The Apéry Numbers As a Stieltjes Moment Sequence

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The ***Apéry sequence*** [2][10, A005259] is

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad n = 0, 1, 2, \cdots .$$

From the reference (or a CAS[1]) we find that it satisfies the recurrence

$$(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0, \tag{1}$$

$$A_0 = 1, \qquad A_1 = 5.$$

We will show that the sequence (A_n) is a ***Stieltjes moment sequence***. In fact:

Theorem 1. *There is $c > 0$ and a positive Lebesgue integrable function φ such that*

$$A_n = \int_0^c x^n \, \varphi(x) \, dx$$

for $n = 0, 1, 2, \cdots$.

Definition 2. We say φ is the ***moment density function*** for (A_n).

Notes

I have tried to make the argument as short as possible. This means many asides and variations have been removed.

Some of the proofs may be done using a computer algebra system (CAS). I used Maple 2015. These are the sort of thing that—until 1980 or later—would have been done by paper-and-pencil computation. I have added some of the Maple as footnotes.

This result arose from a question asked by Alan Sokal. It was posted on the MathOverflow discussion board [6]. Pietro Majer provided the idea to use the differential equation.

[1] `SumTools[Hypergeometric][Zeilberger](binomial(n,k)^2*binomial(n+k,k)^2,n,k,En);`

Notation 3. We will use these values.

$$\tau = 1 + \sqrt{2} \approx 2.4142$$
$$c = \tau^4 = 17 + 12\sqrt{2} \approx 33.9705$$
$$c_0 = \tau^{-4} = \frac{1}{c} = 34 - c = 17 - 12\sqrt{2} \approx 0.0294$$

The Differential Equation

We proceed with a discussion of this[2] third-order holonomic Fuchsian ODE:

$$x^2(x^2 - 34x + 1)u'''(x) + 3x(2x^2 - 51x + 1)u''(x) + (7x^2 - 112x + 1)u'(x) + (x - 5)u(x) = 0. \quad \text{(DE3)}$$

We consider x a complex variable, and sometimes consider solutions in the complex plane.

Differential equation (DE3) has four singularities: $\infty, 0, c_0, c$. They are all regular singular points. Series solutions exist adjacent to each of them. From the Frobenius "series solution" method[3] [4, Ch. 5][3, Ch. 3] we may describe these series solutions:

Proposition 4. *The general solution of* (DE3) *near the complex singular point* ∞ *has the form*

$$A\left(\frac{1}{x} + o(x^{-1})\right) + B\left(\frac{\log x}{x} + o(x^{-1})\right) + C\left(\frac{(\log x)^2}{x} + o(x^{-1})\right)$$

as $x \to \infty$, *for complex constants* A, B, C. *The general solution of* (DE3) *near the singular point* 0 *has the form*

$$A\,(1 + o(1)) + B\,(\log x + o(1)) + C\,\left((\log x)^2 + o(1)\right)$$

as $x \to 0$, *for complex constants* A, B, C. *The general solution of* (DE3) *near the singular point* c_0 *has the form*

$$A\left(1 - \frac{240 + 169\sqrt{2}}{48}(x - c_0) + O(|x - c_0|^2)\right) + B\left((x - c_0)^{1/2} + O(|x - c_0|^{3/2})\right) + C\left((x - c_0) + O(|x - c_0|^2)\right)$$

as $x \to c_0$, *for complex constants* A, B, C. *The general solution of* (DE3) *near the singular point* c *has the form*

$$A\left(1 - \frac{240 - 169\sqrt{2}}{48}(x - c) + O(|x - c|^2)\right)$$

[2] `DE3:=x^2*(x^2-34*x+1)*diff(u(x),x$3)+3*x*(2*x^2-51*x+1)*(diff(u(x),x$2))+(7*x^2-112*x+1)*diff(u(x),x)+(x-5)*u(x);`

[3] `dsolve(DE3,u(x),series,x=c);`

$$+ B\left((x-c)^{1/2} + O(|x-c|^{3/2})\right) + C\left((x-c) + O(|x-c|^2)\right)$$

as $x \to c$, for complex constants A, B, C.

Corollary 5. *If $u(x)$ is any solution of* (DE3) *on* $(0, c_0)$ *or on* (c_0, c)*, then $u(x)$ has at worst logarithmic singularities. So $u(x)$ is (absolutely, Legesgue) integrable.*

Notation 6. Four particular solutions of (DE3) will be named for use here:
- Solution $u_\infty(x) = 1/x + o(x^{-1})$ as $x \to \infty$, defined in the complex plane cut on the real axis interval $[0, c]$.
- Solution $u_0(x) = 1 + o(1)$ as $x \to 0^+$, defined for $0 < x < c_0$.
- Solution $v_0(x) = \log x + o(1)$ as $x \to 0^+$, defined for $0 < x < c_0$.
- Solution $v_2(x) = (c-x)^{1/2} + O(|x-c|^{3/2})$ as $x \to c^-$, defined for $c_0 < x < c$.

Proposition 7. *The Maclaurin series for $u_0(x)$ is the generating function for the Apéry sequence:*

$$u_0(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad |x| < c_0.$$

Proof. This may be checked by your CAS. The recurrence (1)[4] converted to a differential equation[5] yields (DE3). Of course the radius of convergence extends to the nearest singularity at c_0. □

Corollary 8. *$u_0(x) > 0$ for $0 < x < c_0$.*

Determining the signs of v_0 and v_2 will be more difficult.

Proposition 9. *The Laurent coefficients for $u_\infty(z)$ are the Apéry numbers:*

$$u_\infty(z) = \sum_{n=0}^{\infty} \frac{A_n}{z^{n+1}}, \qquad |z| > c.$$

Proof. Check that if $u(x)$ is a solution of (DE3), then $w(z) = u(1/z)/z$ is also a solution of (DE3). Matching the boundary conditions, we get

$$u_\infty(z) = \frac{1}{z} u_0\left(\frac{1}{z}\right).$$

Apply Prop. 7. □

Note: In general, for other similar sequences that can be handled in this same way:
(a) the generating function for the sequence, and
(b) the moment density function for the sequence
satisfy *different* differential equations.

[4] `Rec:=(n+1)^3*Q(n+1)-(34*n^3+51*n^2+27*n+5)*Q(n)+n^3*Q(n-1);`
[5] `gfun[rectodiffeq]({Rec,Q(0)=1,Q(1)=5},Q(n),u(x));`

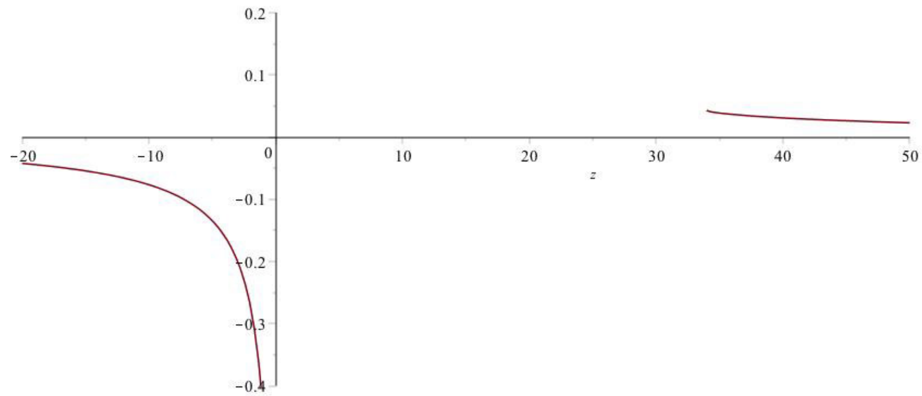

Figure 1: $u_\infty(x)$

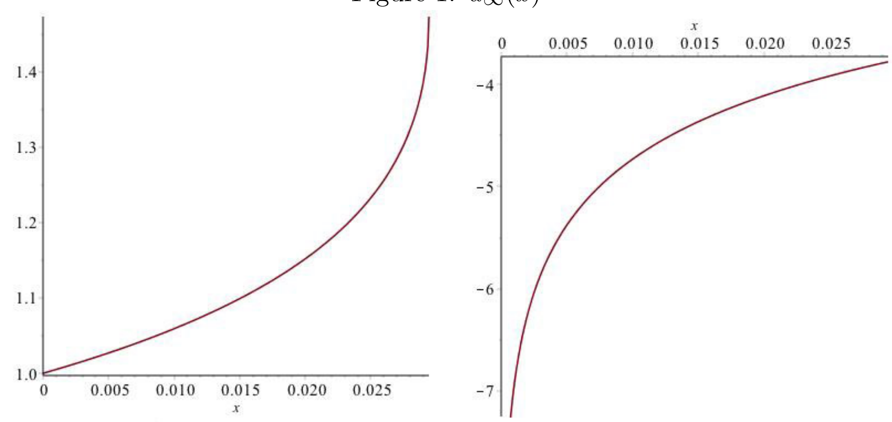

Figure 2: $u_0(x)$ and $v_0(x)$

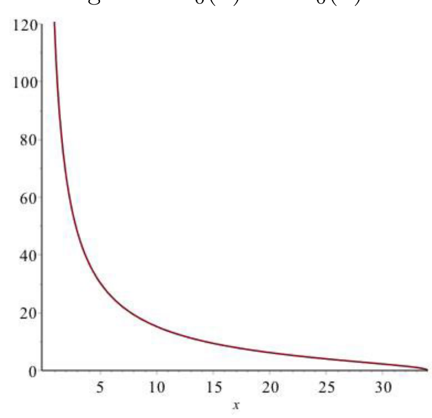

Figure 3: $v_2(x)$

The Function φ

Series solution u_∞ of (DE3) is meromorphic and single-valued near ∞. It continues analytically to the complex plane with a cut on the interval $[0, c]$ of the real axis. We will still use the notation u_∞ for that continuation. Since the Laurent coefficients are all real, we have

$$u_\infty(\,\overline{z}\,) = \overline{u_\infty(z)} \tag{2}$$

near ∞, and therefore on the whole domain. In particular, $u_\infty(z)$ is real for z on the real axis (except the cut, of course). Define upper and lower values on the cut $0 < x < c$:

$$u_\infty(x + i0) = \lim_{\delta\to 0+} u_\infty(x + i\delta), \quad u_\infty(x - i0) = \lim_{\delta\to 0+} u_\infty(x - i\delta).$$

Then from (2) we have

$$u_\infty(x - i0) = \overline{u_\infty(x + i0)}, \qquad 0 < x < c. \tag{3}$$

Notation 10.

$$\varphi(x) = \frac{1}{2\pi i}\big(u_\infty(x - i0) - u_\infty(x + i0)\big).$$

Function u_∞ in the upper half plane extends analytically to a solution in a neighborhood of $(0, c_0)$, and similarly u_∞ in the lower half plane. Thus $\varphi(x)$ restricted to $(0, c_0)$ is a solution of (DE3), since it is a linear combination of solutions. In the same way, $\varphi(x)$ restricted to (c_0, c) is a solution of (DE3).

See Figure 4; an enlargement shows the behavior near the singular point c_0. We will see that φ has square root asymptotics near the right endpoint c (Prop. 25) and logarithmic asymptotics near the left endpoint 0 (Prop. 26).

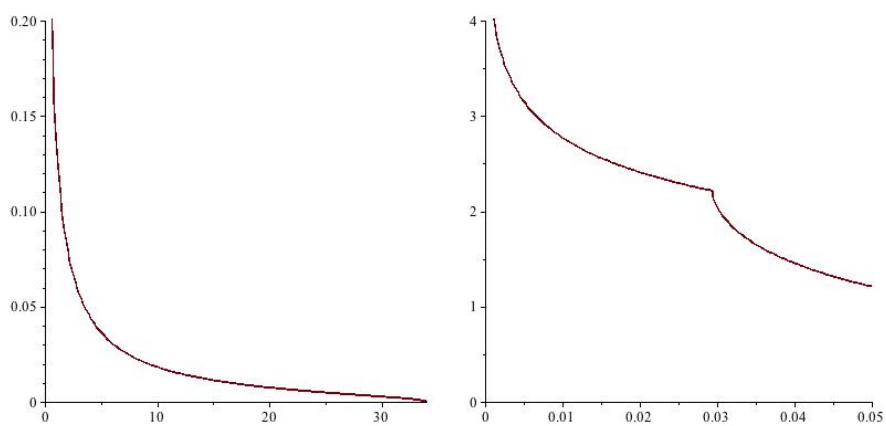

Figure 4: Moment density function $\varphi(x)$

Proposition 11. *The Apéry numbers satisfy* $A_k = \int_0^c x^k\,\varphi(x)\,dx$, $k = 0, 1, 2, \cdots$.

Proof. Fix a nonnegative integer k. For $\delta > 0$, let Γ_δ be the contour in the complex plane at distance δ from $[0, c]$, as in Figure 5. (Two line segments and two semicircles; traced counterclockwise.) Now u_∞ has at worst logarithmic singularities, so we have this limit:

$$\lim_{\delta\to 0+} \oint_{\Gamma_\delta} z^k u_\infty(z)\, dz = \int_0^c x^k \big(u_\infty(x - i0) - u_\infty(x + i0)\big) dx.$$

On the other hand, $z^k u_\infty(z)$ is analytic on and outside the contour Γ_δ, except at ∞ where it has an isolated singularity with residue A_k. Therefore

$$\oint_{\Gamma_\delta} z^k u_\infty(z)\, dz = 2\pi i A_k.$$

Thus

$$A_k = \int_0^c \frac{x^k}{2\pi i} \big(u_\infty(x - i0) - u_\infty(x + i0)\big)\, dx. \qquad \square$$

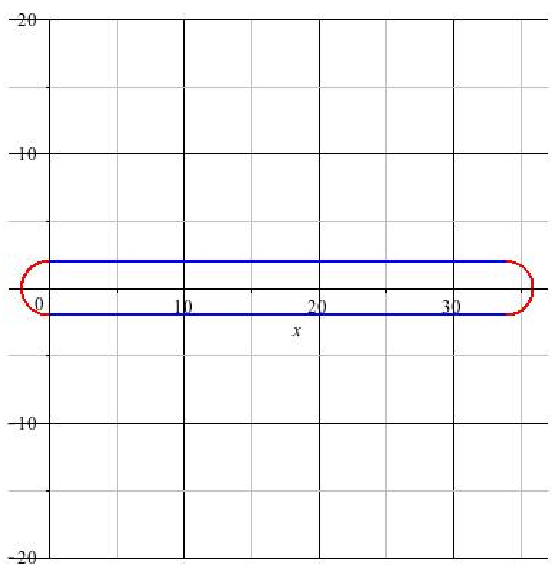

Figure 5: Contour Γ_δ

What remains to be proved: φ is nonnegative on $(0, c)$ (Cor. 27). From (3) we know that $\varphi(x)$ is real on $(0, c)$.

Heun General Functions

Some of the basic solutions in Notation 6 may be represented in terms of Heun functions. The Heun functions are described in [7, 8, 9].

Definition 12. Let complex parameters $a, q, \alpha, \beta, \gamma, \delta, \varepsilon$ be given satisfying $a \neq 0$, $\alpha+\beta+1 = \gamma+\delta+\varepsilon$, $\delta \neq 0$, and $\gamma \neq 0, -1, -2, \cdots$. Define[6] the ***Heun general function***

$$\mathrm{Hn}\left(\begin{array}{c|cc|c} a & \alpha\,, & \beta & \\ q & \gamma\,; & \delta & z \end{array}\right) = \sum_{n=0}^{\infty} p_n z^n, \tag{4}$$

where the Maclaurin coefficients satisfy initial conditions

$$p_0 = 1, \qquad p_1 = \frac{q}{a\gamma},$$

and recurrence

$$R_n p_{n+1} - (q + Q_n) p_n + P_n p_{n-1} = 0,$$

with

$$\begin{aligned} R_n &= a(n+1)(n+\gamma), \\ Q_n &= n\big((n-1+\gamma)(1+a) + a\delta + \varepsilon\big), \\ P_n &= (n-1+\alpha)(n-1+\beta). \end{aligned}$$

This function satisfies the ***Heun general differential equation***

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right) w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)}\, w(z) = 0.$$

(Consult the references, or use your CAS to go from the recurrence to the differential equation.) This DE has singularities at $\infty, 0, 1, a$; all regular singular points. Convergence of the series extends to the nearest singularity, so the radius of convergence in (4) is $\min\{1, |a|\}$.

Proposition 13. *Within the radius of convergence:*

$$\begin{aligned} u_0(x) &= \mathrm{Hn}\left(\begin{array}{c|rc|c} a_2 & 1/2\,, & 1/2 & \\ q_4 & 1\,; & 1/2 & cx \end{array}\right)^2 \\ v_2(x) &= \frac{(x-c_0)(c-x)^{1/2}}{c-c_0}\, \mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 3/2\,, & 3/2 & \\ q_1 & 3/2\,; & 1 & 1-c_0x \end{array}\right) \cdot \\ &\qquad \cdot\ \mathrm{Hn}\left(\begin{array}{c|rc|c} a_1 & 1\,, & 1 & \\ q_2 & 1/2\,; & 1 & 1-c_0x \end{array}\right) \\ u_\infty(z) &= \frac{1}{z}\mathrm{Hn}\left(\begin{array}{c|rc|c} a_2 & 1/2\,, & 1/2 & \\ q_4 & 1\,; & 1/2 & \frac{c}{z} \end{array}\right)^2, \end{aligned}$$

where

$$\begin{aligned} a_1 &= 1 - c_0^2 = -576 + 408\sqrt{2} \approx 0.9991 \\ a_2 &= c^2 = 577 + 408\sqrt{2} \approx 1153.9991 \end{aligned}$$

[6] `HeunG(a,q,alpha,beta,gamma,delta,z)`

$$
\begin{aligned}
q_1 &= -\frac{1317}{4} + 234\sqrt{2} \approx 1.676 \\
q_2 &= \tau^{-1}(1+c_0) = -42 + 30\sqrt{2} \approx 0.4264 \\
q_4 &= \frac{5c}{2} = \frac{85}{2} + 30\sqrt{2} \approx 84.93
\end{aligned}
$$

Proof. In each case verify that it satisfies the differential equation[7] and boundary properties[8] that specify the solution. □

All Coefficients Positive

In some cases we can determine that all Maclaurin coefficients of a Heun general function

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a & \alpha\,, & \beta & \multirow{2}{*}{z} \\ q & \gamma\,; & \delta & \end{array}\right)
$$

are positive. When that is true, then in particular this function will be positive and increasing and convex on $(0, R)$ where $R = \min\{1, |a|\}$ is the radius of convergence.

Lemma 14. *All Maclaurin coefficients are positive in*

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 3/2\,, & 3/2 & \multirow{2}{*}{z} \\ q_1 & 3/2\,; & 1 & \end{array}\right),
$$

where $a_1 = -576 + 408\sqrt{2}$ *and* $q_1 = -\frac{1317}{4} + 234\sqrt{2}$.

Proof. Let p_n be the Maclaurin coefficients. Then

$$
R_n p_{n+1} - (q_1 + Q_n)p_n + P_n p_{n-1} = 0,
$$

with

$$
\begin{aligned}
R_n &= a_1(n+1)(n+\tfrac{3}{2}) \\
Q_n &= n\big((n+\tfrac{1}{2})(1+a_1) + a_1 + \tfrac{3}{2}\big) \\
P_n &= (n+\tfrac{1}{2})^2.
\end{aligned}
$$

Write $r_n = p_n/p_{n-1}$ and rearrange:

$$
r_{n+1} = \frac{q_1 + Q_n}{R_n} - \frac{P_n}{R_n}\,\frac{1}{r_n}.
$$

Recall that $|a_1| < 1$; we expect $r_n \to 1/a_1$. We claim: if

$$
n \geq 45 \qquad \text{and} \qquad 1 - \frac{1}{10n} < r_n < \frac{1}{a_1},
$$

[7] `subs(u(x)=v2,DE3):  simplify(%);`
[8] `MultiSeries[series](v2,x=c,2);`

then also

$$1-\frac{1}{10(n+1)}<r_{n+1}<\frac{1}{a_1}.$$

Once the claim is proved, all that remains is checking that $p_0,\cdots,p_{45}$ are positive, and

$$1-\frac{1}{450}<r_{45}<\frac{1}{a_1}.$$

By induction we conclude that $r_n>0$ for all $n\geq 45$. So p_n with $n>45$ is a product of positive numbers

$$p_{45}\ r_{46}r_{47}r_{48}\cdots r_n,$$

so $p_n>0$.

Proof of the claim. Since

$$r\mapsto\frac{q_1+Q_n}{R_n}-\frac{P_n}{R_n}\,\frac{1}{r}$$

is an increasing function, we need only check

$$1-\frac{1}{10(n+1)}<\frac{q_1+Q_n}{R_n}-\frac{P_n}{R_n}\,a_1<\frac{1}{a_1}$$

and

$$1-\frac{1}{10(n+1)}<\frac{q_1+Q_n}{R_n}-\frac{P_n}{R_n}\,\frac{1}{1-\frac{1}{10n}}<\frac{1}{a_1}$$

where $n\geq 45$. Your CAS can be used for this. □

A warning for the computations. If you do this using 20-digit arithmetic—as I did at first—you may erroneously conclude that it is false. You may see negative coefficients. With exact arithmetic, we find that r_{45} involves integers with more than 100 digits. To compare $\sqrt{2}$ to a rational number with 100-digit numerator and denominator, there are two methods: we can square those 100-digit numbers, or we can use a decimal value of $\sqrt{2}$ accurate to more than 100 places. Of course a modern CAS can do either.

Lemma 15. *All Maclaurin coefficients are positive in*

$$\operatorname{Hn}\left(\begin{array}{c|cc|c} a_1 & 1\,, & 1 & \\ q_2 & 1/2\,; & 1 & z\end{array}\right),$$

where $a_1=-576+408\sqrt{2}$ *and* $q_2=-42+30\sqrt{2}$.

Proof. The proof is similar to Lemma 14. Let p_n be the coefficients, and $r_n=p_n/p_{n-1}$. Then

$$r_{n+1}=\frac{q_2+Q_n}{R_n}-\frac{P_n}{R_n}\,\frac{1}{r_n},$$

with

$$
\begin{aligned}
R_n &= a_1(n+1)(n+\tfrac{1}{2}),\\
Q_n &= n\big((n-\tfrac{1}{2})(1+a_1)+a_1+\tfrac{3}{2}\big),\\
P_n &= n^2.
\end{aligned}
$$

We claim: If

$$
n \geq 18 \qquad \text{and} \qquad 1-\frac{1}{4n} < r_n < \frac{1}{a_1},
$$

then also

$$
1-\frac{1}{4(n+1)} < r_{n+1} < \frac{1}{a_1}.
$$

The remainder of the proof is similar to Lemma 14. □

Proposition 16. *$v_2(x) > 0$ for $c_0 < x < c$.*

Proof. By Lemma 14, all Maclaurin coefficients of

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 3/2\,, & 3/2 & \\ q_1 & 3/2\,; & 1 & z \end{array}\right)
$$

are positive. It has radius of convergence $a_1 = 1 - c_0^2$, so

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 3/2\,, & 3/2 & \\ q_1 & 3/2\,; & 1 & 1-c_0x \end{array}\right) > 0
$$

for all x with $c_0 < x < c$. By Lemma 15, all Maclaurin coefficients of

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 1\,, & 1 & \\ q_2 & 1/2\,; & 1 & z \end{array}\right)
$$

are positive. Again,

$$
\mathrm{Hn}\left(\begin{array}{c|cc|c} a_1 & 1\,, & 1 & \\ q_2 & 1/2\,; & 1 & 1-c_0x \end{array}\right) > 0
$$

for all x with $c_0 < x < c$. Also

$$
\frac{(x-c_0)(c-x)^{1/2}}{c-c_0}
$$

is positive on (c_0, c). The product of three positive factors is $v_2(x)$ on (c_0, c), so $v_2(x) > 0$. □

Hypergeometric Function

Some Heun functions can be expressed in terms of hypergeometric ${}_2F_1$ functions [1, Chap. 2–3]. Here, we will use only one of them.[9]

Definition 17. $${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}\,\frac{z^n}{27^n}$$

Lemma 18. (a) ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right)$ *has radius of convergence* 1. (b) *For* $0<z<1$, *we have* ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right) > 1$. (c) *As* $\delta \to 0^+$,

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\delta\right) = -\frac{\sqrt{3}}{2\pi}\log\delta + \frac{3\sqrt{3}\log 3}{2\pi} + o(1).$$

Proof. (a) Ratio test.

(b) All Maclaurin coefficients are positive, and the constant term is 1.

(c) Due to Gauss (or perhaps Goursat?), see [5, Thm. 2.1.3][11],

$$\begin{aligned}{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\delta\right) &= \frac{\Gamma(1)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}\Big[\log\tfrac{1}{\delta} - 2\gamma - \psi\left(\tfrac{1}{3}\right) - \psi\left(\tfrac{2}{3}\right)\Big] + o(1)\\ &= -\frac{\sqrt{3}}{2\pi}\log\delta + \frac{3\sqrt{3}\log 3}{2\pi} + o(1).\end{aligned}$$

Here γ is Euler's constant and ψ is the digamma function. Use [1, Thm. 1.2.7] to evaluate the digamma of a rational number. ☐

Lemma 19. *Let the degree* 1 *Taylor polynomial for* ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right)$ *at* $z_0 = 1/(2^{3/2}\tau)$ *be* ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right) = S_0 + S_1\cdot(z-z_0) + o(|z-z_0|)$ *as* $z \to z_0$. *Then*

$$S_0\cdot(3S_1+\sqrt{2}S_0) = \frac{3^{3/2}2^{1/2}}{\pi}.$$

Proof. Reference to be supplied? ☐

To complete the proof of Theorem 1, we do not need the exact value in Lemma 19, but only that it is positive; which is clear from the fact that all Maclaurin coefficients of ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;z\right)$ are positive and z_0 is positive.

Notation 20.

$$\begin{aligned}\mu(x) &= \frac{\left(3-3x-\sqrt{x^2-34x+1}\,\right)^{1/2}}{\sqrt{2}\,(x+1)}\\ \mu_2(x) &= \frac{\left(3-3x+\sqrt{x^2-34x+1}\,\right)^{1/2}}{\sqrt{2}\,(x+1)}\\ \lambda(x) &= \frac{x^3+30x^2-24x+1-(x^2-7x+1)\sqrt{x^2-34x+1}}{2(x+1)^3}\end{aligned}$$

[9] `hypergeom([1/3,2/3],[1],z)`

$$\lambda_2(x) = \frac{x^3 + 30x^2 - 24x + 1 + (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x+1)^3}$$

(See Figures 6 and 7.)

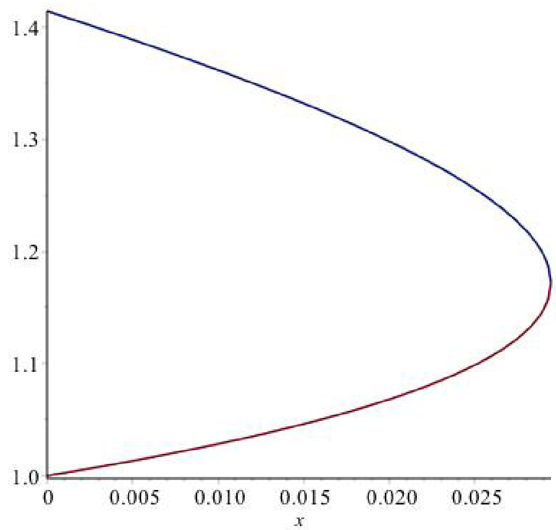

Figure 6: μ (bottom) and μ_2 (top)

Lemma 21. *For $0 < x < c_0$, we have $\mu(x) > 1$, $\mu_2(x) > 1$, $0 < \lambda(x) < 1$, and $0 < \lambda_2(x) < 1$.*

Proof. Elementary inequalities. □

Lemma 22. *As $x \to 0^+$,*

$$\mu(x)\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) = 1 + \frac{5}{2}\,x + O(x^2),$$
$$\mu_2(x)\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right) = \frac{-\sqrt{3}}{\pi\sqrt{2}}\,\log x + o(1).$$

The second one indeed has constant term zero.

Proof. Compute (as $z \to 0$ and $x \to 0$):

$$\mu(x) = 1 + \frac{5}{2}x + O(x^2)$$
$$\lambda(x) = 27x^2 + O(x^3)$$
$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right) = 1 + \frac{2}{9}z + O(z^2)$$
$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) = 1 + 6x^2 + O(x^3)$$

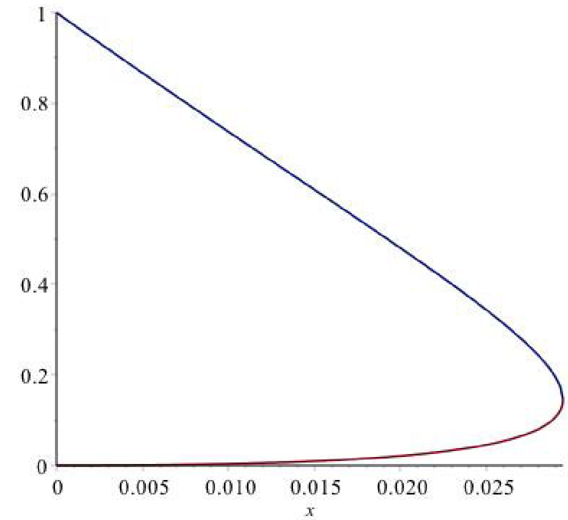

Figure 7: λ (bottom) and λ_2 (top)

$$\mu(x)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda(x)\right) = 1+\frac{5}{2}x+O(x^2)$$

For the second one, we apply Lemma 18(c). As $x \to 0$:

$$\begin{aligned}
\mu_2(x) &= \sqrt{2}-\frac{7}{\sqrt{2}}x+O(x^2) \\
\lambda_2(x) &= 1-27x+O(x^2) \\
{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda_2(x)\right) &= -\frac{\sqrt{3}}{2\pi}\log(27x)+\frac{3\sqrt{3}\log 3}{2\pi}+o(1) \\
&= -\frac{\sqrt{3}}{2\pi}(\log 27+\log x)+\frac{3\sqrt{3}\log 3}{2\pi}+o(1) \\
&= -\frac{\sqrt{3}}{2\pi}\log x+o(1) \\
\mu_2(x){}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda_2(x)\right) &= -\frac{\sqrt{3}}{\sqrt{2}\,\pi}\log x+o(1) \qquad \square
\end{aligned}$$

Proposition 23.

$$\begin{aligned}
u_0(x) &= \mu(x)^2\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda(x)\right)^2, \\
v_0(x) &= -\frac{2\pi}{\sqrt{3}(x+1)}\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda(x)\right)\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda_2(x)\right), \\
u_\infty(z) &= \frac{1}{z}\mu\left(\frac{1}{z}\right)^2\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda\left(\frac{1}{z}\right)\right)^2.
\end{aligned}$$

Proof. Note: $\mu(x)\mu_2(x) = \sqrt{2}/(x+1)$. Verify that these expressions satisfy (DE3) as usual. Then verify the asymptotics using Lemma 22. $\square$

How were these formulas found? The first one is from Mark van Hoeij [10, `A005259`]; I do not know how he found it. But then it is natural to try the other square root, since that will still satisfy the same differential equation.

Proposition 24. *$v_0(x) < 0$ for $0 < x < c_0$.*

Proof. For $0 < x < c_0$: By Lemma 21, $0 < \lambda(x) < 1$, so by Lemma 18(b), ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda(x)\right) > 0$. Similarly, ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda_2(x)\right) > 0$. $\square$

The Two Endpoints

Proposition 25. *On interval (c_0, c) we have exactly $\varphi(x) = v_2(x)/(2^{5/4}\tau^4\pi^2)$.*

Proof. We examine the solution $u_\infty(x)$ of (DE3) on the interval $(c, +\infty)$. As $\delta \to 0^+$, the Frobenius series solution shows that

$$u_\infty(c+\delta) = A + B\sqrt{\delta} + C\delta + O(\delta^{3/2}) \tag{5}$$

for some real constants A, B, C; we will have to evaluate the constant B below. Following (5) around the point c by a half-turn in either direction, we get

$$\begin{aligned}
u_\infty(c-\delta-i0) &= A + B(-i)\sqrt{\delta} - C\delta + O(\delta^{3/2}) \\
u_\infty(c-\delta+i0) &= A + Bi\sqrt{\delta} - C\delta + O(\delta^{3/2}) \\
\varphi(c-\delta) &= \frac{1}{2\pi i}\big(u_\infty(c-\delta-i0) - u_\infty(c-\delta+i0)\big) \\
&= \frac{0A - 2Bi\sqrt{\delta} + 0C\delta}{2\pi i} + O(\delta^{3/2}) \\
&= \frac{-B}{\pi}\sqrt{\delta} + O(\delta^{3/2}).
\end{aligned}$$

Therefore $\varphi(x) = (-B/\pi)v_2(x)$ on (c_0, c).

On interval $(c, +\infty)$, we have

$$u_\infty(x) = \frac{1}{x}\mu\left(\frac{1}{x}\right)^2 {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \lambda\left(\frac{1}{x}\right)\right)^2.$$

Argument $\lambda(1/x)$ stays inside the unit disk, so no analytic continuation is required. Now $\lambda(c_0) = \lambda_2(c_0) = 1/(2^{3/2}\tau)$, called z_0 in Lemma 19. Let S_0, S_1 also be as in Lemma 19. As $\delta \to 0^+$,

$$\begin{aligned}
\frac{1}{c+\delta} &= \frac{1}{\tau^4} + O(\delta) \\
\mu\left(\frac{1}{c+\delta}\right) &= \frac{\tau}{2^{1/4}3^{1/2}} - \frac{1}{4\cdot 3\cdot \tau}\sqrt{\delta} + O(\delta)
\end{aligned}$$

$$\begin{aligned}
\lambda\left(\frac{1}{c+\delta}\right) &= \frac{1}{2^{3/2}\tau} - \frac{\sqrt{3}}{2^{9/4}\tau^2}\sqrt{\delta} + O(\delta) \\
{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda\left(\frac{1}{c+\delta}\right)\right) &= S_0 - \frac{\sqrt{3}}{2^{9/4}\tau^2} S_1\sqrt{\delta} + O(\delta) \\
u_\infty(c+\delta) &= \frac{S_0^2}{3\sqrt{2}\tau^2} - \frac{S_0(3S_1+\sqrt{2}S_0)}{2^{7/4}3^{3/2}\tau^4}\sqrt{\delta} + O(\delta) \\
&= \frac{S_0^2}{3\sqrt{2}\tau^2} - \frac{1}{2^{5/4}\tau^4\pi}\sqrt{\delta} + O(\delta).
\end{aligned}$$

So we get $B = -1/(2^{5/4}\tau^4\pi)$. □

Proposition 26. *On interval* $(0, c_0)$ *we have exactly* $\varphi(x) = -6v_0(x)/\pi^2$.

Proof. We examine the solution $u_\infty(x)$ of (DE3) on the interval $(-\infty, 0)$. As $\delta \to 0^+$, the Frobenius series solution shows that

$$u_\infty(-\delta) = A + B\log\delta + C(\log\delta)^2 + o(1) \tag{6}$$

for some real constants A, B, C; we will have to evaluate the constants B and C below. Following (6) around the point 0 by a half-turn in either direction, we get

$$\begin{aligned}
u_\infty(\delta - i0) &= A + B(\log\delta + i\pi) + C(\log\delta + i\pi)^2 + o(1) \\
u_\infty(\delta + i0) &= A + B(\log\delta - i\pi) + C(\log\delta - i\pi)^2 + o(1) \\
\varphi(\delta) &= \frac{1}{2\pi i}\big(u_\infty(\delta - i0) - u_\infty(\delta + i0)\big) \\
&= \frac{2Bi\pi + 4Ci\pi\log\delta}{2\pi i} + o(1) \\
&= B + 2C\log\delta + o(1)
\end{aligned}$$

Therefore $\varphi(x) = Bu_0(x) + 2Cv_0(x)$ on $(0, c_0)$.

On interval $(-\infty, 0)$, we have

$$u_\infty(x) = \frac{1}{x}\mu\left(\frac{1}{x}\right)^2 {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda\left(\frac{1}{x}\right)\right)^2.$$

Argument $\lambda(1/x)$ stays inside the unit disk, so this is an easy analytic continuation of u_∞. As $\delta \to 0^+$,

$$\begin{aligned}
\frac{1}{-\delta} &= \frac{-1}{\delta} + O(1) \\
\mu\left(\frac{1}{-\delta}\right)^2 &= \delta + O(\delta^2) \\
\lambda\left(\frac{1}{-\delta}\right) &= 1 - 27\delta^2 + O(\delta^3)
\end{aligned}$$

So by Lemma 18(c),

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\lambda\left(\frac{1}{-\delta}\right)\right) &= \frac{-\sqrt{3}}{2\pi}\,2\log\delta + o(1)\\
u_\infty(-\delta) &= \frac{-3}{\pi^2}(\log\delta)^2 + o(1).
\end{aligned}
$$

Thus we get $B = 0$ and $C = -3/\pi^2$. □

Corollary 27. *The moment density φ may be written*

$$
\varphi(x) = \begin{cases} \dfrac{-6}{\pi^2}\,v_0(x), & 0 < x < c_0,\\[2ex] \dfrac{1}{2^{5/4}\tau^4\pi^2}\,v_2(x), & c_0 \le x \le c. \end{cases}
$$

It is positive on $(0, c_0) \cup (c_0, c)$.

Proof. For $0 < x < c_0$, by Prop. 24 $v_0(x) < 0$. For $c_0 < x < c$, by Prop. 16 $v_2(x) > 0$. □

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