THE COHOMOLOGY OF TWISTED COALGEBRAS

an invitation to twisted Koszul duality theory

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Abstract

In this paper, which is based on the author's MSc thesis, we study in detail the cohomology theory for twisted coalgebras introduced in [2] by M. Aguiar and S. Mahajan. We compute it completely in various examples, including those proposed by Aguiar and Mahajan, and obtain structural results: in particular, we study its multiplicative structure, provide a Künneth formula, and succeed in giving an alternative description of this cohomology theory which, in particular, allows for its effective computation.

At the very end of the paper, we briefly outline how *all* the computations done in this paper can be swiftly explained and extended to an arbitrary Koszul twisted coalgebra through their corresponding Koszul duality theory. While doing so, we work out the example of the species of linear orders: we show it is twisted Koszul, compute its dual and the (doubly) graded dimensions of its components, which turn out to be the unsigned Stirling numbers of the first kind.

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Introduction

The present paper serves two purposes. First, it is an illustration of the possibility to completely understand a new cohomology theory, that of coalgebras in the combinatorial species of A. Joyal [11], which are usually known as twisted coalgebras. This theory originated in the work of M. Aguiar and S. Mahajan on deformations of coalgebras in the category of species: their objective, among other, was constructing certain Hopf algebras in this category that encode the combinatorics of structures such as linear orders, graphs and posets, for example.

Second, it is intended to show how homological algebra can complement the field of enumerative and analytic combinatorics: using homological tools, we completely solve the problem of computing the cohomology groups of a coalgebra in the category species and in particular the *second* cohomology group of it, thereby completely solving a problem posed originally by M. Aguiar and S. Mahajan; our solution is effective and can be implemented in a computer, being analogous to the computation of the cohomology groups of a locally finite CW-complex through the use of the cellular cochain complex.

To explain our results, we use the language of representation theory and homological algebra. Concretely, let $\mathcal{E}$ be the species of singletons, which is a Hopf algebra in the category of species. The cohomology theory of Aguiar and Mahajan is then encoded by the derived functor $\mathcal{X} \mapsto \operatorname{Ext}(\mathcal{X}, \mathcal{E})$ in the category of $\mathcal{E}$-bicomodules: it turns out that the full subcategory of the coalgebras that Aguiar and Mahajan are interested in corresponds to the full subcategory of certain "linearized" bicomodules.

It is useful to think of the left bicomodule structure as the data of an operation of restriction on the combinatorial objects encoded by $\mathcal{X}$, and of the right bicomodule structure as the data of an operation on such objects, the compatibility axiom of a bicomodule encoding a compatiblity relation between these two operations. The datum of such a bicomodule $\mathcal{X}$ includes, in particular, the sequence $(\mathcal{X}[0], \mathcal{X}[1], \ldots)$ where for each $p \in \mathbb{N}$ the k-vector space $\mathcal{X}[p]$ is a kS_p-module. We say $\mathcal{X}$ is weakly projective if for each $p \in \mathbb{N}$ the module $\mathcal{X}[p]$ is a projective kS_p-module, and write sgn_p for the one dimensional sign representation of S_p, its appearance which we now explain.

Although the machinery we use in the paper is mainly homological, the appearance of the species of the sign representation is due to a rich interplay between homological algebra and the combinatorics of hyperplane arrangements: the cobar construction of the twisted coalgebra $\mathcal{E}$ is the simplicial cochain complex of the triangulation of the sphere by the Coxeter complex for the braid arrangement, and its cohomology is concentrated in top degree, where it is the sign representation of the symmetric group. The main result of this paper is the following:

Theorem. *Suppose that $\mathcal{X}$ is a weakly projective $\mathcal{E}$-bicomodule. There is a complex $S^*(\mathcal{X})$ so that for each $p \in \mathbb{N}$ we have $S^p(\mathcal{X}) = \operatorname{Hom}_{S_p}(\mathcal{X}[p], \operatorname{sgn}_p)$ that computes the cohomology groups $H^*(\mathcal{X})$. If the ground ring contains $\mathbb{Q}$, the differential $\delta^p : S^p(\mathcal{X}) \longrightarrow S^{p+1}(\mathcal{X})$ is such that for $z \in \mathcal{X}[p+1]$ we have*

$$(\delta^p \phi)(z) = \sum_{j=1}^{p+1} (-1)^j \Big(\phi(z'_j) - \phi(z''_j) \Big),$$

where the assignments $z \mapsto z'_j, z''_j$ restrict and contract, respectively, the "jth label" of z.

In particular, one can compute the second cohomology group $H^2(\mathcal{X})$ using the three representations $\mathcal{X}[1], \mathcal{X}[2]$ and $\mathcal{X}[3]$ and the contraction and restriction operations of $\mathcal{X}$ on such spaces.

We have also addressed the problem of determining when products exist in this cohomology theory and describing them using the complex above. Our result comes paired with a Künneth isomorphism in the category of $\mathcal{E}$-bicomodules: since $\mathcal{E}$ is a Hopf algebra, there is an internal product $\otimes$ in its category of bicomodules, and our result is the following:

Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{E}$-bicomodules, and assume that k is a field and at least one of $\mathcal{X}$ or $\mathcal{Y}$ is locally finite. There is an isomorphism of complexes, natural in $\mathcal{X}$ and $\mathcal{Y}$, of the form $S^*(\mathcal{X}) \otimes S^*(\mathcal{Y}) \longrightarrow S^*(\mathcal{X} \otimes \mathcal{Y})$.*

Moreover, every morphism of $\mathcal{E}$-bicomodules $\mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$ induces a product in cohomology and, in particular, if the bicomodule $\mathcal{X}$ is induced from a coalgebra structure on $\mathcal{X}$, the comultiplication of $\mathcal{X}$ is a morphism of $\mathcal{E}$-bicomodules and induces a cup product in $H^*(\mathcal{X})$.

To conclude this paper and set the stage for future applications, we interpret the main results above in terms of Koszul duality for twisted (co)algebras. As it turns out, the twistes coalgebra $\mathcal{E}$ is Koszul, and the complex $S^*(\mathcal{X})$ that we discovered through a spectral sequence method is, in fact, the Koszul complex $K^*(\mathcal{X}, \mathcal{E})$ of $\mathcal{E}$.

After explaining this, we conclude with an example where we show that the species $\mathcal{L}$ of linear orders considered by Aguiar and Mahajan is Koszul, compute its Koszul dual and its doubly graded Betti numbers: we show these are precisely the unsigned Stirling numbers of the first kind. More, precisely, we have the following result, which follows immediately from the twisted version of the Milnor–Moore theorem over a field of characteristic zero.

Theorem. *The twisted Koszul dual algebra to $\mathcal{L}$ is $\mathcal{L}^{\mathrm{i}} = S(s^{-1}\mathsf{Lie})$, the free twisted commutative algebra on the desuspension of the symmetric sequence Lie. In particular, for each $j, n \in \mathbb{N}$, the component of weight j of the S_n-module $\mathcal{L}^{i}(n)$ is in bijection with the permutations of n consisting of exactly j disjoint cycles, which are enumerated by the unsigned Stirling numbers of the first kind.* ◀

Useful references. We refer the reader to [22] for an introduction to homological algebra and recommend coupling it with [17] for a comprehensive exposition on spectral sequences. As a reference on combinatorial species, we use the seminal article of A. Joyal [11] and the book of Labelle, Leroux y Bergeron [13]. Finally, our reference for the formalism of monoidal categories is C. Kassel's book [12], for the basics on abelian categories the book of Freyd [9], and for the simplicial formalism, the book [22] and that of S. MacLane [16].

Running conventions. Throughout, k is a unital commutative ring, and when we write $\otimes$ y Hom, we will be considering the usual functors on k-modules, unless stated otherwise; an important exception is our use of $\otimes$ for the Cauchy product of species. Since we will write them with lower-case boldfaced letters, while k-modules will always be written in capital italics, no confusion should arise.

A *decomposition* S of *length* q of a set I is an ordered tuple $(S_1,\ldots,S_q)$ of possible empty subsets of I, which we call the *blocks of* S, that are pairwise disjoint and whose union is I. We say S is a *composition* of I if every block of S is nonempty. It is clear that if I has n elements, every composition of I has at most n blocks. We will write $S \vdash I$ to mean that S is a decomposition of I, and if necessary will write $S \vdash_q I$ to specify that the length of S is q. Notice the empty set has exactly one composition which has length zero, the empty composition, and exactly one decomposition of each length $n \in \mathbb{N}_0$. If T is a subset of I and $\sigma : I \longrightarrow J$ is a bijection, we let $\sigma_T : T \longrightarrow \sigma(T)$ be the bijection induced by σ.

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1 Algebras and coalgebras in species

1.1 The category of species

Denote by $\mathsf{Set}^\times$ the category of finite sets and bijections.

Definition 1.1. A *combinatorial species* over a category C is a functor $\mathcal{X} : \mathsf{Set}^\times \longrightarrow \mathsf{C}$. Concretely, a combinatorial species X is obtained by assigning

- **S1.** to each finite set I an object $\mathcal{X}(I)$ in C,
- **S2.** to each bijection $\sigma : I \longrightarrow J$ an arrow $\mathcal{X}(\sigma) : \mathcal{X}(I) \longrightarrow \mathcal{X}[j]$,

in such a way that

- **S3.** for every pair of composable bijections σ and τ, we have $\mathcal{X}(\tau\sigma) = \mathcal{X}(\tau)\mathcal{X}(\sigma)$ and,
- **S4.** for every finite set I, it holds that $\mathcal{X}(\mathrm{id}_I) = \mathrm{id}_{\mathcal{X}(I)}$.

In particular, for every finite set I we have a map $\sigma \in \mathrm{Aut}(I) \longmapsto \mathcal{X}(\sigma) \in \mathrm{Aut}(\mathcal{X}(I))$ which gives an action of the symmetric group with letters in I on $\mathcal{X}(I)$. The category $\mathsf{Set}^\times$ is a grupoid, and it has as skeleton the full subcategory spanned by the sets $[n] = \{1,\dots,n\}$ (in particular, $[0] = \varnothing$) , and a species is determined, up to isomorphism, by declaring its values on the finite sets $[n]$ and on every $\sigma \in S_n$. In view of this, one can think of a combinatorial species as a sequence $(\mathcal{X}(n))_{n\geqslant 0}$ of objects in C endowed with S_n actions $(S_n \times \mathcal{X}(n) \longrightarrow \mathcal{X}(n))_{n\geqslant 0}$.

We denote by $\mathsf{Sp}(\mathsf{C})$ the category $\mathrm{Fun}(\mathsf{Set}^\times, \mathsf{C})$ of species over C, whose morphisms are natural transformations: explicitly, an arrow $\eta : \mathcal{X} \longrightarrow \mathcal{Y}$ is an assignment of a map $\eta_I : \mathcal{X}(I) \longrightarrow \mathcal{Y}(I)$ to each finite set I, in such a way that for any bijection $I \xrightarrow{\sigma} J$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{X}(I) \arrow[r, "\eta_I"] \arrow[d, "\mathcal{X}(\sigma)"'] & \mathcal{Y}(I) \arrow[d, "\mathcal{Y}(\sigma)"] \\
\mathcal{X}[j] \arrow[r, "\eta_J"] & \mathcal{Y}(J)
\end{tikzcd}$$

This says that we must specify, for each finite set I, an $\mathrm{Aut}(I)$-equivariant map $\eta_I : \mathcal{X}(I) \longrightarrow \mathcal{Y}(I)$. If we view species as sequences of objects on which the symmetric grupoid acts, a morphism of species $\mathcal{X} \longrightarrow \mathcal{Y}$ is simply a sequence of equivariant maps $(\eta_n : \mathcal{X}_n \longrightarrow \mathcal{Y}_n)_{n\geqslant 0}$.

Our main interest will lie on species over sets or vector spaces. We write Sp for the category of species over Set, the category of sets and functions, and call its objects *set species.* If a species takes values on the subcategory FinSet of finite sets we call it a *finite* set species, and if $\mathcal{X}(\varnothing)$ is a singleton, we say it is *connected.* We write Sp_k for the category of species over ${}_k\mathrm{Mod}$, the category of modules over k, and call its objects *linear species.* If a species takes values on the subcategory ${}_k\mathrm{mod}$ of finite generated modules we call it a linear species of *finite type*, and we say it is *connected* if $\mathcal{X}(\varnothing)$ is k-free of rank one.

Denote by $k[-]$ the functor $\mathsf{Set} \longrightarrow {}_k\mathrm{Mod}$ that sends a set $\mathcal{X}$ to the free k-module with basis $\mathcal{X}$, which we will denote by $k\mathcal{X}$, and call it the *linearization of* $\mathcal{X}$. By postcomposition, we obtain a functor $\mathscr{L} : \mathsf{Sp} \longrightarrow \mathsf{Sp}_k$ that sends a set species $\mathcal{X}$ to the linear species $k\mathcal{X}$. The species in Sp_k that are in the image of $k[-]$ are called *linearized species*. Thus, a linearized species $\mathcal{X} = k\mathcal{X}_0$ is such that, for every finite set I, the vector space $\mathcal{X}(I)$ has a chosen basis $\mathcal{X}_0(I)$, the morphisms $\mathcal{X}(I) \longrightarrow \mathcal{X}[j]$ map basis elements to basis elements, and the action of $\mathrm{Aut}(I)$ on $\mathcal{X}(I)$ is by permutation of the basis elements.

Definition 1.2. Given a species $\mathcal{X} : \mathsf{Set}^\times \longrightarrow \mathsf{Set}$ and a finite set I, we call $\mathcal{X}(I)$ the *set of structures of species* $\mathcal{X}$ *over* I. If $s \in \mathcal{X}(I)$, we call I the *underlying set* of s, and call s an *element* of $\mathcal{X}$ or an $\mathcal{X}$*-structure*. If $I \xrightarrow{\sigma} J$ is a bijection, the element $\mathcal{X}(\sigma)(s) = t$ is the structure over J obtained by *transporting* s *along* σ, which we will usually denote, for simplicity, by σs.

Definition 1.3. Two $\mathcal{X}$ structures s and t over respective sets I and J are said to be *isomorphic* if there is a bijection $\sigma : I \longrightarrow J$ that transports s to t, and we say σ is a *structure isomorphism* from s to t. A permutation that transports a structure s to itself is said to be an *automorphism* of s.

In most cases, if $\mathcal{X}$ is a species and I is a set, $\mathcal{X}(I)$ consists of a collection of combinatorial structures of some kind labelled in some way by the elements of I. For example, there is a species Pos that assigns to every finite set I the set $\mathsf{Pos}(I)$ of partial orders on I, and to every bijection $\sigma : I \longrightarrow J$ the function $\mathsf{Pos}(\sigma) : \mathsf{Pos}(I) \longrightarrow \mathsf{Pos}(J)$ which assigns to every order on I the unique order on J that makes σ an order isomorphism: in concrete terms, $\mathsf{Pos}(\sigma)$ "relabels" a poset on I according to σ.

Examples

To understand all that follows it useful to have a list of examples in mind. We collect in this section such a list. For a comprehensive treatment of combinatorial species, we refer the reader to [13].

E1. The *exponential* or *uniform* species $\mathcal{E} : \mathsf{Set}^\times \longrightarrow \mathsf{FinSet}$ is the species that assigns to every finite set I the singleton set $\{I\}$, and to any bijection $\sigma : I \longrightarrow J$ the unique bijection $\mathcal{E}(\sigma) : \mathcal{E}(I) \longrightarrow \mathcal{E}(J)$. Remark that $\mathcal{E}$ is the unique species, up to isomorphism, that has exactly one structure over each finite set. For ease of notation, we will write $*_I$ for $\{I\}$.

E2. The *species of partitions* $\mathcal{P}$ assigns to each finite set I the collection of partitions of I: sets $T = \{T_1, \ldots, T_s\}$ of nonempty disjoint subsets of I whose union is I. If $\sigma : I \longrightarrow J$ is a bijection and T is a partition of I, $\mathcal{P}(\sigma)(T) = \{\sigma T_1, \ldots, \sigma T_s\}$ is the partition of J obtained by transporting T along σ.

E3. The *species of compositions* $\mathcal{C}$ assigns to each finite set I the collection of composition of I: ordered tuples $(F_1, \ldots, F_t)$ of nonempty disjoint subsets of I whose union is I. If $\sigma : I \longrightarrow J$ is a bijection and F is a composition of I, $\mathcal{C}(\sigma)(F) = (\sigma F_1, \ldots, \sigma F_t)$ is the composition of J obtained by transporting F along σ.

E4. There is a species Simp that assigns to each set I the collection of simplicial structures on I, this is, collections of finite subsets $S \subseteq 2^I$ that contain all singleton sets of elements of I, and such that whenever $\Delta \in S$ and $\Delta' \subseteq \Delta$, then $\Delta' \in S$. We call the elements of S *simplices*.

E5. Again, let X be a topological space. There is a species $\mathscr{F}_X$ that assigns to each finite set I the configuration space $\mathscr{F}_X(I) \subseteq X^I$ of X with coordinates on I: $\mathscr{F}_X(I)$ consists of tuples $(x_i)_{i \in I}$ with $x_i \neq x_j$ whenever i and j are distinct elements of I. As in the previous example, there is an obvious action of any bijection $\sigma : I \longrightarrow J$ that permutes the coordinates. For each fixed finite set I, the set of types of structures over I is usually called the *unordered configuration space* $\mathscr{E}_X(I)$.

E6. There is a *species of parts* $\wp$ that sends each finite set I to the collection 2^I of parts of I, and sends each bijection $\sigma : I \longrightarrow J$ to the induced bijection $\sigma_* : 2^I \longrightarrow 2^J$. In a similar way, if n is a positive integer, there is a species $\wp_n$ which sends each finite set I to the set $\wp_n$ of its subsets of cardinality n; notice that $\wp_n(I)$ is empty if I has less than n elements, and that $\wp_n$ is a subspecies of $\wp$ for each n.

E7. A graph with vertices on a set I is a pair (I, E) where E is a collection of 2-subsets of I. For each finite set I, let $\mathsf{Gr}(I)$ be the collection of graphs on I. If $\sigma : I \longrightarrow J$ is a bijection and (I, E) is a graph on I, we set $\mathsf{Gr}(\sigma)(I, E) = (J, \sigma(E))$. This defines the *species* Gr *of graphs*.

E8. For each finite set I, let $\mathcal{L}_0(I)$ be the collection of linear orders on I. If $\sigma : I \longrightarrow J$ is a bijection, we let $\mathcal{L}_0(\sigma)$ send a linear order $i_1 i_2 \cdots i_t$ on the set I to the linear order $\sigma(i_1) \cdots \sigma(i_t)$ on J. This defines the *species* $\mathcal{L}_0$ *of linear orders*.

1.2 The Cauchy product

The category Sp_k of species over ${}_k\mathrm{Mod}$ is abelian and monoidal with respect to the "pointwise" Hadamard product given for each pair of species $\mathcal{X}$ and $\mathcal{Y}$ and each finite set I by the formula

$$(\mathcal{X} \otimes_H \mathcal{Y})(I) = \mathcal{X}(I) \otimes \mathcal{Y}(I).$$

It turns out that algebras for this tensor product are rather simple: endowing a species $\mathcal{X}$ with the structure of an algebra for $\otimes_H$ amounts to endowing each *individual* space $\mathcal{X}(I)$

with an algebra structure and, in particular, does not combine in any interesting the sequence of spaces defined by $\mathcal{X}$.

There is another product in Sp_k, called the *Cauchy product* which we will denote by $\otimes$, which will play a central role in all that follows, and which categorifies the usual (Cauchy) product of power series. In particular, it will intertwine into a single object the various pieces of $\mathcal{X}$, and provide us with a richer product and, hence, with a more interesting class of (co)algebras.

Definition 1.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be linear species over k. The Cauchy product $\mathcal{X}\otimes\mathcal{Y}$ is the linear species such that for every finite set I

$$(\mathcal{X}\otimes\mathcal{Y})(I) = \bigoplus_{(S,T)\vdash I} \mathcal{X}(S)\otimes\mathcal{Y}(T),$$

the direct sum running through all decompositions of I of length two, and for every bijection $\sigma : I \longrightarrow J$

$$(\mathcal{X}\otimes\mathcal{Y})(\sigma) = \bigoplus_{(S,T)\vdash I} \mathcal{X}(\sigma_S)\otimes\mathcal{Y}(\sigma_T).$$

As it happens with the Hadamard product, the Cauchy product is better understood when viewing species as the product of representations of the various symmetric groups. Indeed, for each n and each pair (p,q) with $p+q=n$, there is an isomorphism

$$\bigoplus_{S\subseteq I, \#S=p} \mathcal{X}(S)\otimes\mathcal{Y}(T) \simeq \mathrm{Ind}_{S_p\times S_q}^{S_{p+q}}(\mathcal{X}[p]\otimes\mathcal{Y}[q]),$$

and these collect to give an isomorphism

$$(\mathcal{X}\otimes\mathcal{Y})([n]) \simeq \bigoplus_{p+q=n} \mathrm{Ind}_{S_p\times S_q}^{S_{p+q}}(\mathcal{X}[p]\otimes\mathcal{Y}[q]).$$

This construction extends to produce a bifunctor $\otimes : \mathsf{Sp}_k\times\mathsf{Sp}_k \longrightarrow \mathsf{Sp}_k$. In what follows, whenever we speak of the category Sp_k, we will view it as a monoidal category with the monoidal structure given by the Cauchy product.

It is important to notice the construction of the Cauchy product in Sp_k carries over to the category $\mathsf{Sp}(\mathsf{C})$ when C is any monoidal category with finite coproducts which commute with its tensor product. The main example of this phenomenon happens when C is the category Set. If $\mathcal{X}$ and $\mathcal{Y}$ are set species, the species $\mathcal{X}\otimes\mathcal{Y}$ has

$$(\mathcal{X}\otimes\mathcal{Y})(I) = \bigsqcup_{(S,T)\vdash I} \mathcal{X}(S)\times\mathcal{Y}(T),$$

so that a structure z of species $\mathcal{X} \otimes \mathcal{Y}$ over a set I is determined by a decomposition (S, T) of I and a pair of structures (z_1, z_2) of species $\mathcal{X}$ and $\mathcal{Y}$ over S and T, respectively.

The linearization functor $\mathcal{L} : \mathsf{Sp} \longrightarrow \mathsf{Sp}_k$ preserves the monoidal structures we have defined on these categories, in the sense there is a natural isomorphism

$$\mathcal{L}(\mathcal{X} \otimes \mathcal{Y}) \longrightarrow \mathcal{L}\mathcal{X} \otimes \mathcal{L}\mathcal{Y}$$

for each pair of objects $\mathcal{X}, \mathcal{Y}$ in Sp. For details on such *monoidal functors* see [12, Chapter XI §4]. The following will be useful, and we record it for future reference:

Proposition 1.1. *If $\mathcal{X}, \mathcal{Y}_1, \ldots, \mathcal{Y}_r$ are linear species, a map of species $\alpha : \mathcal{X} \longrightarrow \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_r$ determines and is determined by a choice of equivariant k-module maps*

$$\alpha_I : \mathcal{X}(I) \longrightarrow \bigoplus \mathcal{Y}_1(S_1) \otimes \cdots \otimes \mathcal{Y}_r(S_r),$$

one for each finite set I, the direct sum running through decompositions $(S_1, \ldots, S_r)$ of length r of I. ◀

The map α_I is specified uniquely by its *components* at each decomposition $S = (S_1, \ldots, S_r)$, which we denote $\alpha(S_1, \ldots, S_r)$ without further mention to the set I which is implicit, for $\bigcup S$ equals I. Moreover, it suffices to specify α_I for I the sets $[\![n]\!]$ with $n \in \mathbb{N}_0$. This said, we will usually define a map $\alpha : \mathcal{X} \longrightarrow \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_r$ by specifying its components at each decomposition of length r of I.

1.3 Twisted coalgebras and bialgebras

An associative algebra $(\mathcal{X}, \mu, \eta)$ in the category Sp_k, which we will call simply a *twisted algebra*, is determined by a multiplication

$$\mu : \mathcal{X} \otimes \mathcal{X} \longrightarrow \mathcal{X}$$

and a unit $\eta : \mathbf{1} \longrightarrow \mathcal{X}$. Specifying the first amounts to giving its components

$$\mu(S, T) : \mathcal{X}(S) \otimes \mathcal{X}(T) \longrightarrow \mathcal{X}(I)$$

at each decomposition (S, T) of every finite set I, and specifying the latter amounts to a choice of the element $\eta(\varnothing)(1) \in \mathcal{X}(\varnothing)$, which we will denote by 1 if no confusion should arise. We think of the multiplication as an operation that glues partial structures on I, and of the unit as an "empty" structure.

For example, the species of graphs admits a multiplication $k\mathsf{Gr} \otimes k\mathsf{Gr} \longrightarrow k\mathsf{Gr}$ which is the linear extension of the map that takes a pair of graphs $(g_1, g_2) \in \mathsf{Gr}(S) \times \mathsf{Gr}(T)$ and constructs the disjoint union $g_1 \sqcup g_2$ on I. The unit for this multiplication is the empty graph $\varnothing \in \mathsf{Gr}(\varnothing)$. One can readily check μ is associative and unital with respect to η, so we indeed have a algebra $k\mathsf{Gr}$.

Dually, a coalgebra $(\mathcal{X}, \Delta, \varepsilon)$ in Sp_k, which we call a *twisted coalgebra*, is determined by a comultiplication

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$$

and a counit $\varepsilon : \mathcal{X} \longrightarrow \mathbf{1}$. The comultiplication has, at each decomposition (S, T) of I, a component $\Delta(S, T) : \mathcal{X}(I) \longrightarrow \mathcal{X}(S) \otimes \mathcal{X}(T)$, which we think of as breaking up a combinatorial structure on I into substructures on S and T, while the counit is a map of k-modules $\mathcal{X}(\varnothing) \longrightarrow k$.

To continue with our example, the linearization of the species of graphs admits a comultiplication $k\mathsf{Gr} \longrightarrow k\mathsf{Gr} \otimes k\mathsf{Gr}$ that sends a graph g on a set I to $g_S \otimes g_T \in k\mathsf{Gr}(S) \otimes k\mathsf{Gr}(T)$, where g_S and g_T are the subgraphs induced by g on S and T, respectively. This comultiplication admits as counit the morphism $\varepsilon : k\mathsf{Gr} \longrightarrow \mathbf{1}$ that assigns $1 \in k$ to the empty graph. In this way, we obtain a coalgebra structure on $k\mathsf{Gr}$ which is, in fact, compatible with the algebra structure we described in the previous paragraph: we therefore have a bialgebra structure on $k\mathsf{Gr}$.

Our main example of a bialgebra in Sp_k is the provided by the following proposition.

Proposition 1.2. *The linearized exponential species $\mathcal{E}$ is a twisted bialgebra with multiplication and comultiplication with components*

$$\mu(S, T) : \mathcal{E}(S) \otimes \mathcal{E}(T) \longrightarrow \mathcal{E}(I), \qquad \Delta(S, T) : \mathcal{E}(I) \longrightarrow \mathcal{E}(S) \otimes \mathcal{E}(T)$$

at each decomposition (S, T) of a finite set I such that

$$\mu(S, T)(*_S \otimes *_T) = *_I, \qquad \Delta(S, T)(*_I) = *_S \otimes *_T$$

and with unit and counit the morphisms $\varepsilon : \mathcal{E} \longrightarrow \mathbf{1}$ and $\eta : \mathbf{1} \longrightarrow \mathcal{E}$ such that $\varepsilon(*_\varnothing) = 1$ and $\eta(1) = *_\varnothing$.

Proof. The verifications needed to prove this follow immediately from the fact that $\mathcal{E}(I)$ is a singleton for every finite set I. ◀

The exponential species plays a central role in the category of bialgebras, as evinced by the following proposition.

Proposition 1.3.

1. *The exponential species $\mathcal{E}_0$ admits a unique structure of set- theoretic bialgebra.*

2. *If $\mathcal{X}_0$ is a set theoretical coalgebra in* Sp, *the linearization of the unique morphism of species $\mathcal{X}_0 \longrightarrow \mathcal{E}_0$ is a morphism of coalgebras.*
3. *In particular, every twisted coalgebra coming from a set theoretic coalgebra is canonically an $\mathcal{E}$-bicomodule.*

Proof. If s is a singleton set and x is any set, there is a unique function $x \longrightarrow s$, and it follows from this, first, that the bialgebras structure defined on $\mathcal{E}$ is the only linearized bialgebra structure, and, second, that if $\mathcal{X}$ is a species in Sp, there is a unique morphism of species $\mathcal{X} \longrightarrow \mathcal{E}$. If $\mathcal{X}$ is a pre-coalgebra in Sp, the following square commutes because $\mathcal{E}(S) \times \mathcal{E}(T)$ has one element:

$$\begin{array}{ccc} \mathcal{E}(I) & \xrightarrow{\Delta} & \mathcal{E}(S) \times \mathcal{E}(T) \\ \uparrow & & \uparrow \\ \mathcal{X}(I) & \xrightarrow{\Delta} & \mathcal{X}(S) \times \mathcal{X}(T), \end{array}$$

and, by the same reason, $\mathcal{X} \longrightarrow \mathcal{E}$ is pre-counital. All this shows that the exponential species $\mathcal{E}$ is terminal in the category of linearized coalgebras. This completes the proof of the proposition. ◀

We will fix some useful notation to deal with coalgebras. Let $\mathcal{X} = k\mathcal{X}_0$ be a linearized species that is a coalgebra in Sp_k. If z is an element of $\mathcal{X}_0(I)$, we write

$$\Delta(I)(z) = \sum z \mathbin{\backslash\!\backslash} S \otimes z \mathbin{/\!/} T$$

with $z \mathbin{\backslash\!\backslash} S \otimes z \mathbin{/\!/} T$ denoting an element of $\mathcal{X}(S) \otimes \mathcal{X}(T)$ (not necessarily an elementary tensor, à la Sweedler).

Consider now a left $\mathcal{E}$-comodule $\mathcal{X}$ with coaction $\lambda : \mathcal{X} \longrightarrow \mathcal{E} \otimes \mathcal{X}$. Since $\mathcal{E}(S) = k\{*_S\}$, the component $\mathcal{X}(I) \longrightarrow \mathcal{E}(S) \otimes \mathcal{X}(T)$ can canonically be viewed as map $\mathcal{X}(I) \longrightarrow \mathcal{X}(T)$ which we denote by λ^I_T, and call the it *the restriction from I to T to the right.*

In these terms, that λ be counital means λ^I_I is the identity for all finite sets I, and the equality $1 \otimes \lambda \circ \lambda = \Delta \otimes 1 \circ \lambda$, which expresses the coassociativity of λ, translates to the condition that we have $\lambda^I_A = \lambda^B_A \circ \lambda^I_B$ for any chain of finite sets $A \subseteq B \subseteq I$. It follows that, if $\mathsf{FinSet}^{\mathrm{inc}}$ is the category of finite sets and inclusions, a left $\mathcal{E}$-comodule $\mathcal{X}$ in Sp_k can be viewed as a pre-sheaf

$$\mathsf{FinSet}^{\mathrm{inc}} \longrightarrow {}_k\mathrm{Mod}.$$

These are usually called FI-modules in the literature, see for example [7]. When convenient, we will write $z \mathbin{/\!/} S$ for $\lambda^I_S(z)$ without explicit mention to I, which will usually be understood from context. Using this notation, we can write the coaction on $\mathcal{X}$ as

$$\lambda(I)(z) = \sum e_S \otimes z \mathbin{/\!\!/} T.$$

Of course the same consideration apply to a right $\mathcal{E}$-comodule, and we write $z \mathbin{\backslash\!\!\backslash} T$ for $\rho^I_T(z)$. If $\mathcal{X}$ is both a left and a right $\mathcal{E}$-comodule with coactions λ and ρ, the compatilibity condition for it to be an $\mathcal{E}$-bicomodule is that, for any finite set I and pair of non-necessarily disjoint subsets S, T of I, we have $\rho^S_{S\cap T}\lambda^I_S = \lambda^T_{S\cap T}\rho^I_T$. Schematically, we can picture this as follows:

$$\begin{tikzcd}
& S\cap T & \\
S \arrow[ur, "\rho"] & & T \arrow[ul, "\lambda"'] \\
& I \arrow[ul, "\lambda"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

There is a category $\mathsf{FinSet}^{\mathrm{binc}}$ such that an $\mathcal{E}$-bicomodule is exactly the same as a pre-sheaf $\mathsf{FinSet}^{\mathrm{binc}} \longrightarrow \mathsf{Sp}_k$; we leave its construction to the categorically inclined reader. If the structure on $\mathcal{X}$ is cosymmetric, we will write $z \parallel S$ for the common value of $z \mathbin{\backslash\!\!\backslash} S$ and $z \mathbin{/\!\!/} S$. There is a close relation between linearized coalgebras and linearized $\mathcal{E}$-bicomodules, as described in the following proposition.

Proposition 1.4. *Let* $(\mathcal{X}, \Delta)$ *be a linearized coalgebra, and let* $f_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{E}$ *be the unique morphism of linearized coalgebras described in Proposition 1.3. There is on* $\mathcal{X}$ *an* $\mathcal{E}$*-bicomodule structure so that the coactions* $\lambda : \mathcal{X} \longrightarrow \mathcal{E} \otimes \mathcal{X}$ *and* $\rho : \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{E}$ *are obtained from postcomposition of* Δ *with* $f_{\mathcal{X}} \otimes 1$ *and* $1 \otimes f_{\mathcal{X}}$*, respectively.* ◀

We refer the reader to [2, Chapter 8, §3, Proposition 29]. Remark that, with this proposition at hand, the notation introduced for bicomodules and that introduced for coalgebras is consistent.

1.4 Twisted Hopf algebras

Let $\mathcal{X}$ be a twisted bialgebra with structure maps Δ and μ. Recall that a species $\mathcal{X}$ is *connected* if $\mathcal{X}(\varnothing)$ is free of rank one. The following result in [2] states every connected twisted bialgebra is automatically a Hopf algebra, and this automatically endows the various categories of representations of $\mathcal{X}$ with extra structure.

More generally, a twisted bialgebra $\mathcal{X}$ is a Hopf algebra precisely when $\mathcal{X}(\varnothing)$ is a Hopf k-algebra, and the antipode of $\mathcal{X}$ can, in that case, be explicitly constructed from the antipode of $\mathcal{X}(\varnothing)$ —this is a variant of what is known as *Takeuchi's theorem*, see the monograph [3, Proposition 9] for more details.

Theorem 1.1. *Let $(\mathcal{X}, \mu, \Delta)$ be a twisted bialgebra.*

1. *If $\mathcal{X}$ is a Hopf algebra with antipode s, then $\mathcal{X}(\varnothing)$ is a Hopf k-algebra with antipode $s(\varnothing)$.*
2. *If $\mathcal{X}(\varnothing)$ is a Hopf k-algebra with antipode s_0, then $\mathcal{X}$ is a Hopf algebra, and s can be iteratively constructed from s_0, μ and Δ.*
3. *In particular, if $\mathcal{X}$ is a connected bialgebra, $\mathcal{X}$ is a Hopf algebra.*

Proof. For a proof and an explicit formula for s in terms of s_0, we refer the reader to [2, Chapter 8, §3.2, Proposition 8.10, and §4.2, Proposition 8.13]. The third part follows from the second since k is, in a unique way, a Hopf k-algebra. ◀

We define some connected bialgebras that will be of interest in Section 2. In view of the previous result, they are all Hopf algebras in the category of linear species. Remark that, since the algebraal category Sp_k is symmetric, the tensor product of two twisted Hopf algebras is again a twisted Hopf algebra, so the following examples provide further ones by combining them into products. In all cases the unit and counit are the projection and the inclusion of the unit $\mathbf{1}$ in the component of $\varnothing$.

H1. Fix a finite set I and a decomposition (S, T) of I. If ℓ_1 and ℓ_2 are linear orders on S and T respectively, their *concatenation* $\ell_1 \cdot \ell_2$ is the unique linear order on I that restricts to ℓ_1 in S and to ℓ_2 in T, and such that $s < t$ if $s \in S$ and $t \in T$; this operation is in general not commutative. If ℓ is a linear order on I, write $\ell|_S$ for the restriction of ℓ to S, and $\overline{\ell}$ for the reverse order to ℓ. The species of linear orders $\mathcal{L}_0$ admits a bialgebra structure such that

- multiplication is given by concatenation: $\mu(S, T)(\ell_1, \ell_2) = \ell_1 \cdot \ell_2$,
- comultiplication is given by restriction: $\Delta(S, T)(\ell) = \ell|_S \otimes \ell|_T$.

In particular, this endows the linearization $\mathcal{L}$ with a cosymmetric bicomodule structure over $\mathcal{E}$. The map $\mathcal{L} \longrightarrow \mathcal{E}$ that sends a linear order on a finite set I to $*_I = \{I\}$ is a map of bialgebras. The antipode is given, up to sign, by taking the reverse of a linear order: $s(I)(\ell) = (-1)^{\#I}\overline{\ell}$.

H2. If (S, T) is a decomposition of a finite set I, and $F = (F_1, \ldots, F_s)$ and $G = (G_1, \ldots, G_t)$ are compositions of S and of T, respectively, the *concatenation* $F \cdot G$ is the composition $(F_1, \ldots, F_s, G_1, \ldots, G_t)$ of I. If $F = (F_1, \ldots, F_t)$ is a composition of I, the restriction of F to S is the composition $F|_S$ of S obtained from the decomposition $(F_1 \cap S, \ldots, F_t \cap S)$ of S by deleting empty blocks, which usually has shorter length than that of F. Finally, the reverse of a composition F is the composition $\overline{F}$ whose blocks are listed in the reverse order of those in F. The species $\mathcal{C}$ of compositions has a bialgebra structure such that

- multiplication is given by concatenation: $\mu(S, T)(F, G) = F \cdot G$,
- comultiplication is given by restriction: $\Delta(S, T)(F) = F|_S \otimes F|_T$.

This is cocommutative but not commutative. The morphism $\mathcal{L} \longrightarrow \mathcal{C}$ that sends a linear order $i_1 \cdots i_t$ on a set I to the composition $(\{i_1\}, \ldots, \{i_t\})$ is a map of bialgebras. The formula for the antipode is not as immediate as the previous ones. For details, see [3, Section 11].

H3. If (S, T) is a decomposition of a finite set I, and X and Y are partitions of S and T, respectively, the union $X \cup Y$ is a partition of I. If X is a partition of I, then $X|_S = \{x \cap S : x \in X\} \smallsetminus \{\varnothing\}$ is a partition of S, which we call the restriction of X to S. The species $\mathcal{P}$ of partitions admits a bialgebra structure such that

- multiplication is given by the union of partitions: $\mu(S, T)(X, Y) = X \cup Y$,
- comultiplication is given by restriction: $\Delta(S, T)(X) = X|_S \otimes Y|_T$.

This is both commutative and cocommutative. The map $\mathcal{C} \longrightarrow \mathcal{P}$ that sends a decomposition F of a set I to the partition X of I consisting of the blocks of F is a bialgebra map. The morphism $\mathcal{E} \longrightarrow \mathcal{P}$ that sends $*_I = \{I\}$ to the partition of I into singletons is also a map of bialgebras. See [3, Theorem 33] for a formula for the antipode of $\mathcal{P}$.

H4. If p is a poset with underlying set I, and (S, T) is a decomposition of I, we say S *is a lower set of* T *with respect to* p and write $S \prec_p T$ if no element of T is less than an element of S for the order p, and we write p_S for $p \cap (S \times S)$, the *restriction of* p *to* S. The linearized species Pos of posets admits a bialgebra structure so that

- multiplication is given by the disjoint union of posets: if p^1 and p^2 are posets with underlying sets S and T, respectively, $\mu(S, T)(p^1, p^2) = p^1 \sqcup p^2$,
- comultiplication is obtained by lower sets and by restriction: for p a poset defined on $S \sqcup T$, we set $\Delta(S, T)(p) = p_S \otimes p_T$ if $S \prec_p T$, and set $\Delta(S, T)(p) = 0$ if not.

It is worthwhile to remark that one can define another multiplication on this species: if p^1 and p^2 are posets on disjoint sets S and T, respectively, let $p^1 * p^2$ be the usual join of posets. This is associative and has unit the empty poset, and the inclusion of linear orders into posets $\mathcal{L} \longrightarrow P$ is a morphisms of algebras if $\mathcal{L}$ is given the concatenation product. We also remark that the maps described above fit into a commutative diagram of Hopf algebras as illustrated in the figure

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr] & \\
\mathcal{L} \arrow[ur] \arrow[dr] & & \mathcal{P} \\
 & \mathcal{E} \arrow[ur] &
\end{tikzcd}$$

and we will analyse the resulting maps in cohomology in Section 2. For more examples of Hopf algebras in species, and their relation to classical combinatorial results, we refer the reader to [2, Chapter 13].

2 The cohomology of twisted coalgebras

2.1 Definitions and first examples

Let $\mathcal{H}$ be a twisted coalgebra and $\mathcal{X}$ a -bicomodule, and let us define a cosimplicial k-module $C^*(\mathcal{X},\mathcal{H})$ as follows. For each $n \in \mathbb{N}$:

1. Define $C^n(\mathcal{X},\mathcal{H})$ to be $\mathrm{Hom}_{\mathsf{Sp}_k}(\mathcal{X},\mathcal{H}^{\otimes n})$ the set of maps in Sp_k from $\mathcal{X}$ to the iterated tensor product $\mathcal{H}^{\otimes n}$.
2. For $0 < i < n+1$, consider the map $d^i : C^n(\mathcal{X},\mathcal{H}) \longrightarrow C^{n+1}(\mathcal{X},\mathcal{H})$ induced by post-composition with the coproduct of $\mathcal{H}$ at the ith position.
3. For $i = 0, n+1$, let $d^0, d^{n+1} : C^n(\mathcal{X},\mathcal{H}) \longrightarrow C^{n+1}(\mathcal{X},\mathcal{H})$ be the maps obtained post composing with the left and right comodule maps of $\mathcal{X}$, respectively.

It is straightforward to check that, by virtue of the coassociativity of $\mathcal{H}$ and the bicomodule axioms, the maps above satisfy the usual cosimplicial identities. It follows that if for each $n \in \mathbb{N}$ we define the alternated sum $\delta^n = \sum_{i=0}^{n+1}(-1)^i d^i$, we obtain a cohomologically graded complex $(C^*(\mathcal{X},\mathcal{H}),\delta^*)$.

Definition 2.1. The *cohomology of* $\mathcal{X}$ *with values in* $\mathcal{H}$ is the cohomology of the complex $(C^*(\mathcal{X},\mathcal{H}),\delta^*)$, and we denote it by $H^*(\mathcal{X},\mathcal{H})$.

The homologically inclined reader will notice that these cohomology groups are equal to $\mathrm{Ext}^*(\mathcal{X},\mathcal{H})$ with the Ext taken in the category of $\mathcal{H}$-bicomodules. In the following we will mainly consider the case in which $\mathcal{H}$ is the exponential species, but will make it clear when a certain result can be extended to other twisted coalgebras. Usually, it will be necessary that $\mathcal{H}$ is linearized and with a linearized bimonoid structure, and we will usually require that $\mathcal{H}$ be connected. Because of the plethora of relevant examples of such bimonoids found in [2] and other articles by the same authors, such as [3], there is no harm in restricting ourselves to such species.

Fix an $\mathcal{E}$-bicomodule $\mathcal{X}$. The complex $C^*(\mathcal{X},\mathcal{E})$, which we will denote more simply by $C^*(\mathcal{X})$, has in degree q the collection of morphisms of species $\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes q}$. Such a morphism is determined by a collection of k-linear maps $\alpha(I) : \mathcal{X}(I) \longrightarrow \mathcal{E}^{\otimes q}(I)$, one for each finite set I, which is equivariant, in the sense that for each bijection $\sigma : I \longrightarrow J$ between finite sets, and every $z \in \mathcal{X}(I)$, the equality $\sigma(\alpha(I)(z)) = \alpha(J)(\sigma z)$ holds.

Remark 2.1. For each finite set I, the space $\mathcal{E}^{\otimes q}(I)$ is a free k-module with basis the tensors of the form

$$F_1 \otimes \cdots \otimes F_q$$

with $F = (F_1,\dots,F_q)$ a decomposition of I; for simplicity, we use the latter notation for such basis elements. In terms of this basis, we can write

$$\alpha(I)(z) = \sum_{F \vdash_q I} \alpha(F)(z)\, F$$

where $\alpha(F)(z) \in k$.

We recall that the cochain α is completely determined by an equivariant collection of functionals $\alpha(F) : \mathcal{X}(I) \longrightarrow k$, the components of α, one for each decomposition F of a finite set I. The equivariance condition is now that, for a bijection $\sigma : I \longrightarrow J$, and $(F_1,\dots,F_q)$ a decomposition of I, we have

$$\alpha(F_1,\dots,F_q)(z) = \alpha(\sigma(F_1),\dots,\sigma(F_q))(\sigma z)$$

for each $z \in \mathcal{X}(I)$. Recall that when writting $\alpha(F)(z)$ we omit I, recalling that it is always the case $I = \cup F$.

Now fix a q-cochain $\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes q}$ in $C^*(\mathcal{X})$. By the remarks in the last paragraph, to determine the $(q+1)$-cochain $\delta\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes(q+1)}$ it is enough to determine its components.

Lemma 2.1. *For each decomposition $F = (F_0,\dots,F_q)$ of a set I, then the component of the ith coface $d_i\alpha$ at F is given, for $z \in \mathcal{X}(I)$, by*

$$(d^i\alpha)(F_0,\dots,F_q)(z) = \begin{cases} \alpha(F_1,\dots,F_q)(z /\!\!/ F_0^c) & \text{if } i = 0, \\ \alpha(F_0,\dots,F_i \cup F_{i+1},\dots,F_{q+1})(z) & \text{if } 0 < i < q+1, \\ \alpha(F_0,\dots,F_{q-1})(z \backslash\!\!\backslash F_q^c) & \text{if } i = q+1. \end{cases} \tag{1}$$

Proof. Indeed, let us follow the prescription above and compute each coface map explicitly. If $z \in \mathcal{X}(I)$, to compute $d^0\alpha(z)$, we must coact on z to the left and evaluate the result at α, that is

$$(1 \otimes \alpha \circ \lambda)(I)(z) = \sum_{(S,T) \vdash I} *_S \otimes \alpha(T)(z /\!\!/ T),$$

and the coefficient at a decomposition $F = (F_0,\dots,F_q)$ is $\alpha(F_1,\dots,F_q)(z /\!\!/ F_0^c)$. The same argument gives the last coface map. Now consider $0 < i < q+1$, so that we must take $z \in \mathcal{X}(I)$, apply α, and then comultiply the result at coordinate i. Concretely, write

$$\alpha(I)(z) = \sum_{F \vdash_q I} \alpha(I)(F)(z) F$$

and pick a decomposition $F' = (F_0,\dots,F_q)$ into $q+1$ blocks of I. There exists then a unique

$F \vdash_q I$ such that $1^{i-1} \otimes \Delta \otimes 1^{q-i}(F) = F'$, to wit, $F = (F_0, \ldots, F_i \cup F_{i+1}, \ldots, F_q)$, and in this way we obtain the formulas of Equation (1). ◀

Since $\mathcal{E}$ is counital, the complex above admits codegeneracy maps, which are much easier to describe: they are obtained by inserting an empty block into a decomposition. Concretely, for each $j \in \{0, \ldots, q+1\}$,

$$(\sigma^j \alpha)(F_1, \ldots, F_q)(z) = \alpha(F_1, \ldots, F_j, \varnothing, F_{j+1}, \ldots, F_q).$$

As a consequence of this, a cochain $\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes q}$ in $C^*(\mathcal{X})$ is in the normalized subcomplex $\overline{C}^*(\mathcal{X})$ if its components are such that $\alpha(F)(z) = 0 \in k$ whenever F contains an empty block. Alternatively, we can construct a (non-unital) coalgebra $\overline{\mathcal{E}}$ with $\overline{\mathcal{E}}(\varnothing) = 0$ and $\overline{\mathcal{E}}(I) = \mathcal{E}(I)$ whenever I is nonempty, and describe the normalized complex $\overline{C}^*(\mathcal{X})$ as the complex of maps $\mathcal{X} \longrightarrow \overline{\mathcal{E}}^{\otimes *}$ with differential induced by the alternating sum of the coface maps we just described.

Remark 2.2. For each finite set I the space $\overline{\mathcal{E}}^{\otimes q}(I)$ has basis the *compositions* of I into q blocks, while $\mathcal{E}^{\otimes q}(I)$ has basis the *decompositions* of I into q blocks. In particular, $\overline{\mathcal{E}}^{\otimes q}(I) = 0$ if $q > \#I$, while $\mathcal{E}^{\otimes q}(I)$ is always nonzero. This observation will be crucial in Section 3.

2.2 The cobar complex and cup products

Since the twisted coalgebra $\mathcal{E}$ is, in fact, a twisted commutative Hopf algebra, we can endow the complex $C^*(\mathcal{X})$ with the structure of a dga algebra and hence produce on the cohomology groups $H^*(\mathcal{X}, \mathcal{E})$ a structure of a commutative associative algebra, as follows.

First, let us give an alternative way of constructing the complex $C^*(\mathcal{X})$. Let $\Omega^*(\mathcal{E})$ denote the cobar construction on the coalgebra $\mathcal{E}$. This is a dg twisted algebra which is freely generated by $s^{-1}\overline{\mathcal{E}}$, the shift of the species $\mathcal{E}$ without the counit, and whose differential is induced from the coproduct of $\mathcal{E}$: it is the unique coderivation extending the map

$$\Delta : s^{-1}\overline{\mathcal{E}} \longrightarrow (s^{-1}\overline{\mathcal{E}})^{\otimes 2} \subseteq \Omega^*(\mathcal{E}).$$

We can then form the space

$$\mathrm{Hom}_{\mathsf{Sp}_k}(\mathcal{X}, \Omega^*(\mathcal{E}))$$

which, as a graded vector space, coincides with the normalized complex for $C^*(\mathcal{X}, \mathcal{E})$: the way we shifted $\mathcal{E}$ makes sure that maps $\mathcal{X} \longrightarrow \overline{\mathcal{E}}^{\otimes n}$ live in degree n. Observe, moreover, that that hom-set above inherits a differential δ_1 by postcomposition with the differential

$$d : \Omega^*(\mathcal{E}) \longrightarrow \Omega^{*+1}(\mathcal{E}),$$

and this coincides in fact with the internal sum of the coface maps above, omitting the endpoints 0 and $n+1$. To obtain the full differential δ, we consider the canonical degree -1 injection $\tau : \mathcal{E} \longrightarrow \Omega^*(\mathcal{E})$ and the differential δ_2 obtained by the following composition where p is the degree of $\varphi : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes p}$:

$$\delta_2(\varphi) = \mu_{\Omega^*(\mathcal{E})}(\varphi \otimes \tau \circ \lambda + (-1)^p \tau \otimes \varphi \circ \rho).$$

A perhaps involved but straightforward computation shows that $\delta_1 - \delta_2$ coincides with δ, so that we obtain a new description of the complex $C^*(\mathcal{X},\mathcal{E})$ a complex twisted by τ (the summand δ_2 is the twist determined by τ):

$$C^*(\mathcal{X},\mathcal{E}) = \left(\mathrm{Hom}_\tau(\mathcal{X},\Omega^*(\mathcal{E})), \delta_1 - \delta_2\right).$$

Proposition 2.1. *The dg coalgebra $\Omega^*(\mathcal{E})$ is in fact a dg bialgebra if we endow it with the* shuffle coproduct *induced from the commutative product of $\mathcal{E}$, which we will denote by $\Delta_{\Omega^*(\mathcal{E})}$.*

Proof. This statement is completely dual to the classical statement (see for example Chapter 8 in [17]) that if A is a commutative algebra then the bar construction BA is a commutative algebra with the shuffle product induced from the commutative product of A. We remind the reader that it is crucial that A be commutative (and hence, in our case, that $\mathcal{E}$ be cocommutative) for this product to be compatible with the differential of BA. ◀

Definition 2.2. We define the external product

$$- \times - : C^*(\mathcal{X},\mathcal{E}) \otimes C^*(\mathcal{X},\mathcal{E}) \longrightarrow C^*(\mathcal{X} \otimes \mathcal{Y},\mathcal{E})$$

so that for two cochains $\varphi,\psi \in C^*(\mathcal{X},\mathcal{E})$ we have $\varphi \times \psi = \mu_{\Omega^*(\mathcal{E})} \circ (\varphi \otimes \psi)$.

Note that we use the fact $\mathcal{E}$ is a twisted Hopf algebra, which implies that the category of $\mathcal{E}$-bicomodules admits an internal tensor product. Concretely, if $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{E}$-bicomodules, we endow the tensor product $\mathcal{X} \otimes \mathcal{Y}$ with the left and right diagonal actions coming from the product of $\mathcal{E}$.

Remark 2.3. In case we have a comultiplication map $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$ making $\mathcal{X}$ into a coalgebra in the category of $\mathcal{E}$-bicomodules, we can use this external product to obtain a cup product in $C^*(\mathcal{X})$, which we will write

$$- \smile - : C^*(\mathcal{X},\mathcal{E}) \otimes C^*(\mathcal{X},\mathcal{E}) \longrightarrow C^*(\mathcal{X},\mathcal{E}).$$

Remark 2.4. In general, the algebra $H^*(\mathcal{X})$ will be non-commutative: for example, if $\mathcal{X}$ is concentrated in cardinal 0, then the datum of $\mathcal{X}$ really amounts to that of the coalgebra $\mathcal{X}[0]$, a coalgebra in k-modules, and $H^*(\mathcal{X})$ is the algebra dual to it, which may well be non-commutative.

If $\mathcal{X}$ is a $\mathcal{E}$-bicomodule and $\Delta : \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}$ a morphism of $\mathcal{E}$-bicomodules, we write, for each I and each $z \in \mathcal{X}[I]$,

$$\Delta[I](z) = \sum_{(S,T)\vdash I} z_{(S)} \otimes z^{(T)}$$

à la Sweedler, with each summand $z_{(S)} \otimes z^{(T)}$ appearing here standing for an element —not necessarily an elementary tensor— of the submodule $\mathcal{X}[S] \otimes \mathcal{X}[T]$ of $(\mathcal{X} \otimes \mathcal{X})[I]$. If $\alpha : \mathcal{X} \to \mathcal{E}^{\otimes p}$ and $\beta : \mathcal{X} \to \mathcal{E}^{\otimes q}$ are a p- and a q-cochain in the complex $C^*(\mathcal{X})$, then their product $\alpha \smile \beta \in C^{p+q}(\mathcal{X})$ has coefficients given by

$$\alpha \smile \beta(F)(z) = \alpha(F_{1,p})(z_{(F_{1,p})}) \cdot \beta(F_{p+1,p+q})(z^{(F_{p+1,p+q})})$$

for all I, all decompositions $F = (F_1, \ldots, F_{p+q})$ of I and all $z \in \mathcal{X}[I]$. Here we are being succinct and writing $F_{i,i+j}$ for both the decomposition $(F_i, \ldots, F_{i+j})$ obtained from F and for the union of this decomposition. Our main source of examples of coalgebras in $\mathcal{E}$-bicomodules comes from the following simple observation:

Proposition 2.2. *Let $\mathcal{X}'$ be a nonempty set-valued species with left and right restrictions and let $\mathcal{X}$ be the $\mathcal{E}$-bicomodule obtained by linearization from $\mathcal{X}'$. There is a morphism of $\mathcal{E}$-bicomodules $\Delta : \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}$ such that*

$$\Delta[I](z) = \sum_{(S,T)\vdash I} z \mathbin{\backslash\!\backslash} S \otimes z \mathbin{/\!/} T$$

for each finite set I and each $z \in \mathcal{X}'[I]$. ◀

In what follows, we will usually consider every $\mathcal{E}$-bicomodule whose underlying species is a linearization of a twisted coalgebra in the way described in this proposition.

3 An alternative description of cohomology

The objective of this chapter is to obtain an alternative and more useful description of the cohomology groups of an $\mathcal{E}$-bicomodule $\mathcal{X}$. We show that for every $\mathcal{E}$-bicomodule $\mathcal{X}$ there is a filtration on the complex $C^*(\mathcal{X})$ giving rise to a spectral sequence of algebras which converges to $H^*(\mathcal{X})$. If $\mathcal{X}$ is *weakly projective*, that is, if for each non-negative integer j, $\mathcal{X}[j]$ is a projective kS_j-module, this collapses at the E^1-page. Because we can completely describe this page, this provides us with a complex that calculates $H^*(\mathcal{X})$, and which can be used for effective computations.

To be explicit, by this we mean each component of this complex is finitely generated whenever $\mathcal{X}$ has finitely many structures on each finite set, and in that case the differential of an element depends on finite data obtained from it—this is in contrast with the situation of $C^*(\mathcal{X})$. Moreover, the spectral sequence is one of algebras whenever we endow $C^*(\mathcal{X})$ with a cup product arising from a diagonal map $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$, so these remarks apply to the computation of the cup product structure of $H^*(\mathcal{X})$, and we exploit this for the cup product we defined in Section 2.

Some more running conventions. Let $\mathcal{X}$ be a species. The *support* of $\mathcal{X}$ is the set of non-negative integers j for which $\mathcal{X}[j]$ is nontrivial. We say $\mathcal{X}$ is *finitely supported* if is has finite support, and that it is *concentrated in cardinal* j if the support of $\mathcal{X}$ is exactly $\{j\}$. The support of a nontrivial species $\mathcal{X}$ is contained in a smallest interval of non-negative integers, whose length we call the *length of* $\mathcal{X}$. The species $\mathcal{X}$ is of *finite type* if $\mathcal{X}[j]$ is a finitely generated k-module for each nonnegative integer j, and it is *finite* if it is both of finite type and finitely supported.

3.1 The spectral sequence

Let $\mathcal{X}$ be a species in Sp_k and let j be a non-negative integer. We define species $\tau^j\mathcal{X}$ and $\tau_j\mathcal{X}$, which we call the *upper truncation of* $\mathcal{X}$ *at* j and the *lower truncation of* $\mathcal{X}$ *after* j as follows. For every finite set I, we put

$$\tau^j\mathcal{X}(I) = \begin{cases} \mathcal{X}(I) & \text{if } \#I \leqslant j, \\ 0 & \text{else,} \end{cases} \qquad \tau_j\mathcal{X}(I) = \begin{cases} \mathcal{X}(I) & \text{if } \#I \geqslant j, \\ 0 & \text{else.} \end{cases}$$

If $\sigma : I \longrightarrow J$ is a bijection then $(\tau^j\mathcal{X})(\sigma) = \mathcal{X}(\sigma)$ whenever I has at most j elements, while $(\tau^j\mathcal{X})(\sigma)$ is the unique isomorphism $0 \longrightarrow 0$ in the remaining cases. Similarly, $(\tau_j\mathcal{X})(\sigma) = \mathcal{X}(\sigma)$ whenever I has at least j elements, while $(\tau_j\mathcal{X})(\sigma)$ is the unique isomorphism $0 \longrightarrow 0$ in the remaining cases. It is clear both of this constructions depend functorially on $\mathcal{X}$, and that there is a short exact sequence

$$0 \longrightarrow \tau^j\mathcal{X} \longrightarrow \mathcal{X} \longrightarrow \tau_{j+1}\mathcal{X} \longrightarrow 0. \tag{2}$$

By convention, $\tau^j\mathcal{X} = 0$ and $\tau_j\mathcal{X} = \mathcal{X}$ if j is negative. We will write τ_i^j for the composition $\tau_i \circ \tau^j$, which is the same as $\tau^j \circ \tau_i$, and $\mathcal{X}[j]$ instead of τ_j^j; this species is concentrated in cardinal j. This will be of use in Section 3.2. Observe that we can carry out these constructions in the categories of $\mathcal{H}$-(bi)comodules for any twisted coalgebra $\mathcal{H}$. Precisely, we have the following proposition:

Proposition 3.1. *Let $\mathcal{H}$ be a twisted coalgebra, let $\mathcal{X}$ be a left $\mathcal{H}$-comodule, and fix a non-negative integer j.*

T1. *The truncated species $\tau^j\mathcal{X}$ is an $\mathcal{H}$-subcomodule of $\mathcal{X}$ in such a way that the inclusion $\tau^j\mathcal{X} \longrightarrow \mathcal{X}$ is a morphism of $\mathcal{H}$-comodules, and*

T2. *the truncated species $\tau_j\mathcal{X}$ is uniquely an $\mathcal{H}$-comodule in such a way that the morphisms in the short exact sequence (2) in Sp_k are in fact of $\mathcal{H}$-comodules.*

It is clear the above can, first, be extended to $\mathcal{H}$-bicomodules, and second, be dualized to $\mathcal{H}$-modules, and then extended to $\mathcal{H}$-bimodules. This provides a spectral sequence for monoids and modules, which we will not discuss.

Proof. Denote by λ the coaction of $\mathcal{X}$. To see **T1**, we have to show that $\lambda(\tau^j\mathcal{X}) \subseteq \mathcal{H}\otimes\tau^j\mathcal{X}$, which is immediate, and **T2** is deduced from this: we identify $\tau_j\mathcal{X}$ with the quotient $\mathcal{X}/\tau^j\mathcal{X}$, which inherits an $\mathcal{H}$-comodule structure making the maps in the short exact sequence (2) maps of $\mathcal{H}$-comodules. ◀

In what follows, we will need to identify the comodule structure of $\mathcal{X}[j]$. This is done in the following lemma.

Lemma 3.1. *Let $\mathcal{H}$ be a connected twisted coalgebra. An $\mathcal{H}$-(bi)comodule concentrated in one cardinal necessarily has the trivial $\mathcal{H}$-coaction.* ◀

Proof. For trivial reasons, the comodule map(s) land only on the summand that is completely determined by the counitality axioms, meaning that the action is trivial: for any $x\in\mathcal{X}$ we have that $\lambda(x) = x\otimes 1$ or in the right module case, that $\rho(x) = 1\otimes x$. ◀

Let $\mathcal{X}$ be an $\mathcal{E}$-bicomodule. For each integer p, let $F^pC^*(\mathcal{X})$ be the collection of chains that vanish on $\tau^{p-1}\mathcal{X}$. This is a subcomplex because $\tau^p\mathcal{X}$ is a $\mathcal{E}$-subbicomodule of $\mathcal{X}$, so we have a descending filtration of the complex $C^*(\mathcal{X})$. When there is no danger of confusion, we will write F^pC^* instead of $F^pC^*(\mathcal{X})$. This filtration in $C^*(\mathcal{X})$ induces a filtration on $H^* = H^*(\mathcal{X})$ with $F^pH^*(\mathcal{X}) = \operatorname{im}(H^*(F^pC^*) \longrightarrow H^*)$, and we write $E_0(H)$ for the bigraded object with

$$E_0^{p,q}(H) = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}.$$

As explained in detail in [17, Chapter 2, §2], this filtration gives rise to a cohomology spectral sequence $(E_r, d_r)_{r\geqslant 0}$. According to the construction carried out there, the E_0-page has

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p+1}C^{p+q}}$$

and differential $d_0^{pq} : E_0^{pq} \longrightarrow E_0^{p,q+1}$ induced by that of $C^*(\mathcal{X})$, and, in particular, we have $E_0^{pq} = 0$ when $p < 0$ or $p + q < 0$. Moreover:

Proposition 3.2. *Let p be an integer.*

1. *There is a natural isomorphism $F^p C^*(\mathcal{X}) \longrightarrow C^*(\tau_p \mathcal{X})$ that induces, in turn, an isomorphism $(E_0^{p*}, d_0^{p*}) \longrightarrow C^{p+*}(\mathcal{X}(p))$, so that*
2. *for every integer q, there are isomorphisms $E_1^{pq} \longrightarrow H^{p+q}(\mathcal{X}(p))$, and, viewing this as an identification,*
3. *the differential $d_1^{pq} : E_1^{pq} \longrightarrow E_1^{p+1,q}$ is the composition of the connecting homomorphism $H^{p+q}(\mathcal{X}(p)) \longrightarrow H^{p+q+1}(\tau_{p+1}\mathcal{X})$ of the long exact sequence corresponding to the short exact sequence $0 \longrightarrow \mathcal{X}(p) \longrightarrow \tau_p \mathcal{X} \longrightarrow \tau_{p+1}\mathcal{X} \longrightarrow 0$ and the map $H^*(\iota)$ induced by the inclusion $\iota : \mathcal{X}(p+1) \longrightarrow \tau_{p+1}\mathcal{X}$.*

Proof. The exact sequence of $\mathcal{E}$-bicomodules

$$0 \longrightarrow \tau^{p-1}\mathcal{X} \longrightarrow \mathcal{X} \xrightarrow{\pi} \tau_p \mathcal{X} \longrightarrow 0$$

is split in Sp_k, so applying the functor $C^*(-)$ gives an exact sequence

$$0 \longrightarrow C^*(\tau_p \mathcal{X}) \longrightarrow C^*(\mathcal{X}) \longrightarrow C^*(\tau^{p-1}\mathcal{X}) \longrightarrow 0.$$

This gives the desired isomorphism of $F^p C^*(\mathcal{X})$ with $C^*(\tau_p \mathcal{X})$, since the injective map $C^*(\pi)$ has image the kernel of $C^*(\iota)$, which is, by definition, $F^p C^*(\mathcal{X})$. This proves the first claim of the proposition.

Similarly, we have a short exact sequence of bicomodules

$$0 \longrightarrow \mathcal{X}(p) \longrightarrow \tau_p \mathcal{X} \longrightarrow \tau_{p+1}\mathcal{X} \longrightarrow 0,$$

also split in Sp_k, and which gives us the exactness of the second row of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}C^* \arrow[r] \arrow[d] & F^p C^* \arrow[r] \arrow[d] & E_0^{p*} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & C^*(\tau_{p+1}\mathcal{X}) \arrow[r] & C^*(\tau_p \mathcal{X}) \arrow[r] & C^*(\mathcal{X}(p)) \arrow[r] & 0
\end{tikzcd}$$

The desired natural isomorphism $E_0^{p,*} \longrightarrow C^*(\mathcal{X}(p))$ is the unique dashed arrow that extends the commutative diagram, and this proves the second claim of the proposition. To prove the last one, we note the diagram above can be viewed as an isomorphism of *exact sequences*, and so the connecting morphisms are also identified. Moreover, the differential at the E_1-page is induced from the connecting morphism of long exact sequence associated to the exact sequence

$$0 \longrightarrow F^{p+1}C^* \longrightarrow F^pC^* \longrightarrow F^pC^*/F^{p+1}C^* \longrightarrow 0$$

and the projection $F^{p+1}C^* \longrightarrow F^{p+1}C^*/F^{p+2}C^*$ which correspond, under our isomorphisms, to the connecting morphism of the short exact sequence

$$0 \longrightarrow \mathcal{X}(p) \longrightarrow \tau_p\mathcal{X} \longrightarrow \tau_{p+1}\mathcal{X} \longrightarrow 0$$

and to the map $C^*(\iota) : C^*(\tau_{p+1}\mathcal{X}) \longrightarrow C^*(\mathcal{X}(p+1))$ induced by the inclusion. ◀

We will prove in the next section that the spectral sequence just constructed converges to $H^*(\mathcal{X})$. A first step towards this is the following result:

Proposition 3.3. *The filtration is bounded above and complete.*

Proof. Using the identification provided by the isomorphisms $F^pC^* \longrightarrow C^*(\tau_p, \mathcal{X})$ of Proposition 3.2 and the split exact sequences $0 \longrightarrow \tau^p\mathcal{X} \longrightarrow \mathcal{X} \longrightarrow \tau_{p+1}\mathcal{X} \longrightarrow 0$ we are able, in turn, to identify $C^*(\mathcal{X})/F^{p+1}C^*(\mathcal{X})$ with $C^*(\tau^p\mathcal{X})$. In these terms, what the proposition claims is that the canonical map

$$C^*(\mathcal{X}) \longrightarrow \varprojlim C^*(\tau^p\mathcal{X})$$

is an isomorphism, and this is clear: if a cochain vanishes on every $\tau^p\mathcal{X}$ then it is zero, so the map is injective, and if we have cochains $\alpha^p : \tau^p\mathcal{X} \longrightarrow \mathcal{E}^{\otimes *}$ that glue correctly, we obtain a globally defined cochain $\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes *}$, so the map is surjective. ◀

Proposition 3.4. *If $\mathcal{X}$ vanishes in cardinals above N, then the normalized complex $\overline{C}^*(\mathcal{X})$ vanishes in degrees above N, and, a fortiori, the same is true for $H^*(\mathcal{X})$.*

Proof. Let $p > N$, consider a p-cochain α in the normalized complex $\overline{C}^*(\mathcal{X})$, and let us show that α vanishes identically. Indeed, if I is a finite set, the map $\alpha(I) : \mathcal{X}(I) \longrightarrow \overline{E}^{\otimes p}(I)$ is zero: if I has more than p elements, its domain is zero because $\mathcal{X}$ vanishes on I, and if I has at most p elements, then its codomain is zero, since there are no compositions of length p of I. ◀

This has two important consequences, the first of which will be thoroughly exploited in the next sections.

Corollary 3.1. *Fix an integer j.*

1. *We have $H^q(\tau^j\mathcal{X}) = 0$ if $q > j$.*
2. *The E_1-page of the spectral sequence lies in a cone in the fourth quadrant.* ◀

Because the E_1-page of the spectral sequence involves the cohomology of the species $\mathcal{X}(p)$ for $p \geqslant 0$, we turn our attention to the cohomology of species concentrated in a cardinal.

3.2 The E_1-page

This section is devoted to describing the E_1-page of the spectral sequence, and showing it concentrated in one row—so that the spectral sequence degenerates at the E_2-page— when $\mathcal{X}$ is weakly projective in Sp_k. Recall that by this we mean that, for each non-negative integer j, $\mathcal{X}[j]$ is a projective kS_j-module.

For $j \geqslant 1$ and for each integer $p \geqslant -1$, let $\Sigma_p(j)$ be the collection of compositions of length $p+2$ of $[j]$. We will identify the elements of $\Sigma_{j-2}(j)$ with permutations of $[j]$ in the obvious way. There are face maps $\partial_i : \Sigma_p(j) \longrightarrow \Sigma_{p-1}(j)$ for $i \in \{0,\dots,p\}$ given by

$$\partial_i(F_0,\dots,F_i,F_{i+1},\dots,F_{p+1}) = (F_0,\dots,F_i \cup F_{i+1},\dots,F_{p+1})$$

that make the sequence of sets $\Sigma_*(j) = (\Sigma_p(j))_{p\geqslant -1}$ into an augmented semisimplicial set. We write $k\Sigma_*(j)$ for the augmented semisimplicial k-module obtained by linearizing $\Sigma_*(j)$, and $k\Sigma_*(j)'$ for the dual semicosimplicial augmented k-module.

There is an action of S_j on each $\Sigma_p(j)$ by permutation, so that if $\tau \in S_j$ and if $(F_0,\dots,F_t)$ is a composition of $[j]$, then

$$\tau(F_0,\dots,F_t) = (\tau(F_0),\dots,\tau(F_t)).$$

It is straightforward to check the coface maps are equivariant with respect to this action, so $\Sigma_*(j)$ is, in fact, an augmented semisimplicial S_j-set. Consequently, $k\Sigma_*(j)$ and $k\Sigma_*(j)'$ have corresponding S_j-actions compatible with their semi(co)simplicial structures.

This complex $\Sigma_*(j)$ is known in the literature as the *Coxeter complex for the braid arrangement*, and its cohomology can be completely described.

Proposition 3.5. *The complex associated to $k\Sigma_*(j)'$ computes the reduced cohomology of a $(j-2)$-sphere with coefficients in k, that is,*

$$H^p(k\Sigma(j)') = \begin{cases} 0 & \text{if } p \neq j-2 \\ k[\![\xi_j]\!] & \text{if } p = j-2 \end{cases}$$

The non-trivial term is the k-module freely generated by the class of the map $\xi_j : k\Sigma_*(j) \longrightarrow k$ such that $\xi_j(\sigma) = \delta_{\sigma=1}$ and the action of kS_j on $H^{j-2}(k\Sigma_*(j)')$ is the sign representation.

Remark 3.1. In what follows, sgn_j will denote the sign representation of kS_j just described. Note that, when $j = 1$, $S^{j-2} = \varnothing$, and the reduced cohomology of such space is concentrated in degree -1, where it has value k.

Proof. We sketch a proof, and refer the reader to [1] and [5] for details. The braid arrangement $\mathscr{B}_j$ of dimension j in $\mathbb{R}^j$ is the collection of hyperplanes

$$\{H_{s,t} : 1 \leqslant s < t \leqslant j\}, \text{ with } H_{s,t}$$

defined by the equation $x_t = x_s$. This arrangement has rank $j-1$ and its restriction to the hyperplane H with equation $x_1 + \cdots + x_j = 0$ is essential, and defines a triangulation K of the unit sphere $S^{j-2} \subseteq H$. Concretely, the r-dimensional simplices of K are in bijection with compositions of $[j]$ into $r+2$ blocks, so that a composition $F = (F_0, \ldots, F_{r+1})$ corresponds to the r-simplex obtained by intersecting S^{j-2} with the subset defined by the equalities $x_s = x_t$ whenever s, t are in the same block of F and the inequalities $x_s \geqslant x_t$ whenever $t > s$ relative to the order of the blocks of F. It follows that $k\Sigma_*(j)'$ computes the reduced simplicial cohomology of S^{j-2}, and the generator of the top cohomology group is the functional $\xi_j : k\Sigma_*(j) \longrightarrow k$ described in the statement of the proposition. More generally, if $\xi_\sigma : k\Sigma_{j-2}(j) \longrightarrow k$ is the functional that assigns σ to 1 and every other simplex to zero, then $[\![\xi_\sigma]\!] = (-1)^\sigma [\![\xi_j]\!]$. Because the action of S_j on $k\Sigma_{j-2}(j)'$ is such that $\sigma\xi_j = \xi_\sigma$, this proves $H^{j-2}(k\Sigma_*(j)')$ is the sign representation of kS_j. ◀

We can describe the complex that calculates the cohomology of a species concentrated in cardinal j in terms of the Coxeter complex $\Sigma_*(j)$:

Proposition 3.6. *Fix a non-negative integer $j \geqslant 1$, and let $\mathcal{X}$ be an $\mathcal{E}$-bicomodule concentrated in cardinal j. There is an isomorphism of semi-cosimplicial k-modules*

$$\Psi^* : \overline{C}^*(\mathcal{X}) \longrightarrow \mathrm{Hom}_{S_j}(\mathcal{X}[j], k\Sigma_*(j)'[2]).$$

In particular, if $\mathcal{X}[j]$ is a projective kS_j-module, then $H^p(\mathcal{X}) = 0$ when $p \neq j$ and there is an isomorphism

$$\xi : H^j(\mathcal{X}) \longrightarrow \mathrm{Hom}_{S_j}(\mathcal{X}[j], \mathrm{sgn}_j).$$

This isomorphism is such that if $\alpha : \mathcal{X} \longrightarrow E^{\otimes j}$ is a normalized j-cocycle, then

$$\xi([\![\alpha]\!])(z) = \sum_{\sigma \in S_j} (-1)^\sigma \alpha(\sigma)(z)[\![\xi_j]\!], \tag{3}$$

for each $z \in \mathcal{X}[j]$.

If k is a field of characteristic coprime to $j!$ then every kS_j-module is projective by virtue of Maschke's theorem, so the above applies. If k is a field of characteristic zero, then every species $\mathcal{X}$ is weakly projective, and conversely.

Proof. Since $\mathcal{X}$ is concentrated in cardinal j, a normalized p-cochain $\alpha : \mathcal{X} \longrightarrow \overline{\mathcal{E}}^{\otimes p}$ is completely determined by an equivariant k-linear map $\tilde{\alpha} : \mathcal{X}[j] \longrightarrow \overline{\mathcal{E}}^{\otimes p}(j)$. Moreover, $\overline{\mathcal{E}}^{\otimes p}(j)$ is a free k-module with basis the tensors $F_1 \otimes \cdots \otimes F_p$ with $(F_1, \ldots, F_p)$ a composition of $[j]$, that is, $\overline{\mathcal{E}}^{\otimes p}(j)$ can be equivariantly identified with $k\Sigma_{p-2}(j)$. Because $\overline{\mathcal{E}}^{\otimes p}(j)$ is a free k-module, every k-linear map $\beta : \mathcal{X}[j] \longrightarrow \overline{\mathcal{E}}^{\otimes p}(j)$ corresponds uniquely to a map $\beta^t : \mathcal{X}[j] \longrightarrow \overline{\mathcal{E}}^{\otimes p}(j)'$ so that

$$\beta^t(z)(F_1, \ldots, F_p) = \beta(F_1, \ldots, F_p)(z).$$

In this way we obtain a map

$$\Psi^* : \overline{C}^*(\mathcal{X}) \longrightarrow \operatorname{Hom}_{S_j}(\mathcal{X}[j], k\Sigma_*(j)'[2]),$$

which is clearly an isomorphism of graded k-modules, and this map is compatible with the semicosimplicial structure and S_j-equivariant. The non-trivial observation needed to check this is that the first and last coface maps of $\overline{C}^*(\mathcal{X})$ are zero: this follows from Lemma 3.1, which states $\mathcal{X}$ has trivial coactions, so these maps vanish upon normalization.

Assume now that $\mathcal{X}[j]$ is kS_j-projective, so that the functor $\operatorname{Hom}_{S_j}(\mathcal{X}[j], -)$ is exact. The canonical map

$$\theta : H^*(\operatorname{Hom}_{S_j}(\mathcal{X}[j], k\Sigma_*(j)'[2])) \longrightarrow \operatorname{Hom}_{S_j}(\mathcal{X}[j], H^*(k\Sigma_*(j)'[2]))$$

is then an isomorphism, and we can conclude by Lemma 3.5 that $H^p(\mathcal{X})$ is zero except for $p = j$, and that we have a canonical isomorphism induced by Ψ and θ

$$\xi : H^j(\mathcal{X}) \longrightarrow \operatorname{Hom}_{S_j}(\mathcal{X}[j], \operatorname{sgn}_j).$$

It remains to prove the last formula. To this end, consider a j-cocycle $\alpha : \mathcal{X}[j] \longrightarrow \mathcal{E}^{\otimes j}(j)$. This corresponds under Ψ to the map $\mathcal{X}(p) \longrightarrow k\Sigma_{j-2}(j)'$ that assigns to z the functional $\sum_\sigma \alpha(\sigma)(z)\xi_\sigma$. Passing to cohomology and using the equality $[\![\xi_\sigma]\!] = (-1)^\sigma [\![\xi_j]\!]$ valid in view of Lemma 3.5 for all $\sigma \in S_j$, we obtain (3). ◀

Corollary 3.2. *If $\mathcal{X}$ is weakly projective, then E_1 is concentrated in the p-axis, where*

$$E_1^{p,0} \simeq \operatorname{Hom}_{S_p}(\mathcal{X}(p), \operatorname{sgn}_p),$$

so that, in particular, the spectral sequence degenerates at E_2.

This motivates us to consider, independently of convergence matters, the complex $S^*(\mathcal{X})$ that has $S^p(\mathcal{X}) = \mathrm{Hom}_{S_p}(\mathcal{X}(p), \mathrm{sgn}_p)$ and differentials induced from that of the E_1-page. Although this may not compute $H^*(\mathcal{X})$, it provides us with another invariant for $\mathcal{X}$. We call $S^*(\mathcal{X})$ the *small complex of* $\mathcal{X}$. We will give an explicit formula for its differential in Theorem 3.2.

Proof. The above follows for $p \geqslant 1$ by the last proposition, and the case $p = 0$ follows by definition of the E_0-page. ◀

The description of the inverse arrow to ξ will be useful for computations.

Lemma 3.2. *With the hypotheses of Proposition 3.6, the inverse arrow to ξ is the map*

$$\Theta : \mathrm{Hom}_{S_j}(\mathcal{X}[j], \mathrm{sgn}_j) \longrightarrow H^j(\mathcal{X})$$

that assigns to an S_j-equivariant map $f : \mathcal{X}[j] \longrightarrow \mathrm{sgn}_j$ the class of any lift F of f according to the diagram

$$\begin{tikzcd} & \mathcal{X}[j] \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ k\Sigma_{j-2}(j)' \arrow[r, "\pi"] & \mathrm{sgn}_j \arrow[r] & 0 \end{tikzcd}$$

In particular, if k is a field of characteristic coprime to $j!$, we can choose F to be the composition of f with the S_j equivariant map $\Lambda : \mathrm{sgn}_j \longrightarrow k\Sigma_{j-2}(j)'$ such that

$$\Lambda([\![\xi_j]\!]) = \frac{1}{j!} \sum_{\sigma \in S_j} (-1)^\sigma \xi_\sigma .$$

We can now prove, by an easy inductive argument, that the support of the cohomology groups of a weakly projective species of finite length $\mathcal{X}$ is no bigger than the support of $\mathcal{X}$. This is a second step toward proving the convergence of our spectral sequence, which we will completely address in the next section. Concretely:

Proposition 3.7. *Let $\mathcal{X}$ be an $\mathcal{E}$-bicomodule of finite length, which is weakly projective in Sp_k, and let q be a non-negative integer.*

1. *If $\mathcal{X}$ is zero in cardinalities below q, then $H^i(\mathcal{X}) = 0$ for $i < q$.*
2. *In particular, it follows that $H^p(\tau_q \mathcal{X}) = 0$ for $p < q$.*

Proof. Assume $\mathcal{X}$ is a species that vanishes in cardinalities below q, and proceed by induction on the length ℓ of $\mathcal{X}$. The base case in which $\ell = 1$ is part of the content in Proposition 3.6. Indeed, if $\mathcal{X}$ has lenght 1 it is concentrated in some degree p larger than q, and that proposition says $H^j(\mathcal{X}) = 0$ if $j \neq p$.

For the inductive step, suppose $\ell > 1$, and let j be the largest element of the support of $\mathcal{X}$. The long exact sequence corresponding to

$$0 \longrightarrow \tau^{j-1}\mathcal{X} \longrightarrow \mathcal{X} \longrightarrow \tau_j\mathcal{X} \longrightarrow 0.$$

includes the exact segment

$$\underbrace{H^q(\tau_j\mathcal{X})}_{0} \longrightarrow H^q(\mathcal{X}) \longrightarrow \underbrace{H^q(\tau^{j-1}\mathcal{X})}_{0}. \tag{4}$$

The choice of the integer j implies $\tau_j\mathcal{X}$ is of length one, and $\tau^{j-1}\mathcal{X}$ is of length smaller than that of $\mathcal{X}$, so by induction the cohomology groups appearing at the ends of (4) vanish. This proves the first claim, and the second claim is an immediate consequence of it. ◀

Proposition 3.8. *Let $\mathcal{X}$ be an $\mathcal{E}$-bicomodule. For every non-negative integer j, the projection $\mathcal{X} \longrightarrow \tau_{j+1}\mathcal{X}$ induces*

1. *a surjection $H^{j+1}(\tau_{j+1}\mathcal{X}) \longrightarrow H^{j+1}(\mathcal{X})$, and*
2. *isomorphisms $H^q(\tau_{j+1}\mathcal{X}) \longrightarrow H^q(\mathcal{X})$ for $q > j+1$.*

In terms of the filtration on $H^*(\mathcal{X})$, this means that $F^pH^{p+q} = H^{p+q}$ for $q \geqslant 0$.

Proof. Fix a non-negative integer j and consider the exact sequence

$$0 \longrightarrow \tau^j\mathcal{X} \longrightarrow \mathcal{X} \longrightarrow \tau_{j+1}\mathcal{X} \longrightarrow 0.$$

The associated long exact sequence gives an exact sequence

$$H^{j+1}(\tau_{j+1}\mathcal{X}) \longrightarrow H^{j+1}(\mathcal{X}) \longrightarrow \underbrace{H^{j+1}(\tau^j\mathcal{X})}_{0},$$

and exact sequences

$$\underbrace{H^{q-1}(\tau^j\mathcal{X})}_{0} \xrightarrow{\delta} H^q(\tau_{j+1}\mathcal{X}) \longrightarrow H^q(\mathcal{X}) \longrightarrow \underbrace{H^q(\tau^j\mathcal{X})}_{0},$$

for $q > j+1$, with the zeroes explained by Proposition 3.4. This proves both claims and finishes our proof. ◀

3.3 Convergence

The filtration defined on $C^*(\mathcal{X})$ is bounded above, and we have shown it is complete, so it suffices to check the spectral sequence is regular to obtain convergence —see the *Complete*

Convergence Theorem in [22, Theorem 5.5.10]. We have proven the spectral sequences degenerates at the E^2-page when $\mathcal{X}$ is weakly projective, and this implies the spectral sequence is regular, so the cited theorem can be applied. We give a mildly more accessible argument to justify convergence, which the reader can compare with the exposition in [10, pp. 137-140] and [17, pp. 99-102].

Proposition 3.9. *If $\mathcal{X}$ is an $\mathcal{E}$-bicomodule that is weakly projective in Sp_k, then the group $H^p(\tau_{q+1}\mathcal{X})$ vanishes for every integer $p<q$.*

In other words, the filtration on $C^*(\mathcal{X})$ is *regular*, that is, for each integer n, we have that $H^n(F^pC^*)=0$ for large p depending on n; in this case $p>n$ works. This guarantees the spectral sequence is regular, see [6, Chapter XV, §4].

Proof. Let $\mathcal{X}$ be as in the statement. The sequence of inclusions

$$\cdots \longrightarrow \tau^j\mathcal{X} \longrightarrow \tau^{j+1}\mathcal{X} \longrightarrow \cdots \tag{5}$$

gives a tower of cochain complexes $\mathscr{C}=\{C(\tau^j\mathcal{X})\}_{j\geqslant 1}$ of k-modules. We noted, in the proof of Proposition 3.3, that the canonical map $C^*(\mathcal{X}) \longrightarrow \varprojlim_j C^*(\tau^j\mathcal{X})$ is an isomorphism, and furnishes a map

$$\eta : H^*(\mathcal{X}) \longrightarrow \varprojlim{}_j H^*(\tau^j\mathcal{X}).$$

Let us show that this is an isomorphism. Fix $r\geqslant 0$. The tower of cochain complexes $\mathscr{C}$ satisfies the Mittag-Leffler condition since every arrow in it is onto: every inclusion in (5) is split in Sp_k, so there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1_j H^{r-1}(\tau^j\mathcal{X}) \longrightarrow H^r(\mathcal{X}) \xrightarrow{\eta} \varprojlim{}_j H^r(\tau^j\mathcal{X}) \longrightarrow 0.$$

We need only prove $\varprojlim^1_j H^{r-1}(\tau^j\mathcal{X})=0$, and, to do this, that the tower of abelian groups $\{H^{r-1}(\tau^i\mathcal{X})\}_{i\geqslant 0}$ satisfies the Mittag-Leffler condition: let $\iota(k,j) : H^r(\tau^k\mathcal{X}) \longrightarrow H^r(\tau^j\mathcal{X}))$ be the arrow induced by the inclusion for $k\geqslant j$, and let us show that for each j there is some i such that $\operatorname{image}(\iota(k,j)) = \operatorname{image}(\iota(i,j))$ for every $k\geqslant i$. Fix j, and let us show $i=r+2$ works by considering three cases.

- If $j<r$, then for every $k\geqslant j$ the map $\iota(k,j)$ is zero because its codomain is zero, so the claim is true.
- If $j\geqslant r+1$, then for every $k\geqslant j$, the map $\iota(k,j)$ is an isomorphism. In this case, we have the exact sequence

$$0 \longrightarrow \tau^j\mathcal{X} \xrightarrow{i} \tau^k\mathcal{X} \xrightarrow{\pi} \tau^k_{j+1}\mathcal{X} \longrightarrow 0$$

whose corresponding long exact sequence includes the segment

$$\underbrace{H^r(\tau^k_{j+1}\mathfrak{X})}_{0} \longrightarrow H^r(\tau^k\mathfrak{X}) \xrightarrow{\iota(k,j)} H^r(\tau^j\mathfrak{X}) \longrightarrow \underbrace{H^{r+1}(\tau^k_{j+1}\mathfrak{X})}_{0},$$

with the zeroes explained by Proposition 3.7 and the fact $\tau^k_{j+1}\mathfrak{X}$ is zero at cardinals r and $r+1$.

- Finally, suppose $j = r$, and fix $k \geqslant j$. If $k \geqslant r+2$, the map $\iota(k, j+1)$ is an isomorphism, and $\iota(k, j)$ factors as $\iota(j+1, j) \circ \iota(k, j+1)$, so that the image of $\iota(k, j)$ equals the image of $\iota(j+1, j)$.

Fix non-negative integers p and q with $p < q$ as in the statement. For every integer j, the double truncation $\tau^j_{q+1}\mathfrak{X}$ is of finite length and begins in degrees greater than q, so that $H^p(\tau^j_{q+1}\mathfrak{X}) = 0$ by Proposition 3.7. Because we have just shown that

$$\eta : H^p(\tau_{q+1}\mathfrak{X}) \longrightarrow \varprojlim H^p(\tau^j_{q+1}\mathfrak{X})$$

is an isomorphism, we can conclude that $H^p(\tau_{q+1}\mathfrak{X}) = 0$, as we wanted. ◀

Proposition 3.10. *Suppose $\mathfrak{X}$ is a weakly projective $\mathcal{E}$-bicomodule. There is an isomorphism of bigraded objects $E_\infty \longrightarrow E_0(H)$, so that the spectral sequence converges to H, and, as it collapses at the E_1-page, this gives an isomorphism $E_2^{p,0} \longrightarrow H^p$.*

Proof. We have already shown that $E_2 = E_\infty$. Moreover, as we observed after Proposition 3.8, we have $F^pH^{p+q} = H^{p+q}$ if $q \geqslant 0$ while, from Proposition 3.9, $H^{p+q}(\tau_p\mathfrak{X}) = 0$ when $q < 0$, so that $F^pH^{p+q} = 0$ in this case. This means the only non-trivial filtration quotients are exactly $E_0^{p,0}(H) = H^p$, and that there is an isomorphism

$$E_\infty^{p,0} = E_2^{p,0} \longrightarrow E_0^{p,0}(H)$$

which can be explicitly described as follows. Consider the diagram in Figure 1, built from portions of long exact sequences coming from the split exact sequences

$$0 \longrightarrow \mathfrak{X}(i) \longrightarrow \tau_i\mathfrak{X} \longrightarrow \tau_{i+1}\mathfrak{X} \longrightarrow 0$$

for $i \in \{q-1, q, q+1\}$, and in which the horizontal arrows are the differential d_1 of the E_1-page of our spectral sequence. The maps labelled ι^* in the diagram are injective because the diagonals are exact and there are zeros where indicated, and π^* is surjective by the same

$$
\begin{tikzcd}[column sep=small]
 & & & & & H^q(\mathcal{X}(q-1)) = 0 & & & \\
 & & 0 = H^{q-1}(\tau_q\mathcal{X}) \arrow[dr] & & H^q(\tau_{q-1}\mathcal{X}) \arrow[ur] & & & & \\
 & & & H^q(\tau_q\mathcal{X}) \arrow[ur, "\pi^*"] \arrow[dr, "\iota^*"] & & & & & \\
\cdots \arrow[rr] & & H^{q-1}(\mathcal{X}(q-1)) \arrow[rr, "d_1"] \arrow[ur, "\delta"] & & H^q(\mathcal{X}(q)) \arrow[rr, "d_1"] \arrow[dr, "\delta"] & & H^{q+1}(\mathcal{X}(q+1)) \arrow[rr] & & \cdots \\
 & & & & & H^{q+1}(\tau_{q+1}\mathcal{X}) \arrow[ur, "\iota^*"] & & & \\
 & & & & 0 = H^q(\tau_{q+1}\mathcal{X}) \arrow[ur] & & & &
\end{tikzcd}
$$

Figure 1: The diagram used in the proof of Proposition 3.10.

reason. We now calculate:

$$\begin{aligned}
E_\infty^{p,0} = E_2^{p,0} = \frac{\ker d_1}{\operatorname{im} d_1} = \frac{\ker\delta}{\operatorname{im}\iota^*\delta} &= \frac{\iota^*(H^q(\tau_q\mathcal{X}))}{\iota^*\operatorname{im}\delta} \\
&\simeq \frac{H^q(\tau_q\mathcal{X})}{\operatorname{im}\delta} = \frac{H^q(\tau_q\mathcal{X})}{\ker\pi^*} \\
&\simeq H^q(\tau_{q-1}\mathcal{X}) = E_0^{p,0}(H) \\
&= H^q(\mathcal{X}).
\end{aligned}$$

This is what we wanted. ◀

We can summarize the above in the following theorem.

Theorem 3.1. *If $\mathcal{X}$ is an $\mathcal{E}$-bicomodule, weakly projective in Sp_k, the small complex $S^*(\mathcal{X})$ computes $H^*(\mathcal{X})$.* ◀

A useful corollary of this is what follows.

Corollary 3.3. *If $\mathcal{X}$ is an $\mathcal{E}$-bicomodule over a field of characteristic zero, then for every integer q, the dimension of $H^q(\mathcal{X})$ is bounded above by the multiplicity of the irreducible representation sgn_q in $\mathcal{X}(q)$. In particular, the support of $H^*(\mathcal{X})$ is contained in that of $\mathcal{X}$.* ◀

Observation 3.1. Fix a nonnegative integer q and a linearized species $\mathcal{X}$. It is useful to note that an element $f \in \operatorname{Hom}_{S_q}(\mathcal{X}(q), \operatorname{sgn}_q)$ vanishes on every basis structure $z \in \mathcal{X}(q)$ that is fixed by an odd permutation. This improves the last bound on $\dim_k H^q(\mathcal{X})$ and significantly simplifies computations.

3.4 The small complex

The purpose of this section is to give an explicit formula for the differential of the E_1-page of the spectral sequence, equivalently, for the differential of the combinatorial complex, corresponding to a weakly projective $\mathcal{E}$-bicomodule $\mathcal{X}$. Once this is addressed, we show how to use it to calculate $H^*(\mathcal{X})$ for the species considered in Section 2. Throughout the section, we fix a weakly projective $\mathcal{E}$-bicomodule $\mathcal{X}$.

Lemma 3.3. *The connecting morphism $\delta : H^j(\mathcal{X}[j]) \longrightarrow H^{j+1}(\tau_{j+1}\mathcal{X})$ corresponding to the short exact sequence*

$$0 \longrightarrow \mathcal{X}[j] \longrightarrow \tau_j\mathcal{X} \longrightarrow \tau_{j+1}\mathcal{X} \longrightarrow 0$$

is such that, for a cocycle $\alpha : \mathcal{X}[j] \longrightarrow \overline{\mathcal{E}}^{\otimes j}$, we have $\delta[\![\alpha]\!] = [\![d\tilde{\alpha}]\!]$ where

$$\tilde{\alpha} : \tau_j\mathcal{X} \longrightarrow \overline{\mathcal{E}}^{\otimes j}$$

is the cochain that extends α by zero away from cardinal j. Therefore, the differential of the E_1-page is such that

$$d_1[\![\alpha]\!] = [\![d\tilde{\alpha}\circ\iota]\!],$$

that is, $d_1[\![\alpha]\!]$ is the class of the restriction of $d\tilde{\alpha}$ to $\mathcal{X}(j+1)$.

Proof. One follows the construction of the connecting morphism for the diagram of normalized complexes

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{Hom}_{\mathsf{Sp_k}}(\tau_{j+1}\mathcal{X},\overline{\mathcal{E}}^{\otimes j}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathsf{Sp_k}}(\tau_j\mathcal{X},\overline{\mathcal{E}}^{\otimes j}) \arrow[r, "\iota^*"] \arrow[d, "d"] & \operatorname{Hom}_{\mathsf{Sp_k}}(\mathcal{X}[j],\overline{\mathcal{E}}^{\otimes j}) \arrow[d] \\
\operatorname{Hom}_{\mathsf{Sp_k}}(\tau_{j+1}\mathcal{X},\overline{\mathcal{E}}^{\otimes (j+1)}) \arrow[r, "\pi^*"] \arrow[d] & \operatorname{Hom}_{\mathsf{Sp_k}}(\tau_j\mathcal{X},\overline{\mathcal{E}}^{\otimes (j+1)}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathsf{Sp_k}}(\mathcal{X}[j],\overline{\mathcal{E}}^{\otimes (j+1)}) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

If $\alpha : \mathcal{X}[j] \longrightarrow \mathcal{E}^{\otimes j}$ is a normalized cocycle, and if $\tilde{\alpha} : \tau_j\mathcal{X} \longrightarrow \overline{\mathcal{E}}^{\otimes j}$ extends α by zero then certainly $\iota^*\tilde{\alpha} = \alpha$, and $\tilde{\alpha}$ is normalized, and its restriction to $\mathcal{X}[j]$ is zero. So in fact $d\tilde{\alpha}$ is a cochain

$$d\tilde{\alpha} : \tau_{j+1}\mathcal{X} \longrightarrow \mathcal{E}^{\otimes (j+1)}$$

and it is then its own lift for the map π^*. The lemma follows. ◀

Corollary 3.4. *Suppose that $c \in H^j(\mathcal{X}[j])$ is represented by the class of a normalized cocycle $\alpha : \mathcal{X}[j] \longrightarrow \mathcal{E}^{\otimes j}$. Then $d_1(c) \in H^{j+1}(\mathcal{X}(j+1))$ is represented by the class normalized cocycle $\gamma : \mathcal{X}(j+1) \longrightarrow \mathcal{E}^{\otimes (j+1)}$ such that for a permutation σ of a finite set I of $j+1$ elements and $z \in \mathcal{X}(I)$,*

$$\begin{aligned}\gamma(\sigma)(z) = \alpha(\sigma(2),\ldots,\sigma(j+1))(z \mathbin{/\!\!/} (I\smallsetminus\sigma(1))) \\ + (-1)^{j+1}\alpha(\sigma(1),\ldots,\sigma(j))(z \mathbin{\backslash\!\!\backslash} (I\smallsetminus\sigma(j+1))).\end{aligned}$$

Proof. We compute:

$$d\tilde{\alpha}(\sigma(1),\dots,\sigma(j+1))(z) = \tilde{\alpha}(\sigma(2),\dots,\sigma(j+1))(z /\!\!/ (I \smallsetminus \sigma(1)))$$
$$+\sum_{i=1}^{j}(-1)^i\tilde{\alpha}(\sigma(1),\dots,\sigma(i)\cup\sigma(i+1),\dots,\sigma(j+1))(z)$$
$$+(-1)^{j+1}\tilde{\alpha}(\sigma(1),\dots,\sigma(j))(z \setminus\!\!\setminus (I \smallsetminus \sigma(j+1))).$$

Now $\tilde{\alpha}$ equals α on sets of cardinality j so the first and last summands are those of the statement of the corollary, while the sum vanishes, since $\tilde{\alpha}$ vanishes on sets of cardinality different from j. ◀

We have a commutative diagram

$$\begin{tikzcd} H^p(\mathcal{X}(p)) \arrow[r, "\Psi"] \arrow[d, "d_1"] & S^p(\mathcal{X}) \arrow[d, "\partial"] \\ H^{p+1}(\mathcal{X}(p+1)) \arrow[r, "\Psi"'] & S^{p+1}(\mathcal{X}) \end{tikzcd}$$

and we have already identified d_1. We now carefully follow the horizontal isomorphisms to obtain the formula for the differential ∂ of the combinatorial complex. The following notation will be useful.

Definition 3.1. If $j \in [p+1]$, let λ_j be the unique order preserving bijection

$$[p+1] \smallsetminus j \longrightarrow [p],$$

and, given a permutation $\sigma \in S_{p+1}$, we write $\sigma \smallsetminus \sigma(j)$ for the permutation $\lambda_{\sigma(j)}\sigma\lambda_j^{-1}$ in S_p. In simple terms, this permutation is obtained by applying $\lambda_{\sigma j}$ to numbers of the list $\sigma 1 \cdots \hat{\sigma(j)} \cdots \sigma(p+1)$.

Lemma 3.4. *With the notation above,*

1. *the sign of* $\sigma \smallsetminus \sigma(1)$ *is* $(-1)^{\sigma-\sigma(1)-1}$, *and*
2. *the sign of* $\sigma \smallsetminus \sigma(p+1)$ *is* $(-1)^{\sigma+p+1-\sigma(p+1)}$.

Proof. We may obtain the sign of a permutation by counting inversions, that is, if m is the number of inversions in σ, then the sign of σ is $(-1)^m$. By deleting the first number $\sigma(1)$ in σ, we lose $\sigma(1)-1$ inversions coming from those numbers smaller than $\sigma(1)$, and by deleting the last number in σ, we lose $p+1-\sigma(p+1)$ invesions, coming from those numbers larger than $\sigma(p+1)$. ◀

Definition 3.2. Fix a finite set I and a structure $z \in \mathcal{X}(I)$. The *left deck* of z is the set $\mathrm{ldk}(z) = \{z \mathbin{\backslash\!\backslash} (I \smallsetminus i) : i \in I\}$, while the *right deck* of z is the set $\mathrm{rdk}(z) = \{z \mathbin{/\!/} (I \smallsetminus i) : i \in I\}$. If $z \in \mathcal{X}(p)$ and $j \in [p]$, we will write $z'_j \in \mathcal{X}(p-1)$ for $\lambda_j(z \mathbin{\backslash\!\backslash} ([p] \smallsetminus j))$ and $z''_j \in \mathcal{X}(p-1)$ for $\lambda_j(z \mathbin{/\!/} ([p] \smallsetminus j))$.

We now assume k is a field of characteristic zero. With this at hand, we have the following computational result:

Theorem 3.2. *The differential of the small complex $S^*(\mathcal{X})$ is such that if*

$$f : \mathcal{X}(p) \longrightarrow \mathrm{sgn}_p$$

is S_p-equivariant, then $df : \mathcal{X}(p+1) \longrightarrow \mathrm{sgn}_{p+1}$ is the S_{p+1}-equivariant map so that for every $z \in \mathcal{X}(p)$,

$$df(z) = \sum_{j=1}^{p+1} (-1)^{j-1} \Big(f(z'_j) - f(z''_j) \Big).$$

It follows that if $\mathcal{X}$ is a linearization $k\mathcal{X}_0$, the value of $df(z)$ for $f \in S^p(\mathcal{X})$ and an element $z \in \mathcal{X}_0(p+1)$ depends only on the left and right decks of z, and that this data is degree-wise finite if $\mathcal{X}$ is of finite type.

Proof. Fix $f \in S^p(\mathcal{X})$. Following the correspondence described in Lemma 3.2, the normalized cochain $\alpha : \mathcal{X}(p) \longrightarrow \mathcal{E}^{\otimes p}$ representing f is such that $\alpha(\sigma)(z) = \frac{(-1)^\sigma}{p!} f(z)$ for each $\sigma \in S_p$ and each $z \in \mathcal{X}(p)$. By Lemma 3.3 and its corollary, the differential of α is represented by the cochain $\gamma : \mathcal{X}(p+1) \longrightarrow \mathcal{E}^{\otimes (p+1)}$ such that for $z \in \mathcal{X}(p+1)$ and $\sigma \in S_{p+1}$,

$$\gamma(\sigma)(z) = \alpha(\sigma - \sigma(1))(z'_{\sigma,1}) + (-1)^{p+1} \alpha(\sigma - \sigma(p+1))(z''_{\sigma,p+1}).$$

For brevity, we are writing $z'_{\sigma,i}$ for $z \mathbin{/\!/} ([p+1] \smallsetminus \sigma(i))$ and $z''_{\sigma,i}$ for $z \mathbin{\backslash\!\backslash} ([p+1] \smallsetminus \sigma(i))$. We are also writing $F - F_t$ to denote the composition obtained from F by deleting block F_t. Going back to $S^{p+1}(\mathcal{X})$ via Proposition 3.6, we obtain that

$$df(z) = \frac{1}{p!} \sum_{\sigma \in S_{p+1}} (-1)^\sigma \Big(\alpha(\sigma - \sigma(1))(z'_{\sigma,1}) + (-1)^{p+1} \alpha(\sigma - \sigma(p+1))(z''_{\sigma,p+1}) \Big)$$

and we now split the sum according to the value of $\sigma(1)$ and $\sigma(p+1)$ as follows. If $\sigma(1) = j$, then $z'_{\sigma,1} \in \mathcal{X}([p+1] \smallsetminus j)$, so we may transport this to $[p]$ by means of λ_j: using the notation previous to the statement of the theorem, we have

$$\alpha(\sigma - \sigma(1))(z'_{\sigma,1}) = \alpha(\lambda_j(\sigma - \sigma(1)))(z'_j).$$

Now the sign of the permutation corresponding to the composition $\lambda_j(\sigma - \sigma(1))$, which corresponds to the permutation $\sigma \smallsetminus \sigma(1)$, is $(-1)^{\sigma-(j-1)}$ by Lemma 3.4, so that

$$\alpha(\sigma-\sigma(1))(z'_{\sigma,1}) = (-1)^{\sigma-(j-1)} f(z'_j).$$

Because there are $p!$ permutations σ such that $\sigma(1)=j$ for each $j \in [p+1]$, we deduce that

$$\begin{aligned}\frac{1}{p!}\sum_{\sigma\in S_{p+1}}(-1)^{\sigma}\alpha(\sigma-\sigma(1))(z'_{\sigma,1}) &= \frac{p!}{p!}\sum_{j=1}^{p+1}(-1)^{\sigma+\sigma-(j-1)}f(z'_j)\\ &= \sum_{j=1}^{p+1}(-1)^{j-1}f(z'_j)\end{aligned}$$

and this gives the first half of the formula. The second half is completely analogous: the sign $(-1)^{\sigma+p+1}$ partially cancels with $(-1)^{\sigma+p+1-j}$ where $j=\sigma(p+1)$ and we obtain the chain of equalities:

$$\begin{aligned}\frac{1}{p!}\sum_{\sigma\in S_{p+1}}(-1)^{\sigma}(-1)^{p+1}\alpha(\sigma-\sigma(p+1))(z''_{\sigma,p+1}) &= \frac{p!}{p!}\sum_{j=1}^{p+1}(-1)^{j}f(z''_j)\\ &= -\sum_{j=1}^{p+1}(-1)^{j-1}f(z''_j).\end{aligned}$$

This completes the proof of the theorem. ◀

As a consequence of this last theorem, we obtain the following immediate corollaries, which address the structure of the differential of the combinatorial complex for bicomodules that are symmetric or trivial to one side. There is an analogous statement for for bicomodules with trivial right structure, and we denote the corresponding differential by d''.

Corollary 3.5. *For every symmetric bicomodule $\mathcal{X}$ and every nonnegative integer q there is an isomorphism $H^q(\mathcal{X}) \simeq \operatorname{Hom}_{S_q}(\mathcal{X}(q),\operatorname{sgn}_q)$. On the other hand, if $\mathcal{X}$ has a trivial left structure, then the differential in $S^*(\mathcal{X})$ is such that for every functional $f \in S^p(\mathcal{X})$, we have $d'f(z) = \sum_{j=1}^{p+1}(-1)^{j-1}f(z'_j)$.* ◀

3.5 Some computations

To illustrate the use of the combinatorial complex we compute the cohomology groups of some of the twisted coalgebras introduced in Section 4.1 and, in doing so try to convince the reader of the usefulness of the results of this section.

To begin with, we include a new computation that is greatly simplified with the use of the small complex. We remark that, as far as the author knows, the only computation of such cohomology groups that was known previously before the methods in this paper were introduced, are $H^*(\mathcal{E})$ and the first two cohomology groups of $H^*(\mathcal{L})$.

The species of singletons and suspension. Define the species s of singletons so that for every finite set I, $s(I)$ is trivial whenever I is not a singleton, and is k-free with basis I if I is a singleton. By Lemma 3.1, the species s admits unique right and left $\mathcal{E}$-comodule structures, and thus a unique $\mathcal{E}$-bicomodule structure. By induction, it is easy to check that, for each integer $q \geqslant 1$, the species $s^{\otimes q}$, which we write more simply by s^q, is such that $s^q(I)$ is k-free of dimension $q!$ if I has q elements with basis the linear orders on I, and the action of the symmetric group on I is the regular representation, while $s^q(I)$ is trivial in any other case. By convention, set $s^0 = \mathbb{1}$, the unit species. It follows that the sequence of species $\mathfrak{S} = (s^n)_{n\geqslant 0}$ consists of weakly projective species, and we can completely describe their cohomology groups. They are the analogues of spheres for species, its first property consisting of having cohomology concentrated in the right dimension:

Proposition 3.11. *For each integer $n \geqslant 0$, the species s^n has*

$$H^q(s^n) = \begin{cases} k & \text{if } q = n\,, \\ 0 & \text{else.} \end{cases}$$

Proof. Fix $n \geqslant 0$. By the remarks preceding the proposition, it follows that $S^q(s^n)$ always vanishes except when $q = n$, where it equals $\operatorname{Hom}_{S_n}(kS_n, \operatorname{sgn}_n)$, and this is one dimensional. Because each s^n is weakly projective, $S^*(s^n)$ calculates $H^*(s^n)$, and the claim follows. ◀

The above motivates us to check whether $s \otimes -$ acts as a suspension for $H^*(-)$. Assume that $\mathcal{X}$ is weakly projective, so we may use $S^*(\mathcal{X})$ to compute $H^*(\mathcal{X})$. We claim that $S^*(s\mathcal{X})$ identifies with $S^*(\mathcal{X})[-1]$. Indeed, for this it suffices to note, first, that $(s\mathcal{X})(n)$ is isomorphic, as an kS_n-module, to the induced representation $k \otimes \mathcal{X}(n-1)$ from the inclusion $S_1 \times S_{n-1} \hookrightarrow S_n$, and second, that the restriction of the sign representation of S_n under this inclusion is the sign representation of S_{n-1}, so that:

$$\begin{aligned} \operatorname{Hom}_{S_n}((s\mathcal{X})(n), \operatorname{sgn}_n) &= \operatorname{Hom}_{S_n}(\operatorname{Ind}_{S_1 \times S_{n-1}}^{S_n}(k \otimes \mathcal{X}(n-1)), \operatorname{sgn}_n)) \\ &\simeq \operatorname{Hom}_{S_1 \times S_{n-1}}(k \otimes \mathcal{X}(n-1), \operatorname{Res}_{S_1 \times S_{n-1}}^{S_n} \operatorname{sgn}_n) \\ &\simeq \operatorname{Hom}_{S_{n-1}}(\mathcal{X}(n-1), \operatorname{sgn}_{n-1}). \end{aligned}$$

A bit more of a calculation shows the differentials are the correct ones. By induction, of course, we obtain that $s^j\mathcal{X}$ has the cohomology of $\mathcal{X}$, only moved j places up.

Proposition 3.12. *Assume $\mathcal{X}$ is weakly projective. For each j, the suspension $s^j\mathcal{X}$ is also* *weakly projective, and there is a natural suspension isomorphism*

$$H^*(s^j\mathcal{X}) \longrightarrow s^j H^*(\mathcal{X})$$

in cohomology groups. ◀

The exponential species. Every structure on a set of cardinal larger than 1 over the exponential species $\mathcal{E}$ is fixed by an odd permutation: if I is a finite set with more than one element, there is a transposition $I \longrightarrow I$, and it fixes $*_I$. It follows that $S^q(\mathcal{E})$ is zero for $q > 1$, and it is immediate that $S^0(\mathcal{E})$ and $S^1(\mathcal{E})$ are one dimensional, while we already know $d = 0$. Thus $H^q(\mathcal{E},\mathcal{E})$ is zero for $q > 1$ and is isomorphic to k for $q \in \{0,1\}$. The cup product is then completely determined. This is in line with the computations done in the thesis [8] of J. Coppola:

Proposition 3.13 (J. Coppola)**.** *The cohomology algebra of $\mathcal{E}$ is isomorphic to an exterior algebra $k[s]/(s^2)$ in one generator.* ◀

The species of linear orders. Recall the species of linear orders $\mathcal{L}_0$ from Section 3; we endowed its linearization $\mathcal{L}$ with the $\mathcal{E}$-bicomodule structure obtained by restricting a linear order to a subset. The kS_j-module $\mathcal{L}(j)$ is free of rank one for every $j \geqslant 0$, because S_j acts freely and transitively on the set $\mathcal{L}(j)$. It follows that the k-module $\operatorname{Hom}_{S_j}(\mathcal{L}(j), \operatorname{sgn}_j)$ is free of rank one and, by virtue of Theorem 3.2, the computation ends here: the differential on this combinatorial complex is identically zero. We thus deduce that:

Proposition 3.14. *For every integer $j \geqslant 0$ the k-module $H^j(\mathcal{L})$ is free of rank one.* ◀

We will address the multiplicative structure of $H^*(\mathcal{L})$ below.

The species of partitions. The species of partitions $\mathcal{P}$ assigns to each finite set I the collection $\mathcal{P}(I)$ of partitions $\mathcal{X}$ of I, that is, families $\{X_1,\dots,X_t\}$ of disjoint non-empty subsets of I whose union is I. There is a left $\mathcal{E}$-comodule structure on $\mathcal{P}$ defined as follows: if $\mathcal{X}$ is a partition of I and $S \subset I$, $\mathcal{X} \setminus\!\!\setminus S$ is the partition of S obtained from the non-empty blocks of $\{x \cap S : x \in \mathcal{X}\}$. We already noted there is an inclusion $\mathcal{E} \longrightarrow \mathcal{P}$.

Proposition 3.15. *The cohomology group $H^0(\mathcal{P})$ is free of rank one, and $H^1(\mathcal{P})$ is free of rank one generated by the cardinality cocycle. In fact, the inclusion $\mathcal{E} \longrightarrow \mathcal{P}$ induces an isomorphism of commutative algebras $S^*(\mathcal{P}) \longrightarrow S^*(\mathcal{E})$.*

Proof. A partition of a set with at least two elements is fixed by a transposition, and this implies, in view of Observation 3.1, that $S^j(\mathcal{P}) = 0$ for $j \geqslant 2$. On the other hand, $S^0(\mathcal{P})$ and $S^1(\mathcal{P})$ are both k-free of rank one, and we already know from Proposition 3.2 that the differential of $S^*(\mathcal{P})$ is zero. This proves both claims. ◀

The species of compositions. The species of compositions $\mathcal{C}$ is the non-abelian analogue of the species of partitions $\mathcal{P}$. Let us recall its construction: the species of compositions $\mathcal{C}$ assigns to each finite set I the set $\mathcal{C}(I)$ of compositions of I, that is, ordered tuples $(F_1,\dots,F_t)$ of disjoint non-empty subsets of I whose union is I. This has a standard left $\mathcal{E}$-comodule structure such that if $F=(F_1,\dots,F_t)$ is a composition of I and $S\subseteq I$, $F\mathbin{\backslash\!\backslash} S$ is the composition of S obtained from the tuple $(F_1\cap S,\dots,F_t\cap S)$ by deleting empty blocks. We view $\mathcal{C}$ as an $\mathcal{E}$-bicomodule with its cosymmetric structure.

Proposition 3.16. *The morphism $\mathcal{L}\longrightarrow\mathcal{C}$ induces an isomorphism $H^*(\mathcal{C})\longrightarrow H^*(\mathcal{L})$ and, in fact, an isomorphism of commutative algebras $S^*(\mathcal{C})\longrightarrow S^*(\mathcal{L})$.*

Proof. It suffices that we prove the second claim, and, since $S^*(-)$ is a functor, that for a fixed integer q, the map $S^q(\mathcal{C})\longrightarrow S^q(\mathcal{L})$ is an isomorphism of modules. This follows from Observation 3.1: a decomposition F of a set I is fixed by a transposition as soon as it has a block with at least two elements, and therefore an element of $S^q(\mathcal{C})$ vanishes on every composition of $[q]$, except possibly on those into singletons. Thus the surjective map $S^*(\mathcal{C})\longrightarrow S^*(\mathcal{L})$ is injective and it is thus an isomorphism of commutative algebras. ◀

The species of graphs. We have already defined the species Gr of graphs along with its $\mathcal{E}$-bicomodule structure obtained by restriction. We have the following result concerning the cohomology groups of Gr:

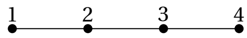
Theorem 3.3. *If k is of characteristic zero then, for each non-negative integer $p\geqslant 0$, $\dim_k H^p(\mathsf{Gr})$ equals the number of isomorphism classes of graphs on p vertices with no odd automorphisms, namely,*

$$1,\quad 1,\quad 0,\quad 0,\quad 1,\quad 6,\quad 28,\quad 252,\quad 4726,\quad 150324,\quad \dots$$

This sequence is [20, A281003].

Proof. Since the structure on Gr is cosymmetric, the differential of $S^*(\mathsf{Gr})$ vanishes, and Observation 3.1 tells us $S^q(\mathsf{Gr})$ has dimension as in the statement of the proposition. The tabulation of the isomorphism classes of graphs in low cardinalities can be done with the aid of a computer —we refer to Brendan McKay's calculation [18] for the final result— and then filter out those graphs with odd automorphisms. ◀

We can exhibit cocycles whose cohomology classes generate $H^1(\mathsf{Gr})$ and $H^4(\mathsf{Gr})$: in degree one, we have the cardinality cocyle κ, and in degree four, the normalized cochain $p^4:\mathsf{Gr}\longrightarrow\mathcal{E}^{\otimes 4}$ such that for a decomposition $F\vdash_4 I$, and a graph g with vertices on I, $p^4(F_1,F_2,F_3,F_4)(g)$ is the number of inclusions $\zeta: p_4\longrightarrow g$, where p_4 is the graph



and $\zeta(i) \in F_i$ for $i \in [4]$. One can check this cochain is in fact a cocycle, and it is normalized by construction.

3.6 Multiplicative matters

We now describe how to exploit the small complex to deduce a Künneth theorem for the Cauchy product. Later, we will address the multiplcative structure of the spectral sequence.

Proposition 3.17. *For each such $p, q \in \mathbb{N}$ there is an isomorphism*

$$\phi^p : \mathrm{Hom}_{S_p \times S_q}(\mathcal{X}[p] \otimes \mathcal{Y}[q], \mathrm{sgn}_p \otimes \mathrm{sgn}_q) \longrightarrow \mathrm{Hom}_{S_{p+q}}((\mathcal{X} \otimes \mathcal{Y}^p[p+q], \mathrm{sgn}_{p+q})$$

where $(\mathcal{X} \otimes \mathcal{Y})^p[p+q]$ is the space of summands $\mathcal{X}[S] \otimes \mathcal{Y}[T]$ with S of cardinality p.

Proof. This is readily described as follows. For each decomposition (S,T) of n, let $u = u_{S,T}$ be the unique bijection that assigns S to $[p]$ and T to $[p+1, p+q]$ in a monotone fashion, and given an element $f \in \mathrm{Hom}_{p,q}(\mathcal{X}[p] \otimes \mathcal{Y}[q], \mathrm{sgn}_p \otimes \mathrm{sgn}_q)$, set

$$\phi^p(f)(z \otimes w) = (-1)^{\mathrm{sch}(S,T)} f(z' \otimes w')$$

where $uz = z'$, $uw = w'$ and $z \otimes w \in \mathcal{X}(S) \otimes \mathcal{Y}(T)$. We claim this is S_{p+q}-equivariant. Note that the sign of u is $\mathrm{sch}(S,T)$. Indeed, if τ is a permutation of n and (S,T) is a decomposition of n, we can write $\tau = \xi\rho$ where $\rho = \tau_1 \times \tau_2$ is a shuffle of (S,T) and ξ is monotone over S and over T. It is clear that if (S',T') is the image of (S,T) under τ and if $u' = u'_{S',T'}$ then $u = u'\xi$. Moreover, note that $u(\tau_1 z \otimes \tau_2 w)$ is transported to $u(z \otimes w)$ by $u\rho^{-1}u^{-1}$, which belongs to $S_p \times S_q$, and we now compute

$$\begin{aligned}
(-1)^\tau \phi^p(f)(\tau(z \otimes w)) &= (-1)^{\tau + u'} f(u'\tau(z \otimes w)) && (6)\\
&= (-1)^{\tau + u'} f(u(\tau_1 z \otimes \tau_2 w)) && (7)\\
&= (-1)^{\tau + u' + \rho} f(u(z \otimes w)) && (8)\\
&= (-1)^{\tau + u' + \rho + u} \phi^p(f)(z \otimes w) && (9)\\
&= \phi^p(f)(z \otimes w) && (10)
\end{aligned}$$

where the signs cancel by virtue of the identities $\xi\rho = \tau$ and $u'\xi = u$. ◄

For each $p, q \in \mathbb{N}$ there are canonical maps

$$\mathrm{Hom}_{S_p}(\mathcal{X}[p], \mathrm{sgn}_p) \otimes \mathrm{Hom}_{S_q}(\mathcal{Y}[q], \mathrm{sgn}_q) \longrightarrow \mathrm{Hom}_{S_{p,q}}(\mathcal{X}[p] \otimes \mathcal{Y}[q], \mathrm{sgn}_p \otimes \mathrm{sgn}_q)$$

that are all isomorphisms if k is a field and $\mathcal{X}$ or $\mathcal{Y}$ is finite in each arity, and they collect along with the maps ϕ to define a map

$$- \times - : S^*(\mathcal{X}) \otimes S^*(\mathcal{Y}) \longrightarrow S^*(\mathcal{X} \otimes \mathcal{Y}).$$

Explicitly, given maps $f_p : \mathcal{X}(p) \longrightarrow \mathrm{sgn}_p$ and $g_q : \mathcal{Y}(q) \longrightarrow \mathrm{sgn}_q$, we have for each decomposition (S, T) and $z \otimes w \in \mathcal{X}(S) \otimes \mathcal{Y}(T)$

$$(f_p \vee g_q)(z \otimes w) = (-1)^{\mathrm{sch}(S,T)} f_p(u_S(z)) \otimes g_q(u_T(w)),$$

where $u = u_{S,T}$. We obtain now the main result of this section, a Künneth formula that allows us to compute the cohomology groups of a product in terms of its factors.

Theorem 3.4 (Künneth formula)**.** *Suppose that k is a field of characteristic zero and $\mathcal{X}$ or $\mathcal{Y}$ is locally finite. The map $- \times - : S^*(\mathcal{X}) \otimes S^*(\mathcal{Y}) \to S^*(\mathcal{X} \otimes \mathcal{Y})$ is an isomorphism of complexes.*

Proof. The only detail to check is that this is a morphism of complexes, since we have already observed it is an equivariant bijection. To see this, we observe that following the definition reveals that this map is exactly the map induced by the external product

$$- \times - : C^*(\mathcal{X}) \otimes C^*(\mathcal{Y}) \longrightarrow C^*(\mathcal{X} \otimes \mathcal{Y})$$

on the E_1-pages of the corresponding spectral sequences for the complexes $C^*(\mathcal{X} \otimes \mathcal{Y})$ and $C^*(\mathcal{X}) \otimes C^*(\mathcal{Y})$, which is what we wanted. ◀

The reader can find details for the computation suggested in the last proof in the next section, where we consider the case of the (interior) cup product $\smile$.

Multiplicative structure of the spectral sequence

We have defined a complete descending filtration

$$S^*(\mathcal{X}) \supseteq F^0 C^*(\mathcal{X}) \supseteq \cdots \supseteq F^p C^*(\mathcal{X}) \supseteq F^{p+1} S^*(\mathcal{X}) \supseteq \cdots$$

on $S^*(\mathcal{X})$ where $F^p C^*(\mathcal{X})$ consists of those cochains that vanish on $\tau^p \mathcal{X}$. Assume now that $\mathcal{X}$ is a linearized coalgebra of the form $k\mathcal{X}_0$, so that there is a cup product defined on $S^*(\mathcal{X})$, as detailed in Subsection 2.2. Remark that the proof of the following proposition adapts immediately to any cup product on $S^*(\mathcal{X})$ induced from a diagonal map $\mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$.

Proposition 3.18. *The cup product on $S^*(\mathcal{X})$ is compatible with the filtration, in the sense that, for every two non-negative integers p and p', we have that $F^p \smile F^{p'} \subseteq F^{p+p'}$.*

Proof. Consider cochains $\alpha \in F^p$ and $\beta \in F^{p'}$. Then $\alpha \smile \beta \in F^{p+p'}$ by a pigeonhole argument: if $F = (F', F'')$ is a decomposition of a finite set with $p + p'$ elements, then F' is a decomposition of a set with at most p elements or F'' is a decomposition of a set with at most p' elements, and the formula

$$(\alpha \smile \beta)(F)(z) = \alpha(F')(z \mathbin{\backslash\!\backslash} F')\beta(F'')(z \mathbin{/\!/} F'')$$

then makes it evident that $\alpha \smile \beta$ is an element of $F^{p+p'}$. ◀

It follows from this proposition that the cup product descends to a product

$$\frac{F^pC}{F^{p+1}C} \otimes \frac{F^{p'}C}{F^{p'+1}C} \longrightarrow \frac{F^{p+p'}C}{F^{p+p'+1}C}$$

so we obtain a multiplicative structure $- \smile - : E_0^{pq} \times E_0^{p'q'} \longrightarrow E_0^{p+p',q+q'}$ induced on the E_0-page of the spectral sequence. This induces in turn a multiplicative structure on our spectral sequence $(E_r, d_r)_{r\geqslant 0}$. Because this spectral sequence degenerates at E^2, we can compute the cup product in $H^*(\mathcal{X})$ from the combinatorial complex $S^*(\mathcal{X})$. We describe how to do so in explicit terms.

If S is a subset of $[n] = \{1,\dots,n\}$ with $m \leqslant n$ elements, and if σ is a permutation of S, we regard σ as a permutation of $[m]$ by means of the unique order preserving bijection $\lambda_S : S \longrightarrow [m]$. We say (σ^1, σ^2) is a (p,q)*-shuffle* of a finite set I with $p+q$ elements whenever σ^1 is a permutation of a p-subset S of I, σ^2 is a permutation of a q-subset T of I, and $S \cap T = \varnothing$. Call (S,T) the *associated composition* of such a shuffle. If (S,T) is a composition of $[n]$, we will write $\operatorname{sch}(S,T)$ for the *Schubert statistic* of (S,T), which counts the number of pairs $(s,t) \in S \times T$ such that $s < t$ according to the canonical ordering of $[n]$. Our result is the following

Theorem 3.5. *The cup product induced by the diagonal* $\mathcal{X} \longrightarrow \mathcal{X} \otimes \mathcal{X}$

$$- \smile - : S^p(\mathcal{X}) \otimes C^q(\mathcal{X}) \longrightarrow S^{p+q}(\mathcal{X})$$

is such that for equivariant maps $f : \mathcal{X}(p) \longrightarrow \operatorname{sgn}_p$ *and* $g : \mathcal{X}(q) \longrightarrow \operatorname{sgn}_q$*, and* $z \in \mathcal{X}(p+q)$*,*

$$(f \smile g)(z) = \sum_{(S,T)\vdash[p+q]} (-1)^{\operatorname{sch}(S,T)} f(\lambda_S(z \mathbin{\backslash\!\backslash} S))g(\lambda_T(z \mathbin{/\!/} T))$$

where the sum runs through decompositions of $[p+q]$ *with* $\#S = p$ *and* $\#T = q$*.*

Before giving the proof, we begin with a few preliminary considerations. First, consider a (p,q)-shuffle (σ^1, σ^2) of $[p+q]$, with associated composition (S,T), and let σ be the permutation of $[p+q]$ obtained by concatenating σ^1 and σ^2.

Lemma 3.5. *For any $\sigma \in S_{p+q}$ and any (p,q)-composition (S,T) of $[p+q]$,*

1. *the sign of σ is $(-1)^{\sigma^1+\sigma^2+\mathrm{sch}(S,T)}$, and*
2. $(-1)^{\mathrm{sch}(S,T)} = (-1)^{\mathrm{sch}(T,S)+pq}$.

Proof. Indeed, by counting inversions, it follows that the number of inversions in σ is precisely $\mathrm{inv}\,\sigma^1 + \mathrm{inv}\,\sigma^2 + \mathrm{sch}(S,T)$, which proves the first assertion. The second claims follows from the first and the fact $\sigma^1\sigma^2$ and $\sigma^2\sigma^1$ differ by pq transpositions. ◀

Recall that if $\alpha : \mathcal{X}(p) \longrightarrow \mathcal{E}^{\otimes p}$ is a cochain, we associate to it the equivariant map $f : \mathcal{X}(p) \longrightarrow \mathrm{sgn}_p$ such that $f(z) = \alpha(\nu_p)(z)$ where $\nu_p = \sum_{\sigma\in S_p}(-1)^\sigma\sigma$ is the antisymmetrization element. Conversely, given such an equivariant map, we associate to it the cochain $\alpha : \mathcal{X}(p) \longrightarrow \mathcal{E}^{\otimes p}$ such that $\alpha(\sigma)(z) = \frac{(-1)^\sigma}{p!} f(z)$. We now proceed to the proof of Theorem 3.5.

Proof. To calculate a representative of the class of $f \smile g$, we lift first lift the maps $f : \mathcal{X}(p) \longrightarrow \mathrm{sgn}_p$ and $g : \mathcal{X}(q) \longrightarrow \mathrm{sgn}_q$ to cochains $\alpha : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes p}, \beta : \mathcal{X} \longrightarrow \mathcal{E}^{\otimes q}$ that are supported in $\mathcal{X}(p)$ and $\mathcal{X}(q)$ respectively, and represent f and g according to the correspondence in the previous paragraph. We compute for any decomposition (F^1, F^2) of a finite set I and any $z \in \mathcal{X}(I)$ that

$$(\alpha \smile \beta)(F^1,F^2)(z) = \alpha(F^1)(z \mathbin{\backslash\!\backslash} F^1)\beta(F^2)(z \mathbin{/\!/} F^2).$$

Now consider $z \in \mathcal{X}(p+q)$. If σ is a permutation of $[p+q]$, write (σ^1,σ^2) for the (p,q)-shuffle obtained by reading $\sigma(1)\cdots\sigma(p)$ as a permutation of $S_\sigma = \{\sigma(1),\ldots,\sigma(p)\}$ and by reading $\sigma(p+1)\cdots\sigma(p+q)$ as a permutation of $T_\sigma = \{\sigma(p+1),\ldots,\sigma(p+q)\}$. Then

$$\begin{aligned}(f \smile g)(z) &= \sum_{\sigma\in S_{p+q}} (-1)^\sigma (\alpha\smile\beta)(\sigma)(z) \\ &= \sum_{\sigma\in S_{p+q}} (-1)^\sigma \alpha(\sigma^1)(z \mathbin{\backslash\!\backslash} S_\sigma)\beta(\sigma^2)(z \mathbin{/\!/} T_\sigma)\end{aligned}$$

Fix a composition (S,T) of $[p+q]$. In the sum above, the permutations σ with $(S_\sigma, T_\sigma) = (S,T)$ are the (p,q)-shuffles with associated composition (S,T). We may then replace the sum throughout S_{p+q} with the sum throughout (p,q)-compositions (S,T) of $[p+q]$ and in turn with the sum throughout shuffles (σ^1,σ^2) of (S,T). This reads

$$(f \smile g)(z) = \sum_{(S,T)\vdash[p+q]} \sum_{(\sigma^1,\sigma^2)} (-1)^{\sigma^1\sigma^2} \alpha(\sigma^1)(z \mathbin{\backslash\!\backslash} S)\beta(\sigma^2)(z \mathbin{/\!/} T).$$

We now note that $\alpha(\sigma^1)(z \mathbin{\backslash\!\backslash} S) = \alpha(\lambda_S(\sigma^1))(\lambda_S(z \mathbin{\backslash\!\backslash} S))$, that the sign of $\lambda_S(\sigma^1) \in S_p$ is $(-1)^{\sigma^1}$, and that the same considerations apply to β, so we obtain that

$$(f \smile g)(z) = \frac{1}{p!q!} \sum_{(S,T)\vdash[p+q]} \sum_{(\sigma^1,\sigma^2)} (-1)^{\sigma^1\sigma^2+\sigma^1+\sigma^2} f(\lambda_S(z \setminus\!\!\setminus S))g(\lambda_T(z /\!\!/ T)).$$

Using Lemma 3.5 finishes the proof: the sum $\sum_{(\sigma^1,\sigma^2)}(-1)^{\sigma^1\sigma^2+\sigma^1+\sigma^2}$ consists of $p!q!$ instances of $(-1)^{\mathrm{sch}(S,T)}$. ◀

Suppose now that $\mathcal{X}$ is a symmetric $\mathcal{E}$-bicomodule. Then Theorem 3.2 proves the differential in $S^*(\mathcal{X})$ is trivial, while Lemma 3.5 along with Proposition 3.5 prove that the cup product in $S^*(\mathcal{X})$ is graded commutative. We obtain the

Theorem 3.6. *Suppose that $\mathcal{X}$ is a cosymmetric $\mathcal{E}$-bicomodule. Then $S^*(\mathcal{X})$ is isomorphic to the cohomology algebra $H = H^*(\mathcal{X})$ via the isomorphism of algebras $E_2 \longrightarrow E_0(H)$. In particular, $H^*(\mathcal{X})$ is graded commutative.* ◀

To illustrate, take $\mathcal{X}$ to be the species of linear orders. Each $S^j(\mathcal{X})$ is one dimensional generated by the map $f_j : L(j) \longrightarrow k$ that assigns $\sigma \longmapsto (-1)^\sigma$. A calculation, which we omit, shows

Proposition 3.19. *The algebra $S^*(\mathcal{L})$ is generated by the elements f_1 and f_2, so that if f_p is the generator of $S^p(\mathcal{L})$, we have*

$$f_{2p} \smile f_{2q} = \binom{p+q}{p} f_{2(p+q)}, \qquad f_1 \smile f_{2p} = f_{2p+1}, \qquad f_1 \smile f_{2p+1} = 0.$$

These relations exhibit $H^*(\mathcal{L})$ as a tensor product of a divided power algebra and an exterior algebra. ◀

For a second example, consider Gr with its cosymmetric $\mathcal{E}$-bicomodule structure. We already know H^4 is one dimensional, and the functional $p^4 : \mathsf{Gr}(4) \longrightarrow \mathrm{sgn}_4$ that assigns the 4-path to 1 and every other graph on four vertices to zero is a generator of S^4. Even more can be said: our formula for the cup product and induction shows that for each $n \geqslant 1$, the product f^n is nonzero on the graph that is the disjoint union of n paths p_4, so that H^{4n} is always nonvanishing for $n \geqslant 1$. Hence the cohomology algebra $H^*(\mathsf{Gr})$ contains both an exterior algebra in degree 1 and a polynomial algebra in degree 4.

4 Coda

4.1 The spectral sequence of a connected bialgebra

One can extend the work done in the first two sections of Section 3 by replacing $\mathcal{E}$ with any linearized connected twisted Hopf algebra $\mathcal{H}$ along the following lines. Let $\mathcal{X}$ be an $\mathcal{H}$-bicomodule. The filtration $F^pC^*(\mathcal{X},\mathcal{H})$ of $C^*(\mathcal{X},\mathcal{H})$ by the subcomplexes

$$\{\alpha:\mathcal{X}\longrightarrow\mathcal{H}^{\otimes *}:\alpha \text{ vanishes on } \tau^{p-1}\mathcal{X}\}$$

is natural with respect to $\mathcal{H}$, and it is complete and bounded above. This yields a spectral sequence starting at $E_0^{p,*}=C^{p+*}(\mathcal{X}(p),\mathcal{H})$ with first page concentrated in a cone in the fourth quadrant, and to prove this is convergent in the sense of [22], it suffices we show this spectral sequence is regular. It should be possible to prove the filtration giving rise to such spectral sequence is regular, which is equivalent to the statement that $H^p(\tau_j\mathcal{X},\mathcal{H})=0$ for large values of j. This is trivially true if $\mathcal{X}$ is of finite length because in such case $\tau_j\mathcal{X}=0$ for large values of j. Independent of convergence matters, we can identify its first page. Indeed, for each natural number q, write $\langle\mathcal{H};q\rangle$[1] for the cosimplicial k-module

$$0\longrightarrow\mathcal{H}^{\otimes 0}([p])\longrightarrow\mathcal{H}^{\otimes 1}([p])\longrightarrow\cdots\longrightarrow\mathcal{H}^{\otimes j}([p])\longrightarrow\cdots$$

with coface maps and codegeneracies induced by Δ and ε, and write $\overline{\langle\mathcal{H};p\rangle}$ for the corresponding normalized complex of $\langle\mathcal{H};p\rangle$. Often we can find a topological space $\langle\mathscr{H};p\rangle$ whose cohomology coincides with that of $\langle\mathcal{H};p\rangle$. Denote by $\mathscr{H}^{p,q}$ the cohomology groups $H^{p+q}(\langle\mathcal{H};p\rangle)$, which are all S_p-modules. If $\mathcal{X}$ is weakly projective, the arguments outlined in Section 3.2 of Section 3 show that the E_1-page of the spectral sequence has

$$E_1^{p,q}\simeq\operatorname{Hom}_{S_p}(\mathcal{X}(p),\mathscr{H}^{p,q}).$$

To illustrate this, we observe that the key point of Section 3, which is the case in which $\mathcal{H}=\mathcal{E}$, is that we may take $\langle\mathscr{H};p\rangle$ to be a sphere S^{p-2}, and $\mathscr{H}^{p,q}=0$ for $q\neq 0$, while $\mathscr{H}^{p,0}$ is the sign representation sgn_p of S_p. In the general case, one must understand the various modules $\mathscr{H}^{p,q}$, hopefully via a geometric construction.

When writing the MSc thesis that then resulted in this paper, we obtained preliminary results for this in the case $\mathcal{H}$ is the species of linear orders. With the aid of a computer, we obtained the rank of $\mathscr{H}^{p,q}$ for $0\leqslant p\leqslant 5$, which we list in Figure 2. The attentive reader might notice this table is nothing else than that of the *unsigned Stirling numbers of the first kind*. We will prove this is the case in the next subsection.

[1] Read "$\mathcal{H}$ evaluated at p".

1	1	1	1	1	1
0	0	1	3	6	10
0	0	0	2	11	35
0	0	0	0	6	50
0	0	0	0	0	24
0	0	0	0	0	0

Figure 2: The Betti numbers for $\Omega^*(\mathcal{L})$.

4.2 An invitation to Koszul duality in twisted coalgebras

We include here a brief way to summarize this paper intended for the readers familiar with the theory of Koszul duality between algebras and coalgebras which works equally well in the symmetric monoidal category of species over k. The two main observations to make are the following:

1. The coalgebra $\mathcal{E}$ is the free cocommutative conilpotent coalgebra in one generator $\mathbb{1}$ —the unit of Sp_k— and as such is trivially Koszul with Koszul dual *algebra* $\mathcal{E}^{¡}$ the free commutative algebra on the desuspension of $\mathbb{1}$: it has the same dimension in each cardinality as $\mathcal{E}$, but the suspension accounts for a change of the trivial representation in each cardinality to the sign representation sgn_* that we used so heavily above.
2. The small complex $S^*(\mathcal{X})$ is nothing but the Koszul complex $\mathcal{K}(\mathcal{X},\mathcal{E}) = \mathrm{Hom}(\mathcal{X},\mathcal{E}^{¡})$ and the resulting cup product in the spectral sequence (that collapses since $\mathcal{E}$ is Koszul) is induced by the comultiplication of $\mathcal{E}^{¡}$. The technical requirement that $\mathcal{X}$ be weakly projective guarantees that homology commutes with the functor $\mathrm{Hom}(\mathcal{X},-)$ in each cardinality, which allows us to replace the twisted dg coalgebra $\Omega^*(\mathcal{E})$ by its homology, the twisted algebra $\mathcal{E}^{¡}$.

Naturally, this observation was done post hoc by the author some time later after finishing writing thesis; we have decided to preserve the work done there to illustrate how one can, without the "heavy machinery" of Koszul duality theory, obtain the complex $S^*(\mathcal{X})$ through the combinatorics of hyperplane arrangements and Coxeter complexes.

It is interesting to observe that this shows us how the purely algebraic theory of Koszul duality can shed light into combinatorics: one can see the observation above implies immediately that the Coxeter complex has the homology of a sphere, for example, and that the representation of this top homology group is the sign representation without doing *any* computation at all.

In particular, the above implies that whenever $\mathcal{H}$ is a Koszul twisted coalgebra, for every $\mathcal{H}$-bicomodule we have available the Koszul complex

$$\mathcal{K}^*(\mathcal{X},\mathcal{H}) = \operatorname{Hom}_{\mathsf{Sp}_k}(\mathcal{X},\mathcal{H}^{\text{¡}})$$

to compute its cohomology groups. Moreover, it often happens that the structure of twisted (co)algebras arising from combinatorial objects can be nicely understood through combinatorial methods. We also remark that in the book [4], V. Dotsenko and M. Bremner explain how apply methods of Gröbner bases to twisted algebras, which can then be effectively used to obtain results on the Koszulness of these.

Remark 4.1. It very often happens that $\mathcal{H}$ is Koszul for trivial reasons: if $\mathcal{H}$ is a cocommutative connected twisted bialgebra, then by the Milnor–Moore theorem [3, Theorem 118] the underlying coalgebra of $\mathcal{H}$ is cocommutative cofree over the collection of its primitives.

We can apply this to the species $\mathcal{L}$ of linear orders to deduce the following result which should aid us in computing $H^*(-,\mathcal{L})$ in the category of $\mathcal{L}$-bicomodules.

Corollary 4.1. *The twisted coalgebra $\mathcal{L}$ is cofree conilpotent over the species underlying the free Lie algebra functor and, in particular, it is twisted Koszul with Koszul dual twisted algebra $\mathcal{L}^{\text{¡}} = S(s^{-1}\mathsf{Lie})$, the free twisted commutative algebra over the desuspension of* Lie.

Proof. The fact that the primitives of $\mathcal{L}$ is equal to Lie contained in Corollary 121, and by the Milnor–Moore theorem in the remark above we have have isomorphism

$$S^c(\mathsf{Lie}) \longrightarrow \mathcal{L}$$

of twisted cocommutative conilpotent coalgebras. Since twisted cocommutative coalgebras are twisted Koszul, the result follows. ◀

With this at hand, we can prove the following result.

Theorem 4.1. *The twisted Koszul dual algebra of the coalgebra $\mathcal{L}$ is the free commutative twisted algebra $S(s^{-1}\mathsf{Lie})$ over the desuspesion of* Lie. *In particular, for each $p,q \in \mathbb{N}$ we have that*

$$H^{p-q}(\Omega^*(\mathcal{L})[p]) = S(s^{-1}\mathsf{Lie})[p]^{p-q}$$

has basis in correspondence with the permutations of p with $p-q$ disjoint cycles and, hence,

$$\dim_k S(s^{-1}\mathsf{Lie})[p]^{p-q} = \begin{bmatrix} p \\ p-q \end{bmatrix}$$

an unsigned Stirling number of the first kind.

Proof. The first equality follows by Koszul duality, since the algebra $S(s^{-1}\mathsf{Lie})$ is Koszul dual to $\mathcal{L}$ and, as such equal to the homology of the cobar construction on $\mathcal{L}$. For the second equality, all that we need to do is observe that for a finite set I of size p, an elementary tensor

$$z_1 \odot \cdots \odot z_{p-q} \in S(s^{-1}\mathsf{Lie})[p]^{p-q}$$

of $S(s^{-1}\mathsf{Lie})$ is in homological degree $p-q$ and corresponds to the datum of an *unordered* partition of I into subsets $(F_1,\ldots,F_{p-q})$ with $z_i \in \mathsf{Lie}(F_i)$. On the other hand, for a finite set $[n]$ we have a basis of $\mathsf{Lie}[n]$ indexed by permutations of n that fix 1. In this way, such an elementary tensor is indeed in bijection with a permutation of I with $p-q$ disjoint cycles, which is what we wanted. ◀

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