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An algorithm for classifying origamis into components of Teichmüller curves

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Abstract Non trivial examples of Veech groups have been studied systematically with the notion of combinatorics coming from coverings. For abelian origamis, coverings of once punctured torus, their Veech groups are described by Schmithüsen in terms of monodromy. Shinomiya applied her method for translation coverings of the surface obtained from regular $2n$-gon. Their results enable us to specify the Veech group in a concrete example by using the Reidemeister-Schreier method. In this paper, we deal with origamis including non abelian origamis using a method inspired to 'comparisons of parallelogram decompositions'. Our algorithms classify all origamis of given degree into natural isomorphism classes and specify their Veech groups in parallel.

Keywords Flat surfaces · Veech groups · Origamis · Teichmüller spaces

1 Introduction

A holomorphic quadratic differential on a Riemann surface induces a flat structure, on which several notions of affine geometry are well-defined. Sometimes a square root of a quadratic differential defines an abelian differential and a translation structure: we call such a case abelian. Affine deformations of a flat structure induces a geometric holomorphic disk on the Teichmüller space. Its projected image in the moduli space is an orbifold isomorphic to the quotient of the unit disk by the Veech group, the group of derivatives of self affine deformations on a flat surface. Veech groups are originally studied by Veech [18] in the context of the billiard flow (a geodesic flow on a surface which represents 'orbits of billiard balls on a billiard table'). We note that that the first non trivial examples of Veech group are presented in his paper.

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Non trivial examples of Veech groups have been studied systematically with the notion of combinatorics coming from coverings. An abelian origami is a covering of once punctured torus equipped with a natural translation structure. Schmithüsen [15] proved that the monodromies of this kind of covering act as automorphisms on the free group F_2 and the Veech group of an abelian origami is described by them. Shinomiya [16] considered the Veech groups of translation coverings of the surface obtained by $2n$-gon (with all opposite sides glued by translation). He proved that the Veech groups of both the $2n$-gon surface and its universal cover are generated by two matrices, which correspond to a Dehn twist and a rotation. By checking the actions of these two matrices on the fundamental group of original surface he renovated Schmithüsen's method for this kind of surfaces. By combining the Reidemeister-Schreier method [12] with their method, we can specify the Veech group for each kind of translation surface.
In this paper we deal with a general *origami*, which is a surface obtained from finitely many unit squares equipped with natural flat structure. (see section 3 for details.) Such a surface correspond to an abelian origami with a sign list of squares (just as data). We can easily see that the Veech group of an origami is a subgroup of the modular group $PSL(2, \mathbb{Z})$.
To study the Veech group, we use a method based on a comparison between two 'parallelogram decompositions' of a surface. The author [10] showed that a flat surface admitting two directional cylinder decompositions is characterized by a parallelogram decomposition, which consists of a sort of abelian origami and some extra data (see Definition 2.10). We can decide whether a matrix belongs to the Veech group of such a surface just by comparing an initial decomposition and the terminal (affinely deformed directional) decomposition. Since for each matrix the terminal decomposition of any origami induces a unique origami again, this result inspires us to define an action of $PSL(2, \mathbb{Z})$ on the set $\tilde{\Omega}_d$ of all classes of origamis of given degree (the number of squares) d. By the 1-1 correspondence between representatives in $PSL(2, \mathbb{Z})$ modulo the Veech group and isomorphism classes of origamis, the orbit decomposition with respect to $PSL(2, \mathbb{Z}) \curvearrowright \tilde{\Omega}_d$ enables us to know about Veech groups of all origamis in $\tilde{\Omega}_d$.
The structure of this paper is as follows: In section 2 we mention background concepts and results to explain the main theory. In section 3 we present a short review of origamis. Main concepts of algorithms are also included. In section 4 we define some notations to deal with origamis and prove key lemmas for algorithms. In section 5 we state the main algorithms.

2 Preliminaries

At first we perpare some definitions related with Teichmüller theory to state what we consider. See [3] and [9] for details.
Let R be a Riemann surface of finite analytic type (g, n) with $3g - 3 + n > 0$.

Definition 2.1 Let R_i $(i = 1, 2)$ be Riemann surfaces homeomorphic to R.

(1) We say two orientation preserving homeomorphisms $f_i : R \to R_i$ $(i = 1, 2)$ are *Teichmüller equivalent* if there is a conformal map $h : R_1 \to R_2$ homotopic to $f_2 \circ f_1^{-1} : R_1 \to R_2$. We denote the Teichmüller equivalence class by $[\bullet]$.

(2) We define the *Teichmüller space* $T(R)$ of Riemann surface R as the space of Teichmüller equivalence classes of homeomorphism from R. We define the *mapping class group* $\mathrm{Mod}(R)$ by the group of homotopy classes of orientation preserving self homeomorphisms on R.
(3) For an orientation preserving self homeomorphism f, we define $\rho_f : T(R) \to T(R)$ by $[g] \mapsto [g \circ f^{-1}]$ for every $[g] \in T(R)$. Then $f \mapsto \rho_f$ factors through $\mathrm{Mod}(R)$. We define the *moduli space* $M(R)$ by the quotient $T(R)/\mathrm{Mod}(R)$.

Definition 2.2 A *holomorphic quadratic differential* ϕ on R is a tensor on R whose restriction to each chart (U, z) on R is of the form $\phi(z)dz^2$ where ϕ is a holomorphic function on U.

Let $p_0 \in R$ be a regular point of ϕ and (U, z) be a chart around p_0. Then ϕ defines a natural coordinate (ϕ*-coordinate*) $\zeta(p) = \int_{p_0}^{p} \sqrt{\phi(z)}dz$ on U, on which $\phi = d\zeta^2$. ϕ-coordinates give an atlas on $R^* = R \setminus \mathrm{Crit}(\phi)$ whose any transition map is of the form $\zeta \mapsto \pm\zeta + c$ ($c \in \mathbb{C}$). It is called a *flat structure* and a pair (R, ϕ) is called a *flat surface*. The space

$$Q(R) := \{\phi : \text{holomorphic quadratic differential on } R \mid \|\phi\| := \int_R |\phi| < \infty\}$$

is known to be identified with a fibre of cotangent bundle of the Teichmüller space, which is a complex vector space of dimension $3g - 3 + n$.

Definition 2.3 Let R, ϕ be as above.

(1) For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, we define $f_A : \mathbb{C} \to \mathbb{C}$ by $\zeta = \xi + i\eta \mapsto (a\xi + c\eta) + i(b\xi + d\eta)$. (The derivative of f_A equals A.)
(2) A homeomorphism $f : R \to R$ is called an *affine map* on (R, ϕ) if f is locally affine (i.e. of the form $f(\zeta) = f_A(\zeta) + k$) with respect to the ϕ-coordinates. We denote the group of all affine map on (R, ϕ) by $\mathrm{Aff}^+(R, \phi)$.
(3) For each $f \in \mathrm{Aff}^+(R, \phi)$, the local derivative A is globally defined up to a factor $\{\pm I\}$ independent of coordinates of u_ϕ. We call the map $D : \mathrm{Aff}^+(R, \phi) \to PSL(2, \mathbb{R}) : f \mapsto \bar{A}$ the *derivative* map and its image $\Gamma(R, \phi) := D(\mathrm{Aff}^+(X, \phi))$ the *Veech group*.
(4) Let R be a Riemann surface of finite analytic type and ϕ be a non-zero, integrable, holomorphic quadratic differential on R. We call such a pair (R, ϕ) a *flat surface*. We say that flat surfaces $(R, \phi), (S, \psi)$ are *isomorphic* if there exists a locally affine homeomorphism $f : R \to S$ with derivative $\bar{I} \in PSL(2, \mathbb{R})$.

Fix a holomorphic quadratic differential $\phi \in Q(R)$ satisfying $\|\phi\| = 1$. For each $t \in \mathbb{D}$ we define $A_t := \left(\begin{smallmatrix} 1+t & 0 \\ 0 & 1-t \end{smallmatrix}\right)$ and $\Delta_\phi := \{[f_{A_t}(R), f_{A_t}] \in T(R) \mid t \in \mathbb{D}\}$. It is known that $\mathbb{D} \ni t \mapsto [f_{A_t}(R), f_{A_t}] \in T(R)$ defines a holomorphic, isometric embedding with respect to the Poincaré metric and the Teichmüller metric (see [3]). We call this embedding $\mathbb{D} \hookrightarrow T(R)$ the *Teichmüller embedding* and its image $\iota_\phi(\mathbb{D})$ the *Teichmüller disk*.

Lemma 2.4 ([2, Theorem1]) *Let (R, ϕ) be a flat surface and $f \in QC(R)$. Then ρ_f maps Δ_ϕ onto itself if and only if f is homotopic to an element in $\mathrm{Aff}^+(R, \phi)$. Furthermore in this case, $\rho_f(\Phi(t\bar{\phi}/|\phi|)) = \Phi(D(f)^*(t)\bar{\phi}/|\phi|)$ for each $t \in \mathbb{D}$ where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^* (\tau) := \frac{-a\tau+b}{c\tau-d}$ for $\tau = \sqrt{-1} \cdot \frac{1+t}{1-t} \in \mathbb{H}$.*

Recall that the Veech group $\Gamma(R, \phi)$ is the group of derivatives of elements in $\mathrm{Aff}^{+}(R, \phi)$. By Lemma 2.4 we can see that the action $\Gamma(R, \phi)$ on $\mathbb{H}$ as Möbius transformations tells us how Δ_ϕ projects into $M(R)$. It is first observed by Veech [18] that Veech group is a discrete group. The Veech group determines the projected image C_ϕ of Δ_ϕ in $M(R)$ in the sense that C_ϕ is isomorphic to the mirror image of $\mathbb{H}/\Gamma(R, \phi)$ as an orbifold. If $\Gamma(R, \phi)$ has a finite covolume then C_ϕ can be seen as a Riemann surface of finite analytic type, called the *Teichmüller curve* induced by ϕ.

Remark 2.5 For $\phi \in Q(R)$ if $\sqrt{\phi}$ (whose restriction to a chart (U, z) is $\sqrt{\phi(z)}dz$) gives an abelian differential on R then natural coordinates define an atlas whose any transition map is of the form $\zeta \mapsto \pm\zeta + c$ $(c \in \mathbb{C})$. It is called a *translation structure* and such a pair $(R, \sqrt{\phi})$ is called a *translation surface*. In this case we say that ϕ is *abelian* and otherwise *non abelian*.
In abelian case the derivative map is well-defined onto $SL(2, \mathbb{R})$ and the Veech group $\Gamma(R, \phi)$ is defined to be a subgroup of $SL(2, \mathbb{R})$.

Considering the geometry induced from a holomorphic quadratic differential is useful to approach the Veech group of a flat surface. We consider the *ϕ-metric*, a flat metric to which the Euclidian metric lifts via ϕ-coordinates. ϕ-geodesics, geodesics of ϕ-metric are locally line segments and their directions are determined uniquely in $[0, \pi)$ up to half-rotarion.

Definition 2.6 $\theta \in [0, \pi)$ is called *Jenkins-Strebel direction* of a flat surface (R, ϕ) if almost every point in R lies on some closed geodesic in the direction θ. We denote the set of Jenkins-Strebel directions by $J(R, \phi)$.

For the existence of a holomorphic quadratic differential with one Jenkins-Strebel direction, the following result is known.

Proposition 2.7 (Strebel [17]) *Let $\gamma = (\gamma_1, ..., \gamma_p)$ be a finite 'admissible' curve system on R, which satisfies bounded moduli condition for γ. Then for any $b = (b_1, ..., b_p) \in \mathbb{R}^p_+$ there exists $\phi \in A(R)$ such that 0 is a Jenkins-Strebel direction of (R, ϕ) and (R, ϕ) is decomposed into cylinders $(V_1, ..., V_p)$ where each V_j has homotopy type γ_j and height b_j.*

Definition 2.8 Let (R, ϕ) be a flat surface of finite analytic type. The *canonical double cover* of (R, ϕ) is the translation surface obtained by a continuation of blanches of locally defined abelian differential $\sqrt{\phi}$.

Remark 2.9 More concretely, the canonical double cover of (R, ϕ) is the surface obtained by taking two copies of R with one of them half-rotated and regluing them in the way respecting directions of vertical and horizontal trajectories. (See [3] for details.) Note that a flat surface is abellian if and only if the canonical double cover is disjoint.

A closed ϕ-geodesic γ generates a cylinder which is the union of all ϕ-geodesics parallel (with same direction) and free homotopic to γ. So for each $\theta \in JS(R, \phi)$ R admits a decomposition in the direction θ. Furthermore, when we assume two Jenkins-Strebel direction Jenkins-Strebel directions $\theta_1, \theta_2 \in J(R, \phi)$, then the surface is decomposed into parallelograms which are intersections of cylinders in the directions θ_1, θ_2. At first we define as follows.

Definition 2.10 (extended origami) Let $N \in \mathbb{N}$, $\Lambda = \{1, 2, ..., N\}$, $\hat{\Lambda} = \Lambda \times \{\pm 1\}$, $M = [M_1, M_2, ..., M_N] \in \mathbb{R}_+ P^{N-1}$. Let $\hat{m} : F_2 \to \mathrm{Sym}(\hat{\Lambda})$ be a homomorphism with following three conditions: For $\hat{G} = \langle \mathbf{x}, \mathbf{y} \rangle := \hat{m}(F_2) < \mathrm{Sym}(\hat{\Lambda})$,

(1) (symmetry) $\hat{m}_w(\hat{\lambda}) = -\hat{m}_{\gamma_{-I}(w)}(\hat{\lambda})$ for any $\hat{\lambda} \in \hat{\Lambda}$ and $w \in F_2$,
(2) (non-branching) $\mathbf{y}(\hat{\lambda}) \neq -\hat{\lambda}$ for any $\hat{\lambda} \in \hat{\Lambda}$, and
(3) (connectivity) the action $\hat{G} \curvearrowright \hat{\Lambda}$ projects to a transitive action on Λ.

Next, we define $K_{\mathcal{O}} = K_{M,\hat{G}} : \hat{\Lambda} \times F_2 \to \mathbb{R}$ by following.

- $K_{\mathcal{O}}(\,\cdot\,, 1) = 1$.
- For any $\hat{\lambda} \in \hat{\Lambda}$, $K_{\mathcal{O}}(\hat{\lambda}, x) = \frac{M_{\hat{\lambda}}}{M_{\hat{m}_x(\hat{\lambda})}}$ and $K_{\mathcal{O}}(\hat{\lambda}, y) = \frac{M_{\hat{m}_y(\hat{\lambda})}}{M_{\hat{\lambda}}}$.
- For any $w_1, w_2 \in F_2$ and $\hat{\lambda} \in \hat{\Lambda}$, $K_{\mathcal{O}}(\hat{\lambda}, w_1 w_2) = K_{\mathcal{O}}(\hat{\lambda}, w_1) K_{\mathcal{O}}(\hat{m}_{w_1}(\hat{\lambda}), w_2)$

Then we call $\mathcal{O} = (M, \hat{G} = \langle \mathbf{x}, \mathbf{y} \rangle)$ an *extended origami* of degree N if $K_{\mathcal{O}}(1, w) = 1$ for all $w \in H_{\hat{G}}$. Extended origamis $\mathcal{O}_i = (M^i = [M^i_1, M^i_2, ..., M^i_N], \hat{G}_i = \langle \mathbf{x}_i, \mathbf{y}_i \rangle)$ $(i = 1, 2)$ of order N are *isomorphic* if there exists a pair (Φ, σ) of $\Phi : G_1 \to G_2$ and $\sigma \in S_{2N}$ such that

- $\Phi : \hat{G}_1 \to \hat{G}_2$ is an isomorphism with $(\Phi(\mathbf{x}_1), \Phi(\mathbf{y}_1)) = (\mathbf{x}_2, \mathbf{y}_2)$,
- $[M^1_{p_1 \circ \sigma(1)}, M^1_{p_1 \circ \sigma(2)}, ..., M^1_{p_1 \circ \sigma(N)}] = [M^2_1, M^2_2, ..., M^2_N]$, and
- $\sigma(\hat{m}_w(\hat{\lambda})) = \hat{m}_{\Phi(w)}(\sigma(\hat{\lambda}))$ for each $\hat{\lambda} \in \hat{\Lambda}$, $w \in \hat{G}$.

We call (Φ, σ) an *isomorphism* between extended origamis $\mathcal{O}_1$ and $\mathcal{O}_2$.

Using the notion of extended origamis, the situation of flat surface decomposed into parallelograms is described as follows.

Lemma 2.11 ([10]) *A flat surface (R, ϕ) with a pair of two distinct Jenkins-Strebel directions $(\theta_1, \theta_2) \in J(R, \phi)^2$ is up to isomorphism uniquely determined by a triple $P(R, \phi, (\theta_1, \theta_2)) = (\Theta, k, \mathcal{O})$ where $\Theta = (\theta_1, \theta_2) \in [0, \pi)^2$ with $\theta_1 \neq \theta_2$, $k > 0$, and $\mathcal{O}$ is an extended origami.*

Furthermore, with this notion we can decide whether a matrix belongs to the Veech group as in the following theorem. Here we define an action of $A \in PSL(2, \mathbb{R})$ on triples in Lemma 2.11 by the deformation of pair Θ of angles and modulus k of a parallelogram under an affine map with derivative A.

Proposition 2.12 ([10]) *Let (R, ϕ) be a flat surface with a pair of two distinct Jenkins-Strebel directions $(\theta_1, \theta_2) \in J(R, \phi)^2$. $\bar{A} \in PSL(2, \mathbb{R})$ belongs to $\Gamma(R, \phi)$ if and only if $A\theta_1, A\theta_2$ belongs to $J(R, \phi)$ and $P(R, \phi, (A\theta_1, A\theta_2))$ is isomorphic to $A \cdot P(R, \phi, (\theta_1, \theta_2))$.*

This result implies that whether a matrix belongs to the Veech group of such a flat surface is completely determined by two conditions: whether the terminal directions are Jenkins-Strebel directions and the correspondence between two decompositions.

3 Origamis

Definition 3.1 An *origami* of degree d is a flat surface obtained by gluing d Euclidian unit squares at edges equipped with the flat structure induced from the natural coordinates of squares.

An origami admits two Jenkins-Strebel directions $0, \frac{\pi}{2}$ and in Theorem 2.11 it has parameters $\Theta = (0, \frac{\pi}{2})$, $k = 1$, and extended origami with $M = [1, 1, ..., 1]$.
Since $M = [1, 1, ..., 1]$ implies $K_{\mathcal{O}} = 1$ in Definition 2.10, an origami of degree d is identified with doubly generated permutation group $\hat{G}$ of $\{1, 2, ..., d\} \times \{\pm 1\}$ satisfying the three conditions in Definition 2.10.
At first we note about the general theory of abelian origamis. (Sometimes they are also called origamis simply.) Abelian origamis are also studied in the context of the Galois action on combinatorial objects as well as *dessins d'enfants*, a crucial result is given by Möller [13] and some of study is described in [7] and [11].
In our definition, an abelian origami of degree d will be returned to a permutation group $G = \langle x, y \rangle$ which acts transitively on a finite set $\{1, 2, ..., d\}$. This can be seen as a monodromy group of a sort of covering. By a general theory of covering maps we have following characterizations for an abelian origami similar to a *dessin d'enfants*. See [7] for details.

Proposition 3.2 *An abelian origami of degree d is up to equivalence uniquely determined by each of the following.*

(1) *A topological covering $p : R \to E$ of degreed from a connected oriented surface R to the torus E ramified at most over one point on E.*
(2) *A finite oriented graph $(\mathcal{V}, \mathcal{E})$ such that $|\mathcal{V}| = d$ and every vertex has precisely two incoming edges and two outgoing edges, with both of them consist of edges labeled with x and y.*
(3) *A monodromy map $m : F_2 \to S_d$ up to conjugation in S_d.*
(4) *A subgroup H of F_2 of index d up to conjugation in F_2.*

Example 3.3 The abelian origami shown in following figure is called the *L-shaped origami* $L(2, 3)$.

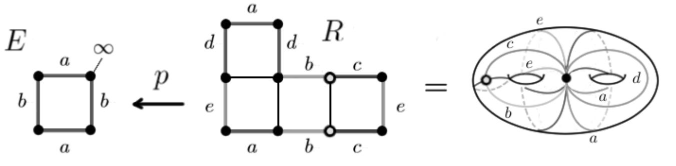

Fig. 1. L-shaped origami $L(2, 3)$: edges with the same letters are glued. (source:[10])

Here F_2 and H are identified with the fundamental group of once punctured torus and the one of R embedded in F_2. The generators $x, y \in G$ correspond to the monodromies of core curves of horizontal and vertical cylinders on E, respectively.

Next we mention about Proposition 2.12 in the cases of origamis. As explained in [10], the permutation group $\hat{G}$ corresponding to an origami is the abelian origami which is the canonical double cover of the original surface.

Remark 3.4 By Proposition 2.12 we may say the followings:

- By taking the universal covering equipped with induced flat structure we easily see that the Veech group of an origami is a subgroup of $PSL(2,\mathbb{Z})$.
- For each $A \in PSL(2,\mathbb{Z})$ and an origami $\mathcal{O}$ the terminal decomposition (in the pair of directions $A(0,\frac{\pi}{2})$) corresponds to an origami $\mathcal{O}_A$ of the same degree as $\mathcal{O}$. Now $\mathcal{O}_A$ is unique for each origami $\mathcal{O}$ up to isomorphism.
- Let $\tilde{\Omega}_d$ be the set of all isomorphism classes of origamis of degree $d \in \mathbb{N}$. $A \in PSL(2,\mathbb{Z})$ stabilizes a class $[\mathcal{O}] \in \tilde{\Omega}_d$ if and only if A belongs to the Veech group $\Gamma(\mathcal{O})$ of $\mathcal{O}$ up to conjugacy in $PSL(2,\mathbb{Z})$.
- For each $[\mathcal{O}] \in \tilde{\Omega}_d$, the set $PSL(2,\mathbb{Z})/\Gamma(\mathcal{O})$ of left cosets is up to conjugacy in $PSL(2,\mathbb{Z})$ identified with $\mathrm{Sym}(\mathrm{Orb}_{PSL(2,\mathbb{Z})}[\mathcal{O}])$ and hence finite. (So the Veech group of an origami has finite index in $PSL(2,\mathbb{Z})$!) In particular, the orbit decomposition with respect to the action $PSL(2,\mathbb{Z}) \curvearrowright \tilde{\Omega}_d$ enables us to know about Veech groups of all origamis in $\tilde{\Omega}_d$.

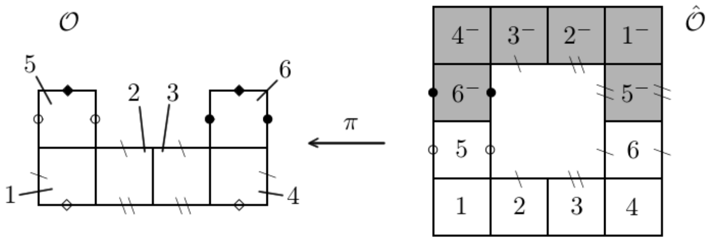

Fig. 2. The canonical double cover $\hat{\mathcal{O}}$ of non abelian origami $\mathcal{O}$:
we cut at all edges where the half-rotation occurs and glue them to the another copies.

4 Notations and key lemmas

Now we prepare some notations and lemmas for describing all isomorphism classes of origamis of given degree $d \in \mathbb{N}$.

Definition 4.1 We define as follows:

$$I_d := \{1,2,...,d\}, \qquad \hat{I}_d := \{\pm 1, \pm 2, ..., \pm d\},$$

$$\mathcal{E}_d := \{\varepsilon : \hat{I}_d \to \{\pm 1\} : \text{odd function}\}, \quad \bar{\mathcal{E}}_d := \{\varepsilon : \hat{I}_d \to \{\pm 1\} : \text{even function}\},$$

$$\mathfrak{S}_d := \mathrm{Sym}(I_d), \qquad \hat{\mathfrak{S}}_d := \mathrm{Sym}(\hat{I}_d), \qquad \hat{\mathfrak{S}}_d^0 := \{\sigma \in \hat{\mathfrak{S}}_d : \text{odd function}\}.$$

For any proposition P, we define $[P] := \begin{cases} 1 & \text{if } P \text{ is true} \\ -1 & \text{if } P \text{ is false} \end{cases}$.

For $\chi \in \hat{\mathfrak{S}}_d$ (or $\mathfrak{S}_d$), $\varepsilon \in \mathcal{E}_d$, $i \in \hat{I}_d$, we define $(\chi^\varepsilon)(i) := \begin{cases} \chi(i) & \text{if } \varepsilon(i) = 1 \\ \chi^{-1}(i) & \text{if } \varepsilon(i) = -1 \end{cases}$.

Definition 4.2 Let $\Omega_d := \mathfrak{S}_d \times \mathfrak{S}_d$, $\Omega_{2d}^0 := \{\mathcal{O} \in \Omega_{2d} \mid \mathcal{O}$ satisfies the three conditions in Definition 2.10$\}$, $\tilde{\Omega}_d := \Omega_d \times \mathcal{E}_d$. Let sign $\in \mathcal{E}_d$ be the sign function on $\hat{I}_d$. For each $\mathcal{O} = (x,y,\varepsilon) \in \tilde{\Omega}_d$, we define $\hat{G}_{\mathcal{O}} = (\mathbf{x},\mathbf{y}) \in \Omega_{2d}^0$ by:

$$\begin{cases} \mathbf{x}(i) = x^{\mathrm{sign}(i)}(i) \\ \mathbf{y}(i) = y^{\varepsilon(i)}(i) \cdot [\varepsilon(i) = \varepsilon(y^{\varepsilon(i)}(i))] \end{cases} \text{for all } i \in \hat{I}_d \text{ with odd extensions of } x, y.$$

Ω_d is the set of all (possibly disconnected) abelian origamis of degree d. For an abelian origami $\mathcal{O} = (x, y) \in \Omega_d$ and $\varepsilon \in \mathcal{E}_d$, we consider the following operation:

Operation A. Do the following for given $\mathcal{O} \in \Omega_d$ and $\varepsilon \in \mathcal{E}_d$:

(1) Cut the resulting surface of $\mathcal{O}$ at all edges. (with the pairings of edges saved)
(2) Apply the vertical reflection to all cells with $\varepsilon = -1$.
(3) Glue all paired edges in the way that with the natural coordinates the quadratic differential dz^2 is well-defined on the resulting surface.

This produces a new origami which can be non abelian. Furthermore we can see that $\hat{G}_{\mathcal{O}}$ is the canonical double cover of this origami (see [10]).
Conversely, for an origami $\mathcal{O}$ of degree d with canonical double cover $\hat{G} = \langle \mathbf{x}, \mathbf{y} \rangle$ we may consider the following operation:

Operation B. Do the following for given $\hat{G}_{\mathcal{O}} = (\mathbf{x}, \mathbf{y}) \in \Omega^0_{2d}$:

(1) Fix directions of core curves of all horizontal and vertical cylinders in the resulting surface of $\mathcal{O}$.
(2) For each $i \in I_d$ we denote the horizontal (resp. vertical) core curve crossing the cell with label i by h_i (resp. v_i) and define as follows.
$$\varepsilon(i) = \begin{cases} 1 & \text{if } v_i \text{ passes } h_i \text{ from right to left} \\ -1 & \text{if } v_i \text{ passes } h_i \text{ from left to right} \end{cases}$$
(3) Do the same operation as Operation A. (i.e. cut all edges, apply reflections to all cells with $\varepsilon < 0$, and reglue them.)

This produces a pair of an abelian origami and $\varepsilon \in \mathcal{E}_d$ which are the inverse image of $\mathcal{O}$ under Operation A. Hence $\tilde{\Omega}_d$ is identified with the set of all (possibly disconnected) origamis of degree d. Remark that ε in Operation B depends on the way to fix directions at step (1) but the resulting surface is uniquely determined.

Lemma 4.3 *Let $\mathcal{O}_j = (x_j, y_j, \varepsilon_j) \in \tilde{\Omega}_d$ and $\xi_j = [\varepsilon_j = \varepsilon_j \circ y_j^{\varepsilon_j}] \in \mathcal{E}_d$ $(j = 1, 2)$. $\mathcal{O}_1, \mathcal{O}_2$ are isomorphic if and only if there exists $\sigma \in \hat{\mathfrak{S}}_d$ such that for all $i \in \hat{I}_d$*

(1) *$\sigma(-i) = -\sigma(i)$. In particular, the projection $\bar{\sigma} \in \mathfrak{S}_d$ of σ is well-defined.*
(2) *$x_2(i) = \bar{\sigma}^* x_1(i)$.*
(3) *$\xi_2(i) = \mathrm{sign}(\sigma^*(y_1^{\varepsilon_1(i)}(i))) \cdot \xi_1(\sigma^{-1}(i))$.*
(4) *$y_2^{\varepsilon_2(i)}(i) = \mathrm{sign}(\sigma^*(y_1^{\varepsilon_1(i)}(i))) \cdot \sigma^*(y_1^{\varepsilon_1(i)}(i))$.*

Proof. ($\Rightarrow$) By Definition 2.10 there exists $\sigma \in \hat{\mathfrak{S}}_d$ such that $\mathbf{x}_2 = \sigma^* \mathbf{x}_1$ and $\mathbf{y}_2 = \sigma^* \mathbf{y}_1$. Because of the symmetry of $\mathbf{y}_1, \mathbf{y}_2$ (see Definition 2.10) it follows that $\mathbf{y}_2(\sigma(-i)) = \mathbf{y}_2(-\sigma(i))$ and we have (1). (2) follows from $\mathbf{x}_2 = \sigma^* \mathbf{x}_1$ and (1).
$\sigma^* \mathbf{y}_1 = \sigma^*(\xi_1 \cdot y_1^{\varepsilon_1}) = \xi_1(\sigma^{-1}(i)) \cdot \sigma^*(y_1^{\varepsilon_1})$ ($\because$ (1)). Since $y_2^{\varepsilon_2} > 0$, if we think in terms of the sign of $\sigma^*(y_1^{\varepsilon_1})$ then (3) and (4) will follow.
($\Leftarrow$) (1) and (2), (3) and (4) imply that $\mathbf{x}_2 = \sigma^* \mathbf{x}_1$, $\mathbf{y}_2 = \sigma^* \mathbf{y}_1$ respectively. □
To describe the isomorphism class of each origami $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, we should take conjugations in all elements in $\hat{\mathfrak{S}}^0_d$. By the condition (2) in Lemma 4.3 we can narrow down the representations of 'x's at first. That is, we only have to know conjugacy classes of 'x's and classes of '(y, ε)'s for each 'x's.

Lemma 4.4 *Let $x \in \mathfrak{S}_d$, $\sigma \in \hat{\mathfrak{S}}^0_d$. Let $\delta \in \mathcal{E}_d$ be the even extension of $\sigma/\bar{\sigma}\restriction_{I_d}$. Then $\sigma^* \mathbf{x} = \mathbf{x}$ if and only if $\delta = \delta \circ x \cdots$* (a) *and $\bar{\sigma}^* x = x^\delta \cdots$* (b) *hold on I_d.*

Proof. On $\hat{I}_d$, $\sigma^*\mathbf{x} = \mathbf{x} \Leftrightarrow \delta(\mathbf{x})\bar{\sigma}(\mathbf{x}) = \mathbf{x}(\delta\bar{\sigma}) = \delta x^{\delta \mathrm{sign}(\bar{\sigma})}(\bar{\sigma})$.

Since $\bar{\sigma}(\mathbf{x})$ and $x^{\delta \mathrm{sign}(\bar{\sigma})}(\bar{\sigma}) > 0$ on I_d, we have $\begin{cases} \delta(\mathbf{x}) = \delta & \ldots \text{(a')} \\ \bar{\sigma}(\mathbf{x}) = x^{\delta \mathrm{sign}(\bar{\sigma})}(\bar{\sigma}) & \ldots \text{(b')} \end{cases}$

$$\begin{aligned} \text{(a')} &\Leftrightarrow \delta(x(i)) = \delta(i),\ \delta(x(-i)) = \delta(-i) \quad \text{for } i \in I_d \\ &\Leftrightarrow \delta(x(i)) = \delta(i),\ \delta(x^{-1}(i)) = \delta(i) \quad \text{for } i \in I_d \ \Leftrightarrow \text{(a)} \end{aligned}$$

$$\text{(b')} \Leftrightarrow \begin{cases} \bar{\sigma}(\mathbf{x}(i)) = x^{\delta(i)\mathrm{sign}(\bar{\sigma}(i))}(\bar{\sigma}(i)) \\ \bar{\sigma}(\mathbf{x}(-i)) = x^{\delta(-i)\mathrm{sign}(\bar{\sigma}(-i))}(\bar{\sigma}(-i)) \end{cases} \text{for } i \in I_d$$

$$\Leftrightarrow \begin{cases} \bar{\sigma}(\mathbf{x}(i)) = x^{\delta(i)}(\bar{\sigma}(i)) \\ -\bar{\sigma}(\mathbf{x}(i)) = -x^{-\delta(i)}(\bar{\sigma}(i)) \end{cases} \text{for } i \in I_d \ \Leftrightarrow \text{(b)}$$

□

For each $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ we may take conjugacy classes of $\hat{G}_{\mathcal{O}} = \langle \mathbf{x}, \mathbf{y} \rangle$ in just elements in $\mathrm{Stab}(\mathbf{x}) := \{\sigma = (\bar{\sigma}, \delta) \in \hat{\mathfrak{S}}_d^0 \mid \delta = \delta \circ x \text{ and } \bar{\sigma}^* x = x^\delta \text{ hold on } I_d\}$. This restriction allows us to obtain all origamis in the class represented by $\mathcal{O}$ with the form (x, y', ε') and $\langle \mathbf{x}, \mathbf{y}' \rangle$. With this argument next we construct an algorithm for describing all classes of origamis neccesary to decide which class the decompositions of each origami in the directions $T(0, \frac{\pi}{2}) = (0, \frac{\pi}{4})$ and $S(0, \frac{\pi}{2}) = (-\frac{\pi}{2}, 0)$ belong to.

5 Algorithm

Algorithm 5.1 For each $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, we can construct the restricted class $[\mathcal{O}] := \{\mathcal{O}' = (x, y', \varepsilon') \in \tilde{\Omega}_d \mid (\mathbf{x}, \mathbf{y}) \sim (\mathbf{x}', \mathbf{y}') \text{ and } \mathbf{x} = \mathbf{x}'\}$ in the following way:

(1) $[\mathcal{O}] := \emptyset$ (initialize).

(2) Take $\sigma = (\bar{\sigma}, \delta) \in \mathrm{Stab}(\mathbf{x})$.

(3) For each $i \in I_d$, we define as follows.

$$\begin{aligned} \mu_i &:= |\sigma^*(y^\varepsilon)(i)| = \bar{\sigma}(y^{\delta(\bar{\sigma}^{-1}(i))\varepsilon(\bar{\sigma}^{-1}(i))}(\bar{\sigma}^{-1}(i))) \\ \nu_i &:= |\sigma^*(y^\varepsilon)(-i)| = \bar{\sigma}(y^{-\delta(\bar{\sigma}^{-1}(i))\varepsilon(\bar{\sigma}^{-1}(i))}(\bar{\sigma}^{-1}(i))) \\ \eta_i &:= [\varepsilon(\bar{\sigma}^{-1}(i)) = \varepsilon(y^{\delta(\bar{\sigma}^{-1}(i))\varepsilon(\bar{\sigma}^{-1}(i))}(\bar{\sigma}^{-1}(i)))] \end{aligned}$$

(4) For each $\varepsilon' \in \mathcal{E}_d$ and $i \in I_d$, we define as follows.

$$\eta'_i := [\varepsilon'(i) = \delta(\bar{\sigma}^{-1}(i)) \cdot \delta(y^{\delta(\bar{\sigma}^{-1}(i))\varepsilon(\bar{\sigma}^{-1}(i))}(\bar{\sigma}^{-1}(i))) \cdot \varepsilon'(\mu_i)]$$

(5) Check whether $\eta = \eta'$. For each $\varepsilon' \in \mathcal{E}_d$ satisfying $\eta = \eta'$ and $i \in I_d$, we define as follows.

$$y^{\sigma,\varepsilon'}(i) := \begin{cases} \mu_i & \text{if } \varepsilon'(i) = 1 \\ \nu_i & \text{if } \varepsilon'(i) = -1 \end{cases}$$

Further we remove ε' such that $y^{\sigma,\varepsilon'} \notin \mathfrak{S}_d$.

(6) Add elements in $\{(x, y^{\sigma,\varepsilon'}, \varepsilon') \mid \eta = \eta', y^{\sigma,\varepsilon'} \in \mathfrak{S}_d\}$ to $[\mathcal{O}]$.

(7) Go back to (2) for other leftover $\sigma = (\bar{\sigma}, \delta) \in \mathrm{Stab}(\mathbf{x})$. When we have been through all elements in $\mathrm{Stab}(\mathbf{x})$, finish the algorithm.

Proof. For each $\sigma = (\bar\sigma, \delta) \in \mathrm{Stab}(\mathbf{x})$ we should take all $(x, y', \varepsilon') \in \tilde\Omega_d$ satisfying the condition (3) and (4) in Lemma 4.3. Let $y_\sigma := \sigma^*(y^\varepsilon)$. For each $i \in \hat I_d$,

$$\text{by Lemma 4.3, } \begin{cases} \xi'(i) = \mathrm{sign}(y_\sigma(i))) \cdot \xi(\sigma^{-1}(i)) & \cdots \text{(a)} \\ y'^{\varepsilon'(i)}(i) = \mathrm{sign}(y_\sigma(i))) \cdot y_\sigma(i) & \cdots \text{(b)} \end{cases}$$

Note that $\sigma^{-1}(i) = \delta(\bar\sigma^{-1}(i)) \cdot \bar\sigma^{-1}(i)$. We have the followings:

$$\begin{aligned} y_\sigma(i) = \sigma^*(y^\varepsilon)(i) &= \sigma(y^{\varepsilon(\sigma^{-1}(i))}(\sigma^{-1}(i))) \\ &= \delta(\bar\sigma^{-1}(i)) \cdot \sigma(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i))) \\ &= \delta(\bar\sigma^{-1}(i)) \cdot \delta(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i))) \cdot \bar\sigma(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i))) \\ &= \delta(\bar\sigma^{-1}(i)) \cdot \delta(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i))) \cdot \mu_i \end{aligned}$$

$$\begin{aligned} \xi(\sigma^{-1}(i)) &= [\varepsilon(\sigma^{-1}(i)) = \varepsilon(y^{\varepsilon(\sigma^{-1}(i))}(\sigma^{-1}(i)))] \\ &= [\delta(\bar\sigma^{-1}(i)\varepsilon(\bar\sigma^{-1}(i)) = \delta(\bar\sigma^{-1}(i)\varepsilon(y^{\delta(\bar\sigma^{-1}(i)\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i)))] \\ &= [\varepsilon(\bar\sigma^{-1}(i)) = \varepsilon(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i)))] \\ &= \eta_i \end{aligned}$$

$$\begin{aligned} \xi'(i) &= [\varepsilon'(i) = \varepsilon'(y'(i))] \\ &= [\varepsilon'(i) = \varepsilon'(\mathrm{sign}(y_\sigma(i)) \cdot y_\sigma^{\varepsilon'(i)}(i))] \quad (\because \text{(b)}) \\ &= [\varepsilon'(i) = \mathrm{sign}(y_\sigma(i)) \cdot \delta(\bar\sigma^{-1}(i)) \cdot \delta(y^{\delta(\bar\sigma^{-1}(i))\varepsilon(\bar\sigma^{-1}(i))}(\bar\sigma^{-1}(i))) \cdot \varepsilon(\mu_i)] \\ &= \mathrm{sign}(y_\sigma(i)) \cdot \eta'_i \end{aligned}$$

Hence $\eta = \eta'$ implies that (b) holds for all $i \in \hat I_d$. If it is true, by (a) y' should coincide with $y^{\sigma,\varepsilon'}$. So the algorithm calculates the expected result. □

To classify origamis of degree d using $\mathrm{Stab}(\mathbf{x}) < \mathfrak{S}_d$, we should start with data of conjugacy classes of $x \in \mathfrak{S}_d$ in $\mathfrak{S}_d$. Each of such classes are characterized by a *partition* [1] (also known to be a *Young tableau*) of d, which is a finite sequence of weakly decreasing positive integers which sum to d. The partition nuumber $p(d)$, which counts the number of partitions of $d \in \{0, 1, 2, ...\}$, defines the rapidly increasing sequence:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, ...$$

(cf. http://oeis.org/A000041.) The asymptotic formula

$$p(d) \sim \tfrac{1}{4d\sqrt{3}} \cdot e^{\sqrt{2d/3}}$$

is proved by Hardy and Ramanujan [6]. Hashiguchi, Niki, and Nakagawa [5] give algorithms for constructing all partitions of given integer. We will start an algorithm with $P(d) = \{(j_1.j_2, ..., j_d) : \text{partition of } d\}$.

Algorithm 5.2 Let $P(d) = \{(j_1.j_2, ..., j_d) : \text{partition of } d\}$. We obtain the set $C\tilde\Omega_d := \{C_n \subset \tilde\Omega_d \mid n \in I_N,\ \cup_{n\in I_N} C_n = \tilde\Omega_d,\ {}^\forall\mathcal{O}, \mathcal{O}' \in C_n \text{ are isomorphic for } {}^\forall n\}$ of all isomorphism classes of origamis of degree d with the following steps.

(1) $C\tilde\Omega_d := \emptyset$ (initialize).

(2) Take $j = (j_1.j_2, ..., j_d) \in P(d)$. Let:

$$\begin{aligned} d'_j &:= \max\{k \mid j_k > 0\} \\ x_j &:= (1, 2, ..., j_1)(j_1 + 1, j_1 + 2, ..., j_1 + j_2) \cdots (\Sigma_{k=1}^{d'-1} j_k + 1, ..., d) \in \mathfrak{S}_d \\ R_j &:= \mathfrak{S}_d \times \mathcal{E}_d. \end{aligned}$$

(3) Take $(y,\varepsilon) \in R_j$. Apply Algorithm 5.1 to $(x_j, y, \varepsilon) \in \tilde{\Omega}_d$ to get $[(x_j, y, \varepsilon)]$.

(4) Denote the second, third projection of $\mathcal{O} \in \tilde{\Omega}_d$ by $y(\mathcal{O}), \varepsilon(\mathcal{O})$ respectively. Add $\sigma^*(x_j, y(\mathcal{O}), \varepsilon(\mathcal{O}))$ to $C\tilde{\Omega}_d$ for each $\mathcal{O} \in [(x_j, y, \varepsilon)]$ and $\sigma \in \mathfrak{S}_d$. Remove $(y(\mathcal{O}), \varepsilon(\mathcal{O}))$ from R_j for each $\mathcal{O} \in [(x_j, y, \varepsilon)]$.

(5) Go back to (3) until $R_j = \emptyset$. If so, go to the next step.

(6) Go back to (2) for other leftover $j \in P(d)$. When we have been through all elements in $P(d)$, finish the algorithm.

It is clear that this algorithm classifies all elements in $\tilde{\Omega}_d$ into classes given by Algorithm 5.1.
Next we calculate the permutations $\varphi_T, \varphi_S \in \mathrm{Sym}(C_0\tilde{\Omega}_d)$ which correspond to $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ acting on $\tilde{\Omega}_d$ as decomposing origamis into pairs of directions $T(0, \frac{\pi}{2}) = (0, \frac{\pi}{4})$ and $S(0, \frac{\pi}{2}) = (-\frac{\pi}{2}, 0)$ respectively. Here we define $\gamma_T, \gamma_S \in \mathrm{Aut}(F_2)$ by:

$$\gamma_T : \begin{cases} x \mapsto x \\ y \mapsto xy \end{cases} \quad \text{and } \gamma_S : \begin{cases} x \mapsto y \\ y \mapsto x^{-1} \end{cases} .$$

Algorithm 5.3 Let $C\tilde{\Omega}_d$ be the result in Algorithm 5.2. Fix a representative $\mathcal{O}_n \in C_n$ for each $C_n \in C\tilde{\Omega}_d$ and denote $C_0\tilde{\Omega}_d := \{\mathcal{O}_n \mid n \in I_N\}$. We obtain the permutations $\varphi_T, \varphi_S \in \mathrm{Sym}(C\tilde{\Omega}_d) \cong \mathfrak{S}_N$ with the following steps.

(1) For each $n \in I_N$, let $\mathcal{O}_n = (x_n, y_n, \varepsilon_n) \in C_0\tilde{\Omega}_d$ and $\hat{G}_{\mathcal{O}_n} = (\mathbf{x}_n, \mathbf{y}_n) \in \Omega^0_{2d}$.

(2) Apply γ_T, γ_S to $\hat{G}_{\mathcal{O}}$. (that is, $\gamma_T(\hat{G}_{\mathcal{O}}) = (\mathbf{x}_n, \mathbf{x}_n\mathbf{y}_n)$ and $\gamma_S(\hat{G}_{\mathcal{O}}) = (\mathbf{y}_n, \mathbf{x}_n^{-1})$.)

(3) For $A = T, S$, let $(\mathbf{x}^A_n, \mathbf{y}^A_n) := \gamma_A(\hat{G}_{\mathcal{O}}) \in \Omega^0_{2d}$ and choose some $\varepsilon^A_n \in \mathcal{E}_d$ with

$$\begin{cases} {}^\forall i \in I_d, {}^\exists j \in I_d \text{ s.t. } \mathbf{x}^A_n(\varepsilon^A_n(i) \cdot i) = \varepsilon^A_n(j) \cdot j \\ {}^\forall i \in I_d, {}^\exists j \in I_d \text{ s.t. } \mathbf{y}^A_n(\varepsilon^A_n(i) \cdot i) = \varepsilon^A_n(j) \cdot j \end{cases} \cdots (\star)$$

$$\text{Farthermore, let } \begin{cases} x^A_n(i) := \mathbf{x}(i) \\ y^A_n(i) := |\mathbf{y}(\varepsilon^A_n(i) \cdot i)| \end{cases} \text{ for each } i \in I_d.$$

(4) For $A = T, S$, search for $C_{n_A} \in C\tilde{\Omega}_d$ satisfying that $(x^A_n, y^A_n, \varepsilon^A_n) \in C_{n_A}$ and let $\varphi_A(n) := n_A$.

(5) Go back to (1) for the next $n \in I_N$. When we have been through all elements in I_N, finish the algorithm.

Proof. $(\star)$ is equivalent to the condition for ε in Operation B in section 4. So in this way choosing ε^A_n we succesfully take $\mathcal{O}_A = (x^A_n, y^A_n, \varepsilon^A_n) \in C_{n_A}$ so that $\hat{G}_{\mathcal{O}_A} = (\mathbf{x}^A_n, \mathbf{y}^A_n)$. □

Finally we calculate the components of Teichmüller curves. As mentioned in Remark 3.4, the next algorithm let us see the Veech groups of all $\mathcal{O} \in \tilde{\Omega}_d$.

Algorithm 5.4 Let $\varphi_T, \varphi_S \in \mathfrak{S}_N$. We obtain the $\langle\varphi_T, \varphi_S\rangle$-orbits in I_N with the following steps.

(1) $R := I_N$ (initialize).

(2) For $t \in \mathbb{N}$, $O_t := \emptyset$ (initialize).

(3) Take $i \in R$ and add i to O_t.

(4) Take $j \in O_t$ and let $O(j) := \{\varphi_T^k(j), \varphi_S^k(j) \mid k \in \mathbb{N}\}$.

(5) Add all elements in $O(j)$ to O_t and remove them from R.

(6) Go back to (4) for other leftover $j \in O_t$. When we have been through all elements in O_t, go th the next step.

(7) Go back to (2) for the next t until $R = \emptyset$. If so, finish the algorithm.

Note that we may apply the Reidemeister-Schreier method [12] to the result of Algorithm 5.4 for the list of generators and the list of representatives of the Veech group of each origami.

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