HOMOLOGICAL STABILITY FOR TEMPERLEY-LIEB ALGEBRAS

RACHAEL BOYD AND RICHARD HEPWORTH

ABSTRACT. This paper studies the homology and cohomology of the Temperley-Lieb algebra $\mathrm{TL}_n(a)$, interpreted as appropriate Tor and Ext groups. Our main result applies under the common assumption that $a = v+v^{-1}$ for some unit v in the ground ring, and states that the homology and cohomology vanish up to and including degree $(n-2)$. To achieve this we simultaneously prove homological stability and compute the stable homology. We show that our vanishing range is sharp when n is even.

Our methods are inspired by the tools and techniques of homological stability for families of groups. We construct and exploit a chain complex of 'planar injective words' that is analogous to the complex of injective words used to prove stability for the symmetric groups. However, in this algebraic setting we encounter a novel difficulty: $\mathrm{TL}_n(a)$ is not flat over $\mathrm{TL}_m(a)$ for $m < n$, so that Shapiro's lemma is unavailable. We resolve this difficulty by constructing what we call 'inductive resolutions' of the relevant modules.

We believe that these results, together with the second author's work on Iwahori-Hecke algebras, are the first time the techniques of homological stability have been applied to algebras that are not group algebras.

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1. Introduction

In this work we prove homological stability for the *Temperley-Lieb algebras*, and we prove that the stable homology vanishes.

A sequence of groups and inclusions $G_0 \to G_1 \to G_2 \to \cdots$ is said to satisfy *homological stability* if for each degree d the induced sequence of homology groups

$$H_d(G_0) \to H_d(G_1) \to H_d(G_2) \to \cdots$$

eventually consists of isomorphisms. Homological stability can also be formulated for sequences of spaces. There are many important examples of groups and spaces for which homological stability is known to hold, such as symmetric groups [Nak60], general linear groups [Cha80, Maa79, vdK80], mapping class groups of surfaces [Har85, RW16] and 3-manifolds [HW10], automorphism groups of free groups [HV98, HV04], diffeomorphism groups of high-dimensional manifolds [GRW18], configuration spaces [Chu12, RW13], Coxeter groups [Hep16], Artin monoids [Boyd20], and many more. In almost all cases, homological stability is one of the strongest things we know about the homology of these families. It is often coupled with computations of the stable homology $\lim_{n\to\infty} H_*(G_n)$, which is equal to the homology of the G_n in the *stable range* of degrees, i.e. those degrees for which stability holds.

The homology and cohomology of a group G can be expressed in the language of homological algebra as

$$H_*(G) = \mathrm{Tor}_*^{RG}(\mathbb{1}, \mathbb{1}), \qquad H^*(G) = \mathrm{Ext}^*_{RG}(\mathbb{1}, \mathbb{1}),$$

where R is the coefficient ring for homology and cohomology, RG is the group algebra of G and $\mathbb{1}$ is its trivial module. Thus the homology and cohomology of a group depend only on the group algebra RG and its trivial module $\mathbb{1}$. It is therefore natural to consider the homology and cohomology of an arbitrary algebra equipped with a 'trivial' module. Moreover, one may ask whether homological stability occurs in this wider context.

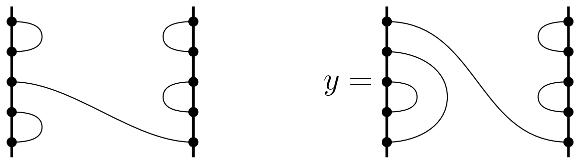
In [Hep20] the second author proved homological stability for *Iwahori-Hecke* algebras of type A. These are deformations of the group rings of the symmetric groups that are important in representation theory, knot theory, and combinatorics. There is a fairly standard suite of techniques used to prove homological stability, albeit with immense local variation, and the proof strategy of [Hep20] followed all the steps familiar from the setting of groups. As is typical, the hardest step was to prove that the homology of a certain (chain) complex vanishes in a large range of degrees.

In the present paper we will prove homological stability for the *Temperley-Lieb algebras*, and we will prove that the stable homology vanishes. However amongst the familiar steps in our proof lies a novel obstacle and — to counter it — a novel construction. At a certain point the usual techniques fail because Shapiro's lemma cannot be applied, as we will explain below. This is a new difficulty that never occurs in the setting of groups, but we are able to resolve it for the algebras at hand, and in fact our solution facilitates the unusually strong results that we are able to obtain. It is not surprising that the Iwahori-Hecke case is more straightforward than the Temperley-Lieb case: Iwahori-Hecke algebras are deformations of group rings, whereas the Temperley-Lieb algebras are significantly different.

To the best of our knowledge, the present paper and [Hep20] are the first time the techniques of homological stability have been applied to algebras that are not group algebras, and together they serve as proof-of-concept for the export of homological stability techniques to the setting of algebras. The moral of [Hep20] is that the 'usual' techniques of homological stability suffice, so long as the algebras involved satisfy a certain flatness condition. The moral of the present paper is that failure of the flatness condition can in some cases be overcome, using new ingredients and techniques, and can even lead to stronger results than in the flat scenario.

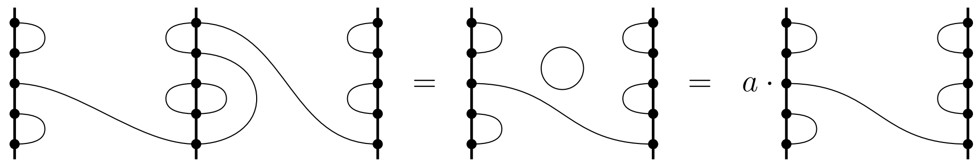
1.1. **Temperley-Lieb algebras.** Let $n \geqslant 0$, let R be a commutative ring, and let $a \in R$. The *Temperley-Lieb algebra* $\mathrm{TL}_n(a)$ is the R-algebra with basis given by the planar diagrams on n strands, taken up to isotopy, and with multiplication given by pasting diagrams and replacing closed loops with factors of a. The last sentence was intentionally brief, but we hope that its meaning becomes clearer with the following illustration of two elements $x, y \in \mathrm{TL}_5(a)$

$x =$ $y =$



and their product $x \cdot y$.

$x \cdot y =$ $=$ $= a \cdot$



The Temperley-Lieb algebras arose in theoretical physics in the 1970s [TL71]. They were later rediscovered by Jones in his work on von Neumann algebras [Jon83], and used in the first definition of the Jones polynomial [Jon85]. Kauffman gave the above diagrammatic interpretation of the algebras in [Kau87] and [Kau90].

The Temperley-Lieb algebra $\mathrm{TL}_n(a)$ is perhaps best studied in the case where $a = v + v^{-1}$, for $v \in R$ a unit. In this case, it is a quotient of the Iwahori-Hecke algebra of type A_{n-1} with parameter $q = v^2$ (so it is closely related to the symmetric group) and it receives a homomorphism from the group algebra of the braid group on n strands. It can also be described as the endomorphism algebra of $V_q^{\otimes n}$, where V_q is a certain 2-dimensional representation of the quantum group $U_q(\mathfrak{sl}_2)$. We recommend [RSA14] and [KT08] for further reading on $\mathrm{TL}_n(a)$.

1.2. **Homological stability for Temperley-Lieb algebras.** The Temperley-Lieb algebra $\mathrm{TL}_n(a)$ has a *trivial module* $\mathbb{1}$ consisting of a copy of R on which all diagrams other than the identity diagram act as multiplication by 0. It therefore has homology and cohomology groups $\mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ and $\mathrm{Ext}^*_{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$.

Our first result is a vanishing theorem in the case that the parameter $a \in R$ is invertible.

Theorem A. *Let R be a commutative ring, and let a be a unit in R. Then $\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ and $\mathrm{Ext}^d_{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ both vanish for $d > 0$.*

The next result holds regardless of whether or not a is invertible, and uses the common assumption that $a = v + v^{-1}$, $v \in R^\times$.

Theorem B. *Let R be a commutative ring, let $v \in R$ be a unit, let $a = v + v^{-1}$, and let $n \geqslant 0$. Then*

$$\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1}) = 0 \quad \textit{and} \quad \mathrm{Ext}^d_{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1}) = 0$$

for $1 \leqslant d \leqslant (n-2)$ if n is even, and for $1 \leqslant d \leqslant (n-1)$ if n is odd.

Thus the map $\mathrm{Tor}_d^{\mathrm{TL}_{n-1}(a)}(\mathbb{1},\mathbb{1}) \to \mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ is an isomorphism for $d \leqslant n-3$, so that we have homological stability, and $\lim_{n\to\infty} \mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1}) = 0$ in positive degrees, so the stable homology is trivial. The last two theorems might lead us to expect that the homology and cohomology of the $\mathrm{TL}_n(a)$ are largely trivial, but in fact the results are as strong as possible, at least for n even:

Theorem C. *In the setting of Theorem B above, suppose further that n is even and that $a = v + v^{-1}$ is not a unit. Then $\mathrm{Tor}_{n-1}^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1}) \neq 0$.*

Thus Theorem A does not extend to the case of a not invertible, and the stable range in Theorem B is sharp. In fact we can say more: $\mathrm{Tor}_{n-1}^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1}) \cong R/bR$ where b is a multiple of a.

Remark. One can compute $\mathrm{Tor}_1^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ directly using the method of [Wei94, Exercise 3.1.3]: it is R/aR for $n = 2$, and vanishes otherwise. We also compute the homology and cohomology of $\mathrm{TL}_2(a)$ by an explicit resolution: $\mathrm{Tor}_*^{\mathrm{TL}_2(a)}(\mathbb{1},\mathbb{1})$ is R/aR in odd degrees, and the kernel R_a of $r \mapsto ar$ in positive even degrees, so that if a is not invertible then $\mathrm{Tor}_*^{\mathrm{TL}_2(a)}(\mathbb{1},\mathbb{1})$ is non-trivial in infinitely many degrees.

The next few sections of this introduction will discuss the proofs of these results in some detail.

1.3. **Planar injective words.** Several proofs of homological stability for the symmetric group [Maa79, Ker05, RW13] make use of the *complex of injective words*. This is a highly-connected complex with an action of the symmetric group $\mathfrak{S}_n$. Our main tool for proving Theorems B and C is the *complex of* *planar injective words* $W(n)$, a Temperley-Lieb analogue of the complex of injective words that we introduce and study here for the first time. It is a chain complex of $\mathrm{TL}_n(a)$-modules, and in degree i it is given by the tensor product module $\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_{n-i-1}(a)}\mathbb{1}$. This is analogous to the complex of injective words, whose i-simplices form a single $\mathfrak{S}_n$-orbit with typical stabiliser $\mathfrak{S}_{n-i-1}$, which is an alternative way of saying that the i-th chain group is isomorphic to $R\mathfrak{S}_n\otimes_{R\mathfrak{S}_{n-i-1}}\mathbb{1}$. We show the following high-acyclicity result. In order to formulate appropriate differentials for $W(n)$ we exploit a homomorphism from the group algebra of the braid group on n strands, which is not necessarily apparent from the definition of $\mathrm{TL}_n(a)$. This is where the restriction of a to $a=v+v^{-1}$ is necessary.

Theorem D. *The homology of $W(n)$ vanishes in degrees $d\leqslant(n-2)$.*

The complex $W(n)$ has rich combinatorial properties, analogous to those of the complex of injective words, that we explore in the companion paper [BH20]. In particular, Theorem D tells us that the homology of $W(n)$ is concentrated in the top degree $H_{n-1}(W(n))$, and in [BH20] we show that the rank of this top homology group is the n-th *Fine number* F_n [DS01], an analogue of the number of derangements on n letters. Furthermore we show that the differentials of $W(n)$ encode the *Jacobsthal numbers* [Slo]. Finally in the semisimple case we show that $H_{n-1}(W(n))$ has descriptions firstly categorifying an alternating sum for the Fine numbers, and secondly in terms of standard Young tableaux. We call the $\mathrm{TL}_n(a)$-module $H_{n-1}(W(n))$ the *Fineberg module*, and we denote it $\mathcal{F}_n(a)$.

The proof of Theorem D is perhaps the most difficult technical result in this paper. It is obtained by filtering $W(n)$ and showing that the filtration quotients are (suspensions of truncations of) copies of $W(n-1)$, and then proceeding by induction.

1.4. **Spectral sequences and Shapiro's lemma.** Let us now outline how we use the complex of planar injective words $W(n)$ to prove Theorems B and C. Following standard approaches to homological stability for groups, we consider a spectral sequence obtained from the complex $W(n)$. The E^1-page of our spectral sequence consists of the groups $\mathrm{Tor}_j^{\mathrm{TL}_n(a)}(\mathbb{1},\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_{n-i-1}(a)}\mathbb{1})$. Furthermore, thanks to Theorem D, the spectral sequence converges to $H_{*-n+1}(\mathbb{1},\mathcal{F}_n(a))$, where $\mathcal{F}_n(a)=H_{n-1}(W(n))$ is the Fineberg module. Our experience from homological stability tells us to apply *Shapiro's lemma*, or in this context a *change-of-rings isomorphism*, to identify

$$\mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1},\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_{n-i-1}(a)}\mathbb{1})\qquad\text{with}\qquad\mathrm{Tor}_*^{\mathrm{TL}_{n-i-1}(a)}(\mathbb{1},\mathbb{1}).$$

This identification applied to the columns of our spectral sequence would allow us to implement an inductive hypothesis. However, such a change-of-rings isomorphism would only be valid if $\mathrm{TL}_n(a)$ were flat as a $\mathrm{TL}_{n-i-1}(a)$-module, and this is not the case. This failure of Shapiro's lemma is a potentially serious obstacle to proceeding further. However, we are able to identify the columns of our spectral sequence by independent means, as follows:

Theorem E. *Let R be a commutative ring and let $a \in R$. Let $0 \leqslant m < n$. Then* $\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1})$ *and* $\mathrm{Ext}^d_{\mathrm{TL}_n(a)}(\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}, \mathbb{1})$ *both vanish for $d > 0$.*

In conjunction with a computation of the $d = 0$ case, this gives us the vanishing results of Theorem B. Moreover, in the case of n even we are able to analyse the rest of the spectral sequence (there is a single differential and a single extension problem) in sufficient detail to prove the sharpness result of Theorem C. This involves a careful study of the Fineberg module $\mathcal{F}_n(a)$. In general, our method identifies $\mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1})$ with $\mathrm{Tor}_{*-n}(\mathbb{1}, \mathcal{F}_n(a))$, except in degrees $* = n-1, n$ when n is even.

1.5. **Inductive resolutions.** It remains for us to discuss the proofs of Theorems A and E. These results are proved by a novel method that exploits the structure of the Temperley-Lieb algebras, and in particular they lie outwith the standard toolkit of homological stability. Moreover, it is Theorem E which allows us to overcome the failure of Shapiro's lemma.

The two theorems are very similar: Theorem A is an instance of the more general statement that $\mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1})$ vanishes in positive degrees for $m \leqslant n$ and a invertible, while Theorem E states that the same groups vanish for $m < n$ and a arbitrary. These are both proved by strong induction on m. The initial cases $m = 0, 1$ are immediate because then $\mathrm{TL}_m(a)$ is the ground ring so that $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ is free. The induction step is proved by constructing and exploiting a resolution of $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ whose terms have the form $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-1}(a)} \mathbb{1}$ and $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-2}(a)} \mathbb{1}$, and then applying the inductive hypothesis. We call these resolutions *inductive resolutions* since they resolve the next module in terms of those already considered.

1.6. **Discussion: Homological stability for algebras.** As stated earlier, we regard the present paper, together with the results of [Hep20] on Iwahori-Hecke algebras, as proof-of-concept for the export of the techniques of homological stability to the setting of algebras, and we hope that it will be a springboard for further research in this direction. Readers with experience in homological stability will be able to think of many new questions in this direction, so we will simply list some that are most prominent in our minds.

The Temperley-Lieb algebra can be regarded as an algebra of 1-dimensional cobordisms embedded in 2 dimensions. The Brauer algebra, an analogue of the Temperley-Lieb algebra in which the arcs are allowed to cross (with no crossing data recorded), can similarly be viewed as an algebra of 1-dimensional cobordisms embedded in infinite dimensions.

Question. Are there analogues of the Temperley-Lieb algebra consisting of d-dimensional cobordisms embedded in n dimensions? Does homological stability hold for these algebras? And can the stability be understood in an essentially geometric way?

This leads us naturally to the following.

Question. Does homological stability hold for other classes of algebras besides Iwahori-Hecke algebras of type A and the Temperley-Lieb algebras?

Some candidate algebras, closely related to the existing cases, are: Iwahori-Hecke and Temperley-Lieb algebras of types B and D; the periodic and dilute Temperley-Lieb algebras; and the blob, partition, Brauer and Birman-Murakami-Wenzl algebras. We invite the reader to think of possibilities from further afield.

There have recently been advances in building frameworks for homological stability proofs. In [RWW17] Randal-Williams and Wahl introduce a categorical framework that encapsulates, improves and extends several of the standard techniques used in homological stability proofs for groups. In [GKRW18] Galatius, Kupers and Randal-Williams introduce a framework that applies to E_k-algebras in simplicial modules. It exploits the notion of cellular E_k-algebras, and incorporates methods for proving *higher stability* results. This invites us to pose the following questions.

Question. Does the general homological stability machinery of Randal-Williams and Wahl [RWW17] generalise to an R-linear version, giving a general framework to prove that a family of R-algebras $A_0 \to A_1 \to A_2 \to \cdots$ satisfy homological stability?

In this question, the most interesting issue is what form the resulting complexes will take. One might expect that for a family of algebras the relevant complexes will be constructed from tensor products, as with our complex $W(n)$. However it may happen, as in this paper, that flatness issues arise, in which case it seems unlikely that complexes built from the honest tensor products will be sufficient.

Question. Can the homological stability machinery of Galatius, Kupers and Randal-Williams [GKRW18] be applied in the setting of algebras?

It seems extremely likely that homology of Temperley-Lieb algebras will indeed fit into the framework of [GKRW18], by using appropriate simplicial models for the $\mathrm{Tor}_*^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1})$, or more precisely for the chain complexes underlying these Tor groups. Again, the difficulty will lie in identifying and computing the associated *splitting complexes*, especially when flatness issues arise.

1.7. **Outline.** In Section 2 we recall the definition of the Temperley-Lieb algebra, the Jones basis, the relationship with Iwahori-Hecke algebras, and we establish results on the induced modules $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ that will be important in the rest of the paper. Section 3 establishes our inductive resolutions and proves Theorems A and E. Section 4 introduces the complex of planar injective words $W(n)$ and the Fineberg module $\mathcal{F}_n(a)$. Sections 5 and 6 then use $W(n)$, in particular its high-acyclicity (Theorem D), to prove Theorems B and C. Section 7 investigates our results in the case of $\mathrm{TL}_2(a)$, computing the homology directly and also in terms of the Fineberg module $\mathcal{F}_2(a)$. Section 8 proves Theorem D.

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2. Temperley-Lieb Algebras

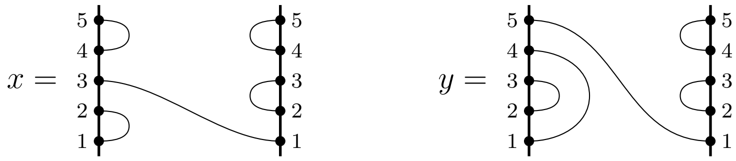
In this section we will cover the basic facts about the Temperley-Lieb algebra that we will need for the rest of the paper. There is some overlap between the material recalled here and in [BH20]. In particular, we cover the definitions by generators and relations and by diagrams; we discuss the Jones basis for $\mathrm{TL}_n(a)$; we look at the induced modules $\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ that will be an essential ingredient in all that follows; and we discuss the homomorphism from the Iwahori-Hecke algebra of type A_{n-1} into $\mathrm{TL}_n(a)$. Historical references on Temperley-Lieb algebras were given in the introduction. General references for readers new to the $\mathrm{TL}_n(a)$ are Section 5.7 of Kassel and Turaev's book [KT08] on the braid groups, and especially Sections 1 and 2 of Ridout and Saint-Aubin's survey on the representation theory of the $\mathrm{TL}_n(a)$ [RSA14].

Definition 2.1 (The Temperley-Lieb algebra $\mathrm{TL}_n(a)$)**.** Let R be a commutative ring and let $a \in R$. Let n be a non-negative integer. The *Temperley-Lieb algebra* $\mathrm{TL}_n(a)$ is defined to be the R-algebra with generators $U_1, \dots, U_{n-1}$ and the following relations:
(1) $U_iU_j = U_jU_i$ for $j \neq i \pm 1$
(2) $U_iU_jU_i = U_i$ for $j = i \pm 1$
(3) $U_i^2 = aU_i$ for all i.

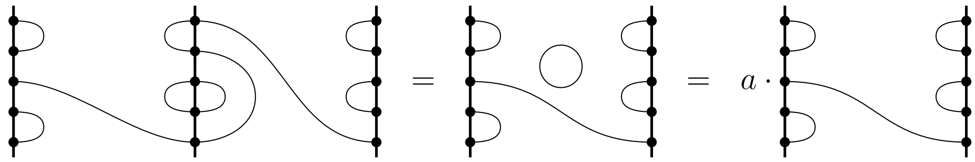
Thus elements of the Temperley-Lieb algebra are formal sums of monomials in the U_i, with coefficients in the ground ring R, modulo the relations above. We often write $\mathrm{TL}_n(a)$ as TL_n. We note here that $\mathrm{TL}_0 = \mathrm{TL}_1 = R$.

There is an alternative definition of TL_n in terms of diagrams. In this description, an element of TL_n is an R-linear combination of *planar diagrams* (or 1-dimensional cobordisms). Each planar diagram consists of two vertical lines in the plane, decorated with n dots labelled $1, \dots, n$ from bottom to top, together with a collection of n arcs joining the dots in pairs. The arcs must lie between the vertical lines, they must be disjoint, and the diagrams are taken up to isotopy.

For example, here are two planar diagrams in the case $n = 5$:

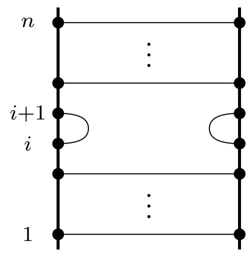


We will often omit the labels on the dots. Multiplication of diagrams is given by placing them side-by-side and joining the ends. Any closed loops created by this process are then erased and replaced with a factor of a. For example, the product xy of the elements x and y above is:



(We have subscribed to the heresy of [RSA14] by drawing planar diagrams that go from left to right rather than top to bottom.)

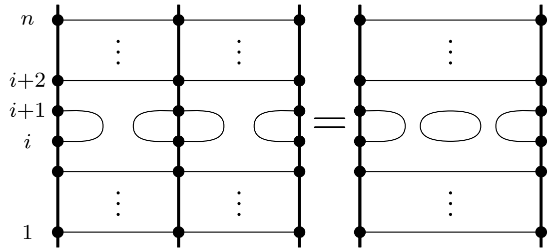
One can pass from the generators-and-relations definition of TL_n in Definition 2.1 to the diagrammatic description of the previous paragraph as follows. For $1 \leqslant i \leqslant n-1$, to each U_i we associate the planar diagram shown below.



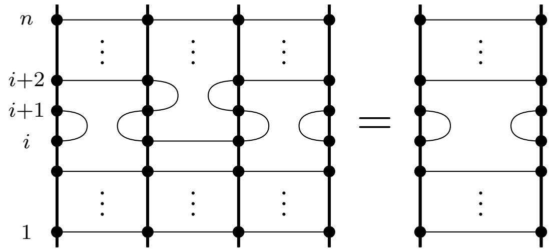
We refer to an arc joining adjacent dots as a *cup*. The relations for the Temperley-Lieb algebras are satisfied, two of them are shown in Figure 1. The fact that this determines an isomorphism between the algebra defined by generators and relations, and the one defined by diagrams, is proved in [RSA14, Theorem 2.4], [KT08, Theorem 5.34], and [Kau05, Section 6].

In the rest of the paper we will refer to the diagrammatic point of view on the Temperley-Lieb algebra, but we will not rely on it for any proofs.

2.1. **The Jones basis.** From the diagrammatic point of view the Temperley-Lieb algebra TL_n has an evident R-basis given by the (isotopy classes of) planar diagrams. This is called the *diagram basis*. We now recall the analogue of the diagram basis given in terms of the U_i, which is called the *Jones basis* for TL_n, and prove some additional facts about it that we will require later. See [KT08, Section 5.7], [RSA14, Section 2] or [Kau05, Section 6], but note that conventions vary, and see Remark 2.5 below in particular.



(a) The relation $U_i^2 = aU_i$.



(b) The relation $U_iU_{i+1}U_i = U_i$.

FIGURE 1. Diagrammatic relations in TL_n.

Definition 2.2 (Jones normal form)**.** The *Jones normal form* for elements of $\mathrm{TL}_n(a)$ is defined as follows. Let

$$n > a_k > a_{k-1} > \cdots > a_1 > 0 \qquad n > b_k > b_{k-1} > \cdots > b_1 > 0$$

be integers such that $b_i \geqslant a_i$ for all i. Let $\underline{a} = (a_k, \ldots a_1)$ and $\underline{b} = (b_k, \ldots b_1)$. Then set

$$x_{\underline{a},\underline{b}} = (U_{a_k} \ldots U_{b_k}) \cdot (U_{a_{k-1}} \ldots U_{b_{k-1}}) \cdots (U_{a_1} \ldots U_{b_1})$$

where the subscripts of the generators increase in each tuple $U_{a_i} \ldots U_{b_i}$. A word written in the form $x_{\underline{a},\underline{b}}$ is said to be written in *Jones normal form* for $\mathrm{TL}_n(a)$.

Example 2.3. In TL_5 the words

$$\begin{aligned}
U_1U_2U_3U_4 &= (U_1U_2U_3U_4) = x_{(1),(4)} \\
U_4U_3U_2U_1 &= (U_4) \cdot (U_3) \cdot (U_2) \cdot (U_1) = x_{(4,3,2,1),(4,3,2,1)} \\
U_3U_4U_1U_2 &= (U_3U_4) \cdot (U_1U_2) = x_{(3,1),(4,2)} \\
U_2U_3U_1U_2 &= (U_2U_3) \cdot (U_1U_2) = x_{(2,1),(3,2)}
\end{aligned}$$

are in Jones normal form. The word $U_2U_1U_4U_2U_3$ is not, but it can be rewritten using the defining relations to give

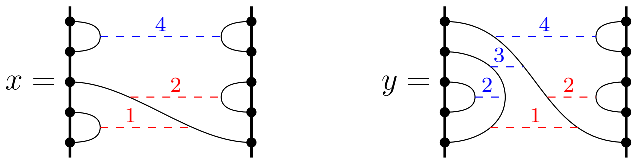
$$U_2U_1U_4U_2U_3 = U_4U_2U_1U_2U_3 = U_4U_2U_3 = (U_4)(U_2U_3) = x_{(4,2),(4,3)}.$$

Denote the subspace of TL_n spanned by all $x_{\underline{a},\underline{b}}$ with $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_k)$ by $\mathrm{TL}_{n,k}$. Then the set

$$\mathrm{TL}_{n,0} \sqcup \mathrm{TL}_{n,1} \sqcup \cdots \sqcup \mathrm{TL}_{n,n-1}$$

is a basis of TL_n, called the *Jones basis*. For a proof of this fact see [KT08, Corollary 5.32], [RSA14, pp.967-969] or [Kau05, Section 6], though we again warn the reader that conventions vary.

There is an algorithm for taking a diagram and writing it as an element of the Jones basis, see [Kau05, Section 6]. We summarise the algorithm here. Take a planar diagram, and ensure that it is drawn in minimal form: all arcs connecting the same side of the diagram to itself are drawn as semicircles, and all arcs from left to right are drawn without any cups, i.e. transverse to all vertical lines. The *i-th row* of the diagram is the horizontal strip whose left and right ends lie between the dots i and $(i+1)$ on each vertical line. Proceed along each row of the diagram, connecting the consecutive arcs encountered with a dotted horizontal line labelled by the row in question. This is done in an alternating fashion: the first arc encountered is connected to the second by a dotted line, then the third is connected to the fourth, and so on. If we start with the elements x and y used earlier in this section, then this gives us the following:



A *sequence* in such a decorated diagram is taken by travelling right along the dotted arcs and up along the solid arcs from one dotted arc to the next, starting as far to the left as possible. The above diagrams each have two sequences, indicated in blue and red. The sequences in a diagram are linearly ordered by scanning from top to bottom and recording a sequence when one of its dotted lines is first encountered. So in the above diagrams the blue sequences precede the red ones. One now obtains a Jones normal form for the element by working through the sequences in turn, writing out the labels from left to right, and then taking the corresponding monomial in the U_i:

$$x = (U_4)(U_1U_2) = x_{(4,1),(4,2)}, \qquad y = (U_2U_3U_4)(U_1U_2) = x_{(2,1),(4,2)}.$$

We now present a proof that the Jones basis spans, adding slightly more detail than we found in the references. The extra detail will be used in the next section.

Definition 2.4. Given a word $w = U_{i_1} \dots U_{i_n}$ in the U_i, define the *terminus* to be the subscript of the final letter of the word appearing, i_n, and denote it $t(w)$. Set $t(e) = \infty$ as a convention. Define the *index* of w to be the minimum subscript i_j appearing, and denote it $i(w)$.

Remark 2.5. Note here that the notions of Jones Normal form and index in $\mathrm{TL}_n(a)$ coincide with those of [KT08], under the bijection which sends the generator e_i of [KT08] to the generator U_{n-i} used in this paper, for $1 \leqslant i \leqslant n-1$.

The following two lemmas are a slight enhancement of Lemmas 5.25 and 5.26 of [KT08].

Lemma 2.6. *Any word $w \in \mathrm{TL}_n(a)$ is equal in $\mathrm{TL}_n(a)$ to a scalar multiple of a word w' in which*
(a) $i(w) = i(w')$ and $U_{i(w)}$ appears exactly once in w';
(b) $t(w') \leqslant t(w)$;
(c) if $t(w') < t(w)$ then $t(w') \leqslant t(w) - 2$.

Proof. Point (a) occurs as [KT08, Lemma 5.25]. We refer the reader to that proof, modifying it in the following way:

- Invoke the bijection of generators of Remark 2.5. This amounts to replacing each occurrence of e_i with U_{n-i}, so for example the subscripts 1 and $n-1$ are interchanged, and inequalities are 'reversed'.
- Whenever the inductive hypothesis is used in [KT08, Lemma 5.25], instead use the statement of the present lemma as a stronger inductive hypothesis.

□

Lemma 2.7. *Any word $w \in \mathrm{TL}_n(a)$ is equivalent in $\mathrm{TL}_n(a)$ to a scalar multiple of a word w' such that*
(a) w' is written in Jones normal form;
(b) $t(w') \leqslant t(w)$;
(c) if $t(w') < t(w)$ then $t(w') \leqslant t(w) - 2$.

Proof. As in the previous Lemma, point (a) occurs as [KT08, Lemma 5.26]. We modify that proof using the same two bullet points as in the proof of Lemma 2.6 and with the following extra modification:

- Whenever [KT08, Lemma 5.25] is used in [KT08, Lemma 5.26], use instead Lemma 2.6. □

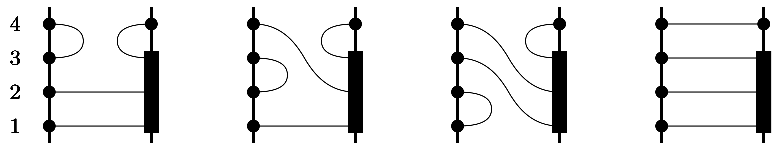
2.2. **Induced modules of Temperley-Lieb Algebras.**

Definition 2.8 (The trivial module $\mathbb{1}$)**.** The *trivial* module $\mathbb{1}$ of the Temperley-Lieb algebra $\mathrm{TL}_n(a)$ is the module consisting of R with the action of $\mathrm{TL}_n(a)$ in which all of the generators $U_1, \ldots, U_{n-1}$ act as 0. We can regard $\mathbb{1}$ as either a left or right module over $\mathrm{TL}_n(a)$, and we will usually do that without indicating so in the notation.

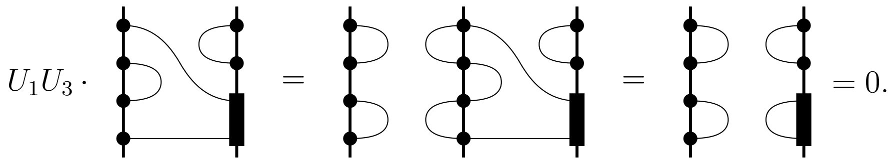
Definition 2.9 (Sub-algebra convention)**.** For $m \leqslant n$, we will regard $\mathrm{TL}_m(a)$ as the sub-algebra of $\mathrm{TL}_n(a)$ generated by the elements $U_1, \ldots, U_{m-1}$. We will often regard $\mathrm{TL}_n(a)$ as a left $\mathrm{TL}_n(a)$-module and a right $\mathrm{TL}_m(a)$-module, so that we obtain the left $\mathrm{TL}_n(a)$-module $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$.

The modules $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ are an essential ingredient in the rest of this paper: they will be the building blocks of all the complexes we construct in order to prove our main results, in particular the complex of planar injective words $W(n)$. The rest of this section will study them in some detail, in particular finding a basis for them analogous to the Jones basis.

Remark 2.10 ($\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ via diagrams)**.** The elements of $\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ can be regarded as diagrams, just like the elements of $\mathrm{TL}_n(a)$, except that now the first m dots on right are encapsulated within a *black box*, and if any cups can be absorbed into the black box, then the diagram is identified with 0. For example, some elements of $\mathrm{TL}_4(a)\otimes_{\mathrm{TL}_3(a)}\mathbb{1}$ are depicted as follows:



The structure of $\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ as a left module for $\mathrm{TL}_n(a)$ is given by pasting diagrams on the left, and then simplifying, as in the following example for $n=4$ and $m=2$:



Definition 2.11 (The ideal I_m)**.** Given $0\leqslant m\leqslant n$, let I_m denote the left ideal of $\mathrm{TL}_n(a)$ generated by the elements $U_1,\dots,U_{m-1}$.

Lemma 2.12. *$\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ and $\mathrm{TL}_n(a)/I_m$ are isomorphic as left $\mathrm{TL}_n(a)$-modules via the maps*

$$\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}\longrightarrow\mathrm{TL}_n(a)/I_m,\qquad y\otimes r\longmapsto yr+I_m$$

and

$$\mathrm{TL}_n(a)/I_m\longrightarrow\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1},\qquad y+I_m\longmapsto y\otimes 1.$$

Proof. Observe that the generators $U_1,\dots,U_{m-1}$ of the left-ideal I_m in TL_n are precisely the generators of the subalgebra TL_m of TL_n. Thus the map $y\otimes r\mapsto yr+I_m$ is well defined because if $i=1,\dots,m-1$ then elements of the form $yU_i\otimes r$ and $y\otimes U_ir$ both map to 0 in TL_n/I_m. And $y+I_m\mapsto y\otimes 1$ is well defined because elements of I_m are linear combinations of ones of the form $x\cdot U_i$ for $i=1,\dots,m-1$, and $(x\cdot U_i)\otimes 1=x\otimes(U_i\cdot 1)=x\otimes 0=0$ for $i=1,\dots,m-1$. One can now check that the two maps are inverses of one another. ☐

Remark 2.13. Lemma 2.12 justifies the description of $\mathrm{TL}_n(a)\otimes_{\mathrm{TL}_m(a)}\mathbb{1}$ in terms of diagrams with 'black boxes' that we gave in Remark 2.10. Indeed, I_m is precisely the span of those diagrams which have a cup on the right between the dots i and $i+1$ for some $i=1,\dots,m-1$. But these are precisely the diagrams which are made to vanish by having a cup fall into the black box. Thus $\mathrm{TL}_n(a)/I_m$ has basis given by the remaining diagrams, i.e. the ones that are not rendered 0 by the black box.

Lemma 2.14. *For $m \leqslant n$, the ideal I_m of $\mathrm{TL}_n(a)$ has basis consisting of those elements of $\mathrm{TL}_n(a)$ written in Jones normal form $x_{\underline{a},\underline{b}}$, which have terminus $b_1 \leqslant m-1$ (and $k \neq 0$).*

Proof. Recall that words of the form $x_{\underline{a},\underline{b}}$ give a basis for TL_n. Then by definition any word $w \in I_m$ is of the form $w = x_{\underline{a},\underline{b}}v$ for $v \in \langle U_1, \ldots, U_{m-1}\rangle$ and $v \neq e$. Then we have that $t(w) \leqslant m-1$. Now apply Lemma 2.7 to w to complete the proof. ☐

Lemma 2.15. *For $m \leqslant n$, $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ has basis given by $x_{\underline{a},\underline{b}} \otimes \mathbb{1}$ such that the terminus $b_1 > m-1$.*

Proof. From Lemma 2.12 $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ is isomorphic to TL_n/I_m. Then elements of the form $x_{\underline{a},\underline{b}}$ give a basis for TL_n and elements of the form $x_{\underline{a},\underline{b}}$, which have terminus $b_1 \leqslant m-1$ give a basis for I_m by Lemma 2.14. Therefore a basis for the quotient is given by $x_{\underline{a},\underline{b}}$ such that the terminus $b_1 > m-1$, and under the isomorphism in Lemma 2.12 this gives the required basis. ☐

Example 2.16. The Jones basis of $\mathrm{TL}_3(a)$ is:

$$1, \quad U_2, \quad U_1U_2, \quad U_1, \quad U_2U_1$$

So $\mathrm{TL}_3(a) \otimes_{\mathrm{TL}_2(a)} \mathbb{1}$ has basis consisting of those elements whose terminus is strictly greater than 1, namely:

$$1, \quad U_2, \quad U_1U_2,$$

(Recall that by convention the terminus of 1 is ∞.)

Lemma 2.17. *For $m \leqslant n$, suppose that $y \in \mathrm{TL}_n(a)$ and that $y \cdot U_{m-1}$ lies in I_{m-1}. Then $y \cdot U_{m-1}$ lies in I_{m-2}.*

Proof. $y \cdot U_{m-1}$ is a linear combination of words ending with U_{m-1}, i.e. of words w with $t(w) = m-1$. By Lemma 2.7, this can be rewritten as a linear combination of Jones basis elements $x_{\underline{a},\underline{b}}$ whose terminus satisfies $t(x_{\underline{a},\underline{b}}) = m-1$ or $t(x_{\underline{a},\underline{b}}) \leqslant m-3$. Since $y \cdot U_{m-1} \in I_{m-2}$, this means that in fact no basis elements with terminus $m-1$ remain after cancellation, and therefore all remaining words have terminus $m-3$ or less, and so lie in I_{m-2}. ☐

2.3. Iwahori-Hecke algebras.

Definition 2.18 (The Iwahori-Hecke algebra)**.** Let $n \geqslant 0$ and let $q \in R^\times$. The *Iwahori-Hecke algebra* $\mathcal{H}_n(q)$ of type A_{n-1} is the algebra with generators

$$T_1, \ldots, T_{n-1}$$

satisfying the following relations:

- $T_iT_j = T_jT_i$ for $i \neq j \pm 1$
- $T_iT_jT_i = T_jT_iT_j$ for $i = j \pm 1$
- $T_i^2 = (q-1)T_i + q$

Definition 2.19 (From Iwahori-Hecke to Temperley-Lieb)**.** Now suppose that there is $v \in R^\times$ such that $q = v^2$. Then there are two natural homomorphisms

$$\theta_1, \theta_2 \colon \mathcal{H}_n(q) \longrightarrow \mathrm{TL}_n(v + v^{-1}),$$

defined by $\theta_1(T_i) = vU_i - 1$ and $\theta_2(T_i) = v^2 - vU_i$ for $i = 1, \ldots, n-1$. They induce isomorphisms

$$\bar{\theta}_1 \colon \mathcal{H}_n(q)/I_1 \xrightarrow{\cong} \mathrm{TL}_n(v + v^{-1}), \qquad \bar{\theta}_2 \colon \mathcal{H}_n(q)/I_2 \xrightarrow{\cong} \mathrm{TL}_n(v + v^{-1}),$$

where I_1 is the two-sided ideal generated by elements of the form

$$T_iT_jT_i + T_iT_j + T_jT_i + T_i + T_j + 1$$

for $i = j \pm 1$, and I_2 is the two-sided ideal generated by elements of the form

$$T_iT_jT_i - qT_iT_j - qT_jT_i + q^2T_i + q^2T_j - q^3$$

for $i = j \pm 1$. See [FG97], Theorem 5.29 of [KT08], and Section 2.3 of [HMR09], though unfortunately conventions change from author to author.

We will take an agnostic approach to the homomorphisms θ_1, θ_2. We will choose one of them and denote it by simply

$$\theta \colon \mathcal{H}_n(q) \longrightarrow \mathrm{TL}_n(v + v^{-1}),$$

and denote by λ the constant term in $\theta(T_i)$, and by μ the coefficient of U_i in $\theta(T_i)$, so that

$$\theta(T_i) = \lambda + \mu U_i.$$

Then θ induces an isomorphism

$$\bar{\theta} \colon \mathcal{H}_n(q)/I \xrightarrow{\cong} \mathrm{TL}_n(v + v^{-1})$$

where I is the two-sided ideal generated by elements of the form

$$T_iT_jT_i - \lambda T_iT_j - \lambda T_jT_i + \lambda^2T_i + \lambda^2T_j - \lambda^3$$

for $i = j \pm 1$. And moreover, the elements $\theta(T_i)$ act on the trivial module $\mathbb{1}$ as multiplication by λ.

Definition 2.20. Let $v \in R^\times$. We define $s_1, \ldots, s_{n-1} \in \mathrm{TL}_n(v + v^{-1})$ by setting

$$s_i = \theta(T_i) = \lambda + \mu U_i$$

and note that these elements satisfy the following properties:

- $s_i^2 = (q-1)s_i + q$ for all i,
- $s_is_j = s_js_i$ for $i \neq j \pm 1$,
- $s_is_js_i = s_js_is_j$ for $i = j \pm 1$,
- $s_is_js_i - \lambda s_is_j - \lambda s_js_i + \lambda^2 s_i + \lambda^2 s_j - \lambda^3 = 0$ for $i = j \pm 1$,
- s_i acts on $\mathbb{1}$ as multiplication by λ.

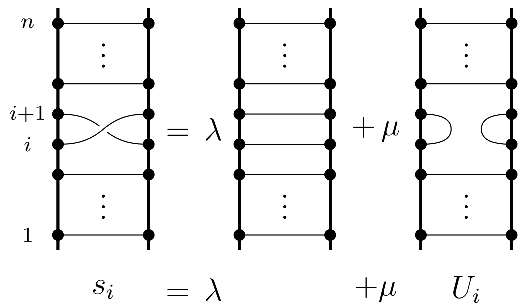


FIGURE 2. Smoothings of s_i

Remark 2.21. There is a homomorphism from (the group algebra of) the braid group into $\mathrm{TL}_n(v+v^{-1})$ given on generators by $s_i \mapsto s_i$. This is the content of the second and third bullet points above, together with the fact that the s_i are invertible. Diagrammatically, this homomorphism can be viewed as a *smoothing expansion* from braided diagrams to planar diagrams: take a braid diagram, and then smooth each crossing in turn in the two possible ways, using appropriate weightings for each smoothing. For example, we can visualise the image of s_i in $\mathrm{TL}_n(v+v^{-1})$ as the standard braid group generator crossing strand i over strand $i+1$. There are two ways this crossing can be resolved to a planar diagram, and we equate s_i to the sum of these two states. They are the identity and U_i, as shown in Figure 2. The coefficient of the identity is λ and the coefficient of U_i is μ, simply because we defined $s_i = \lambda + \mu U_i$. Similarly, we consider the image of s_i^{-1} as strand i crossing under strand $i+1$, and when this is smoothed the coefficient of the identity is λ^{-1} and the coefficient of U_i is μ^{-1}, precisely because one can verify that $s_i^{-1} = \lambda^{-1} + \mu^{-1}U_i$ in $\mathrm{TL}_n(v+v^{-1})$.

In principle we could describe how various Reidemeister moves affect the smoothing expansion, but it will not be necessary for the rest of the paper. Moreover, we will only encounter positive powers of s_i.

3. Inductive resolutions

In this section we prove the following two theorems, which we recall from the introduction.

Theorem A. *Let R be a commutative ring and let a be a unit in R. Then $\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ and $\mathrm{Ext}^d_{\mathrm{TL}_n(a)}(\mathbb{1},\mathbb{1})$ both vanish for $d>0$.*

Theorem E. *Let R be a commutative ring and let $a \in R$. Let $0 \leqslant m < n$. Then $\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1})$ and $\mathrm{Ext}^d_{\mathrm{TL}_n(a)}(\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}, \mathbb{1})$ vanish for $d>0$.*

In fact for Theorem A we will prove the following stronger claim:

Claim 3.1. *Suppose that the parameter $a \in R$ is invertible. Then for any $0 \leqslant m \leqslant n$, the groups $\mathrm{Tor}_d^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1})$ and $\mathrm{Ext}_d^{\mathrm{TL}_n(a)}(\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}, \mathbb{1})$ both vanish for $d > 0$.*

The similarity between Theorem E and Claim 3.1 is now clear. Both will be proved by induction on m, the initial cases $m = 0, 1$ being immediate because then TL_m is the ground ring R so that $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1} \cong \mathrm{TL}_n$ is free. In order to produce an inductive proof, we construct resolutions of the modules $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ whose terms are not free or projective or injective, but instead whose terms are the modules already considered earlier in the induction, specifically $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1}$ and $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$. For this reason we refer to these resolutions as *inductive resolutions*. This approach is inspired by homological stability arguments, in which one considers complexes whose building blocks are induced up from the earlier objects in the sequence. The difference here is that our complexes are actual resolutions — they are acyclic rather than just acyclic up to a point — and because Shapiro's lemma is unavailable we do not change the algebra we are working over, rather we change the algebra from which we are inducing our modules.

3.1. **The inductive resolutions.** In this subsection we establish the resolutions $C(m)$ and $D(m)$ of $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ required to prove Claim 3.1 and Theorem E above.

Definition 3.2 (The complex $C(m)$)**.** Let $m \geqslant 2$. Assume that a is invertible. Given $0 \leqslant m \leqslant n$, we define a chain complex of left $\mathrm{TL}_n(a)$-modules as in Figure 3. The degree is indicated in the right-hand column. The differentials of $C(m)$ are all given by extending the algebra over which the tensor product is taken, by right multiplying in the first factor by the indicated element of $\mathrm{TL}_n(a)$, or by a combination of the two. So, for example, the differential originating in degree 1 sends $x \otimes r \in \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-2}(a)} \mathbb{1}$ to $(x \cdot a^{-1}U_{m-1}) \otimes r \in \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-1}(a)} \mathbb{1}$. The complex is periodic of period 2 in degrees 1 and above, so that all entries are $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-2}(a)} \mathbb{1}$ and the boundary maps between them alternate between $a^{-1}U_{m-1}$ and $(1 - a^{-1}U_{m-1})$. The boundary maps are well defined because U_{m-1} commutes inside $\mathrm{TL}_n(a)$ with all elements of $\mathrm{TL}_{m-2}(a)$.

Definition 3.3 (The complex $D(m)$)**.** Let $2 \leqslant m < n$. We do not assume that a is invertible. Given $0 \leqslant m < n$, we define a chain complex of left $\mathrm{TL}_n(a)$-modules as in Figure 4. The degree is indicated in the right-hand column. The differentials of $D(m)$ are all given by extending the algebra over which the tensor product is taken, by right multiplying in the first factor by the indicated element of $\mathrm{TL}_n(a)$, or by a combination of the two. So, for example, the differential originating in degree 1 sends $x \otimes r \in \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-2}(a)} \mathbb{1}$ to $x \cdot U_{m-1} \otimes r \in \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-1}(a)} \mathbb{1}$. The complex is periodic of period 2 in degrees 1 and above, so that in that range all terms are $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{m-2}(a)} \mathbb{1}$ and the boundary maps between them alternate between $U_{m-1}U_m$ and $(1 - U_{m-1}U_m)$. The boundary maps are well defined

$$\begin{tikzcd}
\vdots \arrow[d, "(1-a^{-1}U_{m-1})"] & \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "a^{-1}U_{m-1}"] & 3 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "(1-a^{-1}U_{m-1})"] & 2 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "a^{-1}U_{m-1}"] & 1 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1} \arrow[d, "1"] & 0 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m}} \mathbb{1} & -1
\end{tikzcd}$$

FIGURE 3. The complex $C(m)$.

$$\begin{tikzcd}
\vdots \arrow[d, "(1-U_{m-1}U_m)"] & \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "U_{m-1}U_m"] & 3 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "(1-U_{m-1}U_m)"] & 2 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1} \arrow[d, "U_{m-1}"] & 1 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1} \arrow[d, "1"] & 0 \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{m}} \mathbb{1} & -1
\end{tikzcd}$$

FIGURE 4. The complex $D(m)$.

because U_{m-1} and $U_{m-1}U_m$ commute inside $\mathrm{TL}_n(a)$ with all elements of $\mathrm{TL}_{m-2}(a)$. Observe that the condition $m < n$ is necessary in order to ensure that U_m is actually an element of $\mathrm{TL}_n(a)$.

Lemma 3.4.

(1) *Let* $2 \leqslant m \leqslant n$ *and let* a *be invertible. Then* $a^{-1}U_{m-1} \in \mathrm{TL}_n(a)$ *is idempotent.*

(2) *Let* $2 \leqslant m < n$ *and let* a *be arbitrary. Then* $U_{m-1}U_m \in \mathrm{TL}_n(a)$ *is idempotent.*

Proof. We calculate

$$(a^{-1}U_i)^2 = a^{-2}U_i^2 = a^{-2}aU_i = a^{-1}U_i$$

and

$$U_{m-1}U_m \cdot U_{m-1}U_m = U_{m-1}U_mU_{m-1} \cdot U_m = U_{m-1}U_m. \qquad \square$$

From now on in this section, we will attempt to talk about $C(m)$ and $D(m)$ at the same time. When we refer to $C(m)$, the relevant assumptions should be understood, namely that $2 \leqslant m \leqslant n$ and that $a \in R$ is a unit. And when we refer to $D(m)$, the assumptions that $2 \leqslant m < n$ but $a \in R$ is arbitrary should be understood. We trust that this will not be confusing.

Lemma 3.5. *$C(m)$ and $D(m)$ are indeed chain complexes.*

Proof. We give the proof for $C(m)$. The proof for $D(m)$ is similar. We must check that consecutive boundary maps of $C(m)$ compose to 0. In the case of the composite from degree 1 to -1, the composition is given by

$$x \otimes r \mapsto (x \cdot a^{-1}U_{m-1}) \otimes r = x \otimes (a^{-1}U_{m-1} \cdot r) = x \otimes 0 = 0$$

this holds because the tensor product is over TL_m, which contains $a^{-1}U_{m-1}$. In the case of the remaining composites, this follows immediately from

$$(a^{-1}U_{m-1}) \cdot (1 - a^{-1}U_{m-1}) = 0 = (1 - a^{-1}U_{m-1}) \cdot (a^{-1}U_{m-1}),$$

which is a consequence of the fact that $a^{-1}U_{m-1}$ is idempotent (from Lemma 3.4). $\square$

Lemma 3.6. *The complexes $C(m)$ and $D(m)$ are acyclic.*

Proof. In degree -1 it is clear that the boundary map is surjective, for both $C(m)$ and $D(m)$.

In degree 0, we will give the proof for $C(m)$, the proof for $D(m)$ being similar. Suppose that $y \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1}$ lies in the kernel of the boundary map, or in other words that $y \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ vanishes. This means that y lies in the left-ideal generated by the elements $U_1, \ldots, U_{m-1}$. Since all but the last of these generators lie in TL_{m-1}, and we started with $y \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1}$, we may assume without loss that $y = y' \cdot U_{m-1}$ for some y'. But then

$$y \otimes 1 = y' \cdot U_{m-1} \otimes 1 = ay' \cdot (a^{-1}U_{m-1}) \otimes 1$$

does indeed lie in the image of the boundary map.

In degree 1, we give the proof for both complexes. First, for $C(m)$, suppose that $y \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$ lies in the kernel of the boundary map. It follows that $y \cdot (a^{-1}U_{m-1}) \otimes 1$ vanishes in $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1}$, which means that $y \cdot (a^{-1}U_{m-1})$ lies in the left ideal I_{m-1} generated by $U_1, \ldots, U_{m-2}$. It follows from Lemma 2.17 that $y \cdot (a^{-1}U_{m-1})$ lies in the left ideal I_{m-2} generated by $U_1, \ldots, U_{m-3}$, so that in $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$ the element $y \cdot (a^{-1}U_{m-1}) \otimes 1$ vanishes. Thus

$$y \otimes 1 = y \cdot (1 - a^{-1}U_{m-1}) \otimes 1$$

does indeed lie in the image of the boundary map. Second, for $D(m)$, suppose that $y \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$ lies in the kernel of the boundary map. Then, as for $C(m)$, $y \cdot U_{m-1}$ lies in I_{m-2}. So $y \cdot U_{m-1}U_m$ also lies in I_{m-2} since U_m commutes with the generators of I_{m-2}. Thus $y \cdot U_{m-1}U_m \otimes 1$ vanishes in $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$, so that $y \otimes 1 = y \cdot (1 - U_{m-1}U_m) \otimes 1$ does indeed lie in the image of the boundary map.

In degrees 2 and higher, acyclicity is an immediate consequence of the fact that $a^{-1}U_{m-1}$ and $U_{m-1}U_m$ are idempotents, as in Lemma 3.4. □

Lemma 3.7. *The complexes* $\mathbb{1} \otimes_{\mathrm{TL}_n(a)} C(m)$*,* $\mathbb{1} \otimes_{\mathrm{TL}_n(a)} D(m)$*,* $\mathrm{Hom}_{\mathrm{TL}_n(a)}(C(m), \mathbb{1})$ *and* $\mathrm{Hom}_{\mathrm{TL}_n(a)}(D(m), \mathbb{1})$ *are acyclic.*

Proof. We give the proof for $\mathbb{1} \otimes_{\mathrm{TL}_n} C(m)$, the proof for the other parts being similar. The terms of $C(m)$ have the form $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-i}} \mathbb{1}$ where $i = 0, 1, 2$ depending on the degree. Thus $\mathbb{1} \otimes_{\mathrm{TL}_n} C(m)$ has terms of the form $\mathbb{1} \otimes_{\mathrm{TL}_n} (\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-i}} \mathbb{1}) \cong \mathbb{1} \otimes_{\mathrm{TL}_{m-i}} \mathbb{1} \cong \mathbb{1}$. Moreover, by tracing through this isomorphism, one sees that if a boundary map in $C(m)$ is labelled by an element $x \in \mathrm{TL}_n$, then the corresponding boundary map in $\mathbb{1} \otimes_{\mathrm{TL}_n} C(m)$ is simply the map $\mathbb{1} \to \mathbb{1}$ given by the action of x on $\mathbb{1}$. Thus $\mathbb{1} \otimes_{\mathrm{TL}_n} C(m)$ is nothing other than the complex in Figure 5. (The right hand column indicates the degree.) This is visibly acyclic, and this completes the proof. □

3.2. **The spectral sequence of a double complex.** Since the spectral sequence of a particular kind of double complex is used several times during this paper, we introduce and discuss it in this subsection.

We begin with the homological version. Suppose we have a chain complex Q_* of left TL_n-modules, such as $C(m)$ or $D(m)$, or the complex of planar injective words $W(n)$ to be introduced later. Then we choose a projective resolution P of $\mathbb{1}$ as a right module over TL_n, and we consider the double complex $P_* \otimes_{\mathrm{TL}_n} Q_*$. This is a homological double complex in the sense that both differentials reduce the grading. Associated to this double complex are two spectral sequences, $\{{}^IE^r\}$ and $\{{}^{II}E^r\}$, which both converge to the homology of the totalisation, $H_*(\mathrm{Tot}(P_* \otimes_{\mathrm{TL}_n} Q_*))$ as in Section 5.6 of [Wei94]. The first spectral sequence has E^1-term given by ${}^IE^1_{i,j} = H_j(P_i \otimes_{\mathrm{TL}_n} Q_*) \cong P_i \otimes_{\mathrm{TL}_n} H_j(Q_*)$ with $d^1 \colon {}^IE^1_{i,j} \to {}^IE^1_{i-1,j}$ induced by the differential $P_i \to P_{i-1}$. The isomorphism

$$\begin{tikzcd} \vdots \arrow[d] & \\ \mathbb{1} \arrow[d, "0"] & 3 \\ \mathbb{1} \arrow[d, "1"] & 2 \\ \mathbb{1} \arrow[d, "0"] & 1 \\ \mathbb{1} \arrow[d, "1"] & 0 \\ \mathbb{1} & -1 \end{tikzcd}$$

FIGURE 5. The complex $\mathbb{1} \otimes C(m)$

above holds because each P_i is projective and therefore flat. It follows that the E^2-term is

$$^{I}E^2_{i,j} = \mathrm{Tor}_i^{\mathrm{TL}_n}(\mathbb{1}, H_j(Q_*))$$

The second spectral sequence has E^1-term given by $^{II}E^1_{i,j} = H_j(P_* \otimes_{\mathrm{TL}_n} Q_i)$, i.e.

$$^{II}E^1_{i,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, Q_i)$$

with $d^1 \colon {}^{II}E^1_{i,j} \to {}^{II}E^1_{i-1,j}$ induced by the boundary maps of Q_*.

We now consider the cohomological version. Suppose we have a chain complex Q_* of left TL_n-modules, again such as $C(m)$, $D(m)$ or $W(n)$ (the latter to be introduced later). Then we choose an injective resolution I^* of $\mathbb{1}$ as a left module over TL_n, and we consider the double complex $\mathrm{Hom}_{\mathrm{TL}_n}(Q_*, I^*)$. This is a cohomological double complex in the sense that both differentials increase the grading. Associated to this double complex are two spectral sequences, $\{^{I}E^r\}$ and $\{^{II}E^r\}$, both converging to the cohomology of the totalisation, $H^*(\mathrm{Tot}(\mathrm{Hom}_{\mathrm{TL}_n}(Q_*, I^*)))$ as in Section 5.6 of [Wei94]. The first spectral sequence has E_1-term given by $^{I}E_1^{i,j} = H^j(\mathrm{Hom}_{\mathrm{TL}_n}(Q_*, I^i)) \cong \mathrm{Hom}_{\mathrm{TL}_n}(H_j(Q_*), I^i)$ with $d^1 \colon {}^{I}E^1_{i,j} \to {}^{I}E^1_{i+1,j}$ induced by the differential of I^*. The isomorphism above holds because each I^i is injective, so that the functor $\mathrm{Hom}_{\mathrm{TL}_n}(-, I^i)$ is exact. It follows that the E_2-term is

$$^{I}E_2^{i,j} = \mathrm{Ext}^i_{\mathrm{TL}_n}(\mathbb{1}, H_j(Q_*))$$

The second spectral sequence has E_1-term ${}^{II}E_1^{i,j} = H^j(\mathrm{Hom}_{\mathrm{TL}_n}(Q_i, I^*))$, i.e.

$$ {}^{II}E_1^{i,j} = \mathrm{Ext}^j_{\mathrm{TL}_n}(\mathbb{1}, Q_i) $$

with differential $d^1\colon {}^{II}E_1^{i,j} \to {}^{II}E_1^{i+1,j}$ induced by the differential of Q_*.

3.3. **Proof of Theorems A and E.** We can now prove Claim 3.1 (which implies Theorem A) and Theorem E. The proofs of the two results will be almost identical except that the former uses the complex $C(m)$ and the latter uses the complex $D(m)$. Moreover, each result has a homological and cohomological part, referring to Tor and Ext respectively. In each case the two parts are proved similarly, by using either the homological or cohomological spectral sequence from Section 3.2 above. We will therefore only prove the homological part of Claim 3.1, leaving the details of the other parts to the reader.

Let us now prove the homological part of Claim 3.1, i.e. we will prove that $\mathrm{Tor}_*^{\mathrm{TL}_n}(\mathbb{1}, \mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1})$ vanishes in positive degrees.

We prove the claim by fixing n and using strong induction on m in the range $n \geqslant m \geqslant 0$. As noted before, the initial cases $m = 0, 1$ of the are immediate since then TL_m is the ground ring and $\mathrm{TL}_n \otimes_{\mathrm{TL}_n} \mathbb{1} \cong \mathrm{TL}_n$ is free. We therefore fix m in the range $2 \leqslant m \leqslant n$.

We now employ the homological spectral sequences $\{{}^I E^r\}$ and $\{{}^{II}E^r\}$ of Section 3.2, in the case $Q = C(m)$. Then ${}^I E^2_{i,j} = \mathrm{Tor}_i^{\mathrm{TL}_n}(\mathbb{1}, H_j(C(m))) = 0$ for all i and j, since $C(m)$ is acyclic by Lemma 3.6. Thus $\{{}^I E^r\}$ converges to zero, and the same must therefore be true of $\{{}^{II}E^r\}$, since both spectral sequences have the same target. In the second spectral sequence the E^1-page

$$ {}^{II}E^1_{i,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, C(m)_i) $$

is largely known to us. The bottom $j = 0$ row of ${}^{II}E^1$ is precisely the complex $\mathbb{1} \otimes_{\mathrm{TL}_n} C(m)$, which is acyclic by Lemma 3.7. And when $i \geqslant 0$, the term $C(m)_i$ is either $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-1}} \mathbb{1}$ or $\mathrm{TL}_n \otimes_{\mathrm{TL}_{m-2}} \mathbb{1}$, and our inductive hypothesis applies to these ($(m-1) < m$ and $(m-2) < m$) to show that ${}^{II}E^1_{i,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, C(m)_i) = 0$ when $j > 0$. See Figure 6 for a visualisation of the E^1 term. Altogether, this tells us that ${}^{II}E^2_{i,j}$ vanishes except for the groups

$$ {}^{II}E^2_{-1,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, C(m)_{-1}) = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, \mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}) $$

for $j > 0$, which are concentrated in a single column and therefore not subject to any further differentials. Thus ${}^{II}E^2 = {}^{II}E^\infty$. But we know that ${}^{II}E^\infty$ vanishes identically, so that the inductive hypothesis is proved, and so, therefore, is the proof of the homological part of Claim 3.1.

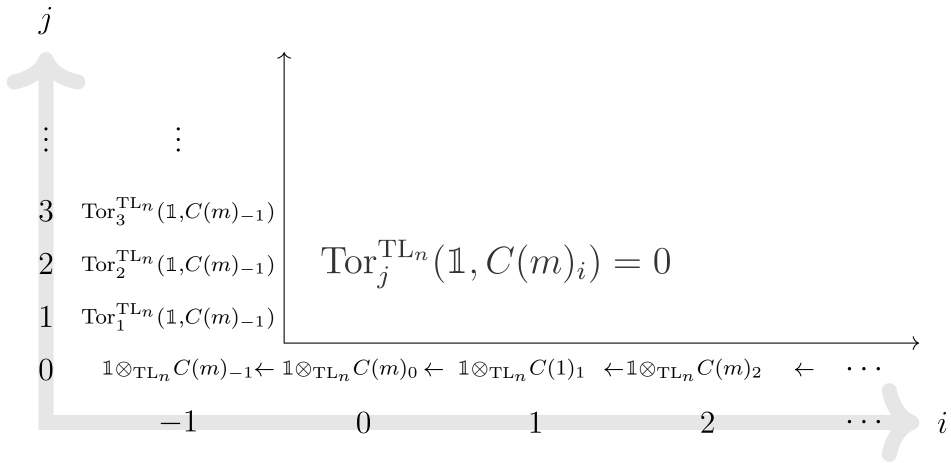


Figure 6. The page ${}^{II}E^1$. The only differentials that affect the ${}^{II}E^2$ page are shown on the $j=0$ row.

4. Planar injective words

Throughout this section we will consider the Temperley-Lieb algebra $\mathrm{TL}_n(a) = \mathrm{TL}_n(v+v^{-1})$, where $v \in R^\times$. We will make use of the elements $s_1,\dots,s_{n-1}$ of Definition 2.20.

Definition 4.1. For $n \geqslant 0$ we define a chain complex $W(n)_*$ of $\mathrm{TL}_n(a)$-modules as follows. For i in the range $-1 \leqslant i \leqslant n-1$, the degree-$i$ part of $W(n)_*$ is defined by

$$W(n)_i = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}$$

and in all other degrees we set $W(n)_i = 0$. Note that

$$W(n)_{-1} = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_n(a)} \mathbb{1} = \mathbb{1}.$$

For $i \geqslant 0$ the boundary map $d^i\colon W(n)_i \to W(n)_{i-1}$ is defined to be the alternating sum $\sum_{j=0}^{i}(-1)^j d^i_j$, where

$$d^i_j\colon W(n)_i \longrightarrow W(n)_{i-1}$$

is the map

$$d^i_j\colon \mathrm{TL}_n(a)\otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1} \longrightarrow \mathrm{TL}_n(a)\otimes_{\mathrm{TL}_{n-i}(a)} \mathbb{1}$$

defined by

$$d^i_j(x\otimes r) = (x\cdot s_{n-i+j-1}\cdots s_{n-i})\otimes \lambda^{-j}r.$$

In the expression $s_{n-i+j-1}\cdots s_{n-i}$, the indices decrease from left to right. For notational purposes we will write $W(n)$ and only use a subscript when identifying a particular degree.

Thus, for example, the product is $s_{n-i+1}s_{n-i}$ when $j=2$, it is s_{n-i} when $j=1$, and it is trivial (the unit element) when $j=0$ (the latter point can be regarded

$$
\begin{array}{ccc}
 & \mathrm{TL}_n \otimes_{\mathrm{TL}_0} \mathbb{1} & n-1 \\
d_0^{n-1} - d_1^{n-1} + \cdots + (-1)^{n-1} d_{n-1}^{n-1} & \downarrow & \\
 & \mathrm{TL}_n \otimes_{\mathrm{TL}_1} \mathbb{1} & n-2 \\
d_0^{n-2} - d_1^{n-2} + \cdots + (-1)^{n-2} d_{n-2}^{n-2} & \downarrow & \\
 & \vdots & \\
 & \downarrow & \\
 & \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-3}} \mathbb{1} & 2 \\
d_0^2 - d_1^2 + d_2^2 & \downarrow & \\
 & \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-2}} \mathbb{1} & 1 \\
d_0^1 - d_1^1 & \downarrow & \\
 & \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-1}} \mathbb{1} & 0 \\
 & \downarrow & \\
 & \mathbb{1} & -1
\end{array}
$$

FIGURE 7. The complex $W(n)$.

as a convention if one wishes). Observe that d_j is well-defined because the elements $s_{n-i}, \ldots, s_{n-i+j-1}$ all commute with all generators of TL_{n-i-1}. We verify in Lemma 4.8 below that iterated differentials vanish. We have depicted $W(n)$ in Figure 7.

Remark 4.2. Let us explain the motivation for the definition of $W(n)$. Let $\mathfrak{S}_n$ denote the symmetric group on n letters. The *complex of injective words* is the chain complex $\mathcal{C}(n)$ of $\mathfrak{S}_n$-modules, concentrated in degrees -1 to $(n-1)$, that in degree i is the free R-module with basis given by tuples $(x_0, \ldots, x_i)$ where $x_0, \ldots, x_i \in \{1, \ldots, n\}$ and no letter appears more than once. We allow the empty word $()$, which lies in degree -1. The differential of $\mathcal{C}(n)$ sends a word $(x_0, \ldots, x_i)$ to the alternating sum $\sum_{j=0}^{i} (-1)^j (x_0, \ldots, \widehat{x_j}, \ldots, x_i)$. A theorem of Farmer [Far79] shows that the homology of $\mathcal{C}(n)$ vanishes in degrees $i \leqslant (n-2)$, and the same result has been proved since then by many authors [Maa79, BW83, Ker05, RW13]. The complex of injective words has been used by several authors to prove homological stability for the symmetric groups [Maa79, Ker05, RW13].

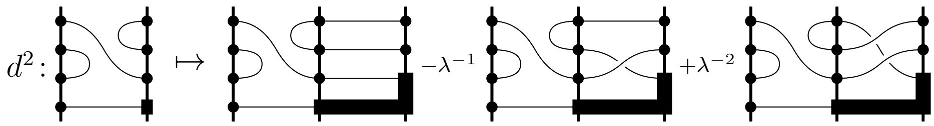
For this paragraph only, let us abuse our established notation and denote by $s_1, \ldots, s_{n-1} \in \mathfrak{S}_n$ the elements defined by $s_i = (i \quad i+1)$, the transposition of i with $i+1$. Then these elements satisfy the braid relations, i.e. the second and third identities of Definition 2.20. The complex of injective words $\mathcal{C}(n)$ can be rewritten in terms of the group ring $R\mathfrak{S}_n$ and the elements s_i. Indeed, it is shown in [Hep20] that $\mathcal{C}(n)_i \cong R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i-1}} \mathbb{1}$, where $\mathbb{1}$ is the trivial module of $R\mathfrak{S}_{n-i-1}$, and that under this isomorphism the differential $d^i\colon \mathcal{C}(n)_i \to \mathcal{C}(n)_{i-1}$ becomes the map

$$d^i\colon R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i-1}} \mathbb{1} \longrightarrow R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i}} \mathbb{1}$$

defined by $d^i(x \otimes 1) = \sum_{j=0}^{i}(-1)^j x \cdot (s_{n-i+j-1} \cdots s_{n-i}) \otimes 1$. (There are no constants λ in this expression). Comparing this description of $\mathcal{C}(n)$ with our definition of $W(n)$, we see that our complex of planar injective words is precisely analogous to the original complex of injective words, after systematically replacing the group algebras of symmetric groups with the Temperley-Lieb algebras. The lack of constants in the differential for $\mathcal{C}(n)$ is explained by the fact that the effect of s_i on $\mathbb{1}$ is multiplication by λ in the Temperley-Lieb setting, and multiplication by 1 in the symmetric group setting.

Since we regard the Temperley-Lieb algebra as the planar analogue of the symmetric group, we chose the name *planar injective words* for our complex $W(n)$. This seemed the least discordant way of giving our complex an appropriate name. See the next remark for a means of picturing the complex.

Remark 4.3. Let us describe a method for visualising $W(n)$. Recall from the diagrammatic description of $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ when $m \leqslant n$ given in Remark 2.10 that elements of $W(n)_i$ can be regarded as diagrams where the first $n-i-1$ dots on right are encapsulated within a black box, and if any cups can be absorbed into the black box, then the diagram is identified with 0. The differential $d^i\colon W(n)_i \to W(n)_{i-1}$ is then given by pasting special elements onto the right of a diagram, followed by taking their signed and weighted sum. These special elements each enlarge the black box by an extra strand, and plumb one of the free strands into the new space in the black box: Here is an example for $n = 4$ and $i = 2$.



The resulting diagrams can be simplified using the smoothing rules for diagrams with crossings described in Remark 2.21. We leave it to the reader to make this description as precise as they wish, and note here that this is where the notion of *braiding*, so often seen in homological stability arguments, fits into our set up.

Remark 4.4. Readers who are familiar with the theory will recognise that $W(n)$ is the chain complex associated to an augmented semi-simplicial $\mathrm{TL}_n(a)$-module. We will not make use of this point of view.

The main result about the complex of planar injective words is the following, which we recall from the introduction. It is analogous to the homological property of the complex of injective words first proved by Farmer [Far79].

Theorem D. *The homology of $W(n)$ vanishes in degrees $d \leqslant (n-2)$.*

The proof of Theorem D is the most technical part of this work, and is proved in Section 8.

The complex of injective words on n letters has rich combinatorial features: its Euler characteristic is the number of derangements of $\{1, \ldots, n\}$; when one works over $\mathbb{C}$, its top homology has a description as a virtual representation that categorifies a well-known alternating sum formula for the number of derangements; and again when one works over $\mathbb{C}$, its top homology has a compact description in terms of Young diagrams and counts of standard Young tableaux. In the associated paper [BH20] we establish analogues of these for the complex of planar injective words. In particular we show that the rank of $H_{n-1}(W(n))$ is the n-th *Fine number* [DS01]. (The rank of the Temperley-Lieb algebra is the n-th *Catalan number*, which is the number of Dyck paths of length $2n$. The n-th Fine number is the number of Dyck paths of length $2n$ whose first peak occurs at an even height, and as we explain in [BH20], it is an analogue of the number of derangements.) We also discover a new feature of the complex: the differentials have a alternate expression in terms not of the s_i but of the U_i. This expression demonstrates a connection with the *Jacobsthal numbers*, and we will briefly explain the result for the top differential below.

The top homology of the Tits building is known as the *Steinberg module*. This inspires the name in the following definition.

Definition 4.5. We define the *n-th Fineberg module* to be the $\mathrm{TL}_n(a)$-module $\mathcal{F}_n(a) = H_{n-1}(W(n))$. We often suppress the a and simply write $\mathcal{F}_n$. The rank of $\mathcal{F}_n$ is the n-th Fine number F_n.

The Fineberg module is an important ingredient in the full statement of our stability result, Theorem 5.1. In order to detect the non-zero homology group appearing in Theorem C we need to study it in more detail using the connection with Jacobsthal numbers from [BH20].

The n-th *Jacobsthal number* J_n [Slo] is (among other things) the number of sequences $n > a_1 > a_2 > \cdots > a_r > 0$ whose initial term has the opposite parity to n. Some examples, when $n = 4$, are 3, 1, $3 > 2$, $3 > 1$ and $3 > 2 > 1$. (We allow the empty sequence, and say that by convention its initial term is $a_1 = 0$ and $r = 0$. Of course this only occurs when n is odd.) Another viewpoint of J_n in terms of compositions of n is given in [BH20].

Definition 4.6. Let $a = v + v^{-1}$ where $v \in R^\times$ is a unit. We define the *Jacobsthal* element in $\mathrm{TL}_n(a)$ as follows:

$$\mathcal{J}_n = \sum_{\substack{n > a_1 > \cdots > a_r > 0 \\ n - a_1 \text{ odd}}} (-1)^{(r-1)+n} \left(\frac{\mu}{\lambda}\right)^r U_{a_1} \cdots U_{a_r}$$

Recall we allow the empty sequence ($a_1 = 0$ and $r = 0$) when n is odd. This corresponds to a constant summand 1 in $\mathcal{J}_n$ for odd n. Note that the number of irreducible terms in $\mathcal{J}_n$ is J_n.

Since $\mathcal{F}_n$ is the homology of $W(n)$ in the top degree, it is simply the kernel of the top differential $d^{n-1}\colon W(n)_{n-1} \to W(n)_{n-2}$. There are identifications

$$W(n)_{n-1} = \mathrm{TL}_n(a)\otimes_{\mathrm{TL}_0(a)}\mathbb{1} \cong \mathrm{TL}_n(a) \text{ and } W(n)_{n-2} \cong \mathrm{TL}_n(a)\otimes_{\mathrm{TL}_1(a)}\mathbb{1} \cong \mathrm{TL}_n(a).$$

The following is shown in [BH20]:

Proposition 4.7. *Under the above identifications, the top differential of $W(n)$ is right-multiplication by $\mathcal{J}_n$. In particular, there is an exact sequence*

$$0 \longrightarrow \mathcal{F}_n(a) \longrightarrow \mathrm{TL}_n(a) \xrightarrow{-\cdot\mathcal{J}_n} \mathrm{TL}_n(a).$$

We conclude this section with the long overdue verification that $W(n)$ is indeed a complex.

Lemma 4.8. *The boundary maps of $W(n)$ satisfy $d^{i-1}\circ d^i = 0$.*

Proof. We will show that if $i \geqslant 1$ and $0 \leqslant j < k \leqslant i$, then the composite maps $d^{i-1}_j d^i_k, d^{i-1}_{k-1}d^i_j\colon W(n)_i \to W(n)_{i-2}$ coincide. The fact that $d\circ d$ vanishes then follows. We have

$$d^{i-1}_j d^i_k(x\otimes r) = [x\cdot(s_{n-i+k-1}\cdots s_{n-i})\cdot(s_{n-i+j}\cdots s_{n-i+1})]\otimes \lambda^{-(j+k)}r$$

and

$$d^{i-1}_{k-1}d^i_j(x\otimes r) = [x\cdot(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i+1})]\otimes \lambda^{-(j+k-1)}r.$$

Now, by repeated use of the braid relations on the s_k, we have

$$\begin{aligned}
&(s_{n-i+k-1}\cdots s_{n-i})\cdot(s_{n-i+j}\cdots s_{n-i+1})\\
=&(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i})\\
=&(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i+1})\cdot s_{n-i}
\end{aligned}$$

so that

$$\begin{aligned}
d^{i-1}_j d^i_k(x\otimes r) &= [x\cdot(s_{n-i+k-1}\cdots s_{n-i})\cdot(s_{n-i+j}\cdots s_{n-i+1})]\otimes\lambda^{-(j+k)}r\\
&= [x\cdot(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i+1})\cdot s_{n-i}]\otimes\lambda^{-(j+k)}r\\
&= [x\cdot(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i+1})]\otimes s_{n-i}\cdot(\lambda^{-(j+k)}r)\\
&= [x\cdot(s_{n-i+j-1}\cdots s_{n-i})\cdot(s_{n-i+k-1}\cdots s_{n-i+1})]\otimes\lambda^{-(j+k-1)}r\\
&= d^{i-1}_{k-1}d^i_j(x\otimes r)
\end{aligned}$$

where the third equality holds because this computation takes place in $W(n)_{i-2} = \mathrm{TL}_n\otimes_{\mathrm{TL}_{n-i+1}}\mathbb{1}$ and $s_{n-i}\in\mathrm{TL}_{n-i+1}$. $\square$

5. Homological stability and stable homology

The aim of this section is to prove the following result. Theorem B is an immediate consequence, and Theorem C will be proved in the next section as a corollary of it.

Theorem 5.1. *Let R be a commutative ring, let $v \in R$ be a unit, and let $a = v + v^{-1}$. Then for n odd we have:*

$$\operatorname{Tor}_i^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & i = 0 \\ 0 & 1 \leqslant i \leqslant (n-1) \\ \operatorname{Tor}_{i-n}^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathcal{F}_n(a)) & i \geqslant n \end{cases}$$

And for n even we have

$$\operatorname{Tor}_i^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & i = 0 \\ 0 & 1 \leqslant i \leqslant (n-2) \\ \operatorname{Tor}_{i-n}^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathcal{F}_n(a)) & i \geqslant (n+1) \end{cases}$$

for $i \neq n-1, n$, while in degrees $(n-1)$ and n there is an exact sequence

$$0 \to \operatorname{Tor}_n^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \to \mathbb{1} \otimes_{\mathrm{TL}_n(a)} \mathcal{F}_n(a) \to \mathbb{1} \to \operatorname{Tor}_{n-1}^{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \to 0 \tag{1}$$

Analogous results hold for the Ext*-groups. For n odd we have:*

$$\operatorname{Ext}^i_{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & i = 0 \\ 0 & 1 \leqslant i \leqslant (n-1) \\ \operatorname{Ext}^{i-n}_{\mathrm{TL}_n(a)}(\mathcal{F}_n(a), \mathbb{1}) & i \geqslant n \end{cases}$$

And for n even we have

$$\operatorname{Ext}^i_{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & i = 0 \\ 0 & 1 \leqslant i \leqslant (n-2) \\ \operatorname{Ext}^{i-n}_{\mathrm{TL}_n(a)}(\mathcal{F}_n(a), \mathbb{1}) & i \geqslant (n+1) \end{cases}$$

for $i \neq n-1, n$, while in degrees $(n-1)$ and n there is an exact sequence

$$0 \to \operatorname{Ext}^{n-1}_{\mathrm{TL}_n(a)}(\mathcal{F}_n(a), \mathbb{1}) \to \mathbb{1} \to \operatorname{Hom}_{\mathrm{TL}_n(a)}(\mathcal{F}_n(a), \mathbb{1}) \to \operatorname{Ext}^n_{\mathrm{TL}_n(a)}(\mathbb{1}, \mathbb{1}) \to 0 \tag{2}$$

The central maps of (1) and (2) are described as follows. Regard $\mathcal{F}_n(a)$ as a left-submodule of $\mathrm{TL}_n(a)$ as in Proposition 4.7. Then the maps are

$$\mathbb{1} \otimes_{\mathrm{TL}_n(a)} \mathcal{F}_n(a) \longrightarrow \mathbb{1}, \qquad x \otimes f \longmapsto x \cdot f$$

and

$$\mathbb{1} \longrightarrow \operatorname{Hom}_{\mathrm{TL}_n(a)}(\mathcal{F}_n(a), \mathbb{1}), \qquad x \longmapsto (f \mapsto f \cdot x)$$

where $x \cdot f$ and $f \cdot x$ denote the action of $f \in \mathcal{F}_n(a) \subseteq \mathrm{TL}_n(a)$ on the right and left of $\mathbb{1}$, respectively.

In order to prove this theorem, we will use the complex of planar injective words $W(n)$ introduced in the previous section. Recall that the Fineberg module $\mathcal{F}_n$ appearing in the statement is the top homology group $H_{n-1}(W(n))$.

Lemma 5.2. *The homology groups of both the complex $\mathbb{1} \otimes_{\mathrm{TL}_n(a)} W(n)$ and the complex $\mathrm{Hom}_{\mathrm{TL}_n(a)}(W(n), \mathbb{1})$ are concentrated in degree $(n-1)$, where in both cases they are given by $\mathbb{1}$ if n is even and 0 if n is odd.*

Proof. We have $W(n)_i = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$, and the boundary map $d^i \colon W(n)_i \to W(n)_{i-1}$ is given by $x \otimes r \mapsto x \cdot D_i \otimes r$, where $D_i = \sum_{j=0}^{i} (-1)^j s_{n-i+j-1} \cdots s_{n-i} \lambda^{-j}$.

Now $\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)_i = \mathbb{1} \otimes_{\mathrm{TL}_n} (\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}) \cong \mathbb{1}$, and under these isomorphisms the boundary map originating in degree i becomes the action on $\mathbb{1}$ of the element D_i. Similarly, $\mathrm{Hom}_{\mathrm{TL}_n}(W(n)_i, \mathbb{1}) = \mathrm{Hom}_{\mathrm{TL}_n}(\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}, \mathbb{1}) \cong \mathbb{1}$, and under these isomorphisms the boundary map originating in degree $(i-1)$ becomes the action of the element D_i on $\mathbb{1}$.

The action of $s_{n-i+j-1} \cdots s_{n-i}$ on $\mathbb{1}$ is simply multiplication by λ^j, with one factor of λ for each s term (recall $s_i = \mu U_i + \lambda$). Thus the action of D_i on $\mathbb{1}$ is nothing other than multiplication by $\sum_{j=0}^{i} (-1)^j$, which is 0 for i odd and 1 for i even.

So altogether $\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)$ and $\mathrm{Hom}_{\mathrm{TL}_n}(W(n), \mathbb{1})$ are isomorphic to complexes with a copy of R in each degree $i = -1, \ldots, (n-1)$ and with boundary maps alternating between the identity map and 0. In $\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)$ the identity maps originate in even degrees, and in $\mathrm{Hom}_{\mathrm{TL}_n}(W(n), \mathbb{1})$ they originate in odd degrees. The claim now follows. $\square$

Proof of Theorem 5.1. We begin with the Tor-case.

In degree $d = 0$ the theorem holds trivially. Recall that P_* is a projective resolution of $\mathbb{1}$ as a right TL_n module. We use the two homological spectral sequences $\{{}^I E^r\}$ and $\{{}^{II} E^r\}$ associated to $W(n)$ as described in Section 3.2.

Let us consider $\{{}^I E^r\}$. We have

$$ {}^I E^2_{i,j} = \begin{cases} \mathrm{Tor}_i^{\mathrm{TL}_n}(\mathbb{1}, \mathcal{F}_n) & j = (n-1) \\ 0 & j \neq (n-1) \end{cases} $$

and consequently the spectral sequence converges to $\mathrm{Tor}^{\mathrm{TL}_n}_{*-n+1}(\mathbb{1}, \mathcal{F}_n)$, for $* = i+j$. The same is therefore true of $\{{}^{II} E^r\}$.

Let us write $\varepsilon_n = H_{n-1}(W(n))$, so that by Lemma 5.2, ε_n is trivial for n odd and $\mathbb{1}$ for n even. Since $\mathcal{F}_n$ consists of the cycles in $W(n)_{n-1}$, the map

$$ \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \mathbb{1} \otimes_{\mathrm{TL}_n} W(n)_{n-1} $$

again lands in the cycles, giving us a map

$$ \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to H_{n-1}(\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)) = \varepsilon_n. $$

When n is even and ε_n is identified with $\mathbb{1}$ as in the lemma, then this map simply becomes the one described in the statement of the theorem.

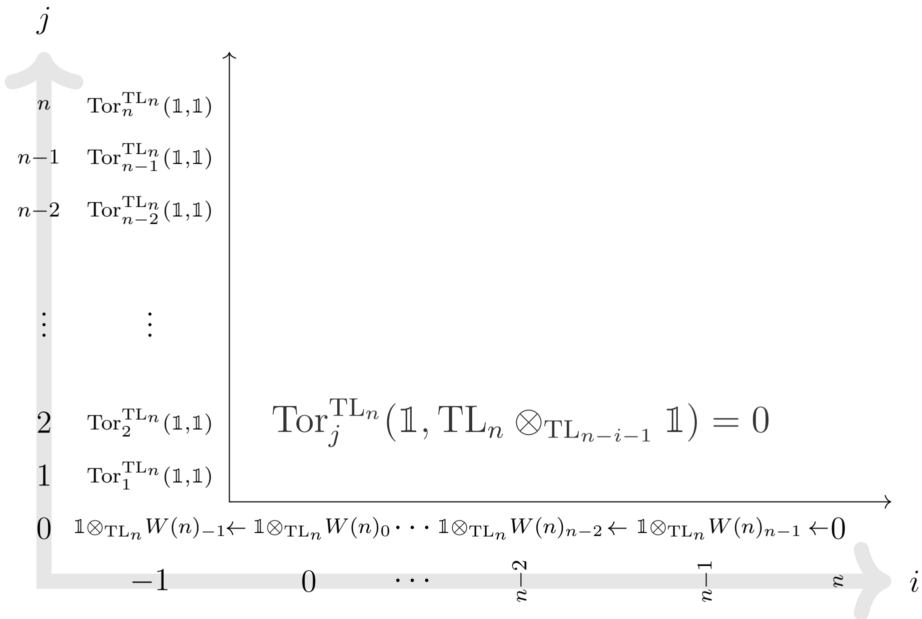


FIGURE 8. The page ${}^{II}E^1$. The only differentials that affect the ${}^{II}E^2$ page are shown on the $j = 0$ row.

We now know that $\{{}^{II}E^r\}$ converges to $\mathrm{Tor}_{*-n+1}^{\mathrm{TL}_n}(\mathbb{1}, \mathcal{F}_n)$. Its E^1-page ${}^{II}E^1_{i,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, W(n)_i)$ is largely known to us. Indeed, when $j = 0$ the terms are $\mathrm{Tor}_0^{\mathrm{TL}_n}(\mathbb{1}, W(n)_i) = \mathbb{1} \otimes_{\mathrm{TL}_n} W(n)_i$, with d^1-maps between them induced by the boundary maps of $W(n)$. In other words, the $j = 0$ part of ${}^{II}E^1_{i,j}$ is precisely the complex $\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)$. When $0 \leqslant i \leqslant (n-1)$, the term $W(n)_i = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$ satisfies $0 \leqslant (n - i - 1) < n$, so that by Theorem E we have

$$ {}^{II}E^1_{i,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}) = 0 $$

for $j > 0$. When $i = -1$ we have $W(n)_{-1} = \mathbb{1}$ so that ${}^{II}E^1_{-1,j} = \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$ for $j > 0$. This is depicted in Figure 8.

By the description in the previous paragraph, we can now identify ${}^{II}E^2_{*,*}$. The only possible differentials are in the $j = 0$ part, which is $\mathbb{1} \otimes_{\mathrm{TL}_n} W(n)$, and whose homology is ε_n concentrated in degree $(n-1)$. Thus ${}^{II}E^2_{*,*}$ is zero except for the following groups:

$$ {}^{II}E^2_{i,j} = \begin{cases} \mathrm{Tor}_j^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}) & i = -1,\ j > 0 \\ \varepsilon_n & i = (n-1),\ j = 0 \end{cases} $$

as depicted in Figure 9.

From the E^2-page onwards there is precisely one possible differential, namely $d^n\colon E^n_{n-1,0} \to E^n_{-1,n-1}$, which is a map $d^n\colon \varepsilon_n \to \mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1})$. It forms part of an

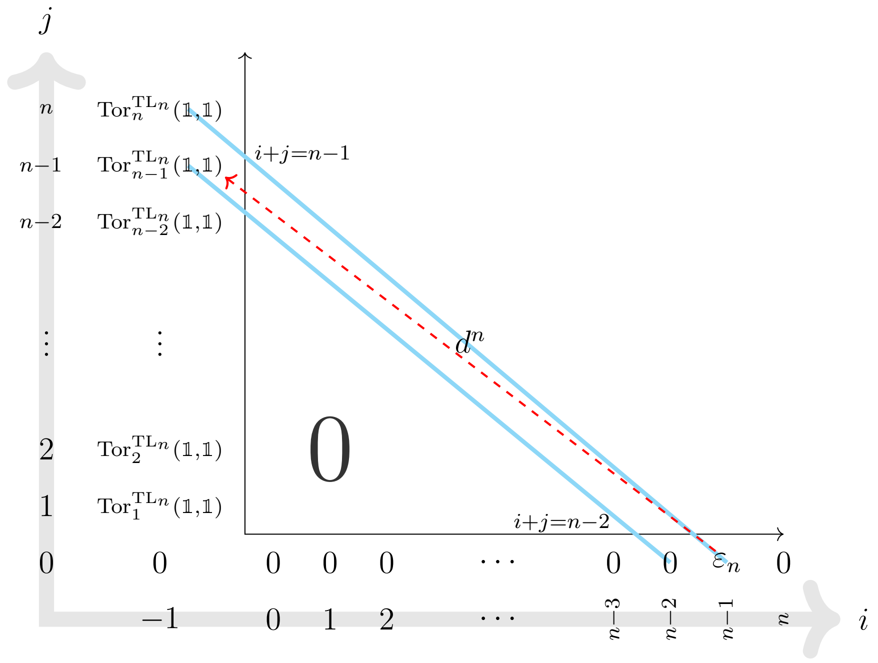


FIGURE 9. The page ${}^{II}E^2$. This page stays constant until ${}^{II}E^n$ where the only possible further differential lies: this is shown in red. The $i+j=n-1$ and $i+j=n-2$ diagonals are indicated in blue.

exact sequence

$$0 \to {}^{II}E^\infty_{n-1,0} \to \varepsilon_n \xrightarrow{d^n} \mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1}) \to {}^{II}E^\infty_{-1,n-1} \to 0$$

In ${}^{II}E^\infty_{*,*}$, each total degree has only one non-zero group, except (possibly) for total degree $(n-1)$, where we have the two groups ${}^{II}E^\infty_{-1,n}$ and ${}^{II}E^\infty_{n-1,0}$. The relationship between the infinity-page of a spectral sequence and the sequence's target now give us a short exact sequence:

$$0 \to {}^{II}E^\infty_{-1,n} \to \mathrm{Tor}_0^{\mathrm{TL}_n}(\mathbb{1},\mathcal{F}_n) \to {}^{II}E^\infty_{n-1,0} \to 0$$

The last two exact sequences combine to give us:

$$0 \to {}^{II}E^\infty_{-1,n} \to \mathrm{Tor}_0^{\mathrm{TL}_n}(\mathbb{1},\mathcal{F}_n) \to \varepsilon_n \to \mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1}) \to {}^{II}E^\infty_{-1,n-1} \to 0$$

The leftmost term is ${}^{II}E^\infty_{-1,n} = {}^{II}E^2_{-1,n} = \mathrm{Tor}_n^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1})$. And ${}^{II}E^\infty_{-1,n-1}$ is the only group in total degree $(n-2)$, and therefore coincides with $\mathrm{Tor}^{\mathrm{TL}_n}_{(n-2)-n+1}(\mathbb{1},\mathcal{F}_n) = \mathrm{Tor}_{-1}^{\mathrm{TL}_n}(\mathbb{1},\mathcal{F}_n) = 0$. And $\mathrm{Tor}_0^{\mathrm{TL}_n}(\mathbb{1},\mathcal{F}_n) = \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n$. So the last exact sequence becomes:

$$0 \to \mathrm{Tor}_n^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1}) \to \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \varepsilon_n \to \mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1},\mathbb{1}) \to 0$$

We claim that the map $\mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \varepsilon_n$ in this sequence is the one described above. Let $\mathcal{F}_n[n-1]$ be the complex consisting of a copy of $\mathcal{F}_n$ concentrated in degree $n-1$. There is a natural inclusion of chain complexes $\mathcal{F}_n[n-1] \hookrightarrow W(n)$, and this leads to a map of double complexes and then of spectral sequences. The map $\mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \varepsilon_n$ can be identified using this map of spectral sequences.

It follows from the sequence that in the case n odd, when $\varepsilon_n = 0$, the final term satisfies $\mathrm{Tor}^{\mathrm{TL}_n}_{n-1}(\mathbb{1}, \mathbb{1}) = 0$, and the first two terms satisfy

$$\mathrm{Tor}^{\mathrm{TL}_n}_{n}(\mathbb{1}, \mathbb{1}) \cong \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n = \mathrm{Tor}^{\mathrm{TL}_n}_{0}(\mathbb{1}, \mathcal{F}_n)$$

as required.

The previous discussion determines what happens in total degrees $(n-1)$ and $(n-2)$. In total degrees d other than $(n-1)$ and $(n-2)$, and when $j > 0$, the only term on the E^∞ page is ${}^{II}E^\infty_{-1,d+1} = \mathrm{Tor}^{\mathrm{TL}_n}_{d+1}(\mathbb{1}, \mathbb{1})$, which must therefore equal $\mathrm{Tor}^{\mathrm{TL}_n}_{d-n+1}(\mathbb{1}, \mathcal{F}_n)$. Thus $\mathrm{Tor}^{\mathrm{TL}_n}_{d}(\mathbb{1}, \mathbb{1}) \cong \mathrm{Tor}^{\mathrm{TL}_n}_{d-n}(\mathbb{1}, \mathcal{F}_n)$ for $d \neq n, n-1$. This completes the proof.

For the Ext-case we use the two cohomological spectral sequences associated to $W(n)$ as in Section 3.2, and then proceed dually to the above. We leave the details to the reader. ☐

6. Sharpness

We recall the statement of Theorem C from the introduction.

Theorem C. *Let n be even and suppose that a is not a unit. Then $\mathrm{Tor}^{\mathrm{TL}_n(a)}_{n-1}(\mathbb{1}, \mathbb{1})$ is non-zero.*

Let $\mathcal{I} \subseteq \mathrm{TL}_n$ denote the left-ideal generated by all diagrams which have a cup on the right in positions other than 1, together with all multiples of a. Thus

$$\mathcal{I} = (\mathrm{TL}_n \cdot a) + (\mathrm{TL}_n \cdot U_2) + \cdots + (\mathrm{TL}_n \cdot U_{n-1}).$$

Lemma 6.1. *Let n be even or odd, and let $1 \leqslant p \leqslant n-1$. Then $U_p \cdot \mathcal{J}_n \in \mathcal{I}$.*

Proof. Recall that the monomials appearing in $\mathcal{J}_n$ are those of the form $U_{i_1} \cdots U_{i_r}$ where $(n-1) \geqslant i_1 > i_2 \cdots > i_r \geqslant 1$ and $i_1 \equiv (n-1) \mod 2$, and that such a monomial appears in $\mathcal{J}_n$ with coefficient $(-1)^{(r-1)+n}(\frac{\mu}{\lambda})^r$. We write $\mathcal{J}_n = K_n + L_n$ where K_n is the part of $\mathcal{J}_n$ featuring monomials of the form $U_i U_{i-1} \cdots U_1$ for $i \equiv n-1 \bmod 2$ in the range $1 \leqslant i \leqslant n-1$, and L_n is the part of $\mathcal{J}_n$ featuring the remaining monomials.

If $U_{i_1} \cdots U_{i_r}$ is a monomial appearing in L_n, then it must either end in U_{i_r} for $n-1 \geqslant i_r > 1$ or end in a monomial of the form $U_{i_j} \cdot U_{i_{j-1}} \cdots U_{i_r} = (U_{i_{j-1}} \cdots U_1) \cdot U_{i_j}$ for some $i_j \geqslant i_{j-1} + 2$, $i_{j-1} \geqslant 1$ and hence must lie in $\mathcal{I}$. Thus $L_n \in \mathcal{I}$, and to prove the lemma it will be sufficient to show that $U_p \cdot K_n \in \mathcal{I}$.

Now observe that

$$K_n = \sum_{\substack{0 \leqslant i \leqslant (n-1) \\ i \equiv n-1 \bmod 2}} (-1)^{(i-1)+n} \left(\frac{\mu}{\lambda}\right)^i \cdot U_i U_{i-1} \cdots U_1.$$

(In the case $i = 0$ the product $U_i \cdots U_1$ is empty and therefore equal to 1. This term only appears in K_n when n is odd.) Suppose that $U_i \cdots U_1$ is a monomial appearing in the above sum. Then:

$$U_p \cdot (U_i \cdots U_1) = \begin{cases} (U_p \cdots U_1) \cdot (U_i \cdots U_{p+2}) & p \leqslant i-2 \\ U_{i-1} \cdots U_1 & p = i-1 \\ a \cdot U_i \cdots U_1 & p = i \\ U_{i+1} \cdots U_1 & p = i+1 \\ (U_i \cdots U_1) \cdot U_p & p \geqslant i+2 \end{cases}$$

Thus $U_p \cdot (U_i \cdots U_1) \in \mathcal{I}$ except for the cases $i = p-1$, $i = p+1$. When $p \equiv (n-1) \bmod 2$ these exceptional cases never occur, since we have assumed $i \equiv (n-1) \bmod 2$, and so $U_p \cdot K_n \in \mathcal{I}$ as required. And when $p \equiv n \bmod 2$, we can compute the contribution from the two exceptional cases to find that, modulo $\mathcal{I}$, $U_p \cdot \mathcal{J}_n$ is equal to

$$\begin{aligned} & (-1)^{(p-2)+n} \left(\frac{\mu}{\lambda}\right)^{p-1} U_p \cdot (U_{p-1} \cdots U_1) + (-1)^{p+n} \left(\frac{\mu}{\lambda}\right)^{p+1} U_p \cdot (U_{p+1} \cdots U_1) \\ = \;& (-1)^{(p-2)+n} \left(\frac{\mu}{\lambda}\right)^{p-1} \cdot (U_p \cdots U_1) + (-1)^{p+n} \left(\frac{\mu}{\lambda}\right)^{p+1} \cdot (U_p \cdots U_1) \\ = \;& (-1)^{p+n} \left(\frac{\mu}{\lambda}\right)^{p} \left[\left(\frac{\mu}{\lambda}\right)^{-1} + \left(\frac{\mu}{\lambda}\right)^{1} \right] \cdot (U_p \cdots U_1) \in \mathcal{I}. \end{aligned}$$

Now from Definition 2.19 we have either $(\mu, \lambda) = (v, -1)$ or $(\mu, \lambda) = (-v, v^2)$. In both cases the square bracket above evaluates to $-a$ (recall $a = v + v^{-1}$). Thus $U_p \cdot K_n$ is a multiple of a and therefore in $\mathcal{I}$ as required. □

Lemma 6.2. *Let n be even or odd. Let $x \in \mathcal{F}_n(a)$, so that $x \cdot \mathcal{J}_n = 0$. Then the constant term of x is a multiple of a.*

Proof. Let b be the constant term of x, so that x is equal to b plus a linear combination of left-multiples of the elements $U_1, \ldots, U_{n-1}$. Thus $x \cdot \mathcal{J}_n$ is equal to $b \cdot \mathcal{J}_n$ plus a linear combination of left-multiples of $U_1 \cdot \mathcal{J}_n, \ldots, U_{n-1} \cdot \mathcal{J}_n$, all of which lie in $\mathcal{I}$ by Lemma 6.1. Thus $x \cdot \mathcal{J}_n = b \cdot \mathcal{J}_n$ modulo $\mathcal{I}$.

As an R-module, the quotient $\mathrm{TL}_n/\mathcal{I}$ is isomorphic to the direct sum of copies of R/aR, with one summand for each monomial whose Jones normal form ends with U_1. We have that

$$\mathcal{J}_n = (-1)^n \left[\left(\frac{\mu}{\lambda}\right) U_1 + \left(\frac{\mu}{\lambda}\right)^3 U_3 U_2 U_1 + \cdots \right] \text{ in } \mathrm{TL}_n/\mathcal{I}$$

and it follows that

$$b \cdot \mathcal{J}_n = (-1)^n \left[b \left(\frac{\mu}{\lambda}\right) U_1 + b \left(\frac{\mu}{\lambda}\right)^3 U_3 U_2 U_1 + \cdots \right] \text{ in } \mathrm{TL}_n/\mathcal{I},$$

so b must vanish in R/aR. □

Lemma 6.3. *Let n be even. Then the image of the map*

$$\mathbb{1} \otimes_{\mathrm{TL}_n(a)} \mathcal{F}_n(a) \to \mathbb{1}, \qquad 1 \otimes x \mapsto 1 \cdot x,$$

is contained in the ideal generated by a.

Proof. Since the elements U_p act on $\mathbb{1}$ as multiplication by 0, the map above simply sends $1 \otimes x$ to the constant term of x. But the previous lemma tells us that the constant term of x is a multiple of a. $\square$

Proof of Theorem C. Let n be even. From Theorem 5.1, we have the (fairly short) exact sequence

$$0 \to \mathrm{Tor}_n^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}) \to \mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \mathbb{1} \to \mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}) \to 0$$

and the image of $\mathbb{1} \otimes_{\mathrm{TL}_n} \mathcal{F}_n \to \mathbb{1}$ is contained in the ideal generated by a, and in particular does not contain the element 1, so that $\mathrm{Tor}_{n-1}^{\mathrm{TL}_n}(\mathbb{1}, \mathbb{1}) \neq 0$. $\square$

7. The case of $\mathrm{TL}_2(a)$

In this section we briefly consider the case $n = 2$, and fully compute the Tor and Ext groups. We do this first by a straightforward computation using a free resolution of our own construction. Then, in order to illustrate the theory developed in the paper, we re-prove the same result by explicitly computing the Fineberg module $\mathcal{F}_2$ and applying Theorem 5.1.

Proposition 7.1. *The homology and cohomology of $\mathrm{TL}_2(a)$ are as follows.*

$$\mathrm{Tor}_i^{\mathrm{TL}_2(a)}(\mathbb{1}, \mathbb{1}) = \begin{cases} R, & i = 0, \\ R/aR, & i > 0,\ i \text{ odd}, \\ R_a, & i > 0,\ i \text{ even}, \end{cases}$$
$$\mathrm{Ext}^i_{\mathrm{TL}_2(a)}(\mathbb{1}, \mathbb{1}) = \begin{cases} R, & i = 0, \\ R_a, & i > 0,\ i \text{ odd}, \\ R/aR, & i > 0,\ i \text{ even}, \end{cases}$$

where R_a denotes the kernel of the map $R \xrightarrow{a} R$. This holds for any choice of ground ring R and any choice of parameter $a \in R$.

Proof. We define a chain complex of left TL_2-modules as follows. The degree is indicated in the right-hand column. The boundary maps are given by right-multiplication by the indicated element of TL_2, except for the last, which is the map $\mathrm{TL}_2 \to \mathbb{1}$, $x \mapsto x \cdot 1$.

$$\begin{array}{cc}
\vdots & \\
\downarrow{\scriptstyle (a-U_1)} & \\
\mathrm{TL}_2 & 3 \\
\downarrow{\scriptstyle U_1} & \\
\mathrm{TL}_2 & 2 \\
\downarrow{\scriptstyle (a-U_1)} & \\
\mathrm{TL}_2 & 1 \\
\downarrow{\scriptstyle U_1} & \\
\mathrm{TL}_2 & 0 \\
\downarrow & \\
\mathbb{1} & -1
\end{array}$$

The composite of consecutive boundary maps is 0, due to the computation

$$U_1 \cdot (a - U_1) = 0 = (a - U_1) \cdot U_1,$$

and the fact that U_1 acts by 0 on $\mathbb{1}$. Moreover, this complex is acyclic, as one sees by considering the bases $1, U_1$ and $1, (a - U_1)$ of TL_2. Thus the non-negative part of the complex above, which we denote by P_*, is a free resolution of the left TL_2-module $\mathbb{1}$. Thus $\mathrm{Tor}_*^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1})$ and $\mathrm{Ext}^*_{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1})$ are the homology of $\mathbb{1} \otimes_{\mathrm{TL}_2} P_*$ and the cohomology of $\mathrm{Hom}_{\mathrm{TL}_2}(P_*, \mathbb{1})$ respectively. Using the isomorphisms $\mathbb{1} \otimes_{\mathrm{TL}_2} \mathrm{TL}_2 \cong \mathbb{1}$, $a \otimes x \mapsto a \cdot x$ and $\mathrm{Hom}^{\mathrm{TL}_2}(\mathrm{TL}_2, \mathbb{1}) \cong \mathbb{1}$, $f \mapsto f(1)$ in every degree, and working out the induced boundary maps, we see that $\mathbb{1} \otimes_{\mathrm{TL}_2} P_*$ and $\mathrm{Hom}_{\mathrm{TL}_2}(P_*, \mathbb{1})$ are isomorphic to the complexes depicted below.

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\downarrow{\scriptstyle a} & \uparrow{\scriptstyle a} & \\
R & R & 3 \\
\downarrow{\scriptstyle 0} & \uparrow{\scriptstyle 0} & \\
R & R & 2 \\
\downarrow{\scriptstyle a} & \uparrow{\scriptstyle a} & \\
R & R & 1 \\
\downarrow{\scriptstyle 0} & \uparrow{\scriptstyle 0} & \\
R & R & 0
\end{array}$$

The homology and cohomology of these complexes are easily computed, and give the claim. □

Proposition 7.2. *When $n = 2$ the Fineberg module satisfies $\mathcal{F}_2(a) \cong \mathbb{1}$, and the map $\mathbb{1} \otimes_{\mathrm{TL}_2(a)} \mathcal{F}_2(a) \to \varepsilon_2 \cong \mathbb{1}$ is multiplication by a.*

Proof. We compute $\mathcal{F}_2$ explicitly. Recall that $\mathcal{F}_2$ is the kernel of the top differential of $W(2)$.

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathrm{TL}_2 \xrightarrow{-\cdot \mathcal{J}_2} \mathrm{TL}_2.$$

In the case $n = 2$ the only sequence $n > i_1 > \cdots > i_r > 0$ with odd initial term is 1 and thus $\mathcal{J}_2 = \mu\lambda^{-1}U_1$. It is relatively straightforward to see that the kernel of right multiplication by $\mathcal{J}_2$ is spanned by $(a - U_1)$, so

$$\mathcal{F}_2 \cong \langle a - U_1 \rangle \cong \mathbb{1}.$$

The map $\mathbb{1} \otimes_{\mathrm{TL}_2} \mathcal{F}_2 \to \varepsilon_2 \cong \mathbb{1}$ is the composite map

$$\mathbb{1} \otimes_{\mathrm{TL}_2} \mathcal{F}_2 \to \mathbb{1} \otimes_{\mathrm{TL}_2} W(2)_1 = \mathbb{1} \otimes_{\mathrm{TL}_2} (\mathrm{TL}_2 \otimes_{\mathrm{TL}_0} \mathbb{1}) \cong \mathbb{1}.$$

Under the central equality the basis element $a - U_1$ of $\mathcal{F}_2 \subset W(2)_1$ gets mapped to $a - U_1 = a$ in the tensor product. Therefore the composite map is given by multiplication by a, as required. □

Corollary 7.3. *Suppose that $v \in R$ is a unit and that $a = v + v^{-1}$. Then the groups $\mathrm{Tor}_i^{\mathrm{TL}_2(a)}(\mathbb{1}, \mathbb{1})$ and $\mathrm{Ext}_i^{\mathrm{TL}_2(a)}(\mathbb{1}, \mathbb{1})$ are as described in Proposition 7.1.*

Proof. In the light of Proposition 7.2, the exact sequence from Theorem 5.1

$$0 \to \mathrm{Tor}_2^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) \to \mathbb{1} \otimes_{\mathrm{TL}_2} \mathcal{F}_2 \to \mathbb{1} \to \mathrm{Tor}_1^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) \to 0$$

now becomes

$$0 \to \mathrm{Tor}_2^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) \to \mathbb{1} \otimes_{\mathrm{TL}_2} \mathbb{1} \xrightarrow{a} \mathbb{1} \to \mathrm{Tor}_1^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) \to 0$$

from which one can compute $\mathrm{Tor}_2^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) = R_a$ and $\mathrm{Tor}_1^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) = R/aR$. For $i \geqslant 3$ we have the recursive formula

$$\mathrm{Tor}_i^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1}) = \mathrm{Tor}_{i-2}^{\mathrm{TL}_2}(\mathbb{1}, \mathcal{F}_2) \cong \mathrm{Tor}_{i-2}^{\mathrm{TL}_2}(\mathbb{1}, \mathbb{1})$$

which completes the proof. The Ext results similarly follow from Theorem 5.1. □

8. High-acyclicity

In this final section we prove high connectivity of $W(n)$, Theorem D.

Theorem D. *The homology of $W(n)$ vanishes in degrees $d \leqslant (n - 2)$.*

8.1. **A filtration.** In this subsection we introduce a filtration of $W(n)$. We state a theorem relating the filtration quotients to $W(n-1)$ (the proof of which is the topic of the next 3 subsections) and therefore by induction prove Theorem D.

Definition 8.1 (The filtration)**.** We define a filtration F of $W(n)$,

$$F^0 \subseteq F^1 \subseteq \cdots \subseteq F^n = W(n)$$

as follows.

- F^0 is defined to be the span of the elements of two kinds. We call elements of the first kind *basic elements* and these are of the form
$$x \otimes 1$$
in degrees i such that $-1 \leqslant i \leqslant n-2$, where x is represented by a monomial in the s_j not involving the letter s_1. Elements of the second kind are those of the form
$$x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1$$
in degrees i such that $0 \leqslant i \leqslant n-1$, where again x is represented by a monomial not involving the letter s_1.
- F^k for $k \geqslant 1$ is defined to be the span of F^{k-1} together with terms of the form
$$x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1$$
in degrees i such that $k \leqslant i \leqslant n-1$, where again x is represented by a monomial not involving s_1.

Remark 8.2. Note that in the description of F^0, it is possible for the product $s_1 \cdots s_{n-i-1}$ to be empty, i.e. the unit element, if the final index $(n-i-1)$ is zero $(i = n-1)$. In contrast, in the description of F^k for $k \geqslant 1$, the product $s_1 \cdots s_{n-i-1+k}$ is never empty. This is one reason why it is important for us to treat F^0 quite separately from the other F^k, as is done in the remainder of this paper.

Definition 8.3. Recall that the *cone* on a chain complex X is the chain complex CX defined by $(CX)_i = X_i \oplus X_{i-1}$, and with differential defined by

$$d^i_{CX}(x,y) = (d^i_X(x) + y, -d^{i-1}_X(y)).$$

The *suspension* of a chain complex X is the complex ΣX defined by

$$(\Sigma X)_i = X_{i-1}$$

and with the same differential as X. The *truncation to degree p* of a chain complex X is the chain complex $\tau_p X$ defined by

$$(\tau_p X)_i = \begin{cases} X_i, & i \leqslant p \\ 0, & i > p \end{cases}$$

and with the same differential as X (in the relevant degrees).

Remark 8.4. Note that our definition of cone and suspension do not seem to match up very well. However, we have chosen our conventions in order to make the proof of the next theorem as direct as possible, and we believe that our choices are the best fit for this purpose.

Definition 8.5. Define the *shift map* σ to be the map

$$\sigma\colon \mathrm{TL}_{n-1}(a) \to \mathrm{TL}_n(a)$$

which sends each U_i to U_{i+1} for $1 \leqslant i \leqslant n-2$, and hence each s_i to s_{i+1}.

Lemma 8.6. *Each F^k consists of $\mathrm{TL}_{n-1}(a)$-submodules of $W(n)$, where $\mathrm{TL}_{n-1}(a)$ acts via the shift map σ.*

Proof. Definition 8.1 defines each F^k as the span of certain 'basis elements' of the form $y \otimes 1$ where $y \in \mathrm{TL}_n$ is represented by a monomial in the s_j subject to certain restrictions. Multiplying any such y on the left by any s_j for $1 < j \leqslant n-1$ does not affect whether it meets these restrictions. Since $s_j = \sigma(s_{j-1})$ for $1 < j \leqslant n-1$, this shows that the generators of TL_{n-1} send the base elements of each F^k to other base elements of F^k, and therefore F^k itself is stable under the action of TL_{n-1}. □

Theorem 8.7. *Each F^k is a subcomplex of $W(n)$. We identify*

$$F^0 \cong C(W(n-1)).$$

And for $k \geqslant 1$, we have

$$F^k/F^{k-1} \cong \tau_{n-1}\Sigma^{k+1}W(n-1).$$

Corollary 8.8 (Theorem D)**.** *For each $n \geqslant 0$ the complex $W(n)$ is $(n-2)$-acyclic, or in other words, its homology vanishes up to and including degree $(n-2)$.*

Proof. We prove this by induction on $n \geqslant 0$. One can verify the claim directly in the case $n = 0$. Fix $n \geqslant 1$ and suppose that the theorem has been proved for the previous case. Now $W(n)$ has the filtration $F^0 \subseteq F^1 \subseteq \cdots \subseteq F^n$. We prove below that F^0 and all filtration quotients F^k/F^{k-1} are $(n-2)$-acyclic, and then it follows (for example by using the short exact sequences $0 \to F^{k-1} \to F^k \to F^k/F^{k-1} \to 0$, or by using the spectral sequence of the filtration) that the same holds for $W(n)$ itself.

Observe that $F^0 \cong C(W(n-1))$, being isomorphic to a cone, is acyclic. Next, for $k \geqslant 1$ we have $F^k/F^{k-1} \cong \tau_{n-1}\Sigma^{k+1}W(n-1)$. The induction hypothesis states that $W(n-1)$ is $(n-3)$-acyclic, so that $\Sigma^{k+1}W(n-1)$ is $(n-3+k)$-acyclic and in particular $(n-2)$-acyclic, so that $\tau_{n-1}\Sigma^{k+1}W(n-1)$ is also $(n-2)$-acyclic. This completes the proof. □

The final three subsections prove Theorem 8.7, by first setting up the required chain map for F^0, then for F^k and then in the final section proving these chain maps are isomorphisms.

8.2. **Proofs for F^0.** In this subsection we prove F^0 is a subcomplex of $W(n)$. We define a map from the cone $C(W(n-1))$ to F^0 and prove this is a well defined chain map.

Lemma 8.9. *F^0 is a subcomplex of $W(n)$.*

Proof. To prove the claim, we must take a generator of F^0 in degree i, and show that under the boundary map $d^i\colon W(n)_i \to W(n)_{i-1}$ this generator is mapped into F^0. Since d^i is the alternating sum $d^i_0 - d^i_1 + \cdots + (-1)^i d^i_i$, it will suffice to fix j in the range $0 \leqslant j \leqslant i$, and show that d^i_j sends our generator into F^0. Recall from Definition 4.1 the definition of d^i_j:

$$d^i_j(y \otimes r) = y \cdot (s_{n-i+j-1} \cdots s_{n-i}) \otimes \lambda^{-j} r.$$

Generators of F^0 come in two kinds. The first kind are the basic elements $x \otimes 1$ in degrees $-1 \leqslant i \leqslant n-2$ where x is represented by a monomial not featuring the letter s_1. The map d^i_j only introduces a letter s_1 in the case $i = n-1$, which is excluded here, so that $d^i_j(x \otimes 1)$ also lies in F^0.

The second kind of generators of F^0 are elements

$$x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1$$

in degrees $0 \leqslant i \leqslant n-1$, where x is represented by a monomial not involving s_1. In the case $j = 0$, we have

$$d^i_0(x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1) = x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1$$

but this lies in $W(n)_{i-1} = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i}} \mathbb{1}$, hence is equal to $x \otimes \lambda^{n-i-1}$, and since x is represented by a monomial not involving s_1, this does indeed lie in F^0. (This argument includes the special case $i = n-1$, where the product $s_1 \cdots s_{n-i-1}$ is empty, but this clearly creates no issues.) In the case $j \geqslant 1$, we have

$$\begin{aligned} d^i_j(x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1) &= x \cdot (s_1 \cdots s_{n-i-1}) \cdot (s_{n-i+j-1} \cdots s_{n-i}) \otimes \lambda^{-j} \\ &= x \cdot (s_1 \cdots s_{n-i-1}) \cdot (s_{n-i+j-1} \cdots s_{n-i+1}) \cdot s_{n-i} \otimes \lambda^{-j} \\ &= x \cdot (s_{n-i+j-1} \cdots s_{n-i+1}) \cdot (s_1 \cdots s_{n-i}) \otimes \lambda^{-j} \\ &= (x \cdot (s_{n-i+j-1} \cdots s_{n-i+1})) \cdot (s_1 \cdots s_{n-(i-1)-1}) \otimes \lambda^{-j} \end{aligned}$$

which lies in F^0 since $(s_{n-i+j-1} \cdots s_{n-i+1})$ does not involve the letter s_1, as required. $\square$

Definition 8.10. Define a map

$$\Phi^0 \colon C(W(n-1)) \longrightarrow F^0$$

as follows. Recall that

$$\begin{aligned} C(W(n-1))_i &= W(n-1)_i \oplus W(n-1)_{i-1} \\ &= \left(\mathrm{TL}_{n-1}(a) \otimes_{\mathrm{TL}_{n-i-2}(a)} \mathbb{1}\right) \oplus \left(\mathrm{TL}_{n-1}(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}\right) \end{aligned}$$

and that

$$F_i^0 \subseteq W(n)_i = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}.$$

We define Φ^0 in degree i by the rule

$$\Phi_i^0(x \otimes \alpha, y \otimes \beta) = \xi_i(x \otimes \alpha) + \eta_i(y \otimes \beta)$$

where

$$\begin{aligned} \xi_i \colon W(n-1)_i &\to W(n)_i \\ x \otimes \alpha &\mapsto \sigma(x) \otimes \lambda^{n-1}\alpha \end{aligned}$$

and

$$\begin{aligned} \eta_i \colon W(n-1)_{i-1} &\to W(n)_i \\ y \otimes \beta &\mapsto \sigma(y) \cdot (s_1 \cdots s_{n-i-1}) \otimes \lambda^i \beta. \end{aligned}$$

It is simple to check that the image of both maps lies in F_i^0.

Lemma 8.11. *The maps ξ_i and η_i are well defined.*

Proof. In the case of ξ_i this is simple to verify, as the map $\sigma \colon \mathrm{TL}_{n-1} \to \mathrm{TL}_n$ is in fact a map of right-modules $\sigma \colon (\mathrm{TL}_{n-1})_{\mathrm{TL}_{n-i-2}} \to (\mathrm{TL}_n)_{\mathrm{TL}_{n-i-1}}$ with respect to the map of algebras $\sigma \colon \mathrm{TL}_{n-i-2} \to \mathrm{TL}_{n-i-1}$.

In the case of η_i, the definition of $\eta_i(y \otimes \beta)$ as presented depends on y and β themselves, and we must check that it depends only on $y \otimes \beta$. Thus we must show that

$$\eta_i(ys_j \otimes \beta) = \eta_i(y \otimes \lambda\beta)$$

whenever $1 \leqslant j \leqslant n-i-2$. And indeed

$$\begin{aligned} \eta_i(ys_j \otimes \beta) &= \sigma(ys_j) \cdot (s_1 \cdots s_{n-i-1}) \otimes \lambda^i \beta \\ &= \sigma(y) \cdot s_{j+1} \cdot (s_1 \cdots s_{n-i-1}) \otimes \lambda^i \beta \\ &= \sigma(y) \cdot (s_1 \cdots s_{n-i-1}) \cdot s_j \otimes \lambda^i \beta \\ &= \sigma(y) \cdot (s_1 \cdots s_{n-i-1}) \otimes \lambda^{i+1} \beta \\ &= \eta_i(y \otimes \lambda\beta) \end{aligned}$$

where the third equality holds since $2 \leqslant j+1 \leqslant n-i-1$, and the fourth holds since $j \leqslant n-i-2$ and the tensor product is over TL_{n-i-1}. $\square$

Lemma 8.12. *The ξ_i and η_i interact with the boundary maps of $W(n)$ in the following way:*

(1) $d_j^i \circ \xi_i = \xi_{i-1} \circ d_j^i$ for i in the range $-1 \leqslant i \leqslant n-2$ and j in the range $0 \leqslant j \leqslant i$.
(2) $d_0^i \circ \eta_i = \xi_{i-1}$ for i in the range $0 \leqslant i \leqslant n-1$.
(3) $d_{j+1}^i \circ \eta_i = \eta_{i-1} \circ d_j^{i-1}$ for i in the range $0 \leqslant i \leqslant n-1$ and j in the range $0 \leqslant j \leqslant i-1$.

Proof. For the first point, we have:

$$\begin{aligned}
d_j(\xi_i(x\otimes\alpha)) &= d_j(\sigma(x)\otimes\lambda^{n-1}\alpha)\\
&= \sigma(x)\cdot(s_{n-i+j-1}\cdots s_{n-i})\otimes\lambda^{-j}\lambda^{n-1}\alpha\\
&= \sigma(x\cdot(s_{n-i+j-2}\cdots s_{n-i-1}))\otimes\lambda^{-j}\lambda^{n-1}\alpha\\
&= \xi_{i-1}(x\cdot(s_{n-i+j-2}\cdots s_{n-i-1})\otimes\lambda^{-j}\alpha)\\
&= \xi_{i-1}(x\cdot(s_{(n-1)-i+j-1}\cdots s_{(n-1)-i})\otimes\lambda^{-j}\alpha)\\
&= \xi_{i-1}(d^i_j(x\otimes\alpha)).
\end{aligned}$$

For the second point, we have:

$$\begin{aligned}
d^i_0(\eta_i(y\otimes\beta)) &= d^i_0(\sigma(y)\cdot(s_1\cdots s_{n-i-1})\otimes\lambda^i\beta)\\
&= \sigma(y)\cdot(s_1\cdots s_{n-i-1})\otimes\lambda^i\beta\\
&= \sigma(y)\otimes\lambda^{n-i-1}\lambda^i\beta\\
&= \sigma(y)\otimes\lambda^{n-1}\beta\\
&= \xi_{i-1}(y\otimes\beta),
\end{aligned}$$

where the third equality holds because the terms lie in $W(n)_{i-1} = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i}} \mathbb{1}$. And for the third point we have:

$$\begin{aligned}
d^i_{j+1}\eta_i(y\otimes\beta) &= d^i_{j+1}(\sigma(y)\cdot(s_1\cdots s_{n-i-1})\otimes\lambda^i\beta)\\
&= \sigma(y)\cdot(s_1\cdots s_{n-i-1})\cdot(s_{n-i+(j+1)-1}\cdots s_{n-i})\otimes\lambda^{-j-1}\lambda^i\beta\\
&= \sigma(y)\cdot(s_1\cdots s_{n-i-1})\cdot(s_{n-i+j}\cdots s_{n-i+1})\cdot s_{n-i}\otimes\lambda^{i-j-1}\beta\\
&= \sigma(y)\cdot(s_{n-i+j}\cdots s_{n-i+1})\cdot(s_1\cdots s_{n-i})\otimes\lambda^{i-j-1}\beta\\
&= \sigma(y\cdot(s_{n-i+j-1}\cdots s_{n-i}))\cdot(s_1\cdots s_{n-(i-1)-1})\otimes\lambda^{i-1}\lambda^{-j}\beta\\
&= \eta_{i-1}(y\cdot(s_{n-i+j-1}\cdots s_{n-i})\otimes\lambda^{-j}\beta)\\
&= \eta_{i-1}(y\cdot(s_{(n-1)-(i-1)+j-1}\cdots s_{(n-1)-(i-1)})\otimes\lambda^{-j}\beta)\\
&= \eta_{i-1}(d^{i-1}_j(y\otimes\beta))
\end{aligned}$$

where for the final equality we recall that the source of η_{i-1} is $W(n-1)_{i-2}$. □

Lemma 8.13. Φ^0 *is a chain map.*

Proof. Referring to the definition of the differential on $C(W(n-1))$ (Definition 8.3), we see that in order to check that $d^i\circ\Phi^0_i = \Phi^0_{i-1}\circ d^i$, it is enough to show that $d^i\circ\xi_i(x\otimes\alpha) = \xi_{i-1}(d^i(x\otimes\alpha))$ and $d^i\circ\eta_i(y\otimes\beta) = \xi_{i-1}(y\otimes\beta) - \eta_{i-1}(d^{i-1}(y\otimes\beta))$.

Using the previous lemma, for the first we have

$$\begin{aligned} d^i \circ \xi_i(x \otimes \alpha) &= \sum_{j=0}^{i} (-1)^j d^i_j(\xi_i(x \otimes \alpha)) \\ &= \sum_{j=0}^{i} (-1)^j \xi_{i-1}(d^i_j(x \otimes \alpha)) \\ &= \xi_{i-1}\left(\sum_{j=0}^{i} (-1)^j d^i_j(x \otimes \alpha) \right) \\ &= \xi_{i-1}(d^i(x \otimes \alpha)). \end{aligned}$$

And for the second we have

$$\begin{aligned} d^i \circ \eta_i(y \otimes \beta) &= \sum_{j=0}^{i} (-1)^j d^i_j(\eta_i(y \otimes \beta)) \\ &= d^i_0(\eta_i(y \otimes \beta)) - \sum_{j=0}^{i-1} (-1)^j d^i_{j+1}\eta_i(y \otimes \beta) \\ &= \xi_{i-1}(y \otimes \beta) - \sum_{j=0}^{i-1} (-1)^j \eta_{i-1} d^{i-1}_j(y \otimes \beta) \\ &= \xi_{i-1}(y \otimes \beta) - \eta_{i-1}\left(\sum_{j=0}^{i-1} (-1)^j d^{i-1}_j(y \otimes \beta) \right) \\ &= \xi_{i-1}(y \otimes \beta) - \eta_{i-1}(d^{i-1}(y \otimes \beta)). \qquad \square \end{aligned}$$

8.3. **Proofs for F^k, $k \geqslant 1$.** In this subsection we prove, for $k \geqslant 1$, that F^k is a subcomplex of $W(n)$. We define a map from $\tau_{n-1}\Sigma^{k+1}W(n-1)$ to F^k/F^{k-1} and prove this is a well defined chain map. We start off with some elementary lemmas involving the s_j, which we require for later proofs.

Lemma 8.14. *Let $m \geqslant 1$ and $p \leqslant m$. Then*

$$s_1 \cdots s_m \cdots s_p = (s_m \cdots s_{p+1}) \cdot (s_1 \cdots s_m).$$

In the case $m = p$ the product $s_m \cdots s_{p+1}$ is empty and therefore equal to 1.

Lemma 8.15. *Let $p \geqslant 1$, $q \geqslant r \geqslant 1$. Then the product $(s_1 \cdots s_p) \cdot (s_q \cdots s_r)$ can be described as follows.*

(1) *When $r - 1 \leqslant p \leqslant q - 1$,*

$$(s_1 \cdots s_p) \cdot (s_q \cdots s_r) = (s_q \cdots s_{r+1}) \cdot (s_1 \cdots s_{p+1}).$$

(2) *When $p = q$, $(s_1 \cdots s_p) \cdot (s_q \cdots s_r)$ is a linear combination of terms of the form $(s_t \cdots s_{r+1}) \cdot (s_1 \cdots s_t)$ for $p \geqslant t \geqslant r+1$, as well as $s_1 \cdots s_r$ and $s_1 \cdots s_{r-1}$.*

(3) *When $p \geqslant q+1$,*

$$(s_1 \cdots s_p) \cdot (s_q \cdots s_r) = (s_{q+1} \cdots s_{r+1}) \cdot (s_1 \cdots s_p).$$

Proof. When $r-1 \leqslant p \leqslant q-1$,

$$\begin{aligned}(s_1 \cdots s_p) \cdot (s_q \cdots s_r) &= (s_1 \cdots s_p) \cdot (s_q \cdots s_{p+2}) \cdot (s_{p+1} \cdots s_r)\\ &= (s_q \cdots s_{p+2}) \cdot (s_1 \cdots s_p) \cdot (s_{p+1} \cdots s_r)\\ &= (s_q \cdots s_{p+2}) \cdot (s_1 \cdots s_{p+1} \cdots s_r)\\ &= (s_q \cdots s_{p+2}) \cdot (s_{p+1} \cdots s_{r+1}) \cdot (s_1 \cdots s_{p+1})\\ &= (s_q \cdots s_{r+1}) \cdot (s_1 \cdots s_{p+1}),\end{aligned}$$

where we used Lemma 8.14 to obtain the fourth equality.

When $p = q$, we claim that

$$(s_1 \cdots s_p) \cdot (s_q \cdots s_r) = (s_1 \cdots s_p) \cdot (s_p \cdots s_r)$$

is a linear combination of terms of the form $(s_t \cdots s_{r+1}) \cdot (s_1 \cdots s_t)$ for $p \geqslant t \geqslant r+1$, as well as $s_1 \cdots s_r$ and $s_1 \cdots s_{r-1}$. We will prove this claim by induction on the difference $p-r$. When $p-r=0$, we have

$$(s_1 \cdots s_p) \cdot (s_p \cdots s_r) = s_1 \cdots s_p \cdot s_p.$$

Now since s_p^2 is a linear combination of s_p and 1, this is a linear combination of $s_1 \cdots s_p = s_1 \cdots s_r$ and $s_1 \cdots s_{p-1} = s_1 \cdots s_{r-1}$ as required. Now let $p-r \geqslant 1$, and assume that the claim holds for all smaller values. Then

$$(s_1 \cdots s_p) \cdot (s_p \cdots s_r) = (s_1 \cdots s_{p-1}) \cdot s_p^2 \cdot (s_{p-1} \cdots s_r)$$

is a linear combination of

$$\begin{aligned}(s_1 \cdots s_{p-1}) \cdot s_p \cdot (s_{p-1} \cdots s_r) &= s_1 \cdots s_p \cdots s_r\\ &= (s_p \cdots s_{r+1}) \cdot (s_1 \cdots s_p)\end{aligned}$$

(where we used Lemma 8.14) and

$$(s_1 \cdots s_{p-1}) \cdot (s_{p-1} \cdots s_r).$$

The former is $(s_t \cdots s_{r+1}) \cdot (s_1 \cdots s_t)$ in the case $t = p$, while the induction hypothesis tells us that the latter is a linear combination of $(s_t \cdots s_{r+1}) \cdot (s_1 \cdots s_t)$ for $p-1 \geqslant t \geqslant r+1$, as well as $s_1 \cdots s_r$ and $s_1 \cdots s_{r-1}$. This completes the proof of the claim.

When $p \geqslant q+1$,

$$\begin{aligned}(s_1 \cdots s_p) \cdot (s_q \cdots s_r) &= (s_1 \cdots s_{q+1}) \cdot (s_{q+2} \cdots s_p) \cdot (s_q \cdots s_r) \\ &= (s_1 \cdots s_{q+1}) \cdot (s_q \cdots s_r) \cdot (s_{q+2} \cdots s_p) \\ &= (s_1 \cdots s_{q+1} \cdots s_r) \cdot (s_{q+2} \cdots s_p) \\ &= (s_{q+1} \cdots s_{r+1}) \cdot (s_1 \cdots s_{q+1}) \cdot (s_{q+2} \cdots s_p) \\ &= (s_{q+1} \cdots s_{r+1}) \cdot (s_1 \cdots s_p)\end{aligned}$$

(where we again used Lemma 8.14 to obtain the fourth equality) as required. □

Lemma 8.16. *For $k \geqslant 1$, F^k is a subcomplex of $W(n)$.*

Proof. We fix $k \geqslant 1$ and take a generator of $F^k \setminus F^{k-1}$ in degree i, where $k \leqslant i \leqslant n-1$, and show that the boundary map $d^i\colon W(n)_i \to W(n)_{i-1}$ sends our generator into F^k. Since d is the alternating sum $d^i_0 - d^i_1 + \cdots + (-1)^i d^i_i$, it will suffice to fix j in the range $0 \leqslant j \leqslant i$, and show that d^i_j sends our generator into F^k. Our generator of $F^k \setminus F^{k-1}$ in degree i is $x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1$, where x does not involve the letter s_1. Note that

$$(n-i-1+k) = (n-1) - i + k \geqslant (n-1) - (n-1) + 1 = 1,$$

so that the product $(s_1 \cdots s_{n-i-1+k})$ is not empty. We have

$$d^i_j(x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1) = x \cdot (s_1 \cdots s_{n-i-1+k}) \cdot (s_{n-i-1+j} \cdots s_{n-i}) \otimes \lambda^{-j},$$

where the factor $(s_{n-i-1+j} \cdots s_{n-i})$ can be empty, in the case $j = 0$.

- First we consider the case $j = 0$. We find that

  $$\begin{aligned}d^i_0(x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1) &= x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1 \\ &= x \cdot (s_1 \cdots s_{n-(i-1)-1+(k-1)}) \otimes 1\end{aligned}$$

  lies in F^{k-1}, and therefore in F^k as required.
- Now we consider the case $1 \leqslant j \leqslant (k-1)$. Then $(n-i-1+k) \geqslant (n-i-1+j)+1$, so that the third item of Lemma 8.15 applies and shows that

  $$\begin{aligned}&d^i_j(x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1) \\ &\quad= x \cdot (s_1 \cdots s_{n-i-1+k}) \cdot (s_{n-i-1+j} \cdots s_{n-i}) \otimes \lambda^{-j} \\ &\quad= x \cdot (s_{n-i+j} \cdots s_{n-i+1}) \cdot (s_1 \cdots s_{n-i-1+k}) \otimes \lambda^{-j} \\ &\quad= x \cdot (s_{n-i+j} \cdots s_{n-i+1}) \cdot (s_1 \cdots s_{n-(i-1)-1+(k-1)}) \otimes \lambda^{-j}\end{aligned}$$

  Since $n-i+1 \geqslant n-(n-1)+1 = 2$, the word $(s_{n-i+j} \cdots s_{n-i+1})$ does not involve s_1, and consequently the element above lies in F^{k-1}, and therefore in F^k.

- Now we consider the case $j = k$. Then $(n-i-1+k) = (n-i-1+j)$ and so the second item of Lemma 8.15 applies and shows that
$$d^i_k(x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1) = \\ x \cdot (s_1 \cdots s_{n-i-1+k}) \cdot (s_{n-i-1+k} \cdots s_{n-i}) \otimes \lambda^{-k}$$
is a linear combination of terms
$$x \cdot (s_t \cdots s_{n-i+1}) \cdot (s_1 \cdots s_t) \otimes \lambda^{-k}$$
for t in the range
$$(n-i+1) \leqslant t \leqslant (n-i-1+k) = (n-(i-1)-1+(k-1))$$
together with
$$x \cdot (s_1 \cdots s_{n-(i-1)-1}) \otimes \lambda^{-k}$$
and
$$x \cdot (s_1 \cdots s_{n-(i-1)-2}) \otimes \lambda^{-k} = x \otimes \lambda^{-k}.$$
Now $(s_t \cdots s_{n-i+1})$ does not involve s_1, so the first of these terms lies in F^{k-1}, while the second and third lie in F^0. So altogether we have the required result.
- Now we consider the case $k+1 \leqslant j$. Here we have
$$(n-i-1) \leqslant (n-i-1+k)+1 \leqslant (n-i-1+j),$$
so that the first item of Lemma 8.15 applies and shows that
$$\begin{aligned} d^i_j(x \cdot (s_1 \cdots s_{n-i-1+k}) \otimes 1) &= x \cdot (s_1 \cdots s_{n-i-1+k}) \cdot (s_{n-i-1+j} \cdots s_{n-i}) \otimes \lambda^{-j} \\ &= x \cdot (s_{n-i-1+j} \cdots s_{n-i+1}) \cdot (s_1 \cdots s_{n-i-1+k+1}) \otimes \lambda^{-j} \\ &= x \cdot (s_{n-i-1+j} \cdots s_{n-i+1}) \cdot (s_1 \cdots s_{n-(i-1)-1+k}) \otimes \lambda^{-j}. \end{aligned}$$
Since $(s_{n-i-1+j} \cdots s_{n-i+1})$ does not involve s_1, the element above lies in F^k as required. □

Definition 8.17. Define a map
$$\Psi^k \colon \tau_{n-1}\Sigma^{k+1}W(n-1) \longrightarrow F^k/F^{k-1}$$
as follows. Note that for i in the range $k \leqslant i \leqslant (n-1)$,
$$\begin{aligned} [\tau_{n-1}\Sigma^{k+1}W(n-1)]_i &= W(n-1)_{i-k-1} \\ &= \mathrm{TL}_{n-1}(a) \otimes_{\mathrm{TL}_{(n-1)-(i-k-1)-1}(a)} \mathbb{1} \\ &= \mathrm{TL}_{n-1}(a) \otimes_{\mathrm{TL}_{n-i-1+k}(a)} \mathbb{1}, \end{aligned}$$
while $(F^k/F^{k-1})_i$ is a quotient of $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}$. Define the degree i part of Ψ to be the map
$$\Psi^k_i \colon \mathrm{TL}_{n-1}(a) \otimes_{\mathrm{TL}_{n-i-1+k}(a)} \mathbb{1} \longrightarrow (F^k/F^{k-1})_i$$
given by
$$\Psi^k_i \colon x \otimes \alpha \longmapsto (-1)^{-i(k+1)}\sigma(x) \cdot (s_1 \cdots s_{n-i-1+k}) \otimes \lambda^i \alpha.$$

For later convenience, we will denote by ψ_i^k the map

$$\psi_i^k\colon x\otimes\alpha \longmapsto \sigma(x)\cdot(s_1\cdots s_{n-i-1+k})\otimes\lambda^i\alpha,$$

so that $\Psi_i^k = (-1)^{-i(k+1)}\psi_i^k$.

Lemma 8.18. *The map ψ_i^k is well defined (and the same therefore holds for Ψ_i^k).*

Proof. As presented above, the value of $\psi_i^k(x\otimes\alpha)$ depends on the choices of x and α, rather than on $x\otimes\alpha$. So to check that ψ_i^k is well-defined, we must check that $\psi_i^k(xs_p\otimes\alpha) = \psi_i^k(x\otimes\lambda\alpha)$ whenever $p\leqslant(n-i-1+k)-1$. Let us write $q=(n-i-1+k)$, so that $p\leqslant q-1$. (In particular we are assuming that $q\geqslant 2$.) Now

$$\begin{aligned}\psi_i^k(xs_p\otimes\alpha) &= \sigma(xs_p)\cdot(s_1\cdots s_q)\otimes\lambda^i\alpha\\ &= \sigma(x)\cdot s_{p+1}\cdot(s_1\cdots s_q)\otimes\lambda^i\alpha\\ &= \sigma(x)\cdot s_{p+1}\cdot(s_1\cdots s_{p-1})\cdot(s_ps_{p+1})\cdot(s_{p+2}\cdots s_q)\otimes\lambda^i\alpha\\ &= \sigma(x)\cdot(s_1\cdots s_{p-1})\cdot(s_{p+1}s_ps_{p+1})\cdot(s_{p+2}\cdots s_q)\otimes\lambda^i\alpha.\end{aligned}$$

Recall from Definition 2.20 that

$$s_{p+1}s_ps_{p+1} = \lambda s_ps_{p+1}+\lambda s_{p+1}s_p-\lambda^2s_p-\lambda^2s_{p+1}+\lambda^3.$$

Now

$$\begin{aligned}(s_1\cdots s_{p-1})\cdot(s_ps_{p+1})\cdot(s_{p+2}\cdots s_q) &= (s_1\cdots s_q)\\ (s_1\cdots s_{p-1})\cdot(s_{p+1}s_p)\cdot(s_{p+2}\cdots s_q) &= (s_{p+1}\cdots s_q)\cdot(s_1\cdots s_p)\\ (s_1\cdots s_{p-1})\cdot s_p\cdot(s_{p+2}\cdots s_q) &= (s_{p+2}\cdots s_q)\cdot(s_1\cdots s_p)\\ (s_1\cdots s_{p-1})\cdot s_{p+1}\cdot(s_{p+2}\cdots s_q) &= (s_{p+1}\cdots s_q)\cdot(s_1\cdots s_{p-1})\\ (s_1\cdots s_{p-1})\cdot 1\cdot(s_{p+2}\cdots s_q) &= (s_{p+2}\cdots s_q)\cdot(s_1\cdots s_{p-1})\end{aligned}$$

so it follows that

$$\begin{aligned}\psi_i^k(xs_p\otimes\alpha) = &\sigma(x)\cdot(s_1\cdots s_q)\otimes\lambda^{i+1}\alpha\\ &+\sigma(x)\cdot(s_{p+1}\cdots s_q)\cdot(s_1\cdots s_p)\cdot\otimes\lambda^{i+1}\alpha\\ &-\sigma(x)\cdot(s_{p+2}\cdots s_q)\cdot(s_1\cdots s_p)\otimes\lambda^{i+2}\alpha\\ &-\sigma(x)\cdot(s_{p+1}\cdots s_q)\cdot(s_1\cdots s_{p-1})\otimes\lambda^{i+2}\alpha\\ &+\sigma(x)\cdot(s_{p+2}\cdots s_q)\cdot(s_1\cdots s_{p-1})\otimes\lambda^{i+3}\alpha.\end{aligned}$$

Now $p<n-i-1+k$, which means that the final four terms above all lie in F^{k-1}, so that in F^k/F^{k-1} we have

$$\begin{aligned}\psi_i^k(xs_p\otimes\alpha) &= \sigma(x)\cdot(s_1\cdots s_q)\otimes\lambda^{i+1}\alpha\\ &= \sigma(x)\cdot(s_1\cdots s_{n-i-1+k})\otimes\lambda^{i+1}\alpha\\ &= \psi_i^k(x\otimes\lambda\alpha)\end{aligned}$$

as required. □

Lemma 8.19. *Let $k \geqslant 1$ and let $k \leqslant i \leqslant n-1$. Then for j in the range $j \geqslant k+1$ we have $\psi^k_{i-1} \circ d^{i-k-1}_{j-k-1} = d^i_j \circ \psi^k_i$.*

Proof. Let $x \otimes \alpha \in W(n-1)_{i-k-1} = \mathrm{TL}_{n-1} \otimes_{\mathrm{TL}_{n-i-1+k}} \mathbb{1}$. Then

$$\begin{aligned} d^i_j(\psi^k_i(x\otimes\alpha)) &= d^i_j(\sigma(x)\cdot(s_1\cdots s_{n-i-1+k})\otimes\lambda^i\alpha) \\ &= \sigma(x)\cdot(s_1\cdots s_{n-i-1+k})\cdot(s_{n-i+j-1}\cdots s_{n-i})\otimes\lambda^{i-j}\alpha. \end{aligned}$$

Since $(n-i-1) \leqslant (n-i-1+k)+1 \leqslant (n-i+j-1)$, we may apply the first part of Lemma 8.15 to obtain

$$\begin{aligned} d^i_j(\psi^k_i(x\otimes\alpha)) &= \sigma(x)\cdot(s_1\cdots s_{n-i-1+k})\cdot(s_{n-i+j-1}\cdots s_{n-i})\otimes\lambda^{i-j}\alpha \\ &= \sigma(x)\cdot(s_{n-i+j-1}\cdots s_{n-i+1})\cdot(s_1\cdots s_{n-i+k})\otimes\lambda^{i-j}\alpha \\ &= \sigma(x)\cdot(s_{n-i+j-1}\cdots s_{n-i+1})\cdot(s_1\cdots s_{n-(i-1)-1+k})\otimes\lambda^{(i-1)}\lambda^{1-j}\alpha \\ &= \sigma(x\cdot(s_{n-i+j-2}\cdots s_{n-i}))\cdot(s_1\cdots s_{n-(i-1)-1+k})\otimes\lambda^{(i-1)}\lambda^{1-j}\alpha \\ &= \psi^k_{i-1}(x\cdot(s_{n-i+j-2}\cdots s_{n-i})\otimes\lambda^{1-j}\alpha). \end{aligned}$$

In the last line of the above computation, $x\cdot(s_{n-i+j-2}\cdots s_{n-i})\otimes\lambda^{1-j}\alpha$ is an element of $W(n-1)_{(i-1)-k-1} = \mathrm{TL}_{n-1}\otimes_{\mathrm{TL}_{n-i+k}}\mathbb{1}$, so we have

$$\begin{aligned} x\cdot(s_{n-i+j-2}&\cdots s_{n-i})\otimes\lambda^{1-j}\alpha \\ &= x\cdot(s_{n-i+j-2}\cdots s_{n-i+k})\cdot(s_{n-i+k-1}\cdots s_{n-i})\otimes\lambda^{1-j}\alpha \\ &= x\cdot(s_{n-i+j-2}\cdots s_{n-i+k})\otimes\lambda^{k}\lambda^{1-j}\alpha \\ &= x\cdot(s_{n-i+j-2}\cdots s_{n-i+k})\otimes\lambda^{-(j-k-1)}\alpha. \end{aligned}$$

Thus

$$\begin{aligned} d^i_j(\psi^k_i(x\otimes\alpha)) &= \psi^k_{i-1}(x\cdot(s_{n-i+j-2}\cdots s_{n-i})\otimes\lambda^{1-j}\alpha) \\ &= \psi^k_{i-1}(x\cdot(s_{n-i+j-2}\cdots s_{n-i+k})\otimes\lambda^{-(j-k-1)}\alpha.) \\ &= \psi^k_{i-1}(x\cdot(s_{(n-1)-(i-k-1)+(j-k-1)-1}\cdots s_{(n-1)-(i-k-1)})\otimes\lambda^{-(j-k-1)}\alpha) \\ &= \psi^k_{i-1}(d^{i-k-1}_{j-k-1}(x\otimes\alpha)) \end{aligned}$$

as required. □

Corollary 8.20. *Ψ^k is a chain map.*

Proof. The boundary map of $\tau_{n-1}\Sigma^{k+1}W(n-1)$ is given in degree i by the boundary map $d^{i-k-1}\colon W(n-1)_{i-k-1} \to W(n-1)_{i-k-2}$, which is itself given by the formula $\sum_{j=0}^{i-k-1}(-1)^j d^{i-k-1}_j$.

The boundary map of F^k/F^{k-1} is given in degree i by the boundary map of $W(n)$ in degree i, which is the alternating sum $\sum_{j=0}^{i}(-1)^j d^i_j$. However, the proof of Lemma 8.16 shows that $d^i_0, \ldots, d^i_k$ all send F^k into F^{k-1}, and hence that they vanish on the quotient F^k/F^{k-1}. Thus the boundary map of F^k/F^{k-1} is $\sum_{j=k+1}^{i}(-1)^j d^i_j$. It follows that

$$\begin{aligned} d^i \circ \Psi^k_i &= \sum_{j=k+1}^{i} (-1)^j d^i_j \circ [(-1)^{-i(k+1)}\psi^k_i] \\ &= \sum_{j=k+1}^{i} (-1)^{j-i(k+1)}\psi^k_{i-1} \circ d^{i-k-1}_{j-k-1} \\ &= \sum_{j=0}^{i-k-1} (-1)^{j+(k+1)-i(k+1)}\psi^k_{i-1} \circ d^{i-k-1}_{j} \\ &= [(-1)^{-(i-1)(k+1)}\psi^k_{i-1}] \circ \sum_{j=0}^{i-k-1} (-1)^{j}\psi^k_{i-1} \circ d^{i-k-1}_{j} \\ &= \Psi^k_{i-1} \circ d^{i-k-1} \end{aligned}$$

as required. □

8.4. **Proof of Theorem 8.7.** In this subsection we prove Theorem 8.7, which in turn completes the proof of Theorem D.

Recall from Definition 2.20 that $s_i = \lambda + \mu U_i$. Recall from Definition 2.4 the definition of index and terminus of a word in TL_n. Then a word $x \in \mathrm{TL}_n$ 'not containing s_1' can instead be described as a word $x \in \mathrm{TL}_n$ such that the index of x satisfies $i(x) \geqslant 2$.

We first focus on the $k = 0$ part of Theorem 8.7. Recall that $F^0_i \subset \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$ is the span of elements of the form $x \otimes 1$ where $i(x) \geqslant 2$ in degrees $-1 \leqslant i \leqslant n-2$, and $x \cdot (s_1 \cdots s_{n-i-1}) \otimes 1$ where $i(x) \geqslant 2$ in degrees $-1 \leqslant i \leqslant n-1$.

Lemma 8.21. *For $1 \leqslant p \leqslant n-1$, the word $s_1 \ldots s_p$ is equal to $\mu^p U_1 \cdots U_p$, plus a linear combination of scalar multiples - by units - of words w with the following properties:*

- *$i(w) \geqslant 2$ and $t(w) \leqslant p$ or*
- *$i(w) = 1$ and $t(w) < p$.*

In particular only the summand $w = \mu^p U_1 \cdots U_p$ satisfies $i(w) = 1$ and $t(w) = p$.

Proof.

$$\begin{aligned} s_1 \ldots s_p &= \sum_{r=0}^{p} \sum_{(1\leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant p)} \lambda^{p-r}\mu^r U_{i_1}U_{i_2}\ldots U_{i_r} \\ &= \mu^p U_1 \cdots U_p + \sum_{r=0}^{p-1} \sum_{(1\leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant p)} \lambda^{p-r}\mu^r U_{i_1}U_{i_2}\ldots U_{i_r} \end{aligned}$$

If $r = 0$ the term is a scalar, which has index ∞ by convention (thus the first point is satisfied). Suppose $0 < r < p$. Then if $i_1 > 1$ it follows that $i(U_{i_1} \dots U_{i_p}) \geqslant 2$. Otherwise $i_1 = 1$ and, since $r < p$, there is some $j \geqslant 2$ such that $i_j \geqslant i_{j-1} + 2$, so that $U_{i_1} \cdots U_{i_r}$ can be written as a word with terminus i_{j-1}, and then the claim follows. Coefficients are given by powers of λ and μ, and multiples of these. The terms λ and μ defined via the homomorphisms in Definition 2.19 and lie in the set $\{-1, \pm v, v^2\}$. Since v is a unit it follows that all coefficients are units. $\square$

Lemma 8.22. *Let $k \geqslant 0$ and $-1 \leqslant i \leqslant n-1$. Then F_i^k has basis consisting of $x_{\underline{a},\underline{b}} \otimes 1$, where $x_{\underline{a},\underline{b}}$ is in Jones normal form and satisfies either:*

- *$i(x_{\underline{a},\underline{b}}) \geqslant 2$ and $t(x_{\underline{a},\underline{b}}) \geqslant n - i - 1$, or*
- *$i(x_{\underline{a},\underline{b}}) = 1$ and $n - i - 1 \leqslant t(x_{\underline{a},\underline{b}}) \leqslant n - i - 1 + k$.*

Proof. This is a subset of the known basis for $\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1} \supseteq F_i^k$ so it is enough to show that F_i^k is spanned by these elements. By definition F_i^k is spanned by elements of the form

- $x \otimes 1$
- $x \cdot (s_1 \cdots s_{n-i-1+k'}) \otimes 1$

where x is a word in the U_i with $i(x) \geqslant 2$ (i.e. containing no U_1s) and $0 \leqslant k' \leqslant k$ (note in the case $i = n-1$ and $k' = 0$ the two kinds coincide). The first kind is spanned by $x_{\underline{a},\underline{b}}$ such that $i(x_{\underline{a},\underline{b}}) \geqslant 2$, the first type of basis elements in the statement of the Lemma. From Lemma 8.21, expanding the product $(s_1 \cdots s_{n-i-1+k'})$ in the second kind gives a linear combination of words $x \cdot w \otimes 1$ such that $t(w) \leqslant n - i - 1 + k'$. Either $i(w)$ will be $\geqslant 2$ or $i(w) = 1$. In the first case, since $i(x) \geqslant 2$ it follows that $i(x \cdot w) \geqslant 2$ and so when written in Jones normal form this will remain the case, giving a basis element of the first type. In the second case, since $i(w) = 1$ and $i(x) \geqslant 2$, then either $i(x \cdot w) \geqslant 2$ and we are done, or $i(x \cdot w) = 1$ and, by Lemma 2.7, when written in Jones normal form the terminus $t(x \cdot w) = t(w) \leqslant n - i - 1 + k' \leqslant n - i - 1 + k$ will either remain the same or reduce. Since $F_i^k \subseteq \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$ any such word written in Jones normal form will vanish if $t(x_{\underline{a},\underline{b}}) \leqslant n - i - 2$ so that all words remaining are of the desired form. $\square$

Proposition 8.23. *The map $\Phi^0 \colon C(W(n-1)) \longrightarrow F^0$ from Definition 8.10 is an isomorphism.*

Proof. Recall that for $-1 \leqslant i \leqslant (n-1)$,

$$\Phi_i^0 : \left(\mathrm{TL}_{n-1} \otimes_{\mathrm{TL}_{n-i-2}} \mathbb{1}\right) \oplus \left(\mathrm{TL}_{n-1} \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}\right) \to F_i^0$$

is given by

$$\Phi_i^0(x \otimes \alpha, y \otimes \beta) = \xi_i(x \otimes \alpha) + \eta_i(y \otimes \beta)$$

where

$$\xi_i(x \otimes \alpha) = \sigma(x) \otimes \lambda^{n-1}\alpha$$

and

$$\eta_i(y \otimes \beta) = \sigma(y) \cdot (s_1 \cdots s_{n-i-1}) \otimes \lambda^i \beta.$$

By Lemma 2.15, a basis for the left hand side is given by elements of either the form $(x_{\underline{a},\underline{b}} \otimes 1, 0)$ such that $t(x_{\underline{a},\underline{b}}) > n-i-3$ or the form $(0, x_{\underline{a}',\underline{b}'} \otimes 1)$ such that $t(x_{\underline{a}',\underline{b}'}) > n-i-2$. Under the map Φ_i^0, $(x_{\underline{a},\underline{b}} \otimes 1, 0)$ is taken to a scalar multiple (by a unit) of $\sigma(x_{\underline{a},\underline{b}}) \otimes 1$, where $\sigma(x_{\underline{a},\underline{b}})$ is a Jones basis element with $i(\sigma(x_{\underline{a},\underline{b}})) \geqslant 2$ and $t(\sigma(x_{\underline{a},\underline{b}})) > n-i-2$. By Lemma 8.21, the element $(0, x_{\underline{a}',\underline{b}'} \otimes 1)$ is taken to a linear combination of scalar multiples (by units) of terms $\sigma(x_{\underline{a}',\underline{b}'}) \cdot w \otimes 1$ such that $t(w) \leqslant n-i-1$. Since $F_i^0 \subseteq \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$ the only non-zero terms in the image will occur when $t(w) = n-i-1$. We consider two cases: $i(w) \geqslant 2$ or $i(w) = 1$. By Lemma 2.7, converting to Jones normal form in the first case gives an element with index $i(\sigma(x_{\underline{a}',\underline{b}'}) \cdot w) > 2$ and terminus $t(\sigma(x_{\underline{a}',\underline{b}'}) \cdot w) = n-i-1$, or zero, since the terminus will either remain the same or reduce when converting. When $i(w) = 1$ and $t(w) = n-i-1$, by Lemma 8.21 the terms will be of the form $\sigma(x_{\underline{a}',\underline{b}'}) \cdot U_1 \ldots U_{n-i-1}$. These elements are already in Jones Normal form, with index 1 and terminus $n-i-1$. Furthermore all Jones basis elements with this index and terminus arise in this way. By Lemma 8.22 a basis for F_i^0 is given by elements $y_{\underline{a},\underline{b}} \otimes 1$ where $y_{\underline{a},\underline{b}}$ is in Jones normal form and satisfies:

- $i(y_{\underline{a},\underline{b}}) \geqslant 2$ and $t(y_{\underline{a},\underline{b}}) \geqslant n-i-1$ or
- $i(y_{\underline{a},\underline{b}}) = 1$ and $t(y_{\underline{a},\underline{b}}) = n-i-1$.

By our analysis, all of these elements lie in the image of Φ_i^0, up to scalar multiplication by units, hence Φ^0 is a bijection on bases and therefore an isomorphism. □

Lemma 8.24. *A basis for $(F^k/F^{k-1})_i$ is given by words $x_{\underline{a},\underline{b}}$ in Jones normal form such that $i(x_{\underline{a},\underline{b}}) = 1$ and $t(x_{\underline{a},\underline{b}}) = n-i-1+k$.*

Proof. This is a direct consequence of taking the quotient of the bases for F^k and F^{k-1} given in Lemma 8.22. □

Proposition 8.25. *The map $\Psi^k : \tau_{n-1}\Sigma^{k+1}W(n-1) \longrightarrow F^k/F^{k-1}$ from Definition 8.17 is an isomorphism.*

Proof. Recall for i in the range $k \leqslant i \leqslant (n-1)$,

$$\Psi_i^k \colon \mathrm{TL}_{n-1} \otimes_{\mathrm{TL}_{n-i-1+k}} \mathbb{1} \longrightarrow (F^k/F^{k-1})_i$$

is given by

$$\Psi_i^k \colon x \otimes \alpha \longmapsto (-1)^{-i(k+1)} \sigma(x) \cdot (s_1 \cdots s_{n-i-1+k}) \otimes \lambda^i \alpha.$$

By Lemma 2.15 a basis for the domain is given by $x_{\underline{a},\underline{b}}$ such that $t(x_{\underline{a},\underline{b}}) > (n-i-1+k)-1$. Note also that $x_{\underline{a},\underline{b}}$ does not contain the letter U_{n-1}. By Lemma 8.21, the image $\Psi_i^k(x_{\underline{a},\underline{b}})$ is a linear combination of scalar multiples (by units) of terms $\sigma(x_{\underline{a},\underline{b}}) \cdot w$ such that $t(w) \leqslant n-i-1+k$. These terms are zero in $(F^k/F^{k-1})_i \subseteq \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1}$ only when w cannot be written as a word with $t(w) < n-i-1$. Rewriting these elements in Jones Normal form will maintain or decrease the terminus, and $i(\sigma(x_{\underline{a},\underline{b}})) \geqslant 2$, so $i(\sigma(x_{\underline{a},\underline{b}}) \cdot w) = 1$ only when $i(w) = 1$. Therefore by Lemma 8.22 quotienting out by F^{k-1} leaves only the term for which $i(w) = 1$ and $t(w) = n - i - 1 + k$. In particular by Lemma 8.21 this term is a scalar multiple (by a unit) of $\sigma(x_{\underline{a},\underline{b}}) \cdot U_1 \ldots U_{n-i-1+k}$.

Since $\sigma(x_{\underline{a},\underline{b}})$ has index $\geqslant 2$ and terminus $> n-i-1+k$, it follows that $\sigma(x_{\underline{a},\underline{b}}) \cdot U_1 \ldots U_{n-i-1+k}$ is in Jones normal form. From Lemma 8.24 this is a Jones basis element for F^k/F^{k-1} and all basis elements arise in this way. Therefore up to unit scalars, the map Ψ^k is a bijection on bases, and hence an isomorphism. □

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