BECK-TYPE IDENTITIES FOR EULER PAIRS OF ORDER r

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Abstract. Partition identities are often statements asserting that the set $\mathcal{P}_X$ of partitions of n subject to condition X is equinumerous to the set $\mathcal{P}_Y$ of partitions of n subject to condition Y. A Beck-type identity is a companion identity to $|\mathcal{P}_X| = |\mathcal{P}_Y|$ asserting that the difference $b(n)$ between the number of parts in all partitions in $\mathcal{P}_X$ and the number of parts in all partitions in $\mathcal{P}_Y$ equals a $c|\mathcal{P}_{X'}|$ and also $c|\mathcal{P}_{Y'}|$, where c is some constant related to the original identity, and X', respectively Y', is a condition on partitions that is a very slight relaxation of condition X, respectively Y. A second Beck-type identity involves the difference $b'(n)$ between the total number of different parts in all partitions in $\mathcal{P}_X$ and the total number of different parts in all partitions in $\mathcal{P}_Y$. We extend these results to Beck-type identities accompanying all identities given by Euler pairs of order r (for any $r \geq 2$). As a consequence, we obtain many families of new Beck-type identities. We give analytic and bijective proofs of our results.

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1. Introduction

The origin of this article is rooted in two conjectures by Beck which appeared in The On-Line Encyclopedia of Integer Sequences [1] on the pages for sequences A090867 and A265251. The conjectures, as formulated by Beck, were proved by Andrews in [3] using generating functions. Certain generalizations and combinatorial proofs appeared in [6] and [11]. Combinatorial proofs of the original conjectures were also given in [5]. Several additional similar identities were proved in the last two years. See for example [4, 7, 8, 9]. In order to define Beck-type identities, we first introduce the necessary terminology and notation.

In this article $\mathbb{N}$ denotes the set of positive integers. Given a non negative integer n, a *partition* λ of n is a non increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ that add up to n, i.e., $\displaystyle\sum_{i=1}^{k} \lambda_i = n$. Thus, if $l = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, we have $l_1 \geq l_2 \geq \ldots \geq l_k$. The numbers λ_i are called the *parts* of λ and n is called the *size* of λ. The number of parts of the partition is called the *length* of λ and is denoted by $\ell(\lambda)$.

If l, μ are two arbitrary partitions, we denote by $l \cup \mu$ the partition obtained by taking all parts of l and all parts of μ and rearranging them to form a partition. For example, if $l = (5, 5, 3, 2, 2, 1)$ and $\mu = (7, 5, 3, 3)$, then $l \cup \mu = (7, 5, 5, 5, 3, 3, 3, 2, 2, 1)$.

When convenient, we use the exponential notation for parts in a partition. The exponent of a part is the multiplicity of the part in the partition. For example, $(7, 5^2, 4, 3^3, 1^2)$ denotes the partition $(7, 5, 5, 5, 4, 3, 3, 3, 1, 1)$. It will be clear from the context when exponents refer to multiplicities and when they are exponents in the usual sense.

Let S_1 and S_2 be subsets of the positive integers. We define $\mathcal{O}_r(n)$ to be be the set of partitions of n with all parts from the set S_2 and $\mathcal{D}_r(n)$ to be the set of partitions of n with parts in S_1 repeated at most $r-1$ times. Subbarao [10] proved the following theorem.

Theorem 1.1. *$|\mathcal{O}_r(n)| = |\mathcal{D}_r(n)|$ for all non-negative integers n if and only if $rS_1 \subseteq S_1$ and $S_2 = S_1 \setminus rS_1$.*

Andrews [2] first discovered this result for $r = 2$ and called a pair (S_1, S_2) such that $|\mathcal{O}_2(n)| = |\mathcal{D}_2(n)|$ an *Euler pair* since the pair $S_1 = \mathbb{N}$ and $S_2 = 2\mathbb{N} - 1$ gives Euler's identity. By analogy, Subbarao called a pair (S_1, S_2) such that $|\mathcal{O}_r(n)| = |\mathcal{D}_r(n)|$ an *Euler pair of order r*.

Example 1 (Subbarao [10])**.** Let

$$S_1 = \{m \in \mathbb{N} : \ m \equiv 1 \ (mod\ 2)\};$$
$$S_2 = \{m \in \mathbb{N} : \ m \equiv \pm 1 \ (mod\ 6)\}.$$

Then (S_1, S_2) is an Euler pair of order 3.

Note that Glaisher's bijection used to prove $|\mathcal{O}_r(n)| = |\mathcal{D}_r(n)|$ when $S_1 = \mathbb{N}$ and $S_2 = 2\mathbb{N} - 1$ can be generalized to any Euler pair of order r. If (S_1, S_2) is an Euler pair of order r, let φ_r be the map from $\mathcal{O}_r(n)$ to $\mathcal{D}_r(n)$ which repeatedly merges r equal parts into a single part until there are no parts repeated more than $r-1$ times. The map φ_r is a bijection and we refer to it as Glaisher's bijection.

Given (S_1, S_2), an Euler pair of order r, we refer to the elements in $S_2 = S_1 \setminus rS_1$ as *primitive* elements and to the elements of $rS_1 = S_1 \setminus S_2$ as *non-primitive* elements. We usually denote primitive parts by bold lower case letters, for example $\mathbf{a}$. Non-primitive parts are denoted by (non-bold) lower case letters. If a is a non-primitive part and we want to emphasize the largest power k of r such that $a/r^k \in S_1$, we write $a = r^k\mathbf{a}$ with $\mathbf{a} \in S_2$.

Let $\mathcal{O}_{1,r}(n)$ be the set of partitions of n with parts in S_1 such that the set of parts in rS_1 has exactly one element. Thus, a partition in $\mathcal{O}_{1,r}(n)$ has exactly one non-primitive part (possibly repeated). Let $\mathcal{D}_{1,r}(n)$ be the set of partitions of n with parts in S_1 in which exactly one part is repeated at least r times.

Let $a_r(n) = |\mathcal{O}_{1,r}(n)|$ and $c_r(n) = |\mathcal{D}_{1,r}(n)|$. Let $b_r(n)$ be the difference between the number of parts in all partitions in $\mathcal{O}_r(n)$ and the number of parts in all partitions in $\mathcal{D}_r(n)$. Thus,

$$b_r(n) = \sum_{l \in \mathcal{O}_r(n)} \ell(l) - \sum_{l \in \mathcal{D}_r(n)} \ell(l).$$

Let $\mathcal{T}_r(n)$ be the subset of $\mathcal{D}_{1,r}(n)$ consisting of partitions of n in which one part is repeated more than r times but less than $2r$ times. Let $c'_r(n) = |\mathcal{T}_r(n)|$. Let $b'_r(n)$ be the difference between the total number of different parts in all partitions in $\mathcal{D}_r(n)$ and the total number of different parts in all partitions in $\mathcal{O}_r(n)$ (i.e., in each partition, parts are counted without multiplicity). If we denote by $\bar{\ell}(l)$ the number of different parts in l, then

$$b'_r(n) = \sum_{l \in \mathcal{D}_r(n)} \bar{\ell}(l) - \sum_{l \in \mathcal{O}_r(n)} \bar{\ell}(l).$$

In [1], Beck conjectured that, if $S_1 = \mathbb{N}$ and $S_2 = 2\mathbb{N} - 1$, then

$$a_2(n) = b_2(n) = c_2(n)$$

and

$$c_2'(n) = b_2'(n).$$

Andrews proved these identities in [3] using generating functions. Combinatorial proofs were given in [5]. For the case $r \geq 2$, $S_1 = \mathbb{N}$, and $S_2 = \{k \in \mathbb{N} : k \not\equiv 0 \pmod{r}\}$, Fu and Tang [6] gave generating function proofs for

$$a_r(n) = \frac{1}{r-1} b_r(n) = c_r(n) \tag{1}$$

and

$$c_r'(n) = b_r'(n). \tag{2}$$

They also proved combinatorially that $a_r(n) = c_r(n)$. In [11], Yang gave combinatorial proofs of (1) and (2) in the case $r \geq 2$, $S_1 = \mathbb{N}$, and $S_2 = \{k \in \mathbb{N} : k \not\equiv 0 \pmod{r}\}$.

Our main theorems establish the analogous result for all Euler pairs. We will prove the theorems both analytically and combinatorially. We refer to the results in Theorem 1.2 as first Beck-type identities and to the result in Theorem 1.3 as second Beck-type identity.

Theorem 1.2. *If n, r are integers such that $n \geq 0$ and $r \geq 2$, and (S_1, S_2) is an Euler pair of order r, then*

(i) $a_r(n) = \dfrac{1}{r-1} b_r(n)$

(ii) $c_r(n) = \dfrac{1}{r-1} b_r(n)$.

Theorem 1.3. *If n, r are integers such that $n \geq 0$ and $r \geq 2$, and (S_1, S_2) is an Euler pair of order r, then $c_r'(n) = b_r'(n)$.*

Example 2. We continue with the Euler pair of order 3 from Example 1. We have

$$\mathcal{O}_3(7) = \{(7), (5, 1^2), (1^7)\};\ \mathcal{D}_3(7) = \{(7), (5, 1^2), (3^2, 1)\};$$

and

$$\mathcal{O}_{1,3}(7) = \{(3^2, 1), (3, 1^4)\};\ \mathcal{D}_{1,3}(7) = \{(1^7), (3, 1^4)\}.$$

Glaisher's bijection is given by

$$\begin{array}{ccc} (7) & \xrightarrow{\varphi_3} & (7) \\ (5,1,1) & \longrightarrow & (5,1,1) \\ (\underbrace{1,1,1},\underbrace{1,1,1},1) & \longrightarrow & (3,\ 3,\ 1) \end{array}$$

We note that

$$a_3(7) = |O_{1,3}(7)| = 2, c_3(7) = |D_{1,3}(7)| = 2, \text{ and } b_3(7) = 11 - 7 = 4.$$

Thus,

$$\frac{1}{r-1} b_3(7) = a_3(7) = c_3(7).$$

If we restrict to counting different parts in partitions, we see that there are a total of 4 diferent parts in the partitions of $\mathcal{O}_3(7)$ and a total of 5 different parts in the partitions of $\mathcal{D}_3(7)$. Since $\mathcal{T}_3(7) = \{(3, 1^4)\}$, we have

$$b'_3(7) = 5 - 4 = 1 = |\mathcal{T}_3(7)|.$$

The analytic proofs of Theorems 1.2 and 1.3 are similar to the proofs in [3] and [6], while the combinatorial proofs follow the ideas of [5]. However, the generalizations of the proofs in the aforementioned articles to Euler pairs of order $r \geq 2$ are important as establishing the theorems in such generality leads to a multitude of new Beck-type identities. We reproduce several Euler pairs listed in [10]. For each identity $|\mathcal{O}_r(n)| = |\mathcal{D}_r(n)|$ holding for the pair below, there are companion Beck-type identities as in Theorems 1.2 and 1.3.

The following pairs (S_1, S_2) are Euler pairs (of order 2).

(i) $S_1 = \{m \in N : m \not\equiv 0 \pmod 3\}$;
$S_2 = \{m \in N : m \equiv 1, 5 \pmod 6\}$.
In this case, the identity $|\mathcal{O}_2(n)| = |\mathcal{D}_2(n)|$ is known as Schur's identity.

(ii) $S_1 = \{m \in N : m \equiv 2, 4, 5 \pmod 6\}$;
$S_2 = \{m \in N : m \equiv 2, 5, 11 \pmod{12}\}$.
In this case, the identity $|\mathcal{O}_2(n)| = |\mathcal{D}_2(n)|$ is known as Göllnitz's identity.

(iii) $S_1 = \{m \in N : m = x^2 + 2y^2 \text{ for some } x, y \in \mathbb{Z}\}$;
$S_2 = \{m \in N : m \equiv 1 \pmod 2 \text{ and } m = x^2 + 2y^2 \text{ for some } x, y \in \mathbb{Z}\}$.

The following is an Euler pair of order 3.

(iv) $S_1 = \{m \in N : m = x^2 + xy + y^2 \text{ for some } x, y \in \mathbb{Z}\}$;
$S_2 = \{m \in N : \gcd(m, 3) = 1 \text{ and } m = x^2 + xy + y^2 \text{ for some } x, y \in \mathbb{Z}\}$.

The following pairs (S_1, S_2) are Euler pairs of order r.

(v) $S_1 = \{m \in N : m \equiv \pm r \pmod{r(r+1)}\}$;
$S_2 = \{m \in N : m \equiv \pm r \pmod{r(r+1)} \text{ and } m \not\equiv \pm r^2 \pmod{r^2(r+1)}\}$.

(vi) $S_1 = \{m \in N : m \equiv \pm r, -1 \pmod{r(r+1)}\}$.
$S_2 = \{m \in N : m \equiv \pm r, -1 \pmod{r(r+1)} \text{ and } m \not\equiv \pm r^2, -r \pmod{r^2(r+1)}\}$.
If $r = 2$, this Euler pair becomes Göllnitz's pair in (ii) above.

(vii) Let $r + 1$ be a prime.
$S_1 = \{m \in N : m \not\equiv 0 \pmod{r+1}\}$;
$S_2 = \{m \in N : m \not\equiv tr, t(r+1) \pmod{r^2 + r} \text{ for } 1 \leq t \leq r\}$.
If $r = 2$, this Euler pair becomes Schur's pair in (i) above.

(viii) Let p be a prime and r a quadratic residue $\pmod p$.
$S_1 = \{m \in \mathbb{N} : m \text{ quadratic residue } \pmod p\}$;
$S_2 = \{m \in \mathbb{N} : m \not\equiv 0 \pmod r \text{ and } m \text{ quadratic residue } \pmod p\}$.

Note that each case (v)-(vii) gives infinitely many Euler pairs and therefore leads to infinitely many new Beck-type identities. We also note that in (vii) we corrected a slight error in (3.4) of [10].

Example 3. Consider the Euler pair in (vii) above with $r = 4$. We have
$S_1 = \{m \in N : m \not\equiv 0 \pmod 5\}$;

$S_2 = \{m \in N : m \not\equiv 4t, 5t \pmod{20} \text{ for } 1 \leq t \leq 4\}$.

Then (S_1, S_2) is an Euler pair of order 4 and we have

$\mathcal{O}_4(7) = \{(7), (6, 1), (3^2, 1), (3, 2^2), (3, 1^4), (3, 2, 1^2), (2^3, 1), (2^2, 1^3), (2, 1^5), (1^7)\}$;

$\mathcal{D}_4(7) = \{(7), (6, 1), (3^2, 1), (3, 2^2), (4, 3), (3, 2, 1^2), (2^3, 1), (2^2, 1^3), (4, 2, 1), (4, 1^3)\}$.

Glaisher's bijection is given by

$$\begin{array}{ccc}
(7) & \xrightarrow{\varphi_4} & (7) \\
(6,1) & \longrightarrow & (6,1) \\
(3,3,1) & \longrightarrow & (3,3,1) \\
(3,2,2) & \longrightarrow & (3,2,2) \\
(3,\underbrace{1,1,1,1}) & \longrightarrow & (4,3) \\
(3,2,1,1) & \longrightarrow & (3,2,1,1) \\
(2,2,2,1) & \longrightarrow & (2,2,2,1) \\
(2,2,1,1,1) & \longrightarrow & (2,2,1,1,1) \\
(2,\underbrace{1,1,1,1},1) & \longrightarrow & (4,2,1) \\
(\underbrace{1,1,1,1},1,1,1) & \longrightarrow & (4,1,1,1)
\end{array}$$

We have $\mathcal{O}_{1,4}(7) = \{(4, 1^3), (4, 2, 1), (4, 3)\}$; $\mathcal{D}_{1,4}(7) = \{(1^7), (2, 1^5), (3, 1^4)\}$.

We note that $a_4(7) = |O_{1,4}(7)| = 3$, $c_3(7) = |D_{1,3}(7)| = 3$, and $b_3(7) = 40 - 31 = 9$, so $\frac{1}{3}b_4(7) = a_4(7) = c_4(7)$.

If we restrict to counting distinct parts, we see that there are 19 distinct parts in the partitions of $\mathcal{O}_4(7)$ and 21 distinct parts in the partitions of $\mathcal{D}_4(7)$. So $b'_4(7) = 21 - 19 = 2 = |\mathcal{T}_4(7)|$ since $\mathcal{T}_4(7) = \{(1^7), (2, 1^5)\}$.

2. Proofs of Theorem 1.2

2.1. **Analytic Proof.** In this article, whenever we work with q-series, we assume that $|q| < 1$. When working with two-variable generating functions, we assume both variables are complex numbers less that 1 in absolute value. Then all series converge absolutely. The generating functions for $|\mathcal{D}_r(n)|$ and $|\mathcal{O}_r(n)|$ are given by

$$\sum_{n=0}^{\infty}|\mathcal{D}_r(n)|q^n = \prod_{a\in S_1}(1+q^a+q^{2a}+\cdots+q^{(r-1)a})$$
$$= \prod_{a\in S_1}\frac{1-q^{ra}}{1-q^a};$$

and

$$\sum_{n=0}^{\infty}|\mathcal{O}_r(n)|q^n = \prod_{\mathbf{b}\in S_2}\frac{1}{1-q^{\mathbf{b}}}.$$

To keep track of the number of parts used, we introduce a second variable z, where $|z|<1$. Let

$$\mathcal{D}_r(n;m) = \{l \in \mathcal{D}_r(n) \mid l \text{ has exactly } m \text{ parts}\}$$

and

$$\mathcal{O}_r(n;m) = \{l \in \mathcal{O}_r(n) \mid l \text{ has exactly } m \text{ parts}\}.$$

Then, the generating functions for $|\mathcal{D}_r(n;m)|$ and $|\mathcal{O}_r(n;m)|$ are

$$f_{\mathcal{D}_r}(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\mathcal{D}_r(n;m)|z^m q^n = \prod_{a\in S_1}(1+zq^a+z^2q^{2a}+\cdots+z^{(r-1)}q^{(r-1)a})$$
$$= \prod_{a\in S_1}\frac{1-z^r q^{ra}}{1-zq^a};$$

and

$$f_{\mathcal{O}_r}(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\mathcal{O}_r(n;m)|z^m q^n = \prod_{\mathbf{b}\in S_2}\frac{1}{1-zq^{\mathbf{b}}}.$$

To obtain the generating function for the total number of parts in all partition in $\mathcal{D}_r(n)$ (respectively $\mathcal{O}_r(n)$), we take the derivative with respect to z of $f_{\mathcal{D}_r}(z,q)$ (respectively $f_{\mathcal{O}_r}(z,q)$), and set $z=1$. We obtain

$$\left.\frac{\partial}{\partial z}\right|_{z=1} f_{\mathcal{D}_r}(z,q)$$
$$= \prod_{a\in S_1}\frac{1-q^{ra}}{1-q^a}\sum_{a\in S_1}\frac{-rq^{ra}(1-q^a)+q^a(1-q^{ra})}{(1-q^a)(1-q^{ra})}$$
$$= \prod_{a\in S_1}\frac{1-q^{ra}}{1-q^a}\sum_{a\in S_1}\left(\frac{q^a}{1-q^a}-\frac{q^{ra}}{1-q^{ra}}-(r-1)\frac{q^{ra}}{1-q^{ra}}\right)$$
$$= \prod_{a\in S_1}\frac{1-q^{ra}}{1-q^a}\left(\sum_{a\in S_1}\sum_{\substack{k=1\\ r\nmid k}}^{\infty}q^{ka}-\sum_{a\in S_1}(r-1)\frac{q^{ra}}{1-q^{ra}}\right);$$

and

$$\left.\frac{\partial}{\partial z}\right|_{z=1} f_{\mathcal{O}_r}(z,q) = \prod_{\mathbf{b}\in S_2}\frac{1}{1-q^{\mathbf{b}}}\sum_{\mathbf{b}\in S_2}\frac{q^{\mathbf{b}}}{1-q^{\mathbf{b}}}.$$

Since $|\mathcal{D}_r(n)| = |\mathcal{O}_r(n)|$, we have

$$\sum_{n=0}^{\infty} b_r(n)q^n = \prod_{\mathbf{b}\in S_2} \frac{1}{1-q^b}\left(\sum_{\mathbf{b}\in S_2} \frac{q^{\mathbf{b}}}{1-q^{\mathbf{b}}} - \sum_{\substack{a\in S_1\\ k\in\mathbb{N}\\ r\nmid k}} q^{ka} + \sum_{a\in S_1}(r-1)\frac{q^{ra}}{1-q^{ra}}\right).$$

Next we show that

$$\sum_{\mathbf{b}\in S_2} \frac{q^{\mathbf{b}}}{1-q^{\mathbf{b}}} = \sum_{\substack{a\in S_1\\ k\in\mathbb{N}\\ r\nmid k}} q^{ka}. \tag{3}$$

The set of exponents of q in the left sum is $C = \{m\mathbf{b} \mid m \in \mathbb{N},\ \mathbf{b} \in S_2\}$. The set of exponents of q in the right sum is $D = \{ka \mid k \in \mathbb{N},\ r \nmid k,\ a \in S_1\}$.

To prove (3), we create a matching of the elements of D and C.

Let d be an element of D. Then $d = ka$ where $k \in \mathbb{N}, r \nmid k$, and $a \in S_1$. Since $a \in S_1 = S_2 \sqcup rS_2 \sqcup r^2S_2 \cdots$, there is some integer $j \geq 0$ and some and $\mathbf{b} \in S_2$ such that $a = r^j\mathbf{b}$. Then $d = (kr^j)\mathbf{b} \in C$.

Conversely, consider $c \in C$. Then $c = m\mathbf{b}$ where $m \in \mathbb{N}$ and $\mathbf{b} \in S_2$. Let j be the largest non-negative integer such that r^j is a factor of m. If $m = r^jm'$, we have $c = m'(r^j\mathbf{b})$. Since $r \nmid m'$ and $r^j\mathbf{b} \in S_1$, this is an element of D. Thus, $C = D$, and (3) holds.

Then the generating function for $b_r(n)$ becomes

$$\begin{aligned}\sum_{n=0}^{\infty} b_r(n)q^n &= \prod_{\mathbf{b}\in S_2} \frac{1}{1-q^{\mathbf{b}}}\left((r-1)\sum_{a\in S_1} \frac{q^{ra}}{1-q^{ra}}\right)\\ &= \prod_{a\in S_1}(1+q^a+q^{2a}+\cdots+q^{(r-1)a})\left((r-1)\sum_{a\in S_1}\frac{q^{ra}}{1-q^{ra}}\right).\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} b_r(n)q^n = \sum_{n=0}^{\infty}(r-1)|\mathcal{O}_{1,r}(n)|q^n = \sum_{n=0}^{\infty}(r-1)|\mathcal{D}_{1,r}(n)|q^n.$$

Equating coefficients results in $a_r(n) = \dfrac{1}{r-1}b_r(n)$ and $c_r(n) = \dfrac{1}{r-1}b_r(n)$.

2.2. **Combinatorial Proof.**

2.2.1. $b_r(n)$ *as the cardinality of a set of marked partitions.* We start with another example of Glaisher's bijection.

Example 4. We continue with the Euler pair of order 3 from Example 1, but this time use $n = 11$.

$\mathcal{O}_3(11) = \{(11), (7, 1^4), (5^2, 1), (5, 1^6), (1^{11})\}$;
$\mathcal{D}_3(7) = \{(11), (9, 1^2), (7, 3, 1), (5^2, 1), (5, 3^2)\}$.

Thus, $b_3(11) = 27 - 13 = 14$.

Glaisher's bijection is given by

$$\begin{array}{ccc}
(11) & \xrightarrow{\varphi_3} & (11) \\
(7, \underbrace{1,1,1}, 1) & \longrightarrow & (7,3,1) \\
(5,5,1) & \longrightarrow & (5,5,1) \\
(5, \underbrace{1,1,1}, \underbrace{1,1,1}) & \longrightarrow & (5,3,3) \\
(\underbrace{\underbrace{1,1,1}, \underbrace{1,1,1}, \underbrace{1,1,1}}, 1, 1) & \longrightarrow & (9,1,1)
\end{array}$$

From Glaisher's bijection, it is clear that each partition $\lambda \in \mathcal{O}_r(n)$ has at least as many parts as its image $\varphi_r(\lambda) \in \mathcal{D}_r(n)$.

When calculating $b_r(n)$, we sum up the differences in the number of parts in each pair $(\lambda, \varphi_r(\lambda))$. Write each part μ_j of $\mu = \varphi_r(\lambda)$ as $\mu_j = r^{k_j}\mathbf{a}$ with $\mathbf{a} \in S_2$. Then, μ_j was obtained by merging r^{k_j} parts equal to $\mathbf{a}$ in λ and thus contributes an excess of $r^{k_j} - 1$ parts to $b_r(n)$. Therefore, the difference between the number of parts of λ and the number of parts of $\varphi_r(\lambda)$ is $\displaystyle\sum_{j=1}^{\ell(\varphi_r(\lambda))} (r^{k_j} - 1)$.

Example 5. In the setting of Example 4, we see that $(7,3,1)$ contributes 2 to $b_3(11)$, $(5,3,3)$ contributes $2+2$ to $b_3(11)$, and $(9,1,1)$ contributes 8 to $b_3(11)$. Thus, $b_3(11) = 2+4+8 = 14$.

Definition 1. Given (S_1, S_2), an Euler pair of order r, we define the set $\mathcal{MD}_{1,r}(n)$ of *marked partitions* of n as the set of partitions in $\mathcal{D}_r(n)$ such that exactly one part of the form $r^k\mathbf{a}$ with $k \geq 1$ and $\mathbf{a} \in S_2$ has as index an integer t satisfying $1 \leq t \leq r^k - 1$. If $\mu \in \mathcal{D}_r(n)$ has parts $\mu_i = \mu_j = r^k\mathbf{a}$, $k \geq 1$, with $i \neq j$, the marked partition in which μ_i has index t is considered different from the marked partition in which μ_j has index t.

Note that marked partitions, by definition, have a non-primitive part. Then, from the discussion above we have the following interpretation for $b_r(n)$.

Proposition 1. *Let n, r be integers such that $n \geq 1$ and $r \geq 2$. Then,*

$$b_r(n) = |\mathcal{MD}_{1,r}(n)|.$$

Definition 2. An *r-word* w is a sequence of letters from the alphabet $\{0, 1, \ldots r-1\}$. The *length* of an r-word w, denoted $\ell(w)$, is the number of letters in w. We refer to *position* i in w as the ith entry from the right, where the most right entry is counted as position 0.

Note that leading zeros are allowed and are recorded. For example, if $r = 5$, the 5-words 032 and 32 are different even though in base 5 they both represent 17. We have $\ell(032) = 3$ and $\ell(32) = 2$. The empty bit string has length 0 and is denoted by $\emptyset$.

Definition 3. Given (S_1, S_2), an Euler pair of order r, we define the set $\mathcal{DD}_r(n)$ of *r-decorated partition* as the set of partitions in $\mathcal{D}_r(n)$ with at least one non-primitive part such that exactly one non-primitive part $r^k\mathbf{a}$ (with $k \geq 1$ and $\mathbf{a} \in S_2$)

is decorated with an index w, where w is an r-word satisfying $0 \leq \ell(w) \leq k-1$. As in Definition 1, if $\mu \in \mathcal{D}_r(n)$ has parts $\mu_i = \mu_j = r^k\mathbf{a}$, $k \geq 1$, with $i \neq j$, the decorated partition in which μ_i has index w is considered different from the decorated partition in which μ_j has index w.

Thus, for each part $\mu_i = r^{k_i}\mathbf{a}$ of $\mu \in \mathcal{DD}_r(n)$ there are $\dfrac{r^{k_i}-1}{r-1}$ possible indices and for each partition $\mu \in \mathcal{DD}_r(n)$ there are precisely $\dfrac{1}{r-1}\displaystyle\sum_{j=1}^{\ell(\mu)}(r^{k_j}-1)$ possible decorated partitions with the same parts as μ.

The discussion above proves the following interpretation for $\dfrac{1}{r-1}b_r(n)$.

Proposition 2. *Let n, r be integers such that $n \geq 1$ and $r \geq 2$. Then,*

$$\frac{1}{r-1}b_r(n) = |\mathcal{DD}_r(n)|.$$

While it is obvious that $|\mathcal{MD}_{1,r}(n)| = (r-1)|\mathcal{DD}_r(n)|$, to see this combinatorially, consider the map $\psi_r : \mathcal{MD}_{1,r}(n) \to \mathcal{DD}_r(n)$ defined as follows. If $l \in \mathcal{MD}_{1,r}(n)$, then $\psi_r(l)$ is the partition in $\mathcal{DD}_r(n)$ in which the r-decorated part is the same as the marked part in l. The index of the part of $\psi_r(l)$ is obtained from the index of the part of l by writing it in base r and removing the leading digit. Clearly, this is a $r-1$ to 1 mapping.

2.2.2. *A combinatorial proof for $a_r(n) = \dfrac{1}{r-1}b_r(n)$.* To prove combinatorially that $a_r(n) = \dfrac{1}{r-1}b_r(n)$ we establishing a one-to-one correspondence between $\mathcal{O}_{1,r}(n)$ and $\mathcal{DD}_r(n)$.

From $\mathcal{DD}_r(n)$ to $\mathcal{O}_{1,r}(n)$:

Start with an r-decorated partition $\mu \in \mathcal{DD}_r(n)$. Suppose the non-primitive part $\mu_i = r^k\mathbf{a}$, with $k \geq 1$ and $\mathbf{a} \in S_2$, is decorated with r-word w of length $\ell(w)$. Then, $0 \leq \ell(w) \leq k-1$. Let d_w be the decimal value of of w. We set $d_\emptyset = 0$. We transform μ into a partition $l \in \mathcal{O}_{1,r}(n)$ as follows.

Define $\bar{\mu}$ to be the partition whose parts are all non-primitive parts of μ of the form $\mu_j = r^t\mathbf{a}$ with $j \leq i$, i.e., all parts $r^t\mathbf{a}$ with $t > k$ and, if μ_i is the pth part of size $r^k\mathbf{a}$ in μ, then $\bar{\mu}$ also has p parts equal to $r^k\mathbf{a}$.

Define $\tilde{\mu}$ to be the partition whose parts are all parts of μ that are not in $\bar{\mu}$.

(1) In $\bar{\mu}$, split one part of size $r^k\mathbf{a}$ into d_w+1 parts of size $r^{k-\ell(w)}\mathbf{a}$ and $r^k-(d_w+1)r^{k-\ell(w)}$ primitive parts of size $\mathbf{a}$. Every other part in $\bar{\mu}$ splits completely into parts of size $r^{k-\ell(w)}\mathbf{a}$. Denote the resulting partition by $\bar{l}$.
(2) Let $\tilde{l} = \varphi_r^{-1}(\tilde{\mu})$. Thus, $\tilde{l}$ is obtained by splitting all parts in $\tilde{\mu}$ into primitive parts.

Let $l = \bar{l} \cup \tilde{l}$. Then $l \in \mathcal{O}_{1,r}(n)$ and its set of non-primitive parts is $\{r^{k-\ell(w)}\mathbf{a}\}$.

Remark 1. Since $d_w + 1 \leq r^{\ell(w)}$, in step 1, the resulting number of primitive parts equal to $\mathbf{a}$ is non-negative. Moreover, in $\bar{l}$ there is at least one non-primitive part.

Example 6. We continue with the Euler pair of order 3 from Example 1. Consider the decorated partition

$$\begin{aligned}\mu &= (1215, 135_{02}, 135, 51, 35, 15, 15, 3)\\ &= (3^5\cdot 5, (3^3\cdot 5)_{02}, 3^3\cdot 5, 3\cdot 17, 35, 3\cdot 5, 3\cdot 5, 3\cdot 1) \in \mathcal{DD}_r(1604).\end{aligned}$$

We have $k = 3, \ell(w) = 2, d_w = 2$, and

$\bar{\mu} = (3^5\cdot 5, 3^3\cdot 5)$;

$\tilde{\mu} = (3^3\cdot 5, 3\cdot 17, 35, 3\cdot 5, 3\cdot 5, 3\cdot 1)$.

To create $\bar{\lambda}$ from $\bar{\mu}$:

(1) Part $135 = 3^3\cdot 5$ splits into three parts of size 15 and eighteen parts of size 5.

(2) Part $1215 = 3^5\cdot 5$ splits into eighty one parts of size 15.

This results in $\bar{\lambda} = (15^{84}, 5^{18})$.

To create $\tilde{\lambda}$ from $\tilde{\mu}$:

All parts in $\tilde{\mu}$ are split into primitive parts. Thus, part $3^3\cdot 5$ splits into twenty seven parts of size 5, part $3\cdot 17$ splits into three parts of size 17, both parts of $3\cdot 5$ split into three parts of size 5 each, and part $3\cdot 1$ splits into three parts of size 1. Part 35 is already primitive so remains unchanged.

This results in $\tilde{\lambda} = (35, 17^3, 5^{33}, 1^3)$. Then, setting $\lambda = \bar{\lambda}\cup\tilde{\lambda}$ results in $\lambda = (35, 17^3, 15^{84}, 5^{51}, 1^3) \in \mathcal{O}_{1,r}(1604)$. The non-primitive part is $15 = 3\cdot 5$.

From $\mathcal{O}_{1,r}(n)$ to $\mathcal{DD}_r(n)$:

Start with a partition $\lambda \in \mathcal{O}_{1,r}(n)$. In l there is one and only one non-primitive part $r^k\mathbf{a}$ with $k \geq 1$ and $\mathbf{a} \in S_2$. Let f be the multiplicity of the non-primitive part of λ. We transform l into an r-decorated partition in $\mathcal{DD}_r(n)$ as follows.

Apply Glaisher's bijection to l to obtain $\mu = \varphi_r(l) \in \mathcal{D}_r(n)$. Since λ has a non-primitive part, μ will have at least one non-primitive part.

Next, we determine the r-decoration of μ. Consider the non-primitive parts μ_{j_i} of μ of the form $r^{t_i}\mathbf{a}$, with $\mathbf{a} \in S_2$ (same $\mathbf{a}$ as in the non-primitive part of λ) and $t_i \geq k$. Assume $j_1 < j_2 < \cdots$. For notational convenience, set $\mu_{j_0} = 0$. Let h be the positive integer such that

$$\sum_{i=0}^{h-1} \mu_{j_i} < f\cdot r^k\mathbf{a} \leq \sum_{i=0}^{h} \mu_{j_i}. \tag{4}$$

Then, we will decorate part $\mu_{j_h} = r^{t_h}\mathbf{a}$. To determine the decoration, let

$$N = \frac{\displaystyle\sum_{i=0}^{h-1} \mu_{j_i}}{r^k\mathbf{a}}.$$

Then, (4) becomes

$$r^k\mathbf{a}N < f\cdot r^k\mathbf{a} \leq r^k\mathbf{a}N + r^{t_h}\mathbf{a},$$

which implies $0 < f - N \leq r^{t_h-k}$.

Let $d = f - N - 1$ and $\ell = t_h - k$. We have $0 \leq \ell \leq t_h - 1$. Consider the representation of d in base r and insert leading zeros to form an r-word w of length ℓ. Decorate μ_{j_h} with w. The resulting decorated partition is in $\mathcal{DD}_r(n)$.

Example 7. We continue with the Euler pair of order 3 from Example 1. Consider the partition $\lambda = (35, 17^3, 15^{84}, 5^{51}, 1^3) \in \mathcal{O}_{1,r}(1604)$. The non-primitive part is 15. We have $k = 1, f = 84$.

Glaisher's bijection produces the partition $\mu = (1215, 135^2, 51, 35, 15^2, 3) = (3^5 \cdot 5, 3^3 \cdot 5, 3^3 \cdot 5, 3 \cdot 17, 35, 3 \cdot 5, 3 \cdot 5, 3 \cdot 1) \in \mathcal{MD}(1604)$. The parts of the form $3^{r_i} \cdot 5$ with $r_i \geq 1$ are $1215, 135, 135, 15, 15$. Since $1215 < 84(3^1 \cdot 5) \leq 1215 + 135$, the decorated part will be the first part $135 = 3^3 \cdot 5$. We have $N = 1215/15 = 81$.

To determine the decoration, let $d_w = 84 - 81 - 1 = 2$ and $\ell = 3 - 1 = 2$. The base 3 representation of d_w is 2. To form an 3-word of length 2, we introduce one leading 0. Thus, the decoration is $w = 02$ and the resulting decorated partition is $(1215, 135_{02}, 135, 51, 35, 15, 15, 3) = (3^5 \cdot 5, (3^3 \cdot 5)_{02}, 3^3 \cdot 5, 3 \cdot 17, 35, 3 \cdot 5, 3 \cdot 5, 3 \cdot 1) \in \mathcal{DD}_r(1604)$.

2.2.3. *A combinatorial proof for* $c_r(n) = \dfrac{1}{r-1} b_r(n)$. We note that one can compose the bijection of section 2.2.2 with the bijection of [6] to obtain a combinatorial proof of part (ii) of Theorem 1.2. However, we give an alternative proof that $c_r(n) = \dfrac{1}{r-1} b_r(n)$ by establishing a one-to-one correspondence between $\mathcal{D}_{1,r}(n)$ and $\mathcal{DD}_r(n)$. This proof does not involve the bijection of [6] and it mirrors the proof of section 2.2.2.

From $\mathcal{DD}_r(n)$ *to* $\mathcal{D}_{1,r}(n)$*:*

Start with an r-decorated partition $\mu \in \mathcal{DD}_r(n)$. Suppose the non-primitive part $\mu_i = r^k \mathbf{a}$, with $k \geq 1$ and $\mathbf{a} \in S_2$, is decorated with r-word w of length $\ell(w)$ and decimal value d_w. Then, $0 \leq \ell(w) \leq k - 1$. We transform μ into a partition $\lambda \in \mathcal{D}_{1,r}(n)$ as follows.

Let $\bar{\bar{\mu}}$ be the partition whose parts are all non-primitive parts of μ of the form $\mu_j = r^t \mathbf{a}$ with $j \geq i$, and $k - \ell(w) - 1 < t \leq k$, i.e., all parts $r^t \mathbf{a}$ with $k - \ell(w) - 1 < t < k$ and, if there are $p - 1$ parts of size $r^k \mathbf{a}$ in μ after the decorated part, then $\bar{\bar{\mu}}$ also has p parts equal to $r^k \mathbf{a}$.

Let $\tilde{\tilde{\mu}}$ be the partition whose parts are all parts of μ that are not in $\bar{\bar{\mu}}$.

In $\bar{\bar{\mu}}$, perform the following steps.

(1) Split one part equal to $r^k \mathbf{a}$ into $r(d_w + 1)$ parts of size $r^{k-\ell(w)-1} \mathbf{a}$ and m primitive parts of size $\mathbf{a}$, where $m = r^k - r(d_w + 1) r^{k-\ell(w)-1}$. Apply Glaisher's bijection φ_r to the partition consisting of m parts equal to $\mathbf{a}$.
(2) Split all remaining parts of $\bar{\bar{\mu}}$ completely into parts of size $r^{k-\ell(w)-1} \mathbf{a}$.

Denote by $\bar{\bar{l}}$ the partition with parts resulting from steps 1 and 2 above.

Let $l = \bar{\bar{l}} \cup \tilde{\tilde{\mu}}$. Since $r(d_w + 1) \geq r$, it follows that $\lambda \in \mathcal{D}_{1,r}(n)$. The part repeated at least r times is $r^{k-\ell(w)-1} \mathbf{a}$.

Remark 2. Since $d_w + 1 \leq r^{\ell(w)}$, the splitting in step 1 can be performed. If $w = \emptyset$ and there are no parts equal to $r^k \mathbf{a}$ after μ_i, then μ_i splits into r equal parts of size $r^{k-1} \mathbf{a}$, and there are no other parts in $\bar{\bar{\mu}}$ to split. Of course, there could also be parts of size $r^{k-1} \mathbf{a}$ in $\tilde{\tilde{\mu}}$.

Example 8. We continue with the Euler pair of order 3 from Example 1. Consider the partition $\mu = (32805, (10935)_{0120}, 10935, 1215, 45, 45, 25, 9, 3) = (3^8 \cdot 5, (3^7 \cdot 5)_{0120}, 3^7 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1) \in \mathcal{DD}_r(56017)$. Then the decorated part is $\mu_2 = 3^7 \cdot 5$ and the decoration is $w = 0120$. We have $k = 7$, $\ell(w) = 4$, $d_w = 15$. So

$\bar{\bar{\mu}} = (3^7 \cdot 5, 3^7 \cdot 5, 3^5 \cdot 5)$;

$\tilde{\tilde{\mu}} = (3^8 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1)$.

(1) $3^7 \cdot 5$ splits into
   - $r(d_w + 1) = 48$ parts of $3^2 \cdot 5$ and
   - $m = r^k - r(d_w + 1)r^{k-\ell(w)-1} = 3^7 - 48(3^2) = 1755$ parts of 5.

   The 1755 parts of 5 merge into two parts of 3645, one part of 1215, and two parts of 135.

(2) $3^7 \cdot 5$ splits into two hundred and forty three parts of $3^2 \cdot 5$ and $3^5 \cdot 5$ splits into twenty seven parts of $3^2 \cdot 5$.

This results in

$\bar{\bar{\lambda}} = (3645^2, 1215, 135^2, 45^{318})$;

$\lambda = \bar{\bar{\lambda}} \cup \tilde{\tilde{\mu}} = (32805, 3645^2, 1215, 135^2, 45^{320}, 25, 9, 3) \in \mathcal{D}_{1,r}(56017)$. The part repeated at least three times is $45 = 3^2 \cdot 5$.

From $\mathcal{D}_{1,r}(n)$ to $\mathcal{DD}_r(n)$:

Start with a partition $\lambda \in \mathcal{D}_{1,r}(n)$. Then, among the parts of λ, there is one and only one part that is repeated at least r times. Suppose the repeated part is $r^k\mathbf{a}$, $k \geq 0$ and $\mathbf{a} \in S_2$, and denote by $f \geq r$ its multiplicity in λ. As in Glaisher's bijection we merge repeatedly parts of l that are repeated at least r times to obtain $\mu \in \mathcal{D}_r(n)$. Since λ has a part repeated at least r times, μ will have at least one non-primitive part.

Next, we determine the decoration of μ. In this case, we want to work with the parts of μ from the right to the left (i.e., from smallest to largest part). Let $\tilde{\mu}_q = \mu_{\ell(\mu)-q+1}$. Consider the parts $\tilde{\mu}_{j_i}$ of the form $r^{t_i}\mathbf{a}$, with $\mathbf{a} \in S_2$ and $t_i \geq k$. If $t_1 < t_2 < \cdots$, we have $j_1 < j_2 < \cdots$.

As before, we set $\tilde{\mu}_{j_0} = 0$. Let h be the positive integer such that

$$\sum_{i=0}^{h-1} \tilde{\mu}_{j_i} < f \cdot r^k\mathbf{a} \leq \sum_{i=0}^{h} \tilde{\mu}_{j_i}. \tag{5}$$

Then, we will decorate part $\tilde{\mu}_{j_h} = r^{t_h}\mathbf{a}$. To determine the decoration, let

$$N = \frac{\sum_{i=0}^{h-1} \tilde{\mu}_{j_i}}{r^k\mathbf{a}}. \tag{6}$$

Then, (5) becomes

$$r^k\mathbf{a}N < f \cdot r^k\mathbf{a} \leq r^k\mathbf{a}N + r^{t_h}\mathbf{a},$$

which implies $0 < f - N \leq r^{t_h - k}$.

Let $d = \dfrac{f-N}{r} - 1$ and $\ell = t_h - k - 1$. We have $0 \leq \ell \leq t_h - 1$. Consider the representation of d in base r and insert leading zeros to form an r-word w of length

ℓ. Decorate $\tilde{\mu}_{j_h}$ with w. The resulting decorated partition (with parts written in non-increasing order) is in $\mathcal{DD}_r(n)$.

Remark 3. To see that $f-N$ above is always divisible by r, note that if $f = qr+t$ with $q, t \in \mathbb{Z}$ and $0 \leq t < r$, then there are t terms equal to $r^k\mathbf{a}$ in the numerator of N. All other terms, if any, are divisible by $r^{k+1}\mathbf{a}$. Therefore, the remainder of N upon division by r is t.

Example 9. We continue with the Euler pair of order 3 from Example 1. Consider the partition $\lambda = (32805, 3645^2, 1215, 135^2, 45^{320}, 25, 9, 3) \in \mathcal{D}_{1,r}(56017)$. The part repeated at least three times is $45 = 3^2 \cdot 5$. We have $k = 2$ and $f = 320$.

Applying Glaisher's bijection to λ results in

$$\mu = \varphi_3(\lambda) = (3^8 \cdot 5, 3^7 \cdot 5, 3^7 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2, 3 \cdot 1) \in \mathcal{D}_r(56017).$$

The parts of the form $3^{t_i} \cdot 5$ with $t_i \geq 2$ are $3^2 \cdot 5, 3^2 \cdot 5, 3^5 \cdot 5, 3^7 \cdot 5, 3^7 \cdot 5, 3^8 \cdot 5$. Since $3^2 \cdot 5 + 3^2 \cdot 5 + 3^5 \cdot 5 + 3^7 \cdot 5 < 320 \cdot 3^2 \cdot 5 \leq 3^2 \cdot 5 + 3^2 \cdot 5 + 3^5 \cdot 5 + 3^7 \cdot 5 + 3^7 \cdot 5$, the decorated part will be the second part (counting from the right) equal to $3^7 \cdot 5 = 10935$. We have $N = \dfrac{3^2 \cdot 5 + 3^2 \cdot 5 + 3^5 \cdot 5 + 3^7 \cdot 5}{3^2 \cdot 5} = 272$. Thus $d = \dfrac{320 - 272}{3} - 1 = 15$ and $\ell = 7 - 2 - 1 = 4$. The base 3 representation of d is 120. To form a 3-word of length 4, we introduce one leading 0. Thus, the decoration is $w = 0120$ and the resulting decorated partition is

$$\begin{aligned}\mu =& (32805, (10935)_{0120}, 10935, 1215, 45, 45, 25, 9, 3)\\ =& (3^8 \cdot 5, (3^7 \cdot 5)_{0120}, 3^7 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1) \in \mathcal{DD}_r(56017).\end{aligned}$$

3. Proofs of Theorem 1.3

3.1. **Analytic Proof.** We create a bivariate generating function to keep track of the number of different parts in partitions in $\mathcal{O}_r(n)$, respectively $\mathcal{D}_r(n)$.

We denote by $\overline{\mathcal{O}}_r(n; m)$ the set of partitions of n with parts from S_2 using m different parts. We denote by $\overline{\mathcal{D}}_r(n; m)$ the set of partitions of n with parts from S_1 using m different parts and allowing parts to repeat no more than $r-1$ times. Then,

$$\begin{aligned} f_{\overline{\mathcal{O}}_r}(z, q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\overline{\mathcal{O}}_r(n; m)| z^m q^n &= \prod_{\mathbf{b} \in S_2} (1 + zq^{\mathbf{b}} + zq^{2\mathbf{b}} + \cdots) \\ &= \prod_{\mathbf{b} \in S_2} \left(1 + \frac{zq^{\mathbf{b}}}{1 - q^{\mathbf{b}}}\right), \end{aligned}$$

and

$$\begin{aligned} f_{\overline{\mathcal{D}}_r}(z, q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\overline{\mathcal{D}}_r(n; m)| z^m q^n &= \prod_{a \in S_1} (1 + zq^a + \cdots + zq^{(r-1)a}) \\ &= \prod_{a \in S_1} \left(1 + \frac{zq^a - zq^{ra}}{1 - q^a}\right). \end{aligned}$$

To obtain the generating function for the total number of different parts in all partition in $\mathcal{O}_r(n)$ (respectively $\mathcal{D}_r(n)$), we take the derivative with respect to z of $f_{\overline{\mathcal{O}}_r}(z,q)$ (respectively $f_{\overline{\mathcal{D}}_r}(z,q)$), and set $z=1$. We obtain

$$\begin{aligned}
\frac{\partial}{\partial z}\Big|_{z=1} f_{\overline{\mathcal{O}}_r}(z,q) &= \sum_{\mathbf{b}\in S_2} \frac{q^{\mathbf{b}}}{1-q^{\mathbf{b}}} \prod_{\mathbf{c}\in S_2, \mathbf{c}\neq\mathbf{b}} \left(1+\frac{q^{\mathbf{c}}}{1-q^{\mathbf{c}}}\right)\\
&= \sum_{\mathbf{b}\in S_2} \frac{q^{\mathbf{b}}}{1-q^{\mathbf{b}}} \prod_{\mathbf{c}\in S_2, \mathbf{c}\neq\mathbf{b}} \left(\frac{1}{1-q^{\mathbf{c}}}\right)\\
&= \prod_{\mathbf{b}\in S_2} \frac{1}{1-q^{\mathbf{b}}} \sum_{\mathbf{b}\in S_2} q^{\mathbf{b}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z}\Big|_{z=1} f_{\overline{\mathcal{D}}_r}(z,q) &= \sum_{a\in S_1} \frac{q^a-q^{ra}}{1-q^a} \prod_{d\in S_1, d\neq a} \left(1+\frac{q^d-q^{rd}}{1-q^d}\right)\\
&= \sum_{a\in S_1} \frac{q^a-q^{ra}}{1-q^a} \prod_{d\in S_1, d\neq a} \frac{1-q^{rd}}{1-q^d}\\
&= \prod_{a\in S_1} \frac{1-q^{ra}}{1-q^a} \sum_{a\in S_1} \frac{q^a-q^{ra}}{1-q^{ra}}
\end{aligned}$$

Since $|\mathcal{D}_r(n)| = |\mathcal{O}_r(n)|$, we have

$$\sum_{n=0}^{\infty} b'_r(n)q^n = \prod_{a\in S_1} \frac{1-q^{ra}}{1-q^a} \left(\sum_{a\in S_1} \frac{q^a}{1-q^{ra}} - \sum_{a\in S_1} \frac{q^{ra}}{1-q^{ra}} - \sum_{\mathbf{b}\in S_2} q^{\mathbf{b}} \right).$$

Moreover,

$$\begin{aligned}
\sum_{a\in S_1} \frac{q^a}{1-q^{ra}} - \sum_{a\in S_1} \frac{q^{ra}}{1-q^{ra}} &= \left(\sum_{a\in S_1} q^a + \sum_{a\in S_1} \frac{q^{(r+1)a}}{1-q^{ra}} \right) - \left(\sum_{a\in rS_1} q^a + \sum_{a\in S_1} \frac{q^{2ra}}{1-q^{ra}} \right)\\
&= \left(\sum_{a\in S_1} q^a - \sum_{a\in rS_1} q^a \right) + \sum_{a\in S_1} \frac{q^{(r+1)a}}{1-q^{ra}} - \sum_{a\in S_1} \frac{q^{2ra}}{1-q^{ra}}\\
&= \sum_{\mathbf{b}\in S_2} q^{\mathbf{b}} + \sum_{a\in S_1} \frac{q^{(r+1)a}-q^{2ra}}{1-q^{ra}},
\end{aligned}$$

the last equality occurring because $S_1 = S_2 \sqcup rS_1$.

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} b'_r(n)q^n &= \prod_{a\in S_1} \frac{1-q^{ra}}{1-q^a} \sum_{a\in S_1} \frac{q^{(r+1)a}-q^{2ra}}{1-q^{ra}} \\
&= \sum_{a\in S_1} \frac{q^{(r+1)a}+q^{(r+2)a}+\cdots+q^{(2r-1)a}}{1+q^a+\cdots+q^{(r-1)a}} \prod_{a\in S_1} (1+q^a+\cdots+q^{(r-1)a}) \\
&= \sum_{a\in S_1} (q^{(r+1)a}+q^{(r+2)a}+\cdots+q^{(2r-1)a}) \prod_{d\in S_1, c\neq a} (1+q^d+\cdots+q^{(r-1)d}) \\
&= \sum_{n=0}^{\infty} c'_r(n)q^n.
\end{aligned}$$

3.2. **Combinatorial Proof.**

3.2.1. *$b'_r(n)$ as the cardinality of a set of overpartitions.* As in section 2.2.1, we use Glaisher's bijection and calculate $b'_r(n)$ by summing up the difference between the number of different parts of $\varphi(\lambda)$ and the number of different parts of λ for each partition $\lambda \in \mathcal{O}_r(n)$. For a given $\mathbf{a} \in S_2$, each part in $\varphi_r(l)$ of the form $r^k\mathbf{a}$, $k \geq 0$, is obtained from l by merging r^k parts equal to $\mathbf{a}$. Therefore, the contribution to $b'_r(n)$ of each $\mu \in \mathcal{D}_r(n)$ equals

$$\sum_{\substack{\mathbf{a}\in S_2 \\ \mathbf{a} \text{ part of } \varphi_r^{-1}(\mu)}} (m_\mu(\mathbf{a}) - 1),$$

where

$$m_\mu(\mathbf{a}) = |\{t \geq 0 \mid r^t\mathbf{a} \text{ is a part of } \mu\}|.$$

Next, we define a set of overpartitions. An overpartition is a partition in which the last appearance of a part may be overlined. For example, $(5, 5, \bar{5}, 3, 3, \bar{2}, 1, 1, \bar{1})$ is an overpartition of 26. We denote by $\overline{\mathcal{D}}_r(n)$ the set of overpartitions of n with parts in S_1 repeated at most $r-1$ times in which *exactly one* part is overlined and such that part $r^s\mathbf{a}$ with $s \geq 0$ and $\mathbf{a} \in S_2$ may be overlined only if there is a part $r^t\mathbf{a}$ with $t < s$. In particular, no primitive part can be overlined. Note that when we count parts in an overpartition, the overlined part contributes to the multiplicity. The discussion above proves the following interpretation of $b_r(n)$.

Proposition 3. *Let $n \geq 1$. Then, $b_r(n) = |\overline{\mathcal{D}}_r(n)|$.*

3.2.2. *A combinatorial proof for $c_r(n) = b_r(n)$.* We establish a one-to-one correspondence between $\overline{\mathcal{D}}_r(n)$ and $\mathcal{T}_r(n)$.

From $\overline{\mathcal{D}}_r(n)$ to $\mathcal{T}_r(n)$:

Start with an overpartition $\mu \in \overline{\mathcal{D}}_r(n)$. Suppose the overlined part is $\mu_i = r^s\mathbf{a}$ for some $s \geq 1$ and $\mathbf{a} \in S_2$. Then there is a part $\mu_j = r^t\mathbf{a}$ of μ with $t < s$. Let k be the largest positive integer such that $r^k\mathbf{a}$ is a part of μ and $k < s$. To obtain $\lambda \in \mathcal{T}_r(n)$ from μ, split μ_i into r parts equal to $r^k\mathbf{a}$ and $r-1$ parts equal to $r^j\mathbf{a}$ for each $j = k+1, k+2, \ldots, s-1$.

Example 10. We continue with the Euler pair of order 3 from Example 1. Let

$$\mu = (3^8 \cdot 5, 3^7 \cdot 5, \overline{3^7 \cdot 5}, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1) \in \overline{\mathcal{D}}_r(56017).$$

Then $k = 5$ and $3^7 \cdot 5$ splits into three parts equal to $3^5 \cdot 5$ and two parts equal to $3^6 \cdot 5$. Thus, we obtain the partition

$$\begin{aligned}\lambda = (&3^8 \cdot 5, 3^7 \cdot 5, 3^6 \cdot 5, 3^6 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1)\\ &\in \mathcal{T}_r(56017).\end{aligned}$$

The part repeated more than three times but less than six times is $3^5 \cdot 5$.

From $\mathcal{T}_r(n)$ to $\overline{\mathcal{D}}_r(n)$:

Start with a partition $\lambda \in \mathcal{T}_r(n)$. Suppose $r^k\mathbf{a}$ is the part repeated more than r times but less than $2r$ times. Let $\mu = \varphi_r(l) \in \mathcal{D}_r(n)$. Overline the smallest part of μ of form $r^t\mathbf{a}$ with $t > k$. The resulting overpartition is in $\overline{\mathcal{D}}_r(n)$.

Example 11. We continue with the Euler pair of order 3 from Example 1. Let

$$\begin{aligned}\lambda = (&3^8 \cdot 5, 3^7 \cdot 5, 3^6 \cdot 5, 3^6 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1)\\ &\in \mathcal{T}_r(56017).\end{aligned}$$

The part repeated more than three times but less than six times is $3^5 \cdot 5$. We have $k = 5$. Merging by Glaisher's bijection, we obtain

$$\mu = (3^8 \cdot 5, 3^7 \cdot 5, 3^7 \cdot 5, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1) \in \mathcal{D}_r(56017).$$

The smallest part of μ of the form $r^t\mathbf{a}$ with $t > k = 5$ is $3^7 \cdot 5$. Thus we obtain the overpartition

$$\mu = (3^8 \cdot 5, 3^7 \cdot 5, \overline{3^7 \cdot 5}, 3^5 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 5, 25, 3^2 \cdot 1, 3 \cdot 1) \in \overline{\mathcal{D}}_r(56017).$$

Remark 4. We could have obtained the transformation above from the combinatorial proof of part (ii) of Theorem 1.2. In the transformation from $\mathcal{D}_{1,r}(n)$ to $\mathcal{DD}_r(n)$, if part $r^k\mathbf{a}$ is the part repeated more than r times but less than $2r$ times, we have $f = r + s$ for some $1 \le s \le r - 1$, $h = s + 1$, and $N = s$. Thus $d = 0$ and the decorated part is the last occurrence of smallest part in the transformed partition μ that is of the form $r^t\mathbf{a}$ with $t > k$. Thus, in μ, the decorated part $r^t\mathbf{a}$ is decorated with an r-word consisting of all zeros and of length $t - k - 1$, one less than the difference in exponents of r of the decorated part and the next smallest part with the same $\mathbf{a}$ factor. Since in this case the decoration of a partition in $\mathcal{DD}_r(n)$ is completely determined by the part being decorated, we can simply just overline the part.

4. Concluding remarks

In this article we proved first and second Beck-type identities for all Euler pairs (S_1, S_2) of order $r \ge 2$. Euler pairs of order r satisfy $rS_1 \subseteq S_1$ and $S_2 = S_1 \setminus rS_1$. Subbarao [10] showed that they completely characterize the pairs of subsets of positive integers for which the following theorem holds.

Theorem 4.1. *For any integer $n \ge 0$, the number of partitions of n with parts in S_2 is equal to the number of partitions of n with parts in S_1 and such that no part is repeated more than $r - 1$ times.*

Thus, we established Beck-type identities accompanying all partition identities of the type given in Theorem 4.1.

At the end of [10], Subbarao mentions that this characterization also holds for vector partitions. Let $n_1, n_2, \ldots, n_s$ be integers summing up to n. A vector partition of $(n_1, n_2, \ldots, n_s)$ is a vector of partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ such that each $l^{(i)}$ is a partition of n_i, $1 \leq i \leq s$. Subbarao extends his theorem as follows.

Theorem 4.2. *The number of vector partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $l^{(i)}$ belong to S_1 and no part is repeated more than $r-1$ times equals the number of vector partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $\mu^{(i)}$ belong to S_2.*

Analogous Beck-type identities hold for vector partitions.

Let (S_1, S_2) be an Euler pair of order $r \geq 2$. Let $b''_r(n)$ be the difference between the total number of parts in all vector partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $l^{(i)}$ belong to S_1 and no part is repeated more than $r-1$ times and the total number of parts in all vector partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $\mu^{(i)}$ belong to S_2. Then the work of this article proves the following first Beck-type identity.

Theorem 4.3. *Suppose (S_1, S_2) is an Euler pair of order $r \geq 2$. Then $\dfrac{1}{r-1}b''_r(n)$ equals the number of vector partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $l^{(i)}$ belong to S_1 and exactly one part is repeated at least r times. Also, $\dfrac{1}{r-1}b''_r(n)$ equals the number of vector partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ exactly one part of $\mu^{(i)}$ (possibly repeated) is from rS_1.*

Let (S_1, S_2) be an Euler pair of order $r \geq 2$. Let $b'''_r(n)$ be the difference in the total number of different parts in all vector partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $l^{(i)}$ belong to S_1 and no part is repeated more than $r-1$ times and the total number of parts in all vector partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $\mu^{(i)}$ belong to S_2. Then we have the following second Beck-type identity.

Theorem 4.4. *Suppose (S_1, S_2) is an Euler pair of order $r \geq 2$. Then $b'''_r(n)$ equals the number of vector partitions $(l^{(1)}, l^{(2)}, \ldots, l^{(s)})$ of $(n_1, n_2, \ldots, n_s)$ such that for each $1 \leq i \leq s$ all parts of $l^{(i)}$ belong to S_1 and exactly one part is repeated more than r times but less than $2r$ times.*

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