## REPORT ON

# ZHÌ－WĚI SŪN＇S 1－3－5 CONJECTURE AND SOME OF ITS REFINEMENTS 

ANTÓNIO MACHIAVELO，ROGÉRIO REIS，AND NIKOLAOS TSOPANIDIS


#### Abstract

We report here on the computational verification of a re－ finement of Zhì－Wěi Sūn＇s＂1－3－5 conjecture＂for all natural numbers up to 105103560126 ．This，together with a result of two of the authors， completes the proof of that conjecture．


## 1．Introduction

In a paper on refinements of Lagrange＇s four squares theorem，Zhì－Wěi Sūn（孙智伟）made the conjecture that any $m \in \mathbb{N}$ can be written as a sum of four squares，$x^{2}+y^{2}+z^{2}+t^{2}$ ，with $x, y, z, t \in \mathbb{N}_{0}$ ，in such a way that $x+3 y+5 z$ is a perfect square．This is Conjecture 4．3（i）in Sūn17， and Zhì－Wěi Sūn called it the＂1－3－5 conjecture＂．Qìng－Hǔ Hóu（侯庆虎） verified it up to $10^{10}$ ：see https：／／oeis．org／A271518．We report here on our computational verification of this conjecture for all $m \leq 105103560126$.

This last number arose from the work［MT20］，in which it was proved that the 1－3－5 conjecture is true for all numbers

$$
m>\left(\frac{10}{\sqrt[4]{35}-\sqrt[4]{34}}\right)^{4} \simeq 105103560126.80255537
$$

Hence，the computation we are here reporting on，together with the main result of MT20］，completes the proof that the 1－3－5 conjecture holds for all natural numbers．

Along the way，we have used another conjecture of Sūn，at his own sug－ gestion，Conjecture 4．9（ii）of［S̄̄n19］，the part whose content is as follows．

Conjecture 1 （Zhì－Wěi Sūn）．Any positive integer can be written as $x^{2}+$ $y^{2}+z^{2}+t^{2}$ with $x, y, z, t \in \mathbb{N}_{0}$ such that $x+3 y+5 z$ is a square，and either $x$ is three times a square，or $y$ is a square，or $z$ is a square．

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As we will see below, we checked this conjecture for all natural numbers up to 105103560 126, which implies the 1-3-5 conjecture for the same range. We have noticed that, actually, for most numbers, one may drop the possibility that $z$ is a square. Moreover, it now seems that after some point on, all numbers have a representation as in the statement of Conjecture 1, but with $x \in\{0,3\}$ or $y \in\{0,1\}$. This is the content of Conjecture 2 below.

## 2. The Modus Operandi

We will say that a quadruple $(x, y, z, t) \in \mathbb{N}_{0}^{4}$ is a 1-3-5 representation of $m$ if $x^{2}+y^{2}+z^{2}+t^{2}=m$ and $x+3 y+5 z$ is a perfect square. Since it is clear that if a number $m$ has the 1-3-5 representation $(x, y, z, t)$, then $(4 x, 4 y, 4 z, 4 t)$ is a 1-3-5 representation of $16 m$, in order to verify the 1-3-5 conjecture, we may disregard multiples of 16 .

Early on, during the first computations we made, it was found out that, apparently, only the following 15 numbers:

$$
31,43,111,151,168,200,248,263,319,456,479,871,1752,1864,3544,
$$

and their multiples by powers of 16 , do not have a $1-3-5$ representation $(x, y, z, t)$ where either $x$ is three times a square or $y$ a square. We used this, together with Conjecture 1, to speed up the search.

Furthermore, while testing the program's speed, and while running it for values up to $10^{7}$, and then up $10^{8}$, it was noted that only 123 numbers have a special 1-3-5 representation requiring $x$ and $y$ bigger than 4 , and that the last one of these was 779832 . This observation motivated the introduction of a "tolerance" input on the program, make it to exit whenever the list of numbers to be checked was smaller than a certain size, and returning the list of those numbers, which can then be checked individually in a faster way.

To tackle the verification of the 1-3-5 conjecture up to the required number, 105103560 126, a program in the C programming language was written that takes into account the above remarks, and runs as follows. Firstly, it allocates the necessary memory for the range one is checking, ignoring the multiples of 16 . Then, it looks for all triplets $(x, y, z)$ such that $x+3 y+5 z$ is a square, and either $x$ is three times a square or $y$ is a square. Each time such a triplet is found, one removes from the appropriate memory the numbers $m=x^{2}+y^{2}+z^{2}+s$, for all squares $s$ with $m$ in the desired range. The program ends when the list of the remaining numbers has size less than a prescribed number, which is part of the input.

Since what is wanted is to check, for every integer below a given bound, if there is an additive decomposition in four squares, the naive implementation of such check would have a $\mathcal{O}\left(n^{3}\right)$ complexity 1 . But the simple observation that we can check the existence of the fourth square summand by subtracting the summation of the first three to the integer that is tested, and check that this result is a square, lowers this complexity to $\mathcal{O}\left(n^{\frac{5}{2}}\right)$. This is still a complexity that makes the algorithm intractable for the desired bound. Since we could not find a canonical ordering for the possible summands that would allow to efficiently prune the search tree, the solution relied in the classic space/time tradeoff. Thus, we represented the whole integer search space as a bitmap, and with a $\mathcal{O}\left(n^{\frac{3}{2}}\right)$ search could sweep this space and verify the conjecture. As a matter of fact, and because, as already mentioned, we can exhaust the whole search space with just a few instances of the variable of the outer cycle, in practice the algorithm finishes in a $\mathcal{O}(n)$ time.

The problem with this approach is that, because the size of the search space is quite considerable, the memory space necessary to store the bitmap representing the referred set was larger than the one available in our laptops. Thus, the program does not try to cover the whole space of considered integers in just one run, but splits this space in various slices, that are searched independently. This has the advantage of a "parallelism for poor people", running the program on different slices in different laptops, but has the drawback that the program for the higher slices, by the nature of the additive decomposition, needs the same time to conclude as the same program would need to run on an unsplitted space of integers. With laptops of 16 GB of RAM, we splitted the search space in 11 slices: the $i$-th slice covering the range $\left[i \times 10^{10},(i+1) \times 10^{10}\right]$, for $0 \leq i \leq 9$, and the 11 -th slice covering the remaing numbers up to 105103560126 . In the process, the range previously checked by Qìng-Hǔ Hóu was rechecked.

The code of the C program we used is given in Appendix A. The input consists of three numbers: the range over which one is checking; the "tolerance", which is the size of the list of the numbers that were not checked yet; and, finally, the interval one is checking.

[^0]
## 3. The results

The different slices were distributed by several machines, and each slice took between 2 to 3.5 days (depending on the machine, and on the extra use that its owner was making of it). While working on slice 1 , it was noticed that only four numbers required the outter "for" cycle to go beyond 1 , and checking these four numbers was taking a huge amount of time. Thus, the program was interruped, and restarted with a tolerance of 10 , and here is the output:

```
135Siever version 1.3
Sieving 135 for 105103560126, with tolerance 10,
in the interval [10000000000,20000000000]
09375000000
1562500004
44
Done!! Lasting numbers: 4
10234584952,11035927288,11051651704,14485001848
```

This output means that there are 9375000000 numbers to be checked (recall that one is ignoring the multiples of 16), that after the first run of the outter cycle (which looks for 1-3-5 representations ( $x, y, z, t$ ) where $x=3 k^{2}$ or $y=k^{2}$, with $k=0,1, \ldots$ ), there remained only 1562500004 numbers, and that after the second run $(k=1)$ only 4 numbers are left. The program then stops, and outputs those numbers.

These four 11-digits-long positive integers were then checked using the PARI/GP functions presented in appendix B which uses an algorithm to write a prime congruent to one modulo 4 as a sum of two squares that is described by John Brillhart in Bri72. Using those functions, one very quickly gets, for example (it is a random algorithm), the following 1-3-5 representations:

$$
\begin{array}{r}
(8524,9502,33094,94744) \\
\text { for }
\end{array} \quad 10234584952, \text {, } \begin{gathered}
\text { for } \\
(13438,32472,12774,98172) \\
\text { for }
\end{gathered} 1035927288,
$$

As a further example, we give here the output of the last slice:

```
135Siever version 1.3
Sieving 135 for 105103560126, with tolerance 10,
in the interval [100000000000,105103560126]
04784587619
1797431269
4 0
Done!! Lasting numbers: 0
132334.83 user 0.20 system 36:45:2 6elapsed 100%CPU
(Oavgtext+0avgdata 623532maxresident)k
Oinputs+8outputs (Omajor+155918minor)pagefaults Oswaps
```


## 4. A new conjecture

As a consequence of the computational results displayed above, we now make the following conjecture.

Conjecture 2. Any $m \in \mathbb{N}$, that is not a multiple of 16 , with the exception of $31,43,111,151,168,200,248,263,319,456,479,871,1752,1864$, 3544, can be represented as a sum of four squares, $x^{2}+y^{2}+z^{2}+t^{2}$, with $x, y, z, t \in \mathbb{N}_{0}$ such that $x+3 y+5 z$ is a square, and either $x$ is three times a square, or $y$ is a square. Moreover, for $m>14485001848$, one has a representation with $x \in\{0,3\}$ or $y \in\{0,1\}$, and (disregarding multiples of 16) exactly $\frac{5}{6}$ of the numbers have a representation with $x=0$ or $y=0$, while the remainder $\frac{1}{6}$ have a representation with $x=3$ or $y=1$.

## Appendix A. The C program

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#define VERSION "1.4"
#define MAX 10000000L
#define LIM 0
```

```
#define MAXS 10000L
#define MINS OL
#define FULLSET (B64)0 xfffefffefffefffe
typedef unsigned long B64;
typedef unsigned long Long;
typedef struct {
    B64 *map;
    Long min, max, nelements;
} BitMap;
BitMap map, squares;
B64 *masks;
Long max = MAX, mins=MINS, maxs=MAX;
B64* buildMasks(void){
    B64 *masks, *pt, val=(B64)1;
    int i;
    masks =(B64*)malloc}(64*\mathrm{ sizeof (B64));
    pt = masks;
    for (i=0; i < 64; i++){
        *(pt++) = val;
        val = val << 1;
    }
    return masks;
}
void newBMapFull(BitMap *bmap, Long min, Long max){
    B64 * pt;
    Long nbytes, i, size;
    size = max - min + 1;
    nbytes = (size / (sizeof (B64)*8)) +1;
    bmap }->\mathrm{ map =(B64 *)malloc(sizeof (B64)* nbytes);
    bmap }->\mathrm{ max = max;
    bmap}->\mathrm{ min = min;
    bmap}->\mathrm{ nelements = max}-(\operatorname{max}/16)-(\operatorname{min}-(\operatorname{min}/16))
    pt = bmap }->\mathrm{ map;
    for (i=0;i<nbytes ; i ++) (* pt++) = FULLSET ;
}
int memberP(BitMap *bmap, Long n){
    B64 byte, foo;
    Long rnumber;
```

```
    if (n > bmap }->\mathrm{ max || n < bmap }->\mathrm{ min) return 0;
    rnumber = n - bmap }->\mathrm{ min;
    byte = rnumber/64;
    foo = rnumber - byte * 64;
    if(*(masks+foo) & *(bmap }>\mathrm{ map+byte)) return 1;
    return 0;
}
void removeM(BitMap *bmap, Long n){
    Long byte, foo, rnumber;
    if (n > bmap ->max || n < bmap ->min) return;
    rnumber = n - bmap }->\mathrm{ min;
    byte = rnumber / 64;
    foo = rnumber - byte * 64;
    if (*(masks+foo) & *(bmap->map+byte)) {
        (bmap->nelements)--;
        *(bmap }->\mathrm{ map + byte ) = *(bmap }->\mathrm{ map+byte) & ~*(masks+foo );
    }
}
void addM(BitMap *bmap, Long n){
    B64 byte, foo;
    Long rnumber;
    if (n > bmap }->\mathrm{ max || n < bmap ->min) return;
    rnumber = n - bmap }->\mathrm{ min;
    byte = rnumber / (64);
    foo = rnumber - byte * 64;
    if(!(*(masks+foo) & *(bmap->map+byte))) {
        (bmap }->\mathrm{ nelements) ++;
        *(bmap }->\mathrm{ map + byte ) = *(bmap }->\mathrm{ map +byte) | *(masks+foo );
    }
}
void printM(BitMap *bmap){
    Long i, size;
    int j;
    size = bmap }->\mathrm{ max - bmap }->\mathrm{ min +1;
    for (i=0; i <= (bmap }>>\mathrm{ max )/64;i++){
        if (*(bmap->map+i)) {
            for ( j=0; j<64; j++){
            if (i*64+j+(bmap }->\mathrm{ min ) > bmap }->\mathrm{ max ) {
                printf("\n");
                return;
            }
```

```
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```

```
                if(*(bmap->map+i) & *(masks+j)) printf("%lu ",i*64+j+bmap->min);
```

                if(*(bmap->map+i) & *(masks+j)) printf("%lu ",i*64+j+bmap->min);
            }
            }
        }
        }
    }
    }
    }
}
void saveM(BitMap *bmap, int ord){
void saveM(BitMap *bmap, int ord){
FILE * file;
FILE * file;
char fname[100];
char fname[100];
Long i;
Long i;
int j;
int j;
sprintf(fname, "c135-%d.csv",ord);
sprintf(fname, "c135-%d.csv",ord);
file = fopen(fname,"w");
file = fopen(fname,"w");
for(i=0; i<= (bmap->max)/64;i++){
for(i=0; i<= (bmap->max)/64;i++){
if (*(bmap->map+i)) {
if (*(bmap->map+i)) {
for (j=0; j<64; j++){
for (j=0; j<64; j++){
if(i i*64+j > bmap->max){
if(i i*64+j > bmap->max){
printf("\n");
printf("\n");
return;
return;
}
}
if(*(bmap->map+i) \& *(masks+j)) fprintf(file,"%lu, ",i*64+j+bmap->min);
if(*(bmap->map+i) \& *(masks+j)) fprintf(file,"%lu, ",i*64+j+bmap->min);
}
}
}
}
}
}
fclose(file);
fclose(file);
}
}
int squarep(Long n){
int squarep(Long n){
Long i;
Long i;
i = (int)(sqrt(n)+0.5);
i = (int)(sqrt(n)+0.5);
return i*i = n;
return i*i = n;
}
}
void dealWTriple(BitMap* map, Long i, Long j, Long k){
void dealWTriple(BitMap* map, Long i, Long j, Long k){
Long foo, n=0, n2=0;
Long foo, n=0, n2=0;
if (squarep (i + 3*j + 5*k)) {
if (squarep (i + 3*j + 5*k)) {
foo = i*i+j*j+k*k;
foo = i*i+j*j+k*k;
while(1){
while(1){
if(foo + n2 > max) break;
if(foo + n2 > max) break;
removeM (map, foo+n2);
removeM (map, foo+n2);
n2 += 2*(n++)+1;
n2 += 2*(n++)+1;
}
}
}
}
}
}
149 int main(int argc, const char * argv[]) {

```
149 int main(int argc, const char * argv[]) {
```

148

```
    Long i 2=0, i=0, j, k, i 4 =0, lim=LIM;
    printf("135 Siever version %s\n",VERSION);
    masks = buildMasks();
    if(\operatorname{argc}==4){
        lim}= atol(argv[3])
        mins=atol(argv[1]);
        maxs = atol(argv[2]);
        max = maxs;
        printf("Sieving 135, with tolerance %lu, in the interval [%lu,%lu]", lim,mins,maxs);
} else {
    printf("Usage c135 lim min max \n");
    exit(-1);
}
newBMapFull(&map, mins, maxs);
while(i4 <= max){
    printf("%lu\t%lu\n",i2,map.nelements );
    if(map.nelements <= lim){
        printf("\n");
        printf("Done!! Lasting numbers: %lu\n",map.nelements );
        printM(&map);
        printf("\n");
        exit(0);
    }
    for (j=0; j * j<= maxs-i4 ; j++){
        for (k=j; k*k<= (maxs-i4-j*j); k++){
            if (k*k+j*j+i2*i2 > maxs) break;
            dealWTriple(&map, 3*i2,j,k);
            dealWTriple(&map, 3*i2,k,j);
            dealWTriple(&map, j, i2,k);
            dealWTriple(&map, k,i2,j);
            }
    }
    i4 += 2*i 2 +1;
        i2 += 2*(i}++)+1
    }
    printf("\n");
    printM(&map);
    printf("\n Done! Lasting numbers: %lu\n",map.nelements );
    return 0;
}
```


## Appendix B. The PARI/GP functions

quat $(a, b, c, d)=[a, b, c, d ;-b, a,-d, c ;-c, d, a,-b ;-d,-c, b, a] ;$
/* The Hurwitz units */
unid $=[$ quat $(1,0,0,0), ~ q u a t(-1,0,0,0), ~ q u a t(0,1,0,0), ~ q u a t(0,-1,0,0)$,
quat $(0,0,1,0)$, quat $(0,0,-1,0)$, quat $(0,0,0,1)$, quat $(0,0,0,-1)$,
quat $(1 / 2,1 / 2,1 / 2,1 / 2), \quad q u a t(-1 / 2,-1 / 2,-1 / 2,-1 / 2)$,
quat $(1 / 2,1 / 2,1 / 2,-1 / 2)$, quat $(-1 / 2,-1 / 2,-1 / 2,1 / 2)$,
quat $(1 / 2,1 / 2,-1 / 2,1 / 2), \quad q u a t(-1 / 2,-1 / 2,1 / 2,-1 / 2)$,
quat $(1 / 2,-1 / 2,1 / 2,1 / 2), \quad q u a t(-1 / 2,1 / 2,-1 / 2,-1 / 2)$,
quat $(1 / 2,1 / 2,-1 / 2,-1 / 2)$, quat $(-1 / 2,-1 / 2,1 / 2,1 / 2)$,
quat $(1 / 2,-1 / 2,1 / 2,-1 / 2), \quad q u a t(-1 / 2,1 / 2,-1 / 2,1 / 2)$,
quat $(1 / 2,-1 / 2,-1 / 2,1 / 2)$, quat $(-1 / 2,1 / 2,1 / 2,-1 / 2)$,
quat $(1 / 2,-1 / 2,-1 / 2,-1 / 2), \quad q u a t(-1 / 2,1 / 2,1 / 2,1 / 2)]$;
/* Fast modular exponentiation */
$\operatorname{expmod}(\mathrm{a}, \mathrm{e}, \mathrm{m})=\{$
local ( $\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{d}$ ) ;
$\mathrm{x}=\mathrm{a} ; \mathrm{y}=1$; $\mathrm{s}=\mathrm{e}$;
while (s, d=s \% 2 ;
$\mathrm{s}=(\mathrm{s}-\mathrm{d}) / 2 ; \quad$ if $(\mathrm{d}, \mathrm{y}=(\mathrm{y} * \mathrm{x}) \% \mathrm{~m}) ; \mathrm{x}=(\mathrm{x} * \mathrm{x}) \% \mathrm{~m}) ;$
return (y) ;
\}
/* Imodp computes the solution of $x^{\wedge} 2=-1(\bmod p)$ with $0<x<p / 2$, for $p=1$
$(\bmod 4) * /$
$\operatorname{Imodp}(\mathrm{p})=\{$
local ( $\mathrm{g}, \mathrm{x}$ );
if (p $\% 4<>1$, return (" not a valid prime!"));
while ( $1, \mathrm{~g}=\mathrm{random}(\mathrm{p})$;
if $(\operatorname{expmod}(\mathrm{g},(\mathrm{p}-1) / 2, \mathrm{p})=\mathrm{p}-1, \mathrm{x}=\operatorname{expmod}(\mathrm{g},(\mathrm{p}-1) / 4, \mathrm{p})$;
if $(x>p / 2, x=p-x) ;$ return ( $x)$ ) ) ;
$\}$
/* Euclid algorithm to compute gcd (a,b) but stopping at the first
remainder that is $<\operatorname{sqrt}(a) * /$
$\operatorname{EuclSp}(\mathrm{a}, \mathrm{b})=\{$
local(r, x, z) ;
$\mathrm{r}=\mathrm{a} \% \mathrm{~b}$;
if $(\mathrm{r}==1$, return $([\mathrm{b}, 1]))$;
$\mathrm{x}=\mathrm{b}$;
while ( $\mathrm{r}>\operatorname{sqrt}(\mathrm{a}), \mathrm{z}=\mathrm{x} \% \mathrm{r} ; \mathrm{x}=\mathrm{r} ; \mathrm{r}=\mathrm{z}$ ) ;
return ([r, x\%r]);
\}

```
/* Writes p = 1 (mod 4) as a sum of two squares, using the algorithm
described in [1] */
PrimeSS(p)=EuclSp(p,Imodp (p));
/* A method to decompose an odd number as a sum of of either four
integer squares, or half integer squares, in a random way */
sum4sqF (a)={
    local(b,c,sq, x,y,z,tt,uu,vv,ct,s);
    if (a==1,return(quat (1,0,0,0)));
    while(1,
        b=random(2*sqrtint(a)-1)+1; if (b% 2== 0,b=b-1);
        c=4*a-b^2;
        sq=floor((sqrtint(c)+1)/2);
        j=random(sq-1)+1;
        z=(c-(2*j - 1)^2)/2;
        if(isprime(z), v=PrimeSS(z); x=v[1];y=v[2];
            uu=x+y;vv=x-y; tt=2*j - 1;
                s=matsolve([1,1,1,1;1,1,-1, -1;1,-1,1, -1;1, - 1, - 1, 1],[b;uu;vv;tt]);
                return(quat(s[1, 1],s[2,1],s[3,1],s[4,1]))
            ));
    }
/* Expanding the sum4sqF function to all natural numbers with results
only in the integers */
v2(n)={
    local(oddp);
    e=0;oddp=n;
    while (oddp%2==0,oddp=oddp / 2; e=e +1);
    return(e);
    };
sum4sqFall(a)={
    local(r,e);
    e=v2(a);
    if (e==0,r=sum4sqF (a),a=a/2^e;r=(quat (1, 1,0,0)^e)*sum4sqF (a));
    while(floor(r[1, 1])<>r[1, 1],
            r=r*unid [random(length(unid))+1];
            );
    return(r);
    }
/* All possible permutations of the numbers 0,1,3,5 as 4D-vectors */
```

```
perm=List ([[0, 1, 3, 5], [0, 1, 5, 3], [0, 3, 1, 5], [0, 3, 5, 1],
[0, 5, 1, 3], [0, 5, 3, 1], [1, 0, 3, 5], [1, 0, 5, 3], [1, 3, 0, 5],
[1, 3, 5, 0], [1, 5, 0, 3], [1, 5, 3, 0], [3, 0, 1, 5], [3, 0, 5, 1],
[3, 1, 0, 5], [3, 1, 5, 0], [3, 5, 0, 1], [3, 5, 1, 0], [5, 0, 1, 3],
[5, 0, 3, 1], [5, 1, 0, 3], [5, 1, 3, 0], [5, 3, 0, 1], [5, 3, 1, 0]]);
/* rep135 gives a solution of the system 1-3-5, returning the
Lipschitz integer whose norm is the imput number, and the permutation
of 0135 with whom its inner product is a square */
rep135(a)={
    local(s,t,f,c);
    if(issquare(a), return([[sqrtint(a),0,0,0],[1,3,5,0]]));
    while(1, s=sum4sqFall(a);
        t=[abs(s[1, 1]), abs (s[1, 2]), abs (s[1, 3]), abs (s[1, 4])];
    for(i=1,length(perm), f=perm[i]*t ~
        if(issquare(f), return([t, perm[i]]))));
    }
```


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Centro de Matemática da Universidade do Porto and Mathematics Department of Faculdade de Ciências do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: ajmachia@fc.up.pt

Centro de Matemática da Universidade do Porto and Computer Science Department of Faculdade de Ciências do Porto, Rua do Campo Alegre 1021/1055, 4169-007 Porto, Portugal

E-mail address: rvreis@fc.up.pt
Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: tsopanidisnikos@gmail.com


[^0]:    ${ }^{1}$ All complexity considerations made here suppose that the cost of arithmetic operations for integers in the range considered has complexity $\mathcal{O}(1)$.

