrightarrow \mathrm{Sp}_{2n}(p)$, see e.g. [Wi, Theorem 1]. As $p \nmid \mathsf{o}(g)$, $\bar{\mathsf{o}}(g)$ is equal to the order of the coset $g\mathbf{Z}(G)R$ in $\mathrm{Out}_1(R) \leq \mathrm{Sp}_{2n}(p)$. On the other hand,

$$\bar{\mathsf{o}}(g) > p^n \tag{8.5.1}$$

as g is an ss-element on V.

(i) First we consider the case $p > 2$. Suppose that $\exp(R) > p$. Then $\mathrm{Out}_1(R)$ is isomorphic to a semidirect product of a p-group of order p^{2n-1} by $\mathrm{Sp}_{2n-2}(p)$ by [Wi, Theorem 1]. As g is a p'-element, $\bar{\mathsf{o}}(g)$ is at most the maximum order of elements in $\mathrm{Sp}_{2n-2}(p)$, which is at most twice of $\mathrm{meo}(\mathrm{PSp}_{2n-2}(p)) < p^n/(p-1)$ by [GMPS, Lemma 2.10] (where the strict inequality holds because $\mathrm{meo}(\cdot)$ is an integer). It follows that $\bar{\mathsf{o}}(g) < 2p^n/(p-1) \leq p^n = \dim(V)$, contradicting (8.5.1).

We have shown that $\exp(R) = p$, i.e. $R \cong p_+^{1+2n}$. In this case, it is known that $\mathrm{Aut}_1(R)$ is a split extension of $\mathrm{Inn}(R) \cong R/\mathbf{Z}(R)$ by $\mathrm{Sp}_{2n}(p)$. Now, $\mathrm{Sp}_{2n}(p)$ as a subgroup of $\mathrm{Aut}_1(R)$ preserves the equivalence class of the representation of R on V, hence it admits a projective representation on V, which must be linearized since $\mathrm{Sp}_{2n}(p)$ has trivial Schur multiplier when $p^n > 9$, and by a faithful representation because $\mathrm{Sp}_{2n}(p)$ acts faithfully on R. Thus we have shown that

$$G \leq \mathbf{N}_{\mathrm{GL}(V)}(R) = \mathbf{Z}(\mathrm{GL}(V))R \rtimes \mathrm{Sp}_{2n}(p);$$

in particular, by conjugating the p'-element g (applying the Schur-Zassenhaus theorem to $\mathbf{Z}(G)R\langle g\rangle$), we can write $g = zh$ for some $z \in \mathbf{Z}(G)$ and some p'-element $h \in \mathrm{Sp}_{2n}(p)$ with $\bar{\mathsf{o}}(h) = \bar{\mathsf{o}}(g) \geq p^n$. If $n = 1$, it follows that $\bar{\mathsf{o}}(h) = \mathsf{o}(h) = p+1$, and we are done in this case.

Assume now that $n \geq 2$, and apply Theorem 8.2(i) to $h \in \mathrm{Sp}_{2n}(p)$ acting on V. In case (α), we have that h generates a cyclic maximal torus $C_{p^n-\epsilon}$ of $\mathrm{Sp}_{2n}(p)$ for some $\epsilon = \pm$. As $\bar{\mathsf{o}}(h) \geq p^n$, we must have that $\epsilon = -$ and $\mathsf{o}(h) = p^n+1 = \bar{\mathsf{o}}(g)$, as stated. In case (β), h belongs to a maximal torus $C_{p^a+1} \times C_{p^b+1} < \mathrm{Sp}_{2a}(p) \times \mathrm{Sp}_{2b}(p)$ with $n = a+b$ and $a, b \in \mathbb{Z}_{\geq 1}$. In this case, $h^{(p^a+1)(p^b+1)/4} \in \mathbf{Z}(\mathrm{Sp}_{2a}(p) \times \mathrm{Sp}_{2b}(p))$ and so $h^{(p^a+1)(p^b+1)/2} = 1$. Thus $\bar{\mathsf{o}}(h) \leq \mathsf{o}(h) \leq (p^a+1)(p^b+1)/2 < p^{a+b} = p^n$, a contradiction.

(ii) Let $\bar{g} \in \mathrm{Sp}_{2n}(2)$ denote the image of g in $G/\mathbf{Z}(G)R$. By [GT1, Lemma 5.8], there is some $\epsilon = \pm$ such that $\bar{g}$ preserves a quadratic form of type $\epsilon \in \{+,-\}$ on the natural module $\mathbb{F}_2^{2n}$ for $\mathrm{Sp}_{2n}(2)$: $\bar{g} \in \mathrm{O}_{2n}^{\epsilon}(2)$. This implies that we may take E to be of type ϵ and g-invariant. Let Sp^+ denote O and let Sp^- denote Sp. As shown in [KT8, Theorem 4.2], the action of E on V then preserves a non-degenerate bilinear form of type ϵ, and

$$\mathbf{N}_{\mathrm{Sp}^{\epsilon}(V)}(E) = E \cdot \mathrm{O}_{2n}^{\epsilon}(2),\ \mathbf{N}_{\mathrm{GL}(V)} = \mathbf{Z}(\mathrm{GL}(V))\mathbf{N}_{\mathrm{Sp}^{\epsilon}(V)}(E).$$

In particular, we can write $g = zh$ with $z \in \mathbf{Z}(\mathrm{GL}(V))$ and $h \in \mathbf{N}_{\mathrm{Sp}^\epsilon(V)}(E)$ having odd order (as $2 \nmid \bar{\mathsf{o}}(g)$). In turn, we can embed the image of h in $\mathrm{O}^\epsilon_{2n}(2)$ in a maximal torus

$$T = C_{2^{a_1}-\epsilon_1} \times C_{2^{a_2}-\epsilon_2} \times \ldots \times C_{2^{a_t}-\epsilon_t} < \mathrm{O}^{\epsilon_1}_{2a_1}(2) \times \mathrm{O}^{\epsilon_2}_{2a_2}(2) \times \ldots \times \mathrm{O}^{\epsilon_t}_{2a_t}(2) \leq \mathrm{O}^\epsilon_{2n}(2),$$

where $a_i \in \mathbb{Z}_{\geq 1}$, $n = \sum_{i=1}^t a_i$, $\epsilon_i = \pm$, and $\epsilon = \prod_{i=1}^t \epsilon_i$. Correspondingly, we can decompose

$$V = V_1 \otimes V_2 \otimes \ldots \otimes V_t,\ E = E_1 \circ E_2 \circ \ldots \circ E_t,$$

where $V_i = \mathbb{C}^{2^{a_i}}$, $E_i \cong 2^{1+2a_i}_{\epsilon_i}$, and $\mathbf{N}_{\mathrm{Sp}^{\epsilon_i}(V_i)}(E_i) \cong E_i \cdot \mathrm{O}^{\epsilon_i}_{2a_i}(2)$, and then put h in

$$\mathbf{N}_{\mathrm{Sp}^{\epsilon_1}(V_1)}(E_1) \otimes \mathbf{N}_{\mathrm{Sp}^{\epsilon_2}(V_2)}(E_2) \otimes \ldots \otimes \mathbf{N}_{\mathrm{Sp}^{\epsilon_t}(V_t)}(E_t).$$

By [KT8, Lemma 4.3], a generator of the maximal torus $C_{2^{a_i}-\epsilon_i}$ of $\mathbf{N}_{\mathrm{Sp}^{\epsilon_i}(V_i)}(E_i)/E_i \cong \mathrm{O}^{\epsilon_i}_{2a_i}(2)$ can be lifted to an element s_i of $\mathrm{Sp}^{\epsilon_i}(V_i)$ that has order $2^{a_i} - \epsilon_i$ and spectrum $\mu_{2^{a_i}+1} \smallsetminus \{1\}$ when $\epsilon_i = -$, and $\mu_{2^{a_i}-1}$ with 1 occurring twice when $\epsilon_i = +$. Thus the $2'$-element h is contained in $E \cdot T = E\langle s_1, s_2, \ldots, s_t\rangle$, with $s_1, \ldots, s_t$ centralizing each other. Applying the Schur-Zassenhaus theorem to $E \cdot T$ with normal Hall subgroup E, we may assume that $h \in \langle s_1, s_2, \ldots, s_t\rangle$.

Recall that g is an ss-element on V, whence so is h. Now, if $\epsilon_i = +$ for some i, then s_i has eigenvalue 1 with multiplicity 2 on V_i, precluding h from being an ss-element. Thus $\epsilon_i = -$ for all i. Now, the order of h is at most $L := \mathrm{lcm}(2^{a_1}+1, 2^{a_2}+1, \ldots, 2^{a_t}+1)$. Denoting by $b_1 > b_2 > \ldots > b_{t'}$ the distinct values among $a_1, a_2, \ldots, a_t$ we have

$$L = \mathrm{lcm}(2^{b_1}+1, 2^{b_2}+1, \ldots, 2^{b_{t'}}+1) < 2^{b_1+b_2+\ldots+b_{t'}+1} \tag{8.5.2}$$

by [GMPS, Lemma 2.9] (with strict inequality because $2 \nmid L$). Now, if some a_j is repeated, then $\sum_{i=1}^{t'} b_i \leq n - a_j \leq n-1$, and so (8.5.2) implies $\mathsf{o}(h) \leq L < 2^n$, whence $\bar{\mathsf{o}}(g) < 2^n$, contradicting (8.5.1). Thus $a_1 > a_2 > \ldots > a_t \geq 1$. If $\gcd(2^{a_i}+1, 2^{a_j}+1) > 1$ for some $i \neq j$, then by [LMT, Lemma 4.1(iii)] we have $\mathsf{o}(h) \leq L \leq \prod_{i=1}^t (2^{a_i}+1)/3 < (2.4)2^n/3 < 2^n$, again a contradiction. We also achieve the same contradiction, if h does not generate $\langle s_1, s_2, \ldots, s_t\rangle$, which is now a cyclic group of order $\prod_{i=1}^t (2^{a_i}+1)$. Hence $\bar{\mathsf{o}}(g) = \prod_{i=1}^t (2^{a_i}+1)$, as stated. □

As one can see, the results in this section leave out the case (i) of Theorem 6.6, where the almost quasisimple group G has $S = \mathrm{PSL}_2(q)$ as its unique non-abelian composition factor. In this case, many complex representations of G, particularly the ones irreducible and nontrivial on $L = G^{(\infty)}$, have dimension $\leq q+1$ always admit ss-elements. On the other hand, if q is not small, say $q \geq 27$, then any hypergeometric sheaf $\mathcal{H}$ admitting G as its (finite) geometric monodromy group, must be in characteristic p dividing q. As a direct application of Theorem 5.1 and results of [KT5], we show that all nontrivial irreducible representations of $\mathrm{GL}_2(q)$ do lead to hypergeometric sheaves.

Theorem 8.6. *Let $q = p^f \geq 4$ be a power of a prime p. Then the following statements hold.*

(i) *Let Φ be any irreducible $\overline{\mathbb{Q}_\ell}$-representation of $G = \mathrm{GL}_2(q)$ of degree > 1. Then, there exists a hypergeometric sheaf $\mathcal{H}$ over $\overline{\mathbb{F}_p}$ that has $G/\mathrm{Ker}(\Phi)$ as its geometric monodromy group.*

(ii) *Let Θ be any irreducible $\overline{\mathbb{Q}_\ell}$-representation of $H = \mathrm{GU}_2(q)$ of degree > 1 that is trivial at $\mathbf{O}_{2'}(\mathbf{Z}(H))$. Then, there exists a hypergeometric sheaf $\mathcal{H}$ over $\overline{\mathbb{F}_p}$ that has $H/\mathrm{Ker}(\Theta)$ as its geometric monodromy group.*

Proof. (i) We use the character table of G as given in [DM, Table 1, p. 155]. In particular, if $T \cong \mu_{q-1} \times \mu_{q-1}$ denotes a diagonal maximal torus of G, then the irreducible representations of G of degree $q+1$ are $R^G_T(\alpha, \beta)$ which are Harish-Chandra induced from $\alpha \boxtimes \beta : T \to \overline{\mathbb{Q}_\ell}^\times$, where α, β are distinct characters of μ_{q-1}. The nontrivial irreducible components of the total Weil representation of G considered in [KT5] are precisely $R^G_T(\alpha, \mathbb{1}) - \delta_{\alpha,\mathbb{1}} 1_G$ with $\alpha \in \mathrm{Irr}(\mu_{q-1})$. Now we pick $\alpha \in \mathrm{Irr}(\mu_{q-1})$ to be faithful. By [KT5, Corollary 8.2], there exists a hypergeometric sheaf

$\mathcal{H}_\alpha$ over $\overline{\mathbb{F}_p}$ that has G as its geometric monodromy group, acting on $\mathcal{H}_\alpha$ via a representation Ψ_α with character $R_T^G(\alpha, \mathbb{1})$. Inspecting the character table of G, we see that

$$\mathrm{Trace}(\Phi(g)) - \mathrm{Trace}(\Psi_\alpha(g)) = \mathrm{Trace}(\Phi(1)) - \mathrm{Trace}(\Psi_\alpha(1))$$

for all p-elements $g \in G$. Hence the statement follows from Theorem 5.1.

(ii) As in (i), we appeal to [KT5, Corollary 8.2] to get a hypergeometric sheaf $\mathcal{H}$ of rank q with geometric monodromy group $\mathrm{PGL}_2(q) \cong \mathrm{PGU}_2(q)$, which utilizes a surjection $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow \mathrm{PGU}_2(q)$ together with a representation $\Phi : \mathrm{PGU}_2(q) \to \mathrm{GL}(\mathcal{H})$. View $\mathrm{PGU}_2(q)$ as H/Z, where $Z := \mathbf{Z}(H) \cong C_{q+1}$, and decompose $Z = Z_1 \times Z_2$, where $Z_1 := \mathbf{O}_{2'}(Z)$ and $Z_2 := \mathbf{O}_2(Z)$. Then observe that $H = Z_1 H^\circ$ and $H^\circ \cap Z = Z_2$, where

$$H^\circ := \{X \in \mathrm{GU}_2(q) \mid \det(X) \in \mathbf{O}_2(\mu_{q+1})\}.$$

It follows that $\mathrm{PGU}_2(q) \cong ZH^\circ/Z \cong H^\circ/Z_2$. Now, the obstruction to lifting ϕ to a homomorphism $\varphi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \to H^\circ$ lies in the group $H^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, Z_2) = 0$, the vanishing because open curves have cohomological dimension ≤ 1, cf. [SGA4t3, Cor. 2.7, Exp. IX and Thm. 5.1, Exp. X]. We claim that φ is surjective. [Indeed, for the image J of φ we have $Z_2 J = H^\circ$, and so

$$J \geq [J, J] = [Z_2 J, Z_2 J] = [H^\circ, H^\circ] = \mathrm{SU}_2(q).$$

Hence, if det maps J onto $\mathbf{O}_2(\mu_{q+1}) =: C_{2^a}$, then $J = H^\circ$ as claimed. Otherwise we have $\det(J) \cong C_{2^b}$ with $0 \leq b \leq a-1$; in particular, q is odd. In this case, one can check that

$$J \cap Z_2 = \left\{ \mathrm{diag}(x, x) \mid x \in \mu_{q+1},\ x^{2^{b+1}} = 1 \right\} \cong C_{2^{b+1}},$$

and so

$$|\mathrm{PGU}_2(q)| = |H^\circ/Z_2| = |JZ_2/Z_2| = |J|/|J \cap Z_2| = 2^b |\mathrm{SU}_2(q)|/2^{b+1} = |\mathrm{PGU}_2(q)|/2,$$

a contradiction.]

Now, consider any irreducible representation Θ of H that is trivial on Z_1. Then we can view Θ as a representation of $H/Z_1 = Z_1 H^\circ/Z_1 \cong H^\circ$, and inflate Φ to a representation of H°. Checking the well-known character table of $\mathrm{GU}_2(q)$, we see that

$$\mathrm{Trace}(\Phi(g)) - \mathrm{Trace}(\Theta(g)) = \mathrm{Trace}(\Phi(1)) - \mathrm{Trace}(\Theta(1))$$

for all p-elements $g \in \mathrm{GU}_2(q)$. Hence the statement follows from Theorem 5.1. $\square$

9. (Non-)existence theorems

In this section we will prove various theorems that rule out the existence of (irreducible) hypergeometric sheaves of type (D, m) with $D > m$ and certain kind of finite monodromy groups $G = G_{\mathrm{geom}}$. For a hypergeometric sheaf $\mathcal{H}$ in question, we will denote by Q the image of $P(\infty)$ on $\mathcal{H}$; note that $Q \neq 1$ as $\mathcal{H}$ is not tame at ∞. We also use the fact that, since $\mathcal{H}$ is tame at 0, $P(0)$ acts trivially on $\mathcal{H}$ and a generator g_0 of the image of the p'-group $I(0)/P(0)$ has simple spectrum on $\mathcal{H}$, if $p = \mathrm{char}(\mathcal{H})$. Furthermore, if $D > 1$, then p divides $|G/\mathbf{Z}(G)|$ by Proposition 4.4(iii).

9A. Alternating groups. First, we rule out the cases (c)–(f) of Theorem 6.2(ii) for hypergeometric sheaves.

Lemma 9.1. *There does not exist any hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D > m$ and $D = 16$ or 32 that has finite geometric monodromy group G such that $G/\mathbf{Z}(G) \cong \mathsf{S}_9$, S_{10}, A_{11}, or S_{12} as listed in Theorem 6.2(ii).*

Proof. Assume the contrary, and let p denote the characteristic of such a sheaf $\mathcal{H}$, and φ denote the character of G acting on $\mathcal{H}$. As mentioned above, a generator g_0 of the image of $I(0)/P(0)$ has simple spectrum on $\mathcal{H}$.

(i) Consider the case of Theorem 6.2(ii)(f), i.e. $D = 32$ and $\bar{\mathsf{o}}(g_0) = 60$. As g_0 is a p'-element, $p > 5$. Now, by Proposition 4.4, $Q \cap \mathbf{Z}(G) = 1$, whence Q embeds in $G/\mathbf{Z}(G) \cong \mathsf{S}_{12}$. It follows that $p = 7$ or 11, and $Q \cong C_p$. By checking the character table of $2\mathsf{S}_{12}$ as given in [GAP], we see that the spectrum of a generator g of Q on $\mathcal{H}$ consists of all p^{th} roots of unity, each with multiplicity at least 4 if $p = 7$ and at least 2 if $p = 11$. On the other hand, the action of g on the wild part Wild yields a (nontrivial) eigenvalue of g with multiplicity 1, a contradiction.

(ii) Now we consider the cases (c)–(e) of Theorem 6.2(ii), i.e. $D = 16$ and $\bar{\mathsf{o}}(g_0) = 20$ or 30, whence $p \neq 2, 5$. Again by Proposition 4.4, $Q \cap \mathbf{Z}(G) = 1$, whence Q embeds in $G/\mathbf{Z}(G) \cong \mathsf{S}_9$, S_{10}, or A_{11}. In all cases, $G/\mathbf{Z}(G)$ embeds in $\mathrm{GL}_{10}(2)$, hence

$$W = \dim \mathsf{Wild} \leq 10 \tag{9.1.1}$$

by Theorem 4.8. Inspecting the character table of $2\mathsf{S}_9$, $2\mathsf{S}_{10}$, and $2\mathsf{A}_{11}$ as given in [GAP], we see that $|\varphi(g)| \leq 8$, whence

$$W \geq 6 \tag{9.1.2}$$

by (7.2.2). It follows from Proposition 4.4(iv), $p \nmid |\mathbf{Z}(G)|$. In turn, this implies that $Q \leq G^{(\infty)} = 2\mathsf{A}_9$, $2\mathsf{A}_{10}$, or $2\mathsf{A}_{11}$. Now, if Q contains an element g of order 3 that projects onto a 3-cycle, then $\varphi(g) = -8$ and g has no eigenvalue 1 on $\mathcal{H}$, whence $W = 16$, contradicting (9.1.1). In all other cases, we have $|\varphi(x)| \leq 4$ for $1 \neq x \in Q$. If moreover $|Q| \geq 7$, then using (7.2.2) we obtain $W \geq 16 \cdot (3/4) \cdot (6/7) > 10$, again contradicting (9.1.1). As $p \neq 2, 5$ and $Q \neq 1$, we conclude that $p = 3$ and $|Q| = 3$. But then Q has at most 2 nontrivial irreducible characters, all of degree 1, and this contradicts (9.1.2). □

We now give a result due to Sawin.

Lemma 9.2. (Sawin) *Given positive integers A, B with $\gcd(A, B) = 1$, and $C := A + B$ consider the polynomial*

$$f(x) := x^A(1-x)^B,$$

viewed as a map from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to $\mathbb{G}_m$. Then we have the following results.

(i) *Let p be a prime with $p|C$. Write $C = C_0p^e$ with C_0 prime to p. Then in characteristic p, we have, for any $\ell \neq p$, and any nontrivial additive character ψ of $\mathbb{F}_p$, the sheaf*

$$f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$$

is geometrically isomorphic to a multiplicative translate of the hypergeometric sheaf

$$\mathcal{H}yp_\psi(\mathsf{Char}(A) \sqcup \mathsf{Char}(B) \setminus \{\mathbb{1}\}; \mathsf{Char}(C_0) \setminus \{\mathbb{1}\}).$$

(ii) *Let p be a prime with $p|A$. Write $A = A_0p^e$ with A_0 prime to p. Then in characteristic p, we have, for any $\ell \neq p$, and any nontrivial additive character ψ of $\mathbb{F}_p$, the sheaf*

$$(1/f)_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$$

is geometrically isomorphic to a multiplicative translate of the hypergeometric sheaf

$$\mathcal{H}yp_\psi(\mathsf{Char}(C) \setminus \{\mathbb{1}\}; \mathsf{Char}(A_0) \sqcup \mathsf{Char}(B) \setminus \{\mathbb{1}\}.$$

Proof. In either of the situations (i) or (ii), we work in the specified characteristic p. Both f and $1/f$ are finite etale maps from $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to $\mathbb{G}_m$, cf. [KRLT3, proof of 1.2]. The constant sheaf $\overline{\mathbb{Q}_\ell}$ has Euler characteristic -1 on $\mathbb{P}^1 \setminus \{0,1,\infty\}$, hence $f_\star\overline{\mathbb{Q}_\ell}$ and $(1/f)_\star\overline{\mathbb{Q}_\ell}$ are lisse sheaves on $\mathbb{G}_m$ with Euler characteristic -1. Each is pure of weight zero, so is geometrically semisimple. By [Ka-ESDE, 8.5.2 and 8.5.3], each of these direct images is the direct sum of a single irreducible hypergeometric sheaf $\mathcal{H}$ with some Kummer sheaves $\mathcal{L}_\chi$. We detect the $\mathcal{H}$ by listing the characters which occur in $f_\star\overline{\mathbb{Q}_\ell}$ and $(1/f)_\star\overline{\mathbb{Q}_\ell}$ respectively at 0 and at ∞, and cancelling those which appear at both 0 and ∞, cf. [Ka-ESDE, 9.3.1]. Because $\gcd(A,B)=1$, the only character to cancel is $\mathbb{1}$, hence the assertion that the $\mathcal{H}$, namely $f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$ or $(1/f)_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$, has the asserted "upstairs" and "downstairs" characters. By [Ka-ESDE, 8.5.5], this local monodromy data at 0 and ∞ determines $\mathcal{H}$ up to multiplicative translation. □

Theorem 9.3. *Let $n \geq 5$. Then the cases listed in Theorem 6.2(i)(a) give rise to hypergeometric sheaves. More precisely,*

(i) *For any prime $p \leq n-3$ with $p \nmid n$, there exists a hypergeometric sheaf $\mathcal{H}$ over $\mathbb{G}_m/\overline{\mathbb{F}_p}$, with $G_{\rm geom} = \mathsf{A}_n$ if $2 \nmid n$ and $G_{\rm geom} = \mathsf{S}_n$ if $2|n$, and with the image of $I(0)$ generated by an n-cycle.*

(ii) *Suppose that $1 \leq k \leq n/2$ is coprime to n, and p is any prime dividing n. If $k=1$, suppose in addition that n is not a p-power, and that $p=3$ if $n=6$ or 24, and $p=2$ if $n=12$. Then there exists a hypergeometric sheaf $\mathcal{H}$ over $\mathbb{G}_m/\overline{\mathbb{F}_p}$, with $G_{\rm geom} = \mathsf{A}_n$ if $2|n$ and $G_{\rm geom} = \mathsf{S}_n$ if $2 \nmid n$, and with the image of $I(0)$ generated by the disjoint product of an $(n-k)$-cycle and a k-cycle.*

Proof. (i) By Sawin's Lemma 9.2(ii), by considering $f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$ with $f(x) = x^{-p}(x-1)^{p-n}$ in characteristic p, we get

$$\mathcal{H} = \mathcal{H}yp(\mathsf{Char}_n \smallsetminus \{\mathbb{1}\}; \mathsf{Char}_{n-p}),$$

with $G_{\rm geom} \leq \mathsf{S}_n$ acting (irreducibly) via the restriction of the deleted natural permutation module of S_n. This irreducibility implies that $G_{\rm geom}$ is a doubly transitive subgroup of S_n, in particular a primitive subgroup. The wild part Wild has dimension $p-1$, whence by Proposition 7.3 the image Q of $P(\infty)$ is of order p. Now a generator $g \in \mathsf{S}_n$ of Q has order p, and it acts trivially on the tame part of dimension $n-p$, and thus it is a p-cycle. By Jordan's theorem [J], $G_{\rm geom} = \mathsf{A}_n$ or S_n. Since a generator of the image of $I(0)$ has its spectrum on $\mathcal{H}$ consisting of all nontrivial $n^{\rm th}$ roots of unity, it must act as an n-cycle. Applying Theorem 4.1, we conclude that $G_{\rm geom} = \mathsf{A}_n$ if $2 \nmid n$ and $G_{\rm geom} = \mathsf{S}_n$ if $2|n$.

(ii) Now we choose any prime $p|n$, and again follow Lemma 9.2(i) to consider $f_\star\overline{\mathbb{Q}_\ell}/\overline{\mathbb{Q}_\ell}$ with $f(x) = x^k(x-1)^{n-k}$ in characteristic p, to get

$$\mathcal{H} := \mathcal{H}yp(\mathsf{Char}_k \smallsetminus \{\mathbb{1}\} \sqcup \mathsf{Char}_{n-k}; \mathsf{Char}_{n_0} \smallsetminus \{\mathbb{1}\}), \tag{9.3.1}$$

where n_0 is the p'-part of n (also see [KRLT3, Proposition 1.2(ii)]). As in (i), $G_{\rm geom} \leq \mathsf{S}_n$ is a doubly transitive subgroup. But now a generator g_0 of the image of $I(0)$ has its (simple) spectrum on $\mathcal{H}$ consisting of all $(n-k)^{\rm th}$ and all $k^{\rm th}$ roots of unity, hence it must act as a product of an $(n-k)$-cycle and a k-cycle. We note that $g_0 \in \mathsf{A}_n$ if and only if $2|n$. Hence, using Theorem 4.1 and assuming $G_{\rm geom} \geq \mathsf{A}_n$, we can say that $G_{\rm geom} = \mathsf{S}_n$ is $2 \nmid n$ and $G_{\rm geom} = \mathsf{A}_n$ if $2|n$.

Since $\gcd(k, n-k) = 1$, g_0^{n-k} is a k-cycle. Suppose in addition that $2 \leq k \leq n/8$. As g_0^{n-k} fixes $n-k$ points, we have that $G_{\rm geom} \geq \mathsf{A}_n$ by Bochert's theorem [Bo]; in fact, the same is true by Manning's theorem [Man] if $k \leq n/3 - 2\sqrt{n/3}$. The same is true by Jordan's theorem if k is a prime. Note that, up until this point of this proof, we have not used the Classification of Finite Simple Groups.

Suppose now that $n/8 < k \leq n/2$ and k is not a prime. As the element g_0^{n-k} of order k fixes $n-k \geq n/2$ points, we can quote either [GM, Theorem 1] or [Jo, Theorem 1.2], which both use the Classification, to conclude that $G_{\rm geom} \geq \mathsf{A}_n$.

Finally, assume that $k = 1$, in which case g_0 is an $(n-1)$-cycle. If n is not a prime power (equivalently, n is not a p-power since $p|n$) and $n-1$ is not a prime, then $G_{\rm geom} \geq \mathsf{A}_n$ by [Jo, Theorem 1.2].

We note that when $n = p^a$, the Kloosterman sheaf $\mathcal{H}$ is Kummer induced. Consider the case $n = r+1$ for a prime $r \geq 5$ and assume that $G_{\rm geom} \not\geq \mathsf{A}_n$. Suppose $(n,p) = (6,3)$. Then $\dim \mathsf{Wild} = 4$ and so $|Q| = 3^2$ by Proposition 7.3, but $\mathrm{rank}(\mathcal{H}) = 5$ divides $|G_{\rm geom}|$, and this forces $G_{\rm geom} \leq \mathsf{S}_6$ to contain A_6 by [Atlas]. Suppose $(n,p) = (12,2)$. By [Jo, Theorem 1.2], $G_{\rm geom} \in \{M_{11}, M_{12}, \mathrm{PSL}_2(11), \mathrm{PGL}_2(11)\}$. As $\dim \mathsf{Wild} = 9$, $G_{\rm geom}$ must contain an element of order divisible by 9 (namely a generator for the tame quotient $I(\infty)/P(\infty)$) by Proposition 7.3(ii), which is impossible in all the four listed groups. Next suppose that $(n,p) = (24,3)$. By [Jo, Theorem 1.2], $G_{\rm geom} \in \{M_{24}, \mathrm{PSL}_2(23), \mathrm{PGL}_2(23)\}$. Now $\dim \mathsf{Wild} = 16$, whence $G_{\rm geom}$ must contain an element of order divisible by 16, which is again impossible in all the three listed groups. Assume now that $r \neq 5, 11, 23$ and $n \neq p^a$. Then by [Jo, Theorem 1.2] we have $\mathrm{PSL}_2(r) \leq G_{\rm geom} \leq \mathrm{PGL}_2(r)$. This last possibility is ruled out by Theorem 7.4, which implies that $p = \mathrm{char}(\mathcal{H})$ must have been equal to r and so coprime to n. □

Remark 9.4. Let us comment on $G_{\rm geom}$ of Sawin's sheaf $\mathcal{H}$ of (9.3.1) in the exceptional cases $(n,p) = (6,2)$, $(12,3)$, and $(24,2)$ of Theorem 9.3(ii) when $k = 1$. If $(n,p) = (12,3)$, then $\mathcal{H}$ is recorded in Table 3 and it is shown in Lemma 9.5 that $G_{\rm geom} = M_{11}$. If $(n,p) = (24,2)$, then $\mathcal{H}$ is recorded in Table 3 and we show that $G_{\rm geom} = M_{24}$. Finally, let $(n,p) = (6,2)$. Then [KT8, Corollary 8.2] yields a hypergeometric sheaf $\mathcal{H}_1$ of type $(4,1)$ in characteristic 2 with $G_{\rm geom} = \mathrm{PGL}_2(4) \cong \mathsf{A}_5$, and with C_5 as the image of $I(0)$. Applying Theorem 5.1 to the two irreducible representations of degree 4 and 5 of A_5, we get a hypergeometric sheaf $\mathcal{H}_2$ of type $(5,2)$ in characteristic 2 with $G_{\rm geom} = \mathsf{A}_5$, and with C_5 as the image of $I(0)$; in particular, the set of "upstairs" characters of $\mathcal{H}_2$ is Char_5. A $2'$-generator $g \in \mathsf{A}_5$ of $I(\infty)/P(\infty)$ has order divisible by $3 = \dim \mathsf{Wild}$, hence $\mathsf{o}(g) = 3$, and the set of "downstairs" characters of $\mathcal{H}_2$ is $\mathsf{Char}_3^\times$. Thus $\mathcal{H} = \mathcal{H}_2$ has $G_{\rm geom} = \mathsf{A}_5$.

9B. Sporadic groups.

Lemma 9.5. *The first three lines of Table 3 give hypergeometric sheaves over $\mathbb{G}_m/\overline{\mathbb{F}_3}$, each with finite geometric monodromy group $G_{\rm geom} = \mathrm{M}_{11}$.*

Proof. We start with the rank 11 sheaf $\mathcal{H}_1$. This can be obtained as a Sawin's sheaf with $(n,k) = (12,11)$; in fact, as Sawin [Sa] kindly explained to us, it follows from previous results of Adler and Abhyankar that this sheaf has $G_{\rm geom} = G \cong \mathrm{M}_{11}$, with the image of $I(0)$ in G being $\langle g_0 \rangle \cong C_{11}$. As $p = 3$ and $\dim \mathsf{Wild} = 8$, we see that the image Q of $P(\infty)$ in G is $Q \cong C_3^2$. Now, the image J of $I(\infty)$ in G permutes cyclically the 8 linear characters of Q on Wild, and checking the character table of G [GAP], we see that $J \cong C_3^2 \rtimes C_8$ (as listed in Table 3).

Let $\Phi_1 : G \to \mathrm{GL}(\mathcal{H}_1)$ denote the representation of G on $\mathcal{H}_1$. Let $\Phi_2 : G \to \mathrm{GL}_{10}(\overline{\mathbb{Q}_\ell})$ and $\Phi_3 : G \to \mathrm{GL}_{10}(\overline{\mathbb{Q}_\ell})$ denote irreducible representations of G that afford a rational, respectively non-real, character of degree 10. Using [GAP] we can check that $\mathrm{Trace}(\Phi_i(g)) - \mathrm{Trace}(\Phi_1(g)) = -1$ for all 3-elements $g \in G$ and $i = 2,3$. It follows from Theorem 5.1 that Φ_i, $i = 2,3$, gives rise to a hypergeometric sheaf $\mathcal{H}_i$ over $\mathbb{G}_m/\overline{\mathbb{F}_3}$ with G as its geometric monodromy group. The "upstairs" and "downstairs" characters of $\mathcal{H}_i$ can be seen by inspecting the spectra of g_0 and an element of order 8 in J in Φ_i, which are precisely those listed in Table 3. We also note that $\mathcal{H}_2$ is Sawin-like, with $(n,k) = (11,9)$, see Lemma 9.2(ii). □

Lemma 9.6. *There does not exist any hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $12 \geq D > m$, that has finite geometric monodromy group G such that $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for $S \cong \mathrm{M}_{12}$.*

Proof. Assume the contrary, and let p and φ denote the characteristic of such a sheaf $\mathcal{H}$ and the character of G acting on $\mathcal{H}$. Then a generator g_0 of the image of $I(0)/P(0)$ has simple spectrum on $\mathcal{H}$, and we can apply Theorem 6.4 to arrive at one of the following cases.

Case 1: $D = 12$, $\bar{\mathsf{o}}(g_0) = 24$, and $G/\mathbf{Z}(G) = S \cdot 2$.

As g_0 is a p'-element, $p \geq 5$, whence $p = 5$ or 11, and moreover $Q/(Q \cap \mathbf{Z}(G))$ embeds in a Sylow p-subgroup which is cyclic of order p. Thus Q is abelian and $Q/(Q \cap \mathbf{Z}(G)) \cong C_p$. Next, observe that $L := G^{(\infty)}$ is a quasisimple cover of S acting on $\mathcal{H}$ of rank 12, whence $L = 2S$ and $x \in \mathbf{Z}(G)L$ for any $x \in Q \smallsetminus \mathbf{Z}(G)$. Checking the character table of L as given in [GAP], we see that $|\varphi(x)|/\varphi(1) \leq 1/6$, and so $W = \dim \mathsf{Wild} \geq 8$ by (7.2.2). As Q is abelian, we see that Q admits at least 8 distinct linear characters on W. This is impossible when $p = 5$, since, with the action of $Q \cap \mathbf{Z}(G)$ fixed on $\mathcal{H}$, Q can have at most $|Q/(Q \cap \mathbf{Z}(G))| = p$ linear characters lying above it. Thus $p = 11$, whence $|\varphi(x)|/\varphi(1) \leq 1/12$ by [GAP] and $W \geq 10$ by (7.2.2). In particular, $p \nmid |\mathbf{Z}(G)|$ by Proposition 4.4(iv), and so $Q \cong C_{11}$ and $W = 10$. Since $\bar{\mathsf{o}}(g_0) = 24$ and $G/\mathbf{Z}(G) = S \cdot 2$, by inspecting the spectrum of such an element on $\mathcal{H}$, we see that the "upstairs" characters of $\mathcal{H}$ must be $\mathsf{Char}_{12}\chi$ for a fixed χ. Next, a p'-element g in the image of $I(\infty)$ permutes cyclically the 10 characters of Q on Wild, hence g is a scalar multiple of an element of class $10B$ or $10C$ of $2S \cdot 2$ in [GAP]. Checking the spectrum of g, we see that the "downstairs" characters of $\mathcal{H}$ must be $\mathsf{Char}_{12}\rho$ for a fixed ρ. Thus $\mathcal{H}$ is stable under multiplication by χ_2, and so it is induced from a sheaf of rank 6. But this is also impossible, since this does not hold for the 12-dimensional representations of $2S.2$.

Case 2: $D = 10$.

Checking the character table of quasisimple covers of S, we see that $L := G^{(\infty)} = 2S$. First we consider the case $p \neq 2$. Then $|\varphi(x)|/\varphi(1) \leq 1/5$ for all $x \in Q \smallsetminus \mathbf{Z}(G)$ and $|Q| \geq 3$, whence $W \geq 6$ by (7.2.2). If $p = 5$, then $Q/(Q \cap \mathbf{Z}(G)) \cong C_5$, and so Q is abelian and cannot have 6 distinct linear characters of Wild, a contradiction. Hence $p = 3$ or 11, and $p \nmid D$. This in turn implies by Proposition 4.4(iv) that $p \nmid |\mathbf{Z}(G)|$. As $p > 2$, we have $Q \leq \mathbf{Z}(G)L$, and so in fact $Q \leq L$. Now observe that $\varphi(y) - \psi(y) = -2$ for all p-elements $y \in L$, if $\psi \in \mathrm{Irr}(L)$ has degree 12. If $G/\mathbf{Z}(G) = S$, then $G = \mathbf{Z}(G)L$ and ψ extends to G. In the remaining case we have $G = \langle \mathbf{Z}(G)L, y \rangle$ where $y^2 \in \mathbf{Z}(G)L$. As y centralizes $\mathbf{Z}(G)$, and ψ extends to $L \cdot 2$, we have that y fixes an extension of ψ to $\mathbf{Z}(G)L$, and so this extension extends to G. Thus in all cases ψ extends to a character $\tilde{\psi}$ of G of degree 12, and $\varphi(y) - \tilde{\psi}(y) = -2$ for all p-elements $y \in G$. This implies by Theorem 5.1 that there exists a rank 12 hypergeometric sheaf realizing G in a representation with character $\tilde{\psi}$, contrary to the result of Case 1.

We have shown that $p = 2$. Since we still have $|\varphi(x)|/\varphi(1) \leq 1/5$ for all $x \in Q \smallsetminus \mathbf{Z}(G)$, $W \geq 4$ by (7.2.2) using $|Q| \geq 2$. This in turn implies that $|Q| \geq 8$, and so in fact $W \geq 7$. If $2 \nmid W$, then a p'-element in the image of $I(\infty)$ permutes W linear characters of Q on Wild and so $\bar{\mathsf{o}}(g)$ is divisible by W. But this is impossible, since $L.2$ does not possess any element of such order modulo $\mathbf{Z}(G)$. Suppose $W = 8$, whence Q acts irreducibly on Wild. Now if $1 \neq z \in \mathbf{Z}(Q)$, then z acts as 1 on Tame and as a scalar on Wild, whence $|\varphi(z)| \geq 8 - 2 = 6$. Checking the character table of $L.2$, we see that $|\varphi(z)| = 10$, and so z acts trivially on Wild and on $\mathcal{H}$, contrary to $z \neq 1$. Thus $W = 10$. In this case, $\mathcal{H}$ is Kloosterman, and the $2'$-element g_0 has order ≥ 10 modulo $\mathbf{Z}(G)$. It follows that $\bar{\mathsf{o}}(g_0) = 11$ and the "upstairs" characters of $\mathcal{H}$ should be $\mathsf{Char}^{\times}_{11}\chi$ for a fixed character χ. Presumably this case should however lead to $\mathrm{SU}_5(2)$ by [KT11].

Case 3: $D = 11$ and $\bar{\mathsf{o}}(g_0) = 11$.

In this case we have $|\varphi(x)|/\varphi(1) \leq 3/11$ for all $x \in Q \smallsetminus \mathbf{Z}(G)$, and so $W \geq 4$ by (7.2.2). Also, checking the representations of quasisimple covers of S, we see that $G^{(\infty)} = S$, and moreover $G = \mathbf{Z}(G) \times S$, as the two 11-dimensional irreducible representations of S are fused by outer automorphisms of S. First we consider the case $p = 11$. Then $Q/(Q \cap \mathbf{Z}(G)) \cong C_p$ and so Q is abelian. This in turn implies that $W \neq 11$ (as otherwise Q is irreducible on Wild of dimension p), whence $Q \cap \mathbf{Z}(G) = 1$ and $Q \cong C_p$ by Proposition 4.4(i). As $\varphi(x) = 0$ for all $x \in Q \smallsetminus \mathbf{Z}(G)$, we now have $W \geq 10$ by (7.2.2), and so in fact $W = 10$. Thus some p'-element g of $\mathbf{Z}(G) \times S$ permutes cyclically the 10 distinct characters of $Q \cong C_{11}$ on Wild. We will write a generator of Q as zh, with $z \in \mathbf{Z}(G)$ and $h \in S$ of order 11. As g normalizes Q, we have $z^i h^i = g(zh)g^{-1} = z(ghg^{-1})$, implying $z^i = z$ and $ghg^{-1} = h^i$. Checking the latter relation in S, we see that g can permute cyclically only 5 eigenspaces for h, and so for zh as well, a contradiction.

We have shown that $p \neq 11$. Then $p \nmid |\mathbf{Z}(G)|$ by Proposition 4.4(iv), and so we may assume $G = S$ by Corollary 5.2. In the cases $p = 3, 5$, we can further lift the surjection $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow S$ to a surjection $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow 2S$ and then consider an irreducible character ψ of $2S$ of degree 12. After inflating φ to a character of $2S$, we get that $\varphi(y) - \psi(y) = -1$ for all p-elements $g \in 2S$. But this leads by Theorem 5.1 to a hypergeometric sheaf of rank 12 realizing $2S$, contradicting the result of Case 1. Thus $p = 2$. As $W \geq 4$, we must have $|Q| \geq 8$. Another application of 7.2.2 now shows that $W \geq 7$. As in Case 2, we can rule out $W = 7, 9$ as G has no elements of order 7 and 9. Likewise, the case $W = 8$ would lead to an element $1 \neq z \in \mathbf{Z}(Q)$ acting as a scalar on Wild and 1 on Tame, whence $|\varphi(z)| \geq 8 - 3 = 5$, which is impossible by [GAP]. If $W = 11$, then, as $\mathbb{F}_2(\zeta_{11}) = \mathbb{F}_{2^10}$, we must have $|Q| = 2^{10}$, too big for a subgroup of S. Thus $W = 10$, and the "upstairs" characters of the sheaf $\mathcal{H}$ is now Char_{11}. Now a $2'$-element g in the image of $I(\infty)$ cyclically permutes the 5 summands of $P(\infty)$ acting on Wild. Since $g \in S$, we see that g has order 5. Checking the spectrum of g, we get that the "downstairs" character of $\mathcal{H}$ is $\mathbb{1}$, which also occurs upstairs, violating the irreducibility of $\mathcal{H}$. $\square$

Lemma 9.7. *There does not exist any hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D = 22 > m$, that has finite geometric monodromy group G such that $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for $S \cong \mathrm{HS}$.*

Proof. Assume the contrary, and let p and φ denote the characteristic of such a sheaf $\mathcal{H}$ and the character of G acting on $\mathcal{H}$. Then a generator g_0 of the image of $I(0)/P(0)$ has simple spectrum on $\mathcal{H}$, and so by Theorem 6.4, $\bar{\mathsf{o}}(g_0) = 30$. As g_0 is a p'-element, $p \geq 7$. On the other hand, p divides $|\mathrm{Aut}(S)| = 2|S|$, whence $p = 7$ or 11, and moreover $Q/(Q \cap \mathbf{Z}(G))$ embeds in a Sylow p-subgroup which is cyclic of order p. Thus Q is abelian and $Q/(Q \cap \mathbf{Z}(G)) \cong C_p$. Next, observe that $G^{(\infty)}$ is a quasisimple cover of S. Since the Schur multiplier of S is C_2 and $2S$ cannot act faithfully on any space of dimension < 56, see [GAP], $G^{(\infty)} \cong S$, and we will identify it with S. Moreover, $[G : \mathbf{Z}(G)S] \leq 2$. Now, for any $1 \neq x \in Q$, x belongs to $\mathbf{Z}(G)S$. Checking the character table of S as given in [GAP], we see that $|\varphi(x)|/\varphi(1) \leq 1/22$ if $x \in Q \smallsetminus \mathbf{Z}(G)$. An application of (7.2.2) then gives $W = \dim \mathsf{Wild} \geq 21(1 - 1/|Q|) \geq 21(1 - 1/7) = 18$. As Q is abelian, we see that Q admits at least 18 distinct linear characters on W. But this is impossible, since, with the action of $Q \cap \mathbf{Z}(G)$ fixed on $\mathcal{H}$, Q can have at most $|Q/(Q \cap \mathbf{Z}(G))| = p$ linear characters lying above it. $\square$

Lemma 9.8. *There does not exist any hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $D = 15 > m$, that has finite geometric monodromy group G such that $G/\mathbf{Z}(G) \cong \mathrm{Sp}_6(2)$.*

Proof. Assume the contrary, and let p and φ denote the characteristic of such a sheaf $\mathcal{H}$ and the character of G acting on $\mathcal{H}$. Then a generator g_0 of the image of $I(0)/P(0)$ has simple spectrum on $\mathcal{H}$, and so by Theorem 6.6, $\bar{\mathsf{o}}(g_0) = 15$. As g_0 is a p'-element, $p \nmid D = 15$. Now, as in the proof of Corollary 9.1, $Q \cap \mathbf{Z}(G) = 1$, whence Q embeds in $S := G/\mathbf{Z}(G) \cong \mathrm{Sp}_6(2)$ and $p = 2$ or 7. As S is simple, $G^{(\infty)}\mathbf{Z}(G) = G$, and so $G^{(\infty)}$ is a quasisimple cover of S which acts irreducibly on $\mathcal{H}$ of rank

15. It follows that $G^{(\infty)}$ is isomorphic to S and so we can identity it with S. Now $S \cap \mathbf{Z}(G) = 1$, so $G = \mathbf{Z}(G) \times S$. Checking the character table of S as given in [GAP], we see that $|\varphi(x)| \leq 7$ for any $1 \neq x \in Q$, whence

$$m \leq 7 + 8/|Q| \leq 11 \tag{9.8.1}$$

by (7.2.1). It follows from Proposition 4.4(iv) that $p \nmid |\mathbf{Z}(G)|$. In turn, the latter and Corollary 5.2 allow us to assume that $G = S$ and so $\mathcal{H}$ is self-dual. Now, G has a faithful irreducible $\mathbb{C}$-representation of degree 7 [GAP], so by Theorem 4.8 and (9.8.1) we now have

$$4 \leq W = \dim \mathsf{Wild} = D - m \leq 7 \tag{9.8.2}$$

Assume $p = 7$. Then $\varphi(x) = 1$ for all $1 \neq x \in Q$ and $Q \cong C_7$. It follows that $m = 3$ and $W = \dim \mathsf{Wild} = 12$, a contradiction. Thus $p = 2$. As $W \geq 4$, Q cannot be (abelian) of order ≤ 4, whence $|Q| \geq 8$, $m \leq 8$ by (9.8.1), and so (9.8.2) implies that $W = 7$ and $|Q| = 8$. Now a generator of the tame quotient $I(\infty)/P(\infty)$ maps onto an element $h \in S$ which permutes cyclically the seven characters of Q on Wild. It follows that $\mathsf{o}(h) = 7$, and so h cannot have 8 distinct eigenvalues on Tame, again a contradiction. $\square$

9C. Symplectic groups. The next result is well known; we recall a proof for the reader's convenience:

Lemma 9.9. *Let q be an odd prime power, $n \in \mathbb{Z}_{\geq 1}$, and let $\omega_n = \xi_n + \eta_n$ denote the character of a total Weil module M of $L := \mathrm{Sp}_{2n}(q)$, so that ξ_n, η_n are irreducible Weil characters of degree $(q^n+1)/2$ and $(q^n-1)/2$, respectively. Then for any $2'$-element $g \in L$, $\xi_n(g) = \eta_n(g) + 1$.*

Proof. Let $\boldsymbol{j}$ denote the central involution of G, and let M_2 denote a reduction modulo 2 of the complex module M. As shown in [GMST, §5], M_2 has a composition series

$$0 < (\boldsymbol{j} - 1_{M_2})M_2 < \mathbf{C}_{M_2}(t) < M_2,$$

with three successive simple quotients X, Y, and X, where Y is trivial and $\dim(X) = (q^n-1)/2 = \eta_n(1)$. It follows that the restrictions to $2'$-elements of η_n and ξ_n must equal to φ and $\varphi + 1_L$, where φ is the Brauer character of Y. Hence $\xi_n(g) = \eta_n(g) + 1$ for all $2'$-elements $g \in L$. $\square$

Proposition 9.10. *Let $q = p^f$ be a power of an odd prime p, $n \in \mathbb{Z}_{\geq 1}$, $(n,q) \neq (1,3)$, and let $\Phi : G \to \mathrm{GL}(V) \cong \mathrm{GL}_{(q^n-1)/2}(\mathbb{C})$ be a faithful irreducible representation of a finite almost quasisimple group G. Suppose that $\det(\Phi(G)) \cong \mu_N$ is a p'-group, $E(G)$ is a quotient of $L := \mathrm{Sp}_{2n}(q)$ by a central subgroup, and that $\Phi|_{E(G)}$ inflated to L, is an irreducible Weil representation of L. Then there exists a finite almost quasisimple group $\hat{G}$, a surjection $\pi : \hat{G} \twoheadrightarrow G$ with kernel a central subgroup, of order 1 if $2|D$ and 2 if $2 \nmid D$, and an irreducible representation $\Psi : \hat{G} \to \mathrm{GL}_{(q^n+1)/2}(\mathbb{C})$ such that*

$$\mathrm{Trace}(\Psi(g)) = \mathrm{Trace}(\Phi(\pi(g))) + 1$$

for all p-elements $g \in \hat{G}$.

Proof. Write $\mathbf{Z}(G) = Z_1 \times Z_2$, where $2 \nmid |Z_1|$ and Z_2 is a 2-group. Note that if $2|D := (q^n-1)/2$, then $E(G) = L = \mathrm{Sp}_{2n}(q)$ and $Z \cap L = Z_2 \cap L = \mathbf{Z}(L)$, whereas if $2 \nmid D$, then $E(G) = L/\mathbf{Z}(L) = \mathrm{PSp}_{2n}(q)$ and $Z \cap E(G) = 1$. By [KT2, Lemma 4.3], we can embed L in $\tilde{L} = \mathrm{Sp}_{2nf}(p)$ and extend $\Phi|_{E(G)}$ to an irreducible Weil representation $\tilde{\Phi} : \tilde{L} \to \mathrm{GL}(V)$. As $E(G) \lhd G$ and no outer automorphism of $\tilde{L}$ fixes the equivalence class of $\tilde{\Phi}$, we have

$$\Phi(G) \leq \mathbf{N}_{\mathrm{GL}(V)}(\Phi(E(G)) \leq \mathbf{Z}(\mathrm{GL}(V))\tilde{\Phi}(\tilde{L}).$$

In fact,

$$\Phi(G) \leq \tilde{Z}\tilde{\Phi}(\tilde{L}), \tag{9.10.1}$$

where $\tilde{Z} \cong \mu_{ND}$ is a cyclic subgroup of order ND of $\mathbf{Z}(\mathrm{GL}(V))$. [Indeed, for any $x \in G$, we can write $\Phi(x) = \alpha\tilde{\Phi}(y)$ for some $\alpha \in \mathbb{C}^\times$ and $y \in \tilde{L}$. By assumption,

$$1 = \det(\Phi(x))^N = \alpha^{ND}\det(\tilde{\Phi}(y))^N = \alpha^{ND}$$

as $\tilde{L}$ is perfect, whence $\alpha^{ND} = 1$.]

Write $\tilde{Z} = T_1 \times T_2$, with $T_1 = \langle t_1 \rangle \geq Z_1$ cyclic of odd order, and $T_2 = \langle t_2 \rangle \geq Z_2$ a cyclic 2-group. We also write $t_i = \gamma_i \cdot 1_V$ with $\gamma_i \in \mathbb{C}^\times$. Let $\tilde{\Psi} : \tilde{L} \to \mathrm{GL}_{D+1}(\mathbb{C})$ be the other constituent of the total Weil representation of $\tilde{L}$ having $\tilde{\Phi}$ as one constituent. Now, if $2|D$, we extend $\tilde{\Psi}$ to $\tilde{Z}$ by letting t_1 act as scalar γ_1 and t_2 act as scalar γ_2^2. We also take $\hat{G} = G$, $\tilde{\pi} := 1_{\tilde{Z}\tilde{L}}$, $\pi := 1_G$, and choose Ψ to be the restriction of $\tilde{\Psi}$ to G. On the other hand, if $2 \nmid D$, then note that $\tilde{\Phi}$ is trivial and $\tilde{\Psi}$ is faithful at $\mathbf{Z}(\tilde{L})$. In this case, we consider $\hat{\tilde{Z}} = \langle t_1, \hat{t}_2 \rangle$, and extend $\tilde{\Psi}$ to $\hat{\tilde{Z}}$ by letting t_1 act as scalar γ_1 and $\hat{t}_2$ act as scalar $\sqrt{\gamma_2}$. We can also define a surjection $\tilde{\pi} : \hat{\tilde{Z}} * L \to \tilde{Z} \times L/\mathbf{Z}(L)$ by sending t_1 to t_1, $\hat{t}_2$ to t_2, and $y \in \tilde{L}$ to $y\mathbf{Z}(L)$, and note that $\mathrm{Ker}(\tilde{\pi}) = \mathbf{Z}(\tilde{L}) = \mathbf{Z}(L)$. Finally, we take $\hat{G} = \tilde{\pi}^{-1}(G)$, and choose Ψ and π to be the restrictions of $\tilde{\Psi}$ and $\tilde{\pi}$ to $\hat{G}$.

Now consider any p-element $g \in \hat{G}$, of order say p^a, and write $\Phi(\pi(g)) = \beta\tilde{\Phi}(\tilde{\pi}(h))$ for some $\beta \in \tilde{Z}$ and $h \in \tilde{L}$, using (9.10.1). Then $\Phi(\pi(g))^{p^a} = 1_V \in \tilde{\Phi}(\tilde{L})$ and

$$\Phi(\pi(g))^{ND} = \beta^{ND}\tilde{\Phi}(\tilde{\pi}(h)^{ND}) = \tilde{\Phi}(\tilde{\pi}(h)^{ND}) \in \tilde{\Phi}(\tilde{L}).$$

As $p \nmid ND$, it follows that $\Phi(\pi(g)) \in \tilde{\Phi}(\tilde{L})$, and so we may assume that $\beta = 1$, h is a p-element, and $\tilde{\Phi}(\pi(g)) = \tilde{\Phi}(\tilde{\pi}(h))$. Recall that $\tilde{\Phi}$ is faithful, so $\tilde{\pi}(g) = \pi(g) = \tilde{\pi}(h)$. But $\mathrm{Ker}(\tilde{\pi}) \leq C_2$, so we now have that $g = h$. Thus

$$\mathrm{Trace}(\Psi(g)) - \mathrm{Trace}(\Phi(\pi(g))) = \mathrm{Trace}(\tilde{\Psi}(h)) - \mathrm{Trace}(\tilde{\Phi}(\tilde{\pi}(h))) = 1$$

by Lemma 9.9 applied to $\tilde{L}$. □

Theorem 9.11. *Let q be an odd prime power, $n \in \mathbb{Z}_{\geq 1}$, $(n,q) \neq (2,3)$, $(3,3)$, and $q \neq 5,7,9,11,25$ if $n = 1$. Then case* (i)(α) *of Theorem 8.2 with $\bar{\mathsf{o}}(g) = (q^n-1)/2$ does not lead to hypergeometric sheaves in dimension $(q^n \pm 1)/2$. More precisely, there is no hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $m < D = (q^n \pm 1)/2$, with finite geometric monodromy group $G = G_{\mathrm{geom}}$ such that G is almost quasisimple with $S = \mathrm{PSp}_{2n}(q)$ as a non-abelian composition factor, and with the image of $I(0)$ being a cyclic group $\langle g \rangle$ where $\bar{\mathsf{o}}(g) = (q^n-1)/2$.*

Proof. The finiteness of G implies that $\Phi(g)$ has simple spectrum on $\mathcal{H}$, where Φ denotes the representation of G on $\mathcal{H}$. In particular, $\bar{\mathsf{o}}(g) \geq D$, and so the case $D = (q^n+1)/2$ is impossible.

Consider the case $D = (q^n-1)/2$ and assume the contrary that $\mathcal{H}$ exists. The assumptions on (n,q) imply by Theorem 7.4 that the characteristic of $\mathcal{H}$ is the prime p dividing q. Now we consider the surjection $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$ underlying $\mathcal{H}$ and apply Theorem 9.10 to get the surjection $\pi : \hat{G} \twoheadrightarrow G$ with $\mathrm{Ker}(\pi) \leq \mathbf{Z}(\hat{G})$ of order 1 or 2 and the representation $\Psi : \hat{G} \to \mathrm{GL}_{D+1}(\mathbb{C})$. If $2|D$, then $\mathrm{Ker}(\pi) = 1$, then we may trivially lift ϕ to a surjection $\varphi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow \hat{G}$ such that $\pi \circ \varphi = \phi$. Assume $2 \nmid D$, so that $\mathrm{Ker}(\pi) \cong C_2$. The obstruction to lifting ϕ to a homomorphism $\varphi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \to \hat{G}$ lies in the group $H^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathrm{Ker}(\pi)) = 0$, the vanishing because open curves have cohomological dimension ≤ 1, cf. [SGA4t3, Cor. 2.7, Exp. IX and Thm. 5.1, Exp. X]. We claim that φ is surjective. Indeed, we have $H \leq \hat{G}$ for $H := \mathrm{Im}(\varphi)$ and $\pi(H) = \phi\big(\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})\big) = G$. Now, if $H \geq \mathrm{Ker}(\pi)$, then $|H| \geq 2|G| = |\hat{G}|$ and so $H = \hat{G}$. Otherwise $H \cap \mathrm{Ker}(\pi) = 1$, $H \cong G$ and so $H^{(\infty)} \cong G^{(\infty)} = E(G) \cong \mathrm{PSp}_{2n}(q)$. Also, $|H| = |G| = |\hat{G}|/2$, so $H \lhd \hat{G}$. Thus $\hat{G}^{(\infty)} = H^{(\infty)} \cong \mathrm{PSp}_{2n}(q)$. On the other hand, the construction of $\hat{G}$ in Theorem 9.10 ensures that $\hat{G}^{(\infty)} \cong \mathrm{Sp}_{2n}(q)$, a contradiction.

Now we can apply Theorem 9.10 and Theorem 5.1 to the surjection $\varphi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow \hat{G}$ together with $\Phi \circ \pi$ and Ψ and obtain another hypergeometric sheaf $\mathcal{H}'$ of type $(D+1, m+1)$, also tame at 0. So the image I of $I(0)$ in $\hat{G}$ is a cyclic group $\langle \hat{g} \rangle$ which projects onto $\langle g \rangle$ via π (seen by the action on $\mathcal{H}$). In the case $\mathrm{Ker}(\pi) = 1$, $\bar{\mathsf{o}}(\hat{g}) = \bar{\mathsf{o}}(g) = (q^n - 1)/2 = D < \mathrm{rank}(\mathcal{H}')$, and so $\mathcal{H}'$ cannot have finite monodromy. In the other case, $2 \nmid D$, recall by Theorem 8.2 that $\bar{g} \in \mathrm{PSp}_{2n}(q) \lhd G$ and so $\mathsf{o}(g) = D$. As $\mathrm{Ker}(\pi) \leq \mathbf{Z}(\hat{G})$ and $\pi(\hat{g}) = g$, we have $\hat{g}^D \in \mathbf{Z}(\hat{G})$, and so again $\bar{\mathsf{o}}(\hat{g}) \leq D < \mathrm{rank}(\mathcal{H}')$, a contradiction. $\square$

9D. Unitary groups. The unitary analogue of Lemma 9.9 was proved in [DT, Theorem 7.2]; we will give a slight extension of it:

Lemma 9.12. *Let q be any prime power, $n \in \mathbb{Z}_{\geq 2}$, and let $\zeta_n = \sum_{i=0}^{q} \zeta_n^i$ denote the character of a total Weil representation of $G := \mathrm{GU}_n(q)$, as described in* [TZ2, §4]. *Then for any $g \in G$ and any $0 \leq i, j \leq q$ we have*

$$\zeta_n^i(g) - \zeta_n^j(g) = \zeta_n^i(1) - \zeta_n^j(1),$$

if at least one of the following two conditions holds.

(a) *ℓ is any prime divisor of $q+1$, $\ell \nmid \mathsf{o}(g)$, and $i - j$ is divisible by the ℓ'-part s of $q+1$.*

(b) *$\mathsf{o}(g)$ is coprime to $q+1$.*

Proof. Let $\xi \in \overline{\mathbb{F}_q}^\times$ and $\tilde{\xi} \in \mathbb{C}^\times$ be primitive $(q+1)^{\mathrm{th}}$ roots of unity. We write $q + 1 = \ell^c s$ with $c \in \mathbb{Z}_{\geq 1}$ and $\ell \nmid s$ in the case of (a), and choose $(\ell, c, s) = (q+1, 1, 1)$ in the case of (b). Also let $V = \mathbb{F}_{q^2}^n$ denote the natural module of G. Then, by [TZ2, Lemma 4.1], $\zeta_n^i(g) - \zeta_n^j(g)$ is equal to

$$\frac{(-1)^n}{q+1} \sum_{0 \leq k \leq q,\ \ell^c \nmid k} (\tilde{\xi}^{ik} - \tilde{\xi}^{jk})(-q)^{\dim \mathrm{Ker}(g - \xi^k \cdot 1_V)} + \frac{(-1)^n}{q+1} \sum_{0 \leq k \leq q,\ \ell^c | k} (\tilde{\xi}^{ik} - \tilde{\xi}^{jk})(-q)^{\dim \mathrm{Ker}(g - \xi^k \cdot 1_V)}.$$

Since $i \equiv j (\mathrm{mod}\ s)$, in the second summation we have $\tilde{\xi}^{ik} = \tilde{\xi}^{jk}$. In the case of (a), the condition $\dim \mathrm{Ker}(g - \xi^k \cdot 1_V) \neq 0$ implies $\ell^c | k$, hence in the first summation we have $\dim \mathrm{Ker}(g - \xi^k \cdot 1_V) = 0$. The latter equality also holds for $1 \leq k \leq q$ in the case of (b). Thus $\zeta_n^i(g) - \zeta_n^j(g)$ is equal to

$$\frac{(-1)^n}{q+1} \sum_{0 \leq k \leq q} (\tilde{\xi}^{ik} - \tilde{\xi}^{jk}) = (-1)^n(\delta_{i,0} - \delta_{j,0}) = \zeta_n^i(1) - \zeta_n^j(1),$$

as stated. $\square$

Theorem 9.13. *Let q be an odd prime power, $2 \nmid n \in \mathbb{Z}_{\geq 3}$, and $(n, q) \neq (3, 3)$, $(3, 5)$. Then case* (ii) *of Theorem 8.3 does not lead to hypergeometric sheaves in dimension $(q^n - q)/(q+1)$. More precisely, there is no hypergeometric sheaf $\mathcal{H}$ of type (D, m) with $m < D = (q^n - q)/(q+1)$, with finite geometric monodromy group $G = G_{\mathrm{geom}}$ such that G is almost quasisimple with $S = \mathrm{PSU}_n(q)$ as a non-abelian composition factor, and with the image of $I(0)$ being a cyclic group $\langle g_0 \rangle$ where $\bar{\mathsf{o}}(g_0) = q^{n-1} - 1$.*

Proof. (i) Assume the contrary that such a hypergeometric sheaf $\mathcal{H}$ exists, and let $\Phi : G \to \mathrm{GL}(V)$ denote the corresponding representation, with $V = \overline{\mathbb{Q}_\ell}^D$, and with character φ. The assumptions on (n, q) imply by Theorem 7.4 that the characteristic p of $\mathcal{H}$ divides q, that is, $q = p^f$ for some $f \in \mathbb{Z}_{\geq 1}$. Recall that V is irreducible over $G^{(\infty)}$, which in turn is a quotient of $\mathrm{SU}_n(q)$. As $D = (q^n - q)/(q+1)$, we see that in fact $G^{(\infty)} = S$. Also recall that $\varphi|_S$ extends to the Weil character ζ_n^0 of $\mathrm{PGU}_n(q)$, which is fixed by, and hence, extends to $A := \mathrm{PGU}_n(q) \rtimes C_{2f} = \mathrm{Aut}(S)$. Thus we can extend Φ to an A-representation on V which we also denote by $\Phi : A \to \mathrm{GL}(V)$.

Certainly, $\Phi(A)$ and $\Phi(G)$ both normalize $\Phi(S)$. Using the finiteness of G, we can find a finite cyclic subgroup μ_N of order N of $\mathbf{Z}(\mathrm{GL}(V))$ such that

$$\Phi(G) \leq \mu_N \times \Phi(A) < \mathbf{Z}(\mathrm{GL}(V))\Phi(A) = \mathbf{N}_{\mathrm{GL}(V)}(\Phi(S)).$$

Here, $\mu_N \cap \Phi(A) = 1$ since $\mathbf{C}_A(S) = 1$ and $\mathrm{soc}(A) = S$. We now define $\Gamma := C_N \times A$ and extend Φ to $C_N = \mathbf{Z}(\Gamma)$ via scalar action, so that $\Phi(\Gamma) = \mu_N \times \Phi(A)$.

We also note that $m \geq 1$. Indeed, if $m = 0$, then $\mathcal{H}$ is Kloosterman, and the D "upstairs" characters of $\mathcal{H}$ can be read off from the spectrum of the image $\langle g_0 \rangle$ of $I(0)$, and seen to be $\big(\mathsf{Char}_{E(q+1)} \smallsetminus \mathsf{Char}_E\big)\chi$, where $E := (q^{n-1} - 1)/(q + 1) = D/q$ and χ is some character. This set is stable under the multiplication by ξ_E, and so $\mathcal{H}$ is induced from a rank q sheaf. But this is impossible, since S has no proper subgroups of index $\leq E$, cf. [KlL, Table 5.2.A].

(ii) Next we consider the Weil character $\zeta_n^{(q+1)/2}$ of $\mathrm{GU}_n(q)$, which restricts irreducibly to $\mathrm{SU}_n(q)$ and in fact factors through $S = \mathrm{PSU}_n(q)$, since its kernel is the subgroup of order $(q + 1)/2$ of $\mathbf{Z}(\mathrm{GU}_n(q))$ and so contains $\mathbf{Z}(\mathrm{SU}_n(q))$ since $2 \nmid n$. Thus we obtain a self-dual representation $\Psi : S \to \mathrm{GL}(W)$, with $W = \overline{\mathbb{Q}_\ell}^{D+1}$, whose character is invariant under A, since $\zeta_n^{(q+1)/2}$ is fixed by the subgroup C_{2f} of field automorphisms. As $\deg(\Psi) = D + 1$ is odd and S is perfect, Ψ extends to a self-dual representation $\Psi : A \to \mathrm{GL}(W)$ by [NT, Theorem 2.3], which we inflate to a self-dual representation $\Psi : \Gamma \to \mathrm{GL}(W)$ by letting $C_N = \mathbf{Z}(\Gamma)$ act trivially.

(iii) Recall that $\mathcal{H}$ gives rise to a surjection $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$. We will now compose ϕ with

$$\Phi : G \hookrightarrow \Gamma \to \mathrm{GL}(V) \text{ and } \Psi : G \hookrightarrow \Gamma \to \mathrm{GL}(W)$$

and compare their traces at elements in $\phi(P(0)) \cup \phi(P(\infty))$. First, if $y \in \phi(P(0))$, then, since $\mathcal{H}$ is tame at 0, y acts trivially in Φ. But the latter is faithful, so $y = 1$, i.e. $\phi(P(0)) = 1$ and we trivially have

$$\mathrm{Trace}(\Phi(y)) - \mathrm{Trace}(\Psi(y)) = -1 \tag{9.13.1}$$

for all $y \in \phi(P(0))$.

Now, let φ and ψ denote the character of the Γ-representations Φ and Ψ, and let φ° and ψ° denote their restrictions to $2'$-elements. By [DT, Theorem 7.2(ii)], $\theta := \varphi^\circ|_S$ is an irreducible 2-Brauer character of S. Next, by Lemma 9.12(a), $\psi^\circ|_S = \theta + 1_S$. As $S \lhd \Gamma$ and $D > 1$, it follows from Clifford's theorem that $\psi^\circ = \alpha + \beta$ is the sum of two irreducible Brauer characters, with α lying above θ and β lying above 1_S. Furthermore, as Ψ is self-dual, we have that α and β are both real-valued. But $\beta(1) = 1$, so in fact $\beta = 1_\Gamma$. We have shown that $\psi^\circ - 1_\Gamma = \alpha$ and φ° are two extensions to Γ of θ. By [N, Cor. (8.20)], there exists a linear character λ of Γ/S such that

$$\varphi^\circ = (\psi^\circ - 1_\Gamma)\lambda. \tag{9.13.2}$$

Taking the complex conjugate and using $\psi = \overline{\psi}$, we obtain

$$\overline{\varphi^\circ} = \overline{(\psi^\circ - 1_\Gamma)\lambda} = (\psi^\circ - 1_\Gamma)\overline{\lambda} = \varphi^\circ\lambda^2. \tag{9.13.3}$$

In particular, we have that $\overline{\varphi}|_Q = \varphi|_Q \cdot \lambda^2|_Q$. Note that $D = (q^n - q)/(q+1)$ is not a p-power and so $|Q| \neq D$. Hence $\lambda^2|_Q = 1_Q$ by Lemma 5.4. But Q has odd order, so in fact $\lambda|_Q = 1_Q$. The relation (9.13.2) applied to $y \in Q$ now implies that (9.13.1) holds for all $y \in Q = \phi(P(\infty))$.

Now we can apply Theorem 5.1 to Φ and Ψ to conclude that Ψ leads to a hypergeometric sheaf $\mathcal{H}'$ of rank $D + 1$ with geometric monodromy group $\Psi(G)$. In particular, the image $\langle g_0 \rangle$ of $I(0)$ in $\Psi(G)$ has simple spectrum on $\mathcal{H}'$. But this contradicts Theorem 8.3, since $\bar{\mathsf{o}}(g_0) = q^{n-1} - 1$. $\square$

Next we prove the q-even analogue of Theorem 9.13.

Theorem 9.14. *Let $q = 2^f$, $2 \nmid n \in \mathbb{Z}_{\geq 3}$, and $(n,q) \neq (3,2)$, $(3,4)$, $(5,2)$. Then case* (ii) *of Theorem 8.3 does not lead to hypergeometric sheaves in dimension $(q^n - q)/(q+1)$. More precisely, there is no hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $m < D = (q^n - q)/(q+1)$, with finite geometric monodromy group $G = G_{\rm geom}$ such that G is almost quasisimple with $S = \mathrm{PSU}_n(q)$ as a non-abelian composition factor, and with the image of $I(0)$ being a cyclic group $\langle g_0 \rangle$ where $\bar{\mathsf{o}}(g_0) = q^{n-1} - 1$.*

Proof. (i) Assume the contrary that such a hypergeometric sheaf $\mathcal{H}$ exists, and let $\Phi : G \to \mathrm{GL}(V)$ denote the corresponding representation, with $V = \overline{\mathbb{Q}_\ell}^D$, and with character φ. The assumptions on (n,q) imply by Theorem 7.4 that the characteristic p of $\mathcal{H}$ divides q, that is, $p = 2$. Recall that V is irreducible over $G^{(\infty)}$, which in turn is a quotient of $\mathrm{SU}_n(q)$. As $D = (q^n - q)/(q+1)$, we see that in fact $G^{(\infty)} = S$ and that $\Phi|_S$ affords the Weil character ζ_n^0 of $\mathrm{SU}_n(q)$, which is real-valued.

First suppose that $\dim \mathsf{Wild} = D - m = 1$. By Proposition 2.22 and Lemma 2.19 of [GT3], $|\varphi(g)|/\varphi(1) \leq (3.95)/4$ for all $1 \neq g \in Q$. It now follows from (7.2.2) that $D \leq 160$, which is ruled out by our assumptions on (n,q), unless possibly $(n,q) = (3,8)$ or $(7,2)$. When $(n,q) = (3,8)$, using the character table of $\mathrm{Aut}(\mathrm{PSU}_3(8))$ given in [GAP] we can check that $|\varphi(g)|/\varphi(1) \leq 1/7$ for all $1 \neq g \in Q$. When $(n,q) = (7,2)$, using the character table of $X := \mathrm{SU}_7(2)$ given in [GAP] we can check that $|\varphi(x)|/\varphi(1) \leq 1/2$ for all $1 \neq x \in X$, whence by [GT3, Lemma 2.19] we have $|\varphi(g)|/\varphi(1) \leq (3.5)/4$ for all $1 \neq g \in Q$. Hence, in these two cases we have $D \leq 16$ by (7.2.2), which is impossible.

Hence we may assume that $D - m \geq 2$, and therefore

$$G = \mathbf{O}^2(G) \tag{9.14.1}$$

by Theorem 4.1. Now both Φ and its dual Φ^* are extensions to G of $\Phi|_S$, hence $\Phi^* \cong \Phi \otimes \Lambda$ for some 1-dimensional representation Λ by Gallagher's theorem [Is, Cor. (6.17)]. By (9.14.1), Λ has odd order. Applying Corollary 5.3, we can find a power Θ of Λ so that $\Phi \otimes \Theta$ is self-dual and giving rise to a hypergeometric sheaf, and $(\Phi \otimes \Theta)(g_0)$ has the same central order $q^{n-1} - 1$. Replacing (G, Φ) by $(G/\mathrm{Ker}(\Phi \otimes \Theta), \Phi \otimes \Theta)$, we may assume that Φ is self-dual, and $\Phi(g_0)$ has central order $q^{n-1} - 1$. This in turn implies that $|\mathbf{Z}(G)| \leq 2$. Note furthermore that $\mathbf{Z}(G) \cap S = 1$ and $G/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$ by Corollary 8.4. Hence, $G/S \cong \mathbf{Z}(G) \cdot C_d$, where $d := \gcd(n, q+1)$ is odd and $C_d \cong \mathrm{PGU}_n(q)/S$. The oddness of d allows us to write $G/S \cong \mathbf{Z}(G) \times C_d$. Applying (9.14.1) again, we get that $\mathbf{Z}(G) = 1$, and thus

$$G \cong \mathrm{PGU}_n(q). \tag{9.14.2}$$

(ii) Let $r_1, \ldots, r_m$ be all the distinct primes divisors of $d = \gcd(n, q+1)$ (with $m = 0$ if $d = 1$). Then we can write

$$n = n_0 r_1^{a_1} \ldots r_m^{a_m} = n_0 n', \; q + 1 = q_0 r_1^{b_1} \ldots r_m^{b_m} = q_0 q'$$

for some integers $n_0, q_0, a_i, b_i \geq 1$, such that $\gcd(n_0, r_1 \ldots r_m) = \gcd(q_0, r_1 \ldots r_m) = 1$. This implies

$$\gcd(q_0, q') = \gcd(q_0, n) = 1. \tag{9.14.3}$$

Here we prove that if $H \leq \Gamma := \mathrm{GU}_n(q)$ is a subgroup that contains the central subgroup $Z_0 \cong C_{q_0}$ of $Z := \mathbf{Z}(\mathrm{GU}_n(q))$ and maps onto $\mathrm{PGU}_n(q)$ under the surjection $\Gamma \twoheadrightarrow \Gamma/Z$, then $H = \Gamma$. Indeed, let $\gamma \in \mathbb{F}_{q^2}^\times$ be of order $q+1$, so that the determinantal map det maps Γ onto $\mu_{q+1} = \langle \gamma \rangle$. Any element of Z_0 is a scalar matrix $x = \gamma^{q'i} \cdot I_n$ in Γ, with $i \in \mathbb{Z}/q_0\mathbb{Z}$ and with $\det(x) = (\gamma^{q'})^{ni}$. As n is coprime to q_0 by (9.14.3), we see that $\det(x)$ runs over the subgroup μ_{q_0} of μ_{q+1}, i.e. $\det(Z_0) = \mu_{q_0}$. Next, the condition that H maps onto $\mathrm{PGU}_n(q)$ implies that $HZ = \Gamma$. In particular,

$$H \geq [H, H] = [HZ, HZ] = [\Gamma, \Gamma] = \mathrm{SU}_n(q), \tag{9.14.4}$$

and there are some $h \in H$ and $j \in \mathbb{Z}/(q+1)\mathbb{Z}$ such that $\gamma = \det(h(\gamma^j \cdot I_n))$, i.e. $\det(h) = \gamma^{1-jn}$. As $n = n_0 n'$, it follows that the order of γ^{1-jn} in μ_{q+1} is divisible by q'. Thus $\det(H)$ has order divisible by both q_0 and q', and so by (9.14.3) we have $\det(H) = \mu_{q+1} = \det(\Gamma)$. But $H \geq \mathrm{SU}_n(q) = \mathrm{Ker}(\det)$ by (9.14.4), hence $H = \Gamma$, as stated.

(iii) Now we consider the surjection $\phi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \twoheadrightarrow G$ underlying $\mathcal{H}$ and recall that $G \cong \mathrm{PGU}_n(q) = \Gamma/Z$ by (9.14.2). Also, consider the surjection $\pi : \hat{G} = \Gamma/Z_0 \twoheadrightarrow G$ with kernel $\mathrm{Ker}(\pi) \cong C_{q'}$. The obstruction to lifting ϕ to a homomorphism $\varpi : \pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p}) \to \hat{G}$ lies in the group $H^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathrm{Ker}(\pi)) = 0$, the vanishing because open curves have cohomological dimension ≤ 1, cf. [SGA4t3, Cor. 2.7, Exp. IX and Thm. 5.1, Exp. X]. Write $\varpi(\hat{G}) = H/Z_0$ for some subgroup $H \leq \Gamma$ containing Z_0. Then

$$\pi(H/Z_0) = (\pi \circ \varpi)\big(\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})\big) = \phi\big(\pi_1(\mathbb{G}_m/\overline{\mathbb{F}_p})\big) = \Gamma/Z,$$

i.e. H maps onto Γ/Z. By (ii), $H = \Gamma$, that is, ϖ is surjective.

(iv) Next, if $q_0 = q+1$, equivalently $\gcd(n, q+1) = 1$, then $\Gamma = S \times Z_0$, $\Gamma/Z_0 \cong G \cong \mathrm{SU}_n(q)$; set $l := 1$ in this case and consider the Weil character ζ_n^l of $\hat{G} = \Gamma/Z_0$. If $q_0 < q+1$, then we set $l := q_0$ and consider the Weil character ζ_n^l of $\mathrm{GU}_n(q)$, which restricts irreducibly to $\mathrm{SU}_n(q)$ and factors through $\hat{G} = \Gamma/Z_0$, since its kernel is the subgroup Z_0 of $Z = \mathbf{Z}(\Gamma)$. Thus in both cases we obtain an irreducible representation $\Psi : \hat{G} \to \mathrm{GL}(W)$, with $W = \overline{\mathbb{Q}_\ell}^{D+1}$, whose character is ζ_n^l. We can also inflate Φ to a $\hat{G}$-representation $\hat{\Phi}$ with kernel Z_0. We will now compose ϖ with

$$\hat{\Phi} : \hat{G} \to \mathrm{GL}(V) \text{ and } \Psi : \hat{G} \to \mathrm{GL}(W)$$

and compare their traces at any element $y \in \varpi(P(0)) \cup \varpi(P(\infty))$. Then y is a 2-element, and so by Lemma 9.12(ii) we have

$$\mathrm{Trace}(\hat{\Phi}(y)) - \mathrm{Trace}(\Psi(y)) = -1.$$

Now we can apply Theorem 5.1 to $\hat{\Phi}$ and Ψ to conclude that Ψ leads to a hypergeometric sheaf $\mathcal{H}'$ of rank $D+1$ with geometric monodromy group $\Psi(\hat{G})$. In particular, the image $\langle g_0 \rangle$ of $I(0)$ in $\Psi(\hat{G})$ has simple spectrum on $\mathcal{H}'$. But this contradicts Theorem 8.3, since $\bar{\mathsf{o}}(g_0) = q^{n-1} - 1$ but $\mathrm{rank}(\mathcal{H}') = (q^n+1)/(q+1)$. $\square$

To handle other unitary cases, we will need some auxiliary statements.

Lemma 9.15. *Let $2|n \geq 4$, $q = p^f$ a power of a prime $p > 2$, and let H be a finite group with $p \nmid |\mathbf{Z}(H)|$ and $H/\mathbf{Z}(H) \cong \mathrm{PGU}_n(q)$. Let $P < H$ be the full inverse image in H of a Siegel parabolic subgroup $\bar{P}$* [i.e. with Levi subgroup $\mathrm{GL}_{n/2}(q^2)/\mathbf{Z}(\mathrm{GU}_n(q))$] *of $\mathrm{PGU}_n(q)$. Let $Q := \mathbf{O}_p(P)$ and let $J = Q \rtimes C < P$ such that $Z := \mathbf{Z}(H) \leq C$ and C/Z projects onto a maximal torus $C_{(q^n-1)/(q+1)}$ of $\mathrm{PGU}_n(q)$. Then there exists a linear character $\theta \in \mathrm{Irr}(Q)$ such that the following statements hold.*

(i) *If $\xi \in \mathrm{Irr}(H)$ is any irreducible character of degree $D := (q^n-1)/(q+1)$, then $\xi|_J$ is irreducible and there exists a linear character $\xi^* \in \mathrm{Irr}(Z)$ such that*

$$\xi|_Z = \xi(1) \cdot \xi^*, \;\; \xi|_J = \mathrm{Ind}_{QZ}^J(\theta \boxtimes \xi^*).$$

(ii) *Moreover, if σ is an automorphism of H of p-power order that fixes $\xi \in \mathrm{Irr}(H)$ of degree D, then there exists a σ-invariant linear character $\tilde{\xi} \in \mathrm{Irr}(C)$ such that*

$$\tilde{\xi}|_Z = \xi^*, \;\; \xi|_J = \tilde{\xi} \cdot \mathrm{Ind}_{QZ}^J(\theta \boxtimes 1_Z).$$

if we inflate $\tilde{\xi}$ to a linear character of J.

Proof. (i) Let $\hat{H} := \mathrm{GU}_n(q) = \mathrm{GU}(V)$ for a Hermitian space $V = \mathbb{F}_{q^2}^n$, $\hat{Z} := \mathbf{Z}(\hat{H})$, so that $H/Z \cong \hat{H}/\hat{Z}$. Now we can write $P/Z = \bar{P} = \hat{P}/\hat{Z}$, where $\hat{P} = \mathrm{Stab}_{\hat{H}}(U)$ for a totally singular subspace $\langle e_1, \ldots, e_{n/2}\rangle$ of V. Then $\hat{Q} := \mathbf{O}_p(\hat{P})$ is elementary abelian of order $q^{n(n+2)/8}$. Moreover, as shown in the proof of [GMST, Lemma 12.5], $\hat{P}$ acts on $\mathrm{Irr}(\hat{Q})$ with exactly one orbit of length 1, namely $\{1_{\hat{Q}}\}$, one orbit $\mathcal{O}_1$ of length D, and all other orbits have length larger than D. Certainly, this action factors through $\hat{Z}$.

Let $\hat{C} \cong C_{q^n-1}$ be a maximal torus of $\hat{H}$ contained in $\hat{P}$. We may assume that $\hat{C} = \langle \hat{g}\rangle$, where $\hat{g}$ has simple spectrum

$$\{\epsilon, \epsilon^{-q}, \ldots, \epsilon^{(-q)^{n-1}}\}$$

on $V \otimes \overline{\mathbb{F}_q}$ for a generator ϵ of $\mu_{q^n-1} = \mathbb{F}_{q^n}^\times$. Note that $\langle \hat{g}^D\rangle = \mathbf{Z}(\hat{H})$ fixes every $\lambda \in \mathcal{O}_1$. Furthermore, as shown in the proof of [GMST, Lemma 12.5], if some power $\hat{g}^m$ fixes some character $\lambda \in \mathcal{O}_1$, then the p'-element $\hat{g}^m$ belongs to a subgroup $\mathrm{GU}_1(q) \times \mathrm{GL}_{n/2-1}(q^2)$ of a Levi subgroup $\mathrm{GL}_{n/2}(q^2)$ of $\hat{P}$. This implies that $\hat{g}^m$ has an eigenvalue belonging to $\mu_{q+1} \subseteq \mathbb{F}_{q^2}^\times$, and the latter is possible only when $D|m$. We have shown that $\hat{C}$ acts transitively on $\mathcal{O}_1$, with any point stabilizer equal to $\mathbf{Z}(\hat{H})$.

Since $p \nmid |Z|$, the full inverse image of $\hat{Q}\hat{Z}/\hat{Z} \cong \hat{Q}$ in H is precisely $Z \times Q$. Hence, without loss of generality, we may identify $\hat{Q}$ with Q, and conclude that P acts on $\mathrm{Irr}(Q)$ with exactly one orbit of length 1, namely $\{1_Q\}$, one orbit $\mathcal{O}_1$ of length D, and all other orbits have length larger than D; furthermore, C acts transitively on $\mathcal{O}_1$, with any point stabilizer equal to Z. Now, as $\xi \in \mathrm{Irr}(H)$ has degree D, it follows that $\xi|_Q = \sum_{\lambda\in\mathcal{O}_1} \lambda$. Fixing $\theta \in \mathcal{O}_1$, we then have by Clifford's theorem that $\xi|_J = \mathrm{Ind}_{QZ}^J(\theta \boxtimes \xi^*)$, since the inertia group of θ in J is precisely $Q \times Z$ and $\xi|_Z = \xi(1)\cdot\xi^*$ by Schur's lemma. In particular, $\xi|_J$ is irreducible.

(ii) First note that C is abelian, since $C/Z \cong C_{(q^n-1)/(q+1)}$ is cyclic. As σ fixes ξ, it also fixes the central character ξ^*. Hence σ acts on the set of $D = |C/Z|$ irreducible constituents of the character $\mathrm{Ind}_Z^C(\xi^*)$, as C is abelian. But $\mathsf{o}(\sigma)$ is a p-power and $p \nmid D$. Therefore, σ fixes some irreducible constituents $\tilde{\xi}$ of $\mathrm{Ind}_Z^C(\xi^*)$, and we have that $\tilde{\xi}|_Z = \xi^*$. By Frobenius' reciprocity, we also have that

$$\tilde{\xi}\cdot \mathrm{Ind}_{QZ}^J(\theta\boxtimes 1_Z) = \mathrm{Ind}_{QZ}^J\big((\tilde{\xi})|_{QZ}\cdot(\theta\boxtimes 1_Z)\big) = \mathrm{Ind}_{QZ}^J(\theta\boxtimes\xi^*) = \xi|_J.$$

□

Proposition 9.16. *Let $q = p^f$ be a power of a prime p, $2|n \geq 4$, $(n,q) \neq (4,2)$, $(4,3)$, $(6,2)$, and let G be a finite almost quasisimple group with a normal subgroup $H \geq \mathbf{Z}(G)$ such that $H/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$ and $p \nmid |\mathbf{Z}(G)|$. Then $L := G^{(\infty)}$ is a quotient of $\mathrm{SU}_n(q)$ by a central subgroup. Suppose that G has a faithful irreducible complex representation $\Phi : G \to \mathrm{GL}(V)$ of degree*

$$D := (q^n-1)/(q+1)$$

such that $\Phi|_L$ induces a Weil representation of $\mathrm{SU}_n(q)$ with character ζ_n^i for some $1 \leq i \leq q$ and $p \nmid |\det(\Phi(G))|$. Then G admits an irreducible complex representation $\Psi : G \to \mathrm{GL}(W)$ of degree $D+1$ such that

$$\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Phi(y)) = 1$$

for all p-elements $y \in G$.

Proof. (i) Note that G has a unique non-abelian composition factor $S \cong \mathrm{PSU}_n(q)$, and

$$\mathrm{PGU}_n(q) \cong H/\mathbf{Z}(G) \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S) = \mathrm{PGU}_n(q) \rtimes C_{2f} \qquad (9.16.1)$$

by hypothesis. In particular, L is a quasisimple cover of S, so L is a quotient of $\mathrm{SU}_n(q)$ by the assumptions on (n,q). Assume in addition that $p = 2$. If $2|[G:H]$, then G induces an involutive field automorphism, namely the transpose-inverse automorphism, on S, which sends the character

ζ_n^i of $\mathrm{SU}_n(q)$ to $\zeta_n^{q+1-i} \neq \zeta_n^i$, and this contradicts the existence of Φ. Hence $2 \nmid [G : H]$. Next, both $|\mathbf{Z}(G)|$ and $|\mathrm{PGU}_n(q)/S| = \gcd(n, q+1)$ are odd, so $2 \nmid [G : L]$ and $y \in L$ for all 2-elements $y \in G$. Now recall that the Weil character ζ_n^0 of $\mathrm{SU}_n(q)$ factors through S and so can be inflated to a real-valued, $\mathrm{Aut}(S)$-invariant, irreducible character with trivial determinant (as L is perfect) of L. By [NT, Lemma 2.1], the latter character extends to the character of some representation $\Psi : G \to \mathrm{GL}(\mathbb{C}^{D+1})$. Now the relation $\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Phi(y)) = 1$ follows from Lemma 9.12(b).

(ii) From now on we will assume $p > 2$. Also let φ denote the character of Φ, and φ° denote its restriction to $2'$-elements of G (and similarly for any character of G). Now, the Weil character ζ_n^0 of $\mathrm{SU}_n(q)$, of *odd* degree $(q^n + q)/(q+1) = D + 1$, factors through S and yields a real-valued, $\mathrm{Aut}(S)$-invariant, irreducible character with trivial determinant (as S is simple) of S. By [NT, Theorem 2.3], the latter character extends uniquely to a real character ψ with trivial determinant, of some representation $\Psi : G \to \mathrm{GL}(\mathbb{C}^{D+1})$ that is trivial at $\mathbf{Z}(G)$.

Here we consider the case $i = (q+1)/2$. By [DT, Theorem 7.2(ii)], $\theta := \varphi^\circ|_L$ is an irreducible 2-Brauer character of L. Next, by Lemma 9.12(a), $\psi^\circ|_L = \theta + 1_L$. As $L \lhd G$ and $D > 1$, it follows from Clifford's theorem that $\psi^\circ = \alpha + \beta$ is the sum of two irreducible Brauer characters, with α lying above θ and β lying above 1_L. Furthermore, as ϑ is real, we have that α and β are both real-valued. But $\beta(1) = 1$, so in fact $\beta = 1_G$. We have shown that $\psi^\circ - 1_G = \alpha$ and φ° are two extensions to L of θ. By [N, Cor. (8.20)], there exists a linear character λ of G/L such that

$$\varphi^\circ = (\psi^\circ - 1_G)\lambda. \tag{9.16.2}$$

Taking the complex conjugate and using $\psi = \overline{\psi}$, we obtain

$$\overline{\varphi^\circ} = \overline{(\psi^\circ - 1_G)\lambda} = (\psi^\circ - 1_G)\overline{\lambda} = \varphi^\circ\lambda^2. \tag{9.16.3}$$

Now we consider any p-element $y \in G$ and let $Y := \langle y \rangle$. Restricting (9.16.3) to Y, we see that $\Phi^*|_Y \cong \Phi|_Y \otimes \Lambda$ for some representation $\Lambda : Y \to \mathrm{GL}_1(\mathbb{C})$ with character $\lambda|_Y$. In particular,

$$\det(\Phi(y))^{-1} = \det(\Phi^*(y)) = \det(\Phi(y))\lambda(y)^D.$$

As $2 < p \nmid |\det(\Phi(G))|$ and $p \nmid D$, it follows that $\lambda(y) \in \mathbb{C}^\times$ is a p'-root of unity. On the other hand, $\mathsf{o}(\lambda(y))$ is a p-power, since y is a p-element. Hence $\lambda(y) = 1$, and so $\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Phi(y)) = 1$ by (9.16.2).

(iii) In the rest of the proof, we consider the case $i \neq (q+1)/2$. Recalling (9.16.1), we can find a p-element $\bar{\sigma} \in G/\mathbf{Z}(G)$, which is induced by a field automorphism of $\mathrm{GU}_n(q)$, such that p does not divide the index of $\langle H/\mathbf{Z}(G), \bar{\sigma}\rangle = H \rtimes \langle \bar{\sigma} \rangle$ in $G/\mathbf{Z}(G)$. Using $p \nmid |\mathbf{Z}(G)|$, we can find a lift σ of p-power order of $\bar{\sigma}$ in G, and note that $G_1 := \langle H, \sigma \rangle = H \rtimes \langle \sigma \rangle$ satisfies $p \nmid [G : G_1]$; in particular, G_1 contains all p-elements of G. The field automorphism action of $\bar{\sigma}$ induces an action of σ on $H_2 := \mathrm{GU}_n(q)/C_{(q+1)/2}$ (where $C_{(q+1)/2}$ has index 2 in $\mathbf{Z}(\mathrm{GU}_n(q))$ and leads to $G_2 := H_2 \rtimes \langle \sigma \rangle$. Also recall that the real-valued Weil character $\zeta_n^{(q+1)/2}$ of $\mathrm{GU}_n(q)$ factors through H_2 and is invariant under all field automorphisms of $\mathrm{GU}_n(q)$, in particular under σ. By [NT, Lemma 2.1], $\zeta_n^{(q+1)/2}$ has a unique real-valued extension ξ to G_2, afforded by a representation $\Xi : G_2 \to \mathrm{GL}(\mathbb{C}^D)$. The analysis in (ii) applied to (G_2, Ξ) then shows that

$$\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Xi(y)) = 1 \tag{9.16.4}$$

for all p-elements $y \in G_2$. (Note that Ψ is trivial at $\mathbf{Z}(G)$ and so can be viewed as defined on $G_2/\mathbf{Z}(G_2)$ and then inflated to G_2.)

Next we can find a $\bar{\sigma}$-stable Siegel parabolic subgroup $\bar{P}$ of $H/\mathbf{Z}(G)$ with unipotent radical $\bar{Q}$ and a $\bar{\sigma}$-stable maximal torus $\bar{C} \cong C_{(q^n-1)/(q+1)}$ in $\bar{P}$ (using the field automorphism action of $\bar{\sigma}$), and embed $\langle \bar{Q}, \bar{\sigma} \rangle$ in a Sylow p-subgroup $\bar{R} \rtimes \langle \bar{\sigma} \rangle$ of $G/\mathbf{Z}(G)$, where $\bar{R} \in \mathrm{Syl}_p(H/\mathbf{Z}(G))$. Let $J_1 = Q_1 \rtimes C_1$ and $J_2 = Q_2 \rtimes C_2$ be the full inverse images of $\bar{Q} \rtimes \bar{C}$ in G_1 and G_2, respectively, with

$Q_i := \mathbf{O}_p(J_i) \cong \bar{Q}$ as p is coprime to $|\mathbf{Z}(G)|$ and $|\mathbf{Z}(G_2)|$. Similarly, there exist a unique Sylow p-subgroup $R_1 > Q_1$ of H and a unique Sylow p-subgroup $R_2 > Q_2$ of H_2 that project isomorphically onto $\bar{R}$, and we may identify R_2 with R_1 and Q_2 with Q_1 (via some fixed isomorphism). Note that $R_1 < L = G^{(\infty)}$ and $R_2 < L_2 := G_2^{(\infty)}$. By hypothesis and by the construction of (G_2, Ξ), $\Phi(L)$ and $\Xi(L_2)$ afford the Weil characters ζ_n^i and $\zeta_n^{(q+1)/2}$, both of degree D. By Lemma 9.12(b), $\mathrm{Trace}(\Phi(y)) = \mathrm{Trace}(\Xi(y))$ for all $y \in R_1$. Conjugating Ξ suitably, we achieve that

$$\Phi(y) = \Xi(y) \tag{9.16.5}$$

for all $y \in R_1$.

(iv) Denote $Z_1 := \mathbf{Z}(G) < C_1$ and $Z_2 := \mathbf{Z}(G_2) < C_2$. Certainly, $\varphi|_H$ and $\xi|_{H_2}$ are both σ-invariant of degree D. By Lemma 9.15(ii), there exist $\theta \in \mathrm{Irr}(Q_1)$ and σ-invariant linear characters $\lambda_1 \in \mathrm{Irr}(J_1/Q_1)$ and $\lambda_2 \in \mathrm{Irr}(J_2/Q_2)$ such that

$$\varphi|_{J_1} = \lambda_1 \cdot \mathrm{Ind}_{Q_1 Z_1}^{J_1}(\theta \boxtimes 1_{Z_1}),\ \xi|_{J_2} = \lambda_2 \cdot \mathrm{Ind}_{Q_2 Z_2}^{J_2}(\theta \boxtimes 1_{Z_2}).$$

Note that $\mathrm{Ind}_{Q_1 Z_1}^{J_1}(\theta \boxtimes 1_{Z_1})$ is trivial at Z_1 and σ-invariant, and similarly $\mathrm{Ind}_{Q_1 Z_2}^{J_2}(\theta \boxtimes 1_{Z_2})$ is trivial at Z_2. So both of them can be viewed as the character of the same representation Θ of $J_1/Z_1 \cong J_2/Z_2 \cong \bar{Q} \rtimes \bar{C}$. Let Λ_i denote the one-dimensional representation of J_i with character λ_i. Then $\Phi|_{J_1 \rtimes \langle\sigma\rangle}$ is an extension of $\Phi|_{J_1} = \Lambda_1 \otimes \Theta$ to $J_1 \rtimes \langle\sigma\rangle$, and $\Xi|_{J_2 \rtimes \langle\sigma\rangle}$ is an extension of $\Xi|_{J_2} = \Lambda_2 \otimes \Theta$ to $J_2 \rtimes \langle\sigma\rangle$ (with Θ being viewed as representations of J_1 and J_2, respectively). Thus, for all $x \in J_1$ we have, using σ-invariance of λ_1,

$$\lambda_1(x)\Theta(x^\sigma) = \Lambda_1(x^\sigma) \otimes \Theta(x^\sigma) = \Phi(x^\sigma) = \Phi(\sigma)\Phi(x)\Phi(\sigma)^{-1} = \lambda_1(x)\Phi(\sigma)\Theta(x)\Phi(\sigma)^{-1},$$

and so $\Theta(x^\sigma) = \Phi(\sigma)\Theta(x)\Phi(\sigma)^{-1}$ for all $x \in J_1/Z_1$. Similarly, $\Theta(x^\sigma) = \Xi(\sigma)\Theta(x)\Xi(\sigma)^{-1}$ for all $x \in J_2/Z_2$. Since $\Phi|_{J_1}$ is irreducible by Lemma 9.15(i), Θ is irreducible. Hence, the equality $\Phi(\sigma)\Theta(x)\Phi(\sigma)^{-1} = \Xi(\sigma)\Theta(x)\Xi(\sigma)^{-1}$ for all $x \in J_1/Z_1$ implies by Schur's lemma that

$$\Xi(\sigma) = \alpha\Phi(\sigma) \tag{9.16.6}$$

for some $\alpha \in \mathbb{C}^\times$. As σ is a p-element, we see that $\mathsf{o}(\alpha)$ is a p-power. On the other hand, $\xi = \bar{\xi}$, so Ξ is self-dual, whence $\det(\Xi(\sigma)) = \pm 1$. As $p \nmid |\det(\Phi(G))|$, we also have that $\mathsf{o}(\Phi(\sigma))$ is coprime to p. Taking the determinant of (9.16.6), we now see that α^D has p'-order, whence so does α, since $p \nmid D$. Consequently, $\alpha = 1$.

Now, (9.16.5) and (9.16.6) show that $\Phi(y) = \Xi(y)$ for all $y \in R_2 \cup \{\sigma\}$. It follows that $\Phi(y) = \Xi(y)$ for all $y \in R_1 \rtimes \langle\sigma\rangle$, a Sylow p-subgroup of G_2. Hence, $\mathrm{Trace}(\Phi(y)) = \mathrm{Trace}(\Xi(y))$ for all p-elements $y \in G_2$, and, together with (9.16.4), this implies that

$$\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Phi(y)) = 1$$

for all p-elements $y \in G$. ☐

Theorem 9.17. *Let q be a prime power, $2|n \geq 4$, and $(n,q) \neq (4,2)$, $(4,3)$, $(6,2)$. Then case* (i) *of Theorem 8.3 does not lead to hypergeometric sheaves in dimension $(q^n-1)/(q+1)$. More precisely, there is no hypergeometric sheaf $\mathcal{H}$ of type (D,m) with $m < D = (q^n-1)/(q+1)$, with finite geometric monodromy group $G = G_{\mathrm{geom}}$ such that G is almost quasisimple with $S = \mathrm{PSU}_n(q)$ as a non-abelian composition factor, and with the image of $I(0)$ being a cyclic group $\langle g_0 \rangle$ where $\bar{\mathsf{o}}(g_0) = D$.*

Proof. Assume the contrary that such a hypergeometric sheaf $\mathcal{H}$ exists, and let $\Phi : G \to \mathrm{GL}(V)$ denote the corresponding representation, with $V = \overline{\mathbb{Q}_\ell}^D$, and with character φ. The assumptions on (n,q) imply by Theorem 7.4 that the characteristic p of $\mathcal{H}$ divides q, that is, $q = p^f$ for some $f \in \mathbb{Z}_{\geq 1}$.

The existence of g_0 implies by Theorem 8.3(i) that G contains a normal subgroup $H > \mathbf{Z}(G)$ such that $H/\mathbf{Z}(G) \cong \mathrm{PGU}_n(q)$.

Assume in addition that $D-m \geq 2$. Then both $\det(\Phi(G))$ and $\mathbf{Z}(G)$ are p'-groups by Proposition 4.4(iv). Now, Proposition 9.16 implies that G admits an irreducible representation $\Psi : G \to \mathrm{GL}(\overline{\mathbb{Q}_\ell}^{D+1})$ with $\mathrm{Trace}(\Psi(y)) - \mathrm{Trace}(\Phi(y)) = 1$ for all p-elements $y \in G$. Applying Theorem 5.1 to Φ and Ψ, we conclude that Ψ leads to a hypergeometric sheaf $\mathcal{H}'$ of rank $D+1$ with geometric monodromy group $\Psi(G)$. In particular, the image $\langle g_0 \rangle$ of $I(0)$ in $\Psi(G)$ has simple spectrum on $\mathcal{H}'$. But this is impossible, since $\bar{\mathsf{o}}(g_0) = D < \mathrm{rank}(\mathcal{H}')$.

It remains to consider the case $\dim \mathsf{Wild} = D - m = 1$. By Proposition 2.22 and Lemma 2.19 of [GT3], $|\varphi(g)|/\varphi(1) \leq (3.95)/4$ for all $1 \neq g \in Q$. It now follows from (7.2.2) that $D \leq 160$, whence $(n,q) = (4,4)$, $(4,5)$, $(8,2)$. However, if $2|q$ then p. (i) of the proof of Proposition 9.16 shows that $Q \leq \mathbf{Z}(G)G^{(\infty)}$, and so we have $|\varphi(g)|/\varphi(1) \leq 0.95$ for all $1 \neq g \in Q$, whence $D \leq 40$, ruling out two of these possible exceptions. The same arguments apply to the remaining exception $(n,q) = (4,5)$. □

9E. Extraspecial normalizers. We will now show that case (i) of Theorem 8.5 cannot lead to hypergeometric sheaves of rank > 9. First we need the following technical result:

Lemma 9.18. *Let $p > 2$ be a prime, and let $n \in \mathbb{Z}_{\geq 1}$ with $p^n \geq 11$. Let $E < \mathrm{GL}(V)$ be an irreducible extraspecial p-subgroup of order p^{1+2n} for $V := \mathbb{C}^{p^n}$, and let $Q \leq \mathbf{N}_{\mathrm{GL}(V)}(E)$ be a nontrivial p-subgroup. Then the following statements hold for $W := p^n - \dim \mathbf{C}_V(Q)$.*

(i) *$W \geq 7$.*
(ii) *If $|Q| \geq 9$, then $W > p^n/2$.*

Proof. (a) It is well known, see e.g. [Wi, Theorem 1], that

$$\mathbf{N}_{\mathrm{GL}(V)}(E)/ZE \hookrightarrow \mathrm{Sp}(E/\mathbf{Z}(E)) \cong \mathrm{Sp}_{2n}(p),\ \mathbf{C}_{\mathbf{N}_{\mathrm{GL}(V)}(E)}(E/\mathbf{Z}(E)) = ZE \tag{9.18.1}$$

for $Z := \mathbf{Z}(\mathrm{GL}(V))$. The statements are obvious in the case $Q \cap Z \neq 1$, so we may assume $Q \cap Z = 1$. Let φ denote the character of Q acting on V. Now, if $\varphi(x) = 0$ for some $1 \neq x \in Q$ of order p, then $\varphi|_X$ contains 1_X with multiplicity p^{n-1} for $X := \langle x \rangle$, and so $W \geq p^{n-1}(p-1)$, yielding both statements. We also note that $\varphi(y) = 0$ for all $y \in ZE \smallsetminus Z$ of order p. Hence, arguing by contradiction, we may assume that

$$Q \cap ZE = 1,\ W < p^n/2,\ \varphi(x) \neq 0 \text{ for all } x \in Q \text{ of order } p. \tag{9.18.2}$$

Consider any element $1 \neq g \in Q$ and let $\bar{g}$ denote its image in $\mathrm{Sp}(E/\mathbf{Z}(E))$. We write $|\mathbf{C}_{E/\mathbf{Z}(E)}(g)| = p^{e(g)}$ for $e(g) \in \mathbb{Z}$. As $g \notin ZE$ by (9.18.2), the second part of (9.18.1) implies that $0 \leq e(g) \leq 2n-1$. Hence

$$|\varphi(g)| \leq p^{e(g)/2} \leq p^{n-1/2} \tag{9.18.3}$$

by [GT1, Lemma 2.4]. Applying (7.2.2), (9.18.3), and using $p^n \geq 11$ but assuming $p > 3$, we again obtain both (i) and (ii). We also obtain (i) when $p = 3$, since $p^n \geq 27$ in this case.

(b) It remains to prove (ii) for $p = 3$, in which case we have $p^n \geq 27$ and may assume $|Q| = 9$. First suppose that Q contains some g with $e(g) \leq 2n-3$. If $\mathsf{o}(g) = 3$, then as $|\varphi(g)| \leq 3^{n-3/2}$ by (9.18.3), we again have $W > p^n/2$ from (7.2.2). If $\mathsf{o}(g) = 9$, then Q contains 6 elements x with $e(x) \leq 2n-3$ and two more elements y with $e(y) \leq 2n-1$. Using the bound (9.18.3) for x and y, we can see that $[\varphi|_Q, 1_Q]/3^n < 0.37$, contradicting (9.18.2). Hence $e(g) \geq 2n-2$ for all $1 \neq g \in Q$. Since $\bar{g} \in \mathrm{Sp}_{2n}(p)$, it follows that g can have either one Jordan block of size 2 (and so $\bar{g}$ is a transvection), or two Jordan blocks of size 2, and all other blocks of size 1 while acting on

$E/\mathbf{Z}(E)$. In particular, g^3 centralizes $E/\mathbf{Z}(E)$, whence $g^3 \in Q \cap GE = 1$ by (9.18.1) and (9.18.2), and so $\exp(Q) = 3$.

We have shown that $Q \cong C_3 \times C_3$ and Q consists of $2a$ elements x with $\bar{x}$ being transvections and $8 - 2a$ elements y with $e(y) = 2n - 2$, where $0 \leq a \leq 4$. If $a \leq 1$, then (7.2.2) implies $W > 3^n/2$, contradicting (9.18.2). Hence Q contains at least 4 elements x with $\bar{x}$ being transvections. As Q embeds in $\mathrm{Sp}_{2n}(3)$ by (9.18.2) and $|Q| = 9$, we have that $Q = \langle g, h\rangle$ with $\bar{g}$ and $\bar{h}$ two distinct, commuting transvections in $\mathrm{Sp}_{2n}(3)$. By [GMST, Lemma 4.5], this pair $(\bar{g}, \bar{h})$ is unique in $\mathrm{Sp}_{2n}(3)$, up to conjugacy. Now we can readily check that $a = 2$.

Recall by [GT1, Lemma 2.4] and (9.18.2) that $|\varphi(g)| = 3^{n-1/2}$; moreover, g centralizes an extraspecial subgroup E_1 of order 3^{2n-1} of E. The E_1-module V is the sum of 3 copies of a simple module of dimension 3^{n-1}. On the other hand, E_1 preserves each of g-eigenspaces on V, and moreover the 1-eigenspace has dimension $> 3^n/2$ by (9.18.2). Replacing g by g^{-1} if necessary, it follows that g has eigenvalues 1 with multiplicity $2 \cdot 3^{n-1}$ and $\zeta := \zeta_3$ with multiplicity 3^{n-1} on V, whence

$$\varphi(g) + \varphi(g^{-1}) = 3^n, \tag{9.18.4}$$

and the same holds for h.

Next we look at $u := gh$, for which we have $e(u) = 2$ and $\mathsf{o}(u) = 3$. As $\varphi(u) \neq 0$ by (9.18.2), [GT1, Lemma 2.4] implies that u acts trivially on the inverse image of order 3^{2n-1} of $\mathbf{C}_{E/\mathbf{Z}(E)}(u)$ in E, and this contains an extraspecial subgroup E_2 of order 3^{2n-3} of E. The E_2-module V is the sum of 9 copies of a simple module of dimension 3^{n-2}. On the other hand, E_2 preserves each of u-eigenspaces on V. Hence, we may denote by $3^{n-2}b$, $3^{n-2}c$, and $3^{n-2}d$ the dimensions of the u-eigenspaces for eigenvalues 1, ζ, and $\bar{\zeta}$, respectively, with $b, c, d \in \mathbb{Z}_{\geq 0}$ and $b + c + d = 9$. We also have by [GT1, Lemma 2.4] that

$$9 = 3^{e(u)}/3^{2n-4} = |\varphi(u)|^2/3^{2n-4} = |b + c\zeta + d\bar{\zeta}|^2 = \big((b-c)^2 + (c-d)^2 + (d-b)^2\big)/2.$$

Note that $18 = 16 + 1 + 1 = 9 + 9 + 0$ are the only two ways to write 18 as the sum of three squares. One can now readily check that $\{b, c, d\} = \{5, 2, 2\}$ or $\{4, 4, 1\}$. But $b > 4$ by (9.18.2), so $(b, c, d) = (5, 2, 2)$, and

$$\varphi(gh) + \varphi((gh)^{-1}) = 2 \cdot 3^{n-1}, \tag{9.18.5}$$

and the same holds for gh^{-1}. Now using (9.18.4) and (9.18.5), we can compute $[\varphi_Q, 1_Q]$ to be $3^n \cdot (13/27)$, i.e. $W/3^n = 14/27$, contradicting (9.18.2). $\square$

Theorem 9.19. *Let $\mathcal{H}$ be an irreducible hypergeometric sheaf of type (D, m) in characteristic p with $D > m$, $D \geq 11$, such that its geometric monodromy group $G = G_{\mathrm{geom}}$ is a finite extraspecial normalizer in some characteristic r. Then $p = r$, $D = p^n$ for some $n \in \mathbb{Z}_{\geq 1}$, and the following statements hold.*

(i) *Suppose $p > 2$. Then $\mathcal{H}$ is Kloosterman, in fact the sheaf $\mathcal{K}l(\mathsf{Char}_{p^n+1} \smallsetminus \{\mathbb{1}\})$ (studied by Pink* [Pink] *and Sawin* [KT1, p. 841]*).*

(ii) *Suppose $p = 2$. Then $\mathbf{Z}(G) \cong C_2$, and so in Lemma 1.1(i)(c) we have that $R = E$ is a normal extraspecial 2-group 2^{1+2n}_ϵ of G for some $\epsilon = \pm$.*

Proof. Since G is a finite extraspecial normalizer, $E \lhd G < \mathbf{N}_{\mathrm{GL}_{r^n}(\mathbb{C})}(E)$ for an irreducible extraspecial r-group $E < \mathrm{GL}_{r^n}(\mathbb{C})$ of order r^{1+2n}, and $D = r^n$. By Theorem 7.5, $p = r$. If furthermore $p = 2$, then, since $\mathbf{Z}(E) \leq \mathbf{Z}(R) \leq \mathbf{Z}(G)$ in Lemma 1.1(c), (ii) follows from Proposition 4.4(v).

From now on, assume $p > 2$, and let $Q \neq 1$ be the image in G of $P(\infty)$. By Lemma 9.18(i), $W := \dim \mathsf{Wild} \geq 7$. Now, if $|Q| < 9$, then the p-group Q has order $p \leq 7$, whence Q affords at most $p - 1$ distinct, nontrivial, irreducible characters on Wild, which are all linear, and so $W \leq p - 1 \leq 6$ by Proposition 7.3, a contradiction. Hence $|Q| \geq 9$, and so $W > D/2$ by Lemma 9.18(ii).

Now, (9.18.1) implies that $G/(G \cap ZE)$ embeds in $\mathrm{Sp}_{2n}(p)$, and so admits a complex representation Λ of degree $(p^n-1)/2 < W$, with kernel K of of order at most 2. Applying Theorem 4.6 to $\Gamma := G$ and Λ, we conclude that $\Lambda(G/(G \cap ZE))$ is a finite cyclic p'-group. As $|K| \leq 2$ and $p > 2$, it follows that $G/(G \cap ZE)$ is an abelian p'-group. Also, note that $G \cap ZE = \mathbf{Z}(G)E$ since $G \geq E$. Now, applying Theorem 8.5 to a generator g_0 of the image of $I(0)$ in G, we see that the coset $g_0\mathbf{Z}(G)E$ generates a cyclic, self-centralizing, maximal torus C_{p^n+1} of $\mathrm{Sp}_{2n}(p)$. It follows that $G = \mathbf{Z}(G)E\langle g_0\rangle$. In fact, since G normalizes $E\langle g_0\rangle$ and $E\langle g_0\rangle$ contains g_0, by Theorem 4.1 we have $G = E\langle g_0\rangle \cong E \rtimes C_{p^n+1}$.

As g_0 is a generator of $I(0)$, the "upstairs" characters of $\mathcal{H}$ are determined by the spectrum of g_0 on $\mathcal{H}$, which consists of all nontrivial $(p^n+1)^{\mathrm{th}}$ roots of unity, hence they are just $\mathsf{Char}_{p^n+1} \smallsetminus \{\mathbb{1}\}$. Suppose $\mathcal{H}$ is not Kloosterman, and we look at the image $Q \rtimes \langle g_\infty\rangle$ of $I(\infty)$ in G for some p'-element g_∞. By Hall's theorem applied to the solvable group G, $\langle g_\infty\rangle$ is contained in a conjugate of the Hall subgroup $\langle g_0\rangle$. In particular, the spectrum of g_∞ on $\mathcal{H}$ is the spectrum of some power g_0^i on $\mathcal{H}$. But $\mathcal{H}$ is irreducible, so the "downstairs" characters of $\mathcal{H}$, which are determined by the spectrum of g_∞ on the tame part Tame, must be disjoint from the "upstairs", whence $m = \dim \mathsf{Tame} = 1$ and the single "downstairs" character is $\mathbb{1}$. Thus $\mathcal{H}$ is the sheaf $\mathcal{H}_1$ studied in [KT5, Corollary 8.2], which, however, was shown therein to have $G_{\mathrm{geom}} \cong \mathrm{PGL}_2(p^n)$, a contradiction. □

Note that the case $(p, \mathbf{Z}(G)) = (2, C_2)$ in Theorem 9.19(ii) can lead to non-Kloosterman sheaves – indeed in [KT8] we constructed hypergeometric sheaves with $G_{\mathrm{geom}} = 2^{1+2n}_{-} \cdot \Omega^{-}_{2n}(2)$ for any $n \geq 4$.

10. Converse theorems

Let us recall that, in Theorems 6.2, 6.4, and 6.6 we have classified all pairs (G, V), where G is a finite almost quasisimple group and V a faithful irreducible $\mathbb{C}G$-representation of G in which some element g has simple spectrum. Next, in Theorem 7.4 we show that if such a group G occurs as G_{geom} for a hypergeometric sheaf in characteristic p and in addition G is a finite group of Lie type in characteristic r, then $p = r$ unless $\dim(V) \leq 22$. Theorem 7.5 gives an analogous result in the case G is an extraspecial normalizer. The classification of triples (G, V, g) that satisfy the Abhyankar condition at p, with G being almost quasisimple or an extraspecial normalizer, is completed in Theorems 8.1, 8.2, 8.3, and 8.5.

However, as shown in §9, not all of these almost quasisimple triples (G, V, g) can give rise to hypergeometric sheaves. The cases that can possibly occur hypergeometrically are the following:

(a) $G/\mathbf{Z}(G) \cong \mathsf{A}_n$ or S_n with $n \geq 5$, and g is as described in Theorem 6.2(i);
(b) G is a quotient of $\mathrm{GL}_n(q)$ in a Weil representation V of degree $(q^n-q)/(q-1)$ or $(q^n-q)/(q-1)$, $q = p^f$, and $\bar{\mathsf{o}}(g) = (q^n-1)/(q-1)$ with $n \geq 3$ (cf. Theorem 8.1), or G is a quotient of $\mathrm{GL}_2(q)$ or $\mathrm{GU}_2(q)$;
(c) G is a quotient of $\mathrm{Sp}_{2n}(q)$ in a Weil representation V of degree $(q^n \pm 1)/2$ with $n \geq 2$, $2 \nmid q = p^f$, and g is of type (α) or type (β) as described in Theorem 8.2(i);
(d) G is a quotient of $\mathrm{GU}_n(q)$ in a Weil representation V of degree $(q^n-q)/(q+1)$ or $(q^n+1)/(q+1)$ with $2 \nmid n \geq 3$, $q = p^f$, and $\bar{\mathsf{o}}(g) = (q^n+1)/(q+1)$, cf. Theorem 8.3(i);
(e) G is a quotient of $\mathrm{GU}_n(q)$ in a Weil representation V of degree $(q^n+q)/(q+1)$ or $(q^n-1)/(q+1)$ with $2|n \geq 4$, $q = p^f$, and g is is as described in Theorem 8.3(iii);
(f) A finite and explicit list of "non-generic" cases, including sporadic groups, as listed in Table 1.

The cases (a)–(e) are indeed shown to occur. Namely, the respective hypergeometric sheaves $\mathcal{H}$ (in characteristic p in (b)–(e)) are explicitly constructed in Theorem 9.3 for case (a), in [KT5] and Theorem 8.6 for case (b), in [KT6] for type (α) in (c) and for (d) with $2 \nmid q$, in [KT7] for type (β) in (c) and for (e), and in [KT8] for case (d) with $2|q$. The extraspecial normalizers, and the sporadic and non-generic cases in (f), are handled in [KT8] and [KRL], [KRLT1]–[KRLT4].

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Department of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address: `nmk@math.princeton.edu`

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: `tiep@math.rutgers.edu`