COMBINATORICS OF INJECTIVE WORDS FOR TEMPERLEY-LIEB ALGEBRAS

RACHAEL BOYD AND RICHARD HEPWORTH

Abstract. This paper studies combinatorial properties of the *complex of planar injective words*, a chain complex of modules over the Temperley-Lieb algebra that arose in our work on homological stability. Despite being a linear rather than a discrete object, our chain complex nevertheless exhibits interesting combinatorial properties. We show that the Euler characteristic of this complex is the n-th Fine number. We obtain an alternating sum formula for the representation given by its top-dimensional homology module and, under further restrictions on the ground ring, we decompose this module in terms of certain standard Young tableau. This trio of results — inspired by results of Reiner and Webb for the complex of injective words — can be viewed as an interpretation of the n-th Fine number as the 'planar' or 'Dyck path' analogue of the number derangements of n letters. This interpretation has precursors in the literature, but here emerges naturally from considerations in homological stability. Our final result shows a surprising connection between the boundary maps of our complex and the Jacobsthal numbers.

Contents

1. Introduction

In this work we study combinatorial properties of a highly connected complex that arose in our study of the *Temperley-Lieb algebra* in [BH20]. Highly connected

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complexes arise naturally in many areas of mathematics. In combinatorics they arise as matroid complexes and order complexes of geometric lattices [Bjö92], as order complexes of Cohen-Macaulay posets [BGS82], and in the theory of shellability in its various forms [Bjö92, BW83, Koz08], to name just a few. For the authors, highly connected complexes arise in the theory of *homological stability*. This subject is motivated by the study of homology and cohomology of groups and spaces, and makes extensive use of complexes such as buildings, split buildings, complexes of partial bases (of vector spaces, modules, and free groups), complexes of arcs in surfaces, and many more besides. (Though no standard introductory reference currently exists for homological stability, we recommend [Wah13]. The introduction of [RWW17] may also give a good impression of the theory's scope.)

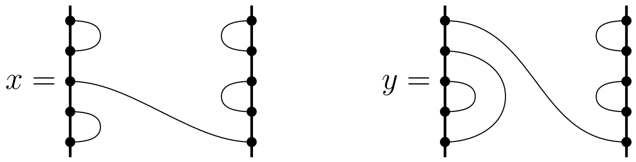
The *complex of injective words* is much studied in both combinatorics and topology. Its high-connectivity has been proved using various methods, by authors including Farmer [Far79], Maazen [Maa79], Björner-Wachs [BW83], Kerz [Ker05], and Randal-Williams [RW13], and is an important ingredient in proofs of homological stability for the symmetric groups [Maa79, Ker05, RW13]. Reiner and Webb [RW04] studied the complex of injective words from a combinatorial point of view. They showed that its Euler characteristic is the number of derangements of n letters, and they described its top-dimensional homology representation in two ways: as an alternating sum, and in terms of standard Young tableaux. A further decomposition of the top-dimensional homology was given by Hanlon and Hersh in [HH04].

In our work on homological stability for Temperley-Lieb algebras [BH20], we introduced and studied the *complex of planar injective words*, a chain complex of modules over the Temperley-Lieb algebra on n strands, closely analogous to the (chain complex of the) complex of injective words. In particular we proved that the homology of our complex is concentrated in degree $(n-1)$, as is the case for the complex of injective words.

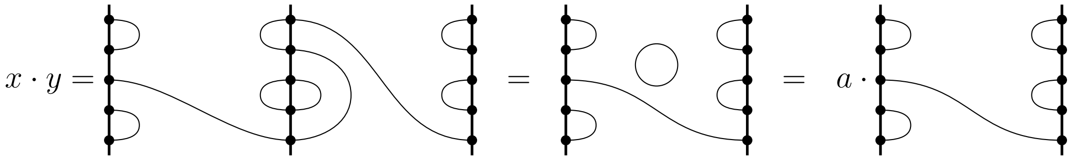
In this paper we study the complex of planar injective words from a combinatorial viewpoint, inspired by the results of Reiner and Webb. We will see that the role of the number of derangements is now taken by the n-th *Fine number*. We will also expose an unexpected appearance of the *Jacobsthal numbers*.

1.1. **Temperley-Lieb algebras and planar injective words.** Let $n \geqslant 0$, let R be a commutative ring, and let $a \in R$. The *Temperley-Lieb algebra* $\mathrm{TL}_n(a)$ is the R-algebra with basis given by the planar diagrams on n strands, taken up to isotopy, and with multiplication given by pasting diagrams and replacing closed loops with factors of a. The last sentence was intentionally brief, we hope that its meaning becomes clearer with the following illustration of two elements $x, y \in$

$\mathrm{TL}_5(a)$



and their product $x \cdot y$.



The Temperley-Lieb algebras arose in theoretical physics in the 1970s [TL71]. They were later rediscovered by Jones in his work on von Neumann algebras [Jon83], and used in the first definition of the Jones polynomial [Jon85]. Kauffman gave the diagrammatic interpretation of the algebras in [Kau87] and [Kau90]. The rank of $\mathrm{TL}_n(a)$ as an R-module is the n-th Catalan number C_n [Jon87].

Now let $a = v + v^{-1}$ where $v \in R^\times$ is a unit (the most commonly studied case in the literature). The *complex of planar injective words* $W(n)$ is a chain complex of $\mathrm{TL}_n(a)$-modules. In degree i it is given by the tensor product module $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}$, where $\mathbb{1}$ is the trivial module for $\mathrm{TL}_{n-i-1}(a)$. In the original complex of injective words the i-simplices are words $(x_0, \ldots, x_i)$ on the alphabet $\{1, \ldots, n\}$ with no repeated entries. The action of $\mathfrak{S}_n$ on these simplices is transitive, and the typical stabiliser is $\mathfrak{S}_{n-i-1}$, so that the i-th chain group is isomorphic to $R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i-1}} \mathbb{1}$. Thus $W(n)$ is an analogue of (the chain complex of) the complex of injective words, in which the role of $\mathfrak{S}_n$ is now played by $\mathrm{TL}_n(a)$. In [BH20] we showed that $H_d(W(n)) = 0$ for $d \leqslant n-2$, and since the complex is concentrated in degrees from -1 to $n-1$, it follows that its only homology group is $H_{n-1}(W(n))$. The restriction to the case $a = v + v^{-1}$ is necessary for $\mathrm{TL}_n(a)$ to receive a homomorphism from the group algebra of the braid group, which is required in order to define the differentials of $W(n)$.

1.2. **Results.** The n-th *Fine number* F_n is the number of Dyck paths of length $2n$ whose first peak has even height. This is the second of 11 descriptions of the Fine numbers given by Deutsch and Shapiro in their survey [DS01]. Deutsch and Shapiro also state the following alternating sum formula for F_n:

$$F_n = \frac{1}{n+1}\left[\binom{2n}{n} - 2\binom{2n-1}{n} + 3\binom{2n-2}{n} - \cdots + (-1)^n(n+1)\binom{n}{n}\right] \quad (1)$$

(See [DS01, Section 4] and also [Deu99, Moo79, Rob04].) We show that this alternating sum has a very simple interpretation: its m-th term counts the Dyck paths whose first peak has height at least m.

Remarkably, the complex of planar injective words $W(n)$ embodies a representation theoretical 'lifting' of the Fine numbers and of this alternating sum formula.

Theorem A. *Let R be a commutative ring, let $v \in R^\times$, and let $a = v+v^{-1}$. Then the Euler characteristic of $W(n)$ is the n-th Fine number, up to sign:*

$$\chi(W(n)) = (-1)^{n-1} F_n.$$

The $\mathrm{TL}_n(a)$-module $H_{n-1}(W(n))$ therefore has rank equal to the Fine number F_n. We call it the *Fineberg module*, and we denote it by $\mathcal{F}_n(a)$. (Such top-dimensional homology groups are often called Steinberg modules, after the top-dimensional homology of the Tits building of a vector space.) As a consequence of Theorem A we obtain the following representation-theoretic lifting of (1) in terms of induced modules.

Corollary B. *Under the assumptions of Theorem A, the alternating sum formula*

$$[\mathcal{F}_n(a)] = \sum_{m=0}^{n} (-1)^m \left[\mathbb{1} \uparrow_{\mathrm{TL}_m(a)}^{\mathrm{TL}_n(a)} \right]$$

holds in the Grothendieck group $K_0(\mathrm{TL}_n(a))$. (If $\mathrm{TL}_n(a)$ is semisimple, then $K_0(\mathrm{TL}_n(a))$ is the module of virtual representations of $\mathrm{TL}_n(a)$.)

We now consider the case $R = \mathbb{C}$, so that $a = v+v^{-1}$ with $v \in \mathbb{C}^\times$. Then $\mathrm{TL}_n(a)$ is semisimple unless $q = v^2$ is an ℓ-th root of unity for $2 \leqslant \ell \leqslant n$. In the case of semisimplicity the irreducible representations V_λ of $\mathrm{TL}_n(a)$ are indexed by partitions $\lambda \vdash n$ with at most two columns. We prove the following description of $\mathcal{F}_n(a)$ in terms of counts of standard Young tableaux (SYT).

Theorem C. *Let $R = \mathbb{C}$, let $v \in \mathbb{C}^\times$ be such that v^2 is not an ℓ-th root of unity for $2 \leqslant \ell \leqslant n$, and let $a = v + v^{-1}$. Then*

$$\mathcal{F}_n(a) \cong \bigoplus_{\substack{\lambda \vdash n \\ \leqslant 2 \text{ columns}}} |\{\text{SYT } Q \text{ of shape } \lambda \text{ with top entry of second column odd}\}| \cdot V_\lambda.$$

(In the case $\lambda = 1^n$, the unique SYT of shape λ has no second column, and so we declare that the top entry of its second column is $(n+1)$.)

The three results listed above are the direct analogues of Reiner and Webb's results relating the complex of injective words to the number of derangements of n letters [RW04, Propositions 2.1–2.3]. This suggests an interpretation of the n-th Fine number as the number of 'planar derangements' of n letters. This interpretation has several precursors in the literature: One precursor is the fact that the n-th Fine number is equal to the number of Dyck paths of length $2n$ whose first peak has even height, while Désarménien [Dés83] showed that the number of derangements of n is equal to the number of permutations whose first ascent $\pi(i) < \pi(i+1)$ occurs for i even. Another precursor is that Dyck paths can be interpreted as permutations that avoid the pattern 321 [Sta99, p.224], and the Fine number is the number of derangements that avoid 321 [DS01, Section 8]. It is striking that the same interpretation has arisen naturally through our work on homological stability and injective words.

We now turn to a feature of planar injective words that does not have a precursor in the case of injective words. The n-th *Jacobsthal number* J_n is the number of compositions of n that end with an odd number. It is equal to the number of sequences $n > a_1 > a_2 > \cdots > a_r > 0$ whose initial term has the opposite parity to n. The l^{th} *Jacobsthal element* in $\mathrm{TL}_n(a)$ is defined to be

$$\mathcal{J}_l^n = \sum_{\substack{l>a_1>\cdots>a_r>0 \\ l-a_1 \text{ odd}}} (-1)^{(r-1)+l} \left(\frac{\mu}{\lambda}\right)^r U_{a_1+n-l} \cdots U_{a_r+n-l}.$$

(Here λ and μ are constants involved in the definition of $W(n)$, and $\frac{\mu}{\lambda}$ is equal to either $-v$ or $-v^{-1}$.) Observe that the number of irreducible terms in $\mathcal{J}_l^n$ is J_l. We prove the following identification of the boundary maps of $W(n)$.

Theorem D. *Under the assumptions of Theorem A, for $0 \leqslant i \leqslant n-1$ the boundary map $d^i\colon W(n)_i \to W(n)_{i-1}$ acts as right multiplication by $\mathcal{J}_{i+1}^n$ in the following sense:*

$$d^i(x \otimes r) = x \cdot \mathcal{J}_{i+1}^n \otimes r.$$

In particular, the image of the boundary map has number of irreducible terms given by J_{i+1}.

The formula for the differentials in Theorem D is convenient for explicit computations, and is used in [BH20] to describe (aspects of) the Fineberg module $\mathcal{F}_n(a)$ in the case of n even.

1.3. **Complexes from algebras.** We hope that the results of this paper will encourage others to consider constructing and studying chain complexes of algebra modules from a combinatorial point of view.

The general idea is that one can combine combinatorial complexes with $\mathfrak{S}_n$-action (such as the complex of injective words) with finite-dimensional algebras (such as the Temperley-Lieb algebras) and construct algebraic analogues of the complexes. Examples of possible complexes with $\mathfrak{S}_n$-action include the complex of injective words, the realisation of the poset of ordered partitions of $\{1, \ldots, n\}$ with $k \geqslant 2$ parts (this can be naturally identified with the permutahedron), and the realisation of the *partition poset*, which consists of partitions of $\{1, \ldots, n\}$ with $1 < k < n$ parts. Examples of possible algebras include the Temperley-Lieb algebras studied here, Temperley-Lieb algebras of types B and D, variants such as the dilute and periodic Temperley-Lieb algebras, and cousins such as the Brauer, blob and partition algebras. (Here we only list examples that are somewhat close to the $\mathrm{TL}_n(a)$; there will be many other candidates besides.)

In all cases, one can undertake the following 'process' that has as input a complex C with $\mathfrak{S}_n$-action and a family of algebras A_n, and as output a chain complex of A_n-modules. The process takes a $\mathfrak{S}_n$-orbit of simplices, and replaces it with the A_n-module induced from the trivial module modulo the subalgebra corresponding to the stabiliser of the 'original' orbit. (This is of course only a vague process, and its success depends on the nature of C and the A_n.)

1.4. **Outline.** In Section 2 we recall the basics of Temperley-Lieb algebras that we require in the rest of the paper. Section 3 recalls the definition of the complex of planar injective words $W(n)$ from [BH20]. Section 4 recalls Dyck paths, Dyck words, Catalan numbers and Fine numbers, and refines the usual relationship between them to take into account the height of the first peak, ending with a new account of the alternating sum in Equation (1). In Section 5 we recall the relationship between planar diagrams and Dyck words, and prove Theorem A and Corollary B. In Section 6 we prove Theorem C. And in Section 7 we recall the Jacobsthal numbers and prove Theorem D.

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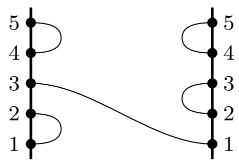
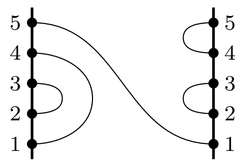
2. Temperley-Lieb algebras

In this section we will cover the basic facts about Temperley-Lieb algebras that we require in the rest of the paper. There is some overlap between the material recalled here and in [BH20]. General references for readers new to the $\mathrm{TL}_n(a)$ are Section 5.7 of Kassel and Turaev's book [KT08] on the braid groups, and especially Sections 1 and 2 of Ridout and Saint-Aubin's survey on the representation theory of the $\mathrm{TL}_n(a)$ [RSA14].

2.1. **Definitions.** A *planar diagram on n strands* consists of two vertical lines in the plane, decorated with n dots labelled $1, \ldots, n$ from bottom to top, together with a collection of n arcs joining the dots in pairs. The arcs must lie between the vertical lines, they must be disjoint, and the diagrams are taken up to isotopy. For example, here are two planar diagrams in the case $n = 5$:

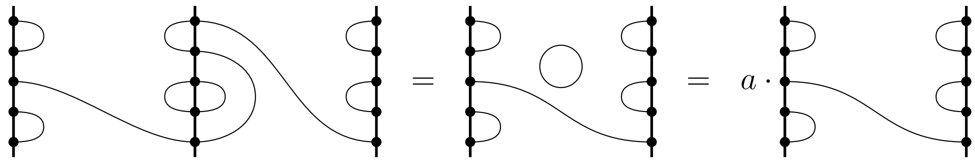
$x =$ 

$y =$ 

We will often omit the labels on the dots.

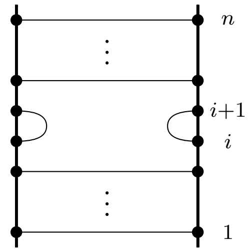
Definition 2.1 (The Temperley-Lieb algebra $\mathrm{TL}_n(a)$)**.** Let R be a commutative ring and let $a \in R$. The *Temperley-Lieb algebra* $\mathrm{TL}_n(a)$ is the R-module with basis given by the planar diagrams on n strands, and with multiplication defined by placing diagrams side-by-side and joining the ends. Any closed loops created by this process are then erased and replaced with a factor of a.

For example, the product xy of the elements x and y above is:



We have subscribed to the heresy of [RSA14] by drawing planar diagrams that go from left to right rather than top to bottom. The identity element of $\mathrm{TL}_n(a)$ is the planar diagram in which each dot on the left is joined to the corresponding dot on the right by a straight horizontal line.

For $1 \leqslant i \leqslant n-1$, we define $U_i \in \mathrm{TL}_n(a)$ to be the planar diagram shown below.

$U_i =$ 

We refer to an arc joining adjacent dots as a *cup*. Thus U_i has a single cup on left and right joining dots i and $i+1$. The elements U_i satisfy the following relations:

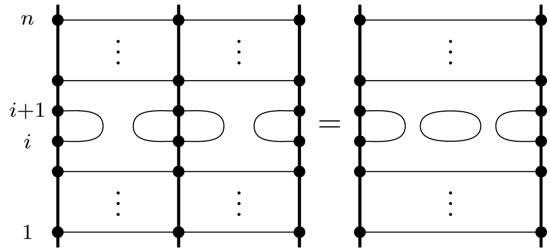
(1) $U_iU_j = U_jU_i$ for $j \neq i \pm 1$
(2) $U_iU_jU_i = U_i$ for $j = i \pm 1$
(3) $U_i^2 = aU_i$ for all i.

The reader can easily verify these relations for themselves; two of them are shown in Figure 1. In fact, the generators U_i together with the three relations above form a presentation of $\mathrm{TL}_n(a)$ as an R-algebra: Elements of the Temperley-Lieb algebra are formal sums of monomials in the U_i, with coefficients in the ground ring R, modulo the relations above. This is proved in [RSA14, Theorem 2.4], [KT08, Theorem 5.34], and [Kau05, Section 6]. We often write $\mathrm{TL}_n(a)$ as TL_n. We note here that $\mathrm{TL}_0 = \mathrm{TL}_1 = R$.

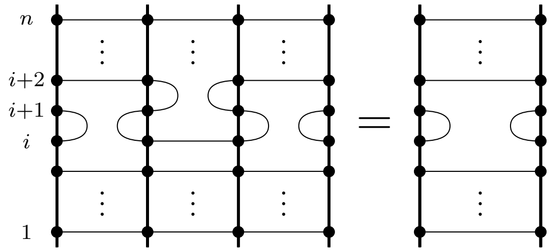
2.2. **Induced modules.**

Definition 2.2 (The trivial module $\mathbb{1}$)**.** The *trivial* module $\mathbb{1}$ of the Temperley-Lieb algebra $\mathrm{TL}_n(a)$ is the module consisting of R with the action of $\mathrm{TL}_n(a)$ in which every diagram acts as multiplication by 0, except for the identity diagram. Equivalently, it is the module on which all of the generators $U_1, \dots, U_{n-1}$ act as 0. We can regard $\mathbb{1}$ as either a left or right module, and we will usually do that without indicating so in the notation.

Definition 2.3 (Sub-algebra convention)**.** For $m \leqslant n$, we will regard TL_m as the sub-algebra of TL_n generated by the elements $U_1, \dots, U_{m-1}$, or equivalently, the subalgebra in which dots $m+1, \dots, n$ on the left are joined to the corresponding dots on the right by horizontal straight lines. We will often regard TL_n



(a) The relation $U_i^2 = aU_i$.



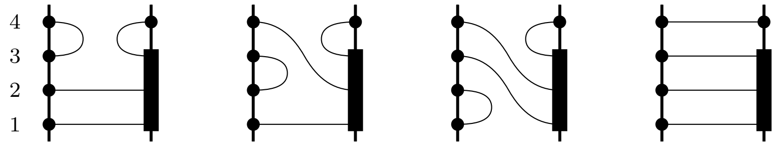
(b) The relation $U_i U_{i+1} U_i = U_i$.

FIGURE 1. Diagrammatic relations in $\mathrm{TL}_n(a)$.

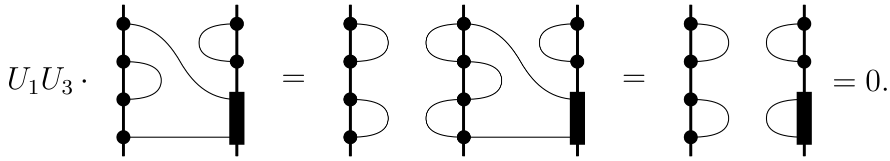
as a left TL_n-module and a right TL_m-module, so that we obtain the left TL_n-module $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$.

The induced modules $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ will be the building blocks of the complex $W(n)$.

A *planar diagram on n strands with black box of size m* is a planar diagram on n strands with the dots $1, \ldots, m$ on the right encapsulated within a *black box*, such that there are no cups with endpoints in the black box. For example, the planar diagrams with 4 strands and black box of size 3 are shown below.



The R-linear span of the planar diagrams on n strands with black box of size m has the structure of a left $\mathrm{TL}_n(a)$-module. If x is a planar diagram on n strands, and y is a planar diagram on n strands with black box of size m, then the product $x \cdot y$ is defined by pasting the diagrams in the usual way, subject to the condition that if the pasting produces a cup attached to the black box, then the diagram is identified with 0. For example:



Proposition 2.4. *Given* $0 \leqslant m \leqslant n$*,* $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ *is isomorphic to the module of planar diagrams on* n *strands with black box of size* m*.*

Proof. Let I_m denote the left ideal of TL_n generated by the elements $U_1, \ldots, U_{m-1}$. In other words, I_m is the span of all diagrams which have a cup on the right among dots $1, \ldots, m$. It is shown in [BH20] that $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ is isomorphic to TL_n/I_m under the isomorphism that sends $x \otimes 1 \in \mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ to $x + I_m \in \mathrm{TL}_n/I_m$. Since TL_n has basis given by planar diagrams, and I_m has basis given by planar diagrams with cup among dots $1, \ldots, m$ on the right, the quotient TL_n/I_m has basis given by the diagrams with no cups among dots $1, \ldots, m$ on the right, which we can identify with the n-planar diagrams with black box of size m. This determines an R-linear isomorphism between $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ and the module of planar diagrams on n strands with black box of size m, and it is simple to see that this respects the module structures. □

2.3. **The braiding elements.** Now we suppose that $a = v + v^{-1}$ where $v \in R$ is a unit.

Definition 2.5 (The braiding elements)**.** Define $s_1, \ldots, s_{n-1} \in \mathrm{TL}_n(v + v^{-1})$ by setting

$$s_i = \lambda + \mu U_i$$

where $\lambda, \mu \in R$ are defined by one of the following two options:

(1) $\lambda = -1$ and $\mu = v$, so that $s_i = vU_i - 1$
(2) $\lambda = v^2$ and $\mu = -v$, so that $s_i = v^2 - vU_i$.

It is now easy to verify that the elements s_i satisfy the *braid relations*:

- $s_i s_j = s_j s_i$ for $i \neq j \pm 1$
- $s_i s_j s_i = s_j s_i s_j$ for $i = j \pm 1$.

Moreover, the s_i are invertible and satisfy the rule:

$$s_i^{-1} = \lambda^{-1} + \mu^{-1} U_i.$$

It is also immediate to verify that s_i acts on $\mathbb{1}$ as multiplication by λ.

The s_i in fact form the generators in a presentation of $\mathrm{TL}_n(v+v^{-1})$ as a quotient of the *Iwahori-Hecke algebra* of type A_{n-1}. In particular, they satisfy further relations of degree 2 and 3, that we will not list here. See [BH20] for more details.

Remark 2.6. There is a homomorphism from (the group algebra of) the braid group into $\mathrm{TL}_n(v+v^{-1})$ given on generators by $s_i \mapsto s_i$. This can be regarded as a

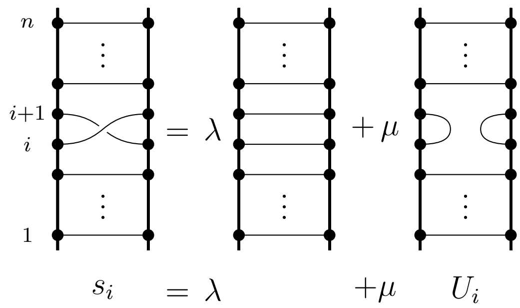


FIGURE 2. Smoothings of s_i.

Kauffman bracket-style smoothing operation from braids to planar diagrams: The formula for s_i tells us to smooth a positive crossing in the two possible ways with weights λ and μ as in Figure 2, and the formula for s_i^{-1} tells us to smooth a negative crossing in the two possible ways with weights λ^{-1} and μ^{-1}. In general, given a braid diagram with p crossings, each crossing is smoothed in the 2 possible ways, with appropriate weights, to obtain a linear combination of 2^p planar diagrams. We may also consider hybrid diagrams obtained by concatenating planar and braid diagrams, or obtained by partially smoothing braid diagrams, though this will not be important in the rest of the paper.

3. Injective words and planar injective words

Throughout this section we will consider the Temperley-Lieb algebra $\mathrm{TL}_n(a)$, where $a = v + v^{-1}$ for $v \in R$ a unit. We will make use of the elements $s_1, \ldots, s_{n-1}$ of Definition 2.5.

An *injective word* on the letters $\{1, \ldots, n\}$ is a tuple $(x_0, \ldots, x_i)$ whose entries come from the set $\{1, \ldots, n\}$, with no repeated entries in the tuple. Injective words form a poset under the subword relation: $w \geqslant v$ if v is a subword of w. The complex of injective words is most commonly defined as the realisation of this poset, as in [Far79] or [BW83]. However, note that for any injective word w, the poset of elements $v \leqslant w$ is Boolean. It follows that the poset of injective words is a simplicial poset, and its realisation admits a cell structure in which the cells are simplices, with an i-simplex for each word $(x_0, \ldots, x_i)$. This cell complex can be obtained from a semi-simplicial set, as in [RW13]. For us, the complex of injective words will be the augmented cellular chains of the cell complex described above, studied for example in [Ker05]. We define it explicitly now.

Definition 3.1 (The complex of injective words)**.** The *complex of injective words* is the chain complex $\mathcal{C}(n)$ of $\mathfrak{S}_n$-modules, concentrated in degrees -1 to $(n-1)$, that in degree i is the free R-module with basis given by tuples $(x_0, \ldots, x_i)$ where $x_0, \ldots, x_i \in \{1, \ldots, n\}$ and no letter appears more than once. We allow the empty word (), which lies in degree -1. The differential of $\mathcal{C}(n)$ sends a word $(x_0, \ldots, x_i)$ to the alternating sum $\sum_{j=0}^{i}(-1)^j(x_0, \ldots, \widehat{x_j}, \ldots, x_i)$.

We can rewrite $\mathcal{C}(n)$ in terms of the group algebra $R\mathfrak{S}_n$. Denote by $s_1, \ldots, s_{n-1} \in \mathfrak{S}_n$ the adjacent transpositions $s_i = (i \;\; i+1)$. These elements satisfy the braid relations listed beneath Definition 2.5. There is an isomorphism

$$\mathcal{C}(n)_i \cong R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i-1}} \mathbb{1},$$

where $\mathbb{1}$ is the trivial module of $R\mathfrak{S}_{n-i-1}$. Under this isomorphism the word $(x_0, \ldots, x_i)$ is sent to $\sigma \otimes 1$ where $\sigma \in \mathfrak{S}_n$ is a permutation such that $\sigma(n-i+j) = x_j$. Furthermore, the differential $d\colon \mathcal{C}(n)_i \to \mathcal{C}(n)_{i-1}$ becomes the map

$$d\colon R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i-1}} \mathbb{1} \longrightarrow R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-i}} \mathbb{1}$$

defined by $d(x \otimes 1) = \sum_{j=0}^{i}(-1)^j x \cdot (s_{n-i+j-1} \cdots s_{n-i}) \otimes 1$. (See [Hep20].) This description inspires the following definition of the planar analogue.

Definition 3.2 (The complex of planar injective words [BH20])**.** Let R be a commutative ring, let $v \in R^\times$, let $a = v + v^{-1}$, and let $n \geqslant 0$. The complex of *planar injective words* is the chain complex $W(n)_*$ of $\mathrm{TL}_n(a)$-modules defined as follows. For i in the range $-1 \leqslant i \leqslant n-1$, the degree-$i$ part of $W(n)_*$ is defined by

$$W(n)_i = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1}$$

and in all other degrees we set $W(n)_i = 0$. Note that

$$W(n)_{-1} = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_n(a)} \mathbb{1} = \mathbb{1}.$$

For $i \geqslant 0$ the boundary map $d^i\colon W(n)_i \to W(n)_{i-1}$ is defined to be the alternating sum $\sum_{j=0}^{i}(-1)^j d^i_j$, where

$$\begin{aligned} d^i_j\colon \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}(a)} \mathbb{1} &\to \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i}(a)} \mathbb{1} \\ x \otimes r &\mapsto (x \cdot s_{n-i+j-1} \cdots s_{n-i}) \otimes \lambda^{-j} r. \end{aligned}$$

In the expression $s_{n-i+j-1} \cdots s_{n-i}$, the indices decrease from left to right. Observe that d_j is well-defined because the elements $s_{n-i}, \ldots, s_{n-i+j-1}$ all commute with all generators of $\mathrm{TL}_{n-i-1}(a)$. We have depicted $W(n)_*$ in Figure 3. For notational purposes we will write $W(n)$ and only use a subscript when identifying a particular degree.

Remark 3.3 (Visualising $W(n)$)**.** Recall from the diagrammatic description of the induced module $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ when $m \leqslant n$ given in Section 2.2 that elements of $W(n)_i$ can be regarded as diagrams where the first $n-i-1$ dots on right are encapsulated within a black box, and if any cups can be absorbed into the black box, then the diagram is identified with 0. The differential $d\colon W(n)_i \to W(n)_{i+1}$ is then given by pasting special elements onto the right of a diagram, followed by taking their signed and weighted sum. These special elements each enlarge the black box by an extra strand, and plumb one of the free strands into the new space

$$\begin{array}{cc}
\mathrm{TL}_n \otimes_{\mathrm{TL}_0} \mathbb{1} & n-1 \\
\downarrow{\scriptstyle d^{n-1}} & \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_1} \mathbb{1} & n-2 \\
\downarrow{\scriptstyle d^{n-2}} & \\
\vdots & \\
\downarrow{\scriptstyle d^{2}} & \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-2}} \mathbb{1} & 1 \\
\downarrow{\scriptstyle d^{1}} & \\
\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-1}} \mathbb{1} & 0 \\
\downarrow{\scriptstyle d^{0}} & \\
\mathbb{1} & -1
\end{array}$$

FIGURE 3. The complex $W(n)$

in the black box, see Figure 4 The resulting diagrams can be simplified using the

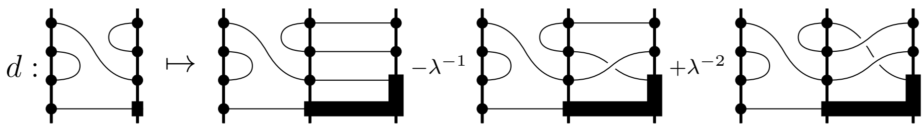


FIGURE 4. Example: $D : W(4)_2 \to W(4)_1$

smoothing rules for diagrams with crossings described in Remark 2.6. We leave it to the reader to make this description as precise as they wish, and note here that this is where the notion of *braiding*, so often seen in homological stability arguments, fits into our set up.

In [BH20] we showed the following analogue of the high-connectivity of the complex of injective words. It was the main technical underpinning of our proof of homological stability for Temperley-Lieb algebras.

Theorem 3.4 ([BH20])**.** *$H_d(W(n)) = 0$ for $d \leqslant (n-2)$.*

The top homology of the Tits building is known as the *Steinberg module*. This inspires the name in the following definition.

Definition 3.5. The *n-th Fineberg module* is the $\mathrm{TL}_n(a)$-module

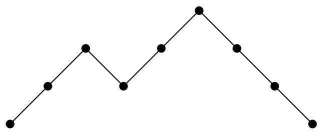
$$\mathcal{F}_n(a) = H_{n-1}(W(n)).$$

We often suppress the a and simply write $\mathcal{F}_n$. The rank of $\mathcal{F}_n$ is the n-th Fine number F_n.

4. Dyck paths, Catalan numbers, and Fine numbers

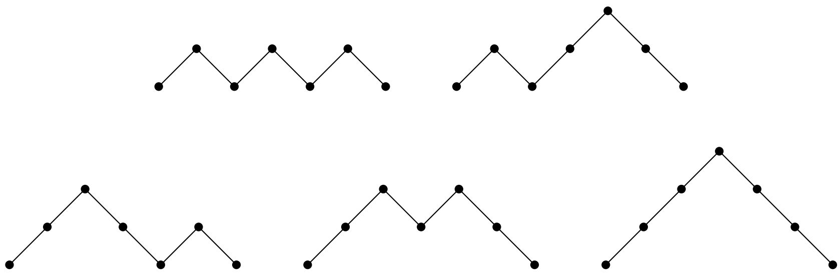
We now recall Dyck paths, Catalan numbers and Fine numbers. We also recall the familiar formula for the Catalan numbers and extend it to take the height of the first peak into account, leading to a new proof of Equation (1) from the introduction.

A *Dyck path* is a path starting and ending on the horizontal axis, built using the steps $(1,1)$ and $(1,-1)$, and never falling below the horizontal axis. We abbreviate the steps $(1,1)$ and $(1,-1)$ by u and d respectively. Thus Dyck paths are in correspondence with *Dyck words*, i.e. words in the letters u and d containing equal numbers of us and ds, and such that no initial segment contains more ds than us. The following figure shows a Dyck path and its corresponding Dyck word.



$uuduuddd$

The n-th *Catalan number* C_n is the number of Dyck paths of length $2n$. For example, $C_3 = 5$:



See Corollary 6.2 of [Sta99], and the paragraphs before and after it, for a discussion of the Catalan numbers. A *peak* in a Dyck path is a sequence of consecutive steps up, followed by a step down. The n-th *Fine number* F_n is the number of Dyck paths of length $2n$ in which the first peak has even height. For example, $F_3 = 2$ as the previous set of diagrams demonstrates. See [DS01] for a nice discussion of the Fine numbers.

Proposition 4.1. *The number of Dyck paths of length $2n$ whose first peak has height m or greater is*

$$\binom{2n-m}{n-m} - \binom{2n-m}{n-m-1} = \frac{m+1}{n+1}\binom{2n-m}{n}$$

In particular, taking $m = 0$ gives the familiar result

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. We begin by recalling the proof of the case $m = 0$ by the 'reflection trick'. See Lemma 5.27 of [KT08]. We will then adapt this to the general case.

Consider the set of all paths built from the steps $(1,1)$ and $(1,-1)$, starting at $(0,0)$ and ending at $(2n,0)$. The Dyck paths are those that do not go below the x-axis, and the rest we call *bad* paths. Given a bad path, we locate the first point at which it meets the line $y = -1$, and reflect the remainder of the path through that line. The result is a path from $(0,0)$ to $(2n,-2)$. And indeed, this establishes a bijection between the set of bad paths, and the set of paths from $(0,0)$ to $(2n,-2)$. A path from $(0,0)$ to $(2n,0)$ has n ups and n downs, so that there are $\binom{2n}{n}$ in total. And a path from $(0,0)$ to $(2n,-2)$ has $n-1$ ups and $n+1$ downs, so there are $\binom{2n}{n+1}$ in total. Therefore the total number of Dyck paths (C_n) is $\binom{2n}{n} - \binom{2n}{n+1}$.

For general m we now repeat the procedure, but only consider paths that begin with at least m up steps. Then the number of paths from $(0,0)$ to $(2n,0)$ is $\binom{2n-m}{n-m}$, and the number from $(0,0)$ to $(2n,-2)$ is $\binom{2n-m}{n-m-1}$, as we see by considering the distribution of the up moves *after* the first m. □

Now let us fix n. Given $0 \leqslant m$, we write B_m for the number of Dyck paths whose first peak occurs at height m or greater. Thus $B_m = 0$ for $m > n$. Then $(B_m - B_{m+1})$ is the number of Dyck paths whose first peak has height exactly m, and so the Fine number F_n is nothing other than

$$F_n = (B_0 - B_1) + (B_2 - B_3) + \cdots = \sum_{m=0}^{n} (-1)^m B_m.$$

In particular, using Proposition 4.1 above we recover the formula in Equation (1):

$$\begin{aligned} F_n &= \sum_{m=0}^{n} (-1)^m \frac{m+1}{n+1} \binom{2n-m}{n} \\ &= \frac{1}{n+1} \left[\binom{2n}{n} - 2\binom{2n-1}{n} + 3\binom{2n-2}{n} - \cdots + (-1)^n (n+1) \binom{n}{n} \right] \end{aligned}$$

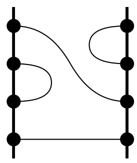
5. Planar diagrams and Dyck paths

We now recall the familiar relationship between planar diagrams and Dyck paths, and we extend it to take the height of the first peak into account.

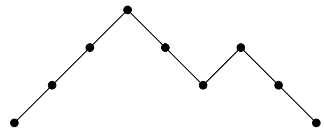
Proposition 5.1. *The set of planar diagrams on n strands is in bijection with the set of Dyck paths (or words) of length $2n$.*

Corollary 5.2. *The rank of $\mathrm{TL}_n(a)$ as an R-module is the Catalan number C_n.*

There are several choices for such a bijection; the one that is relevant to us is as follows: Take a planar diagram on n strands, and work through the dots in order, starting with $1, \ldots, n$ on the right, followed by $n, \ldots, 1$ on the left. At each dot we encounter an arc, either for the first time or for the second time: if it is the first time, record a u, and if it is the second time, record a d. For example, here is a planar diagram, the corresponding Dyck word, and the corresponding Dyck path.



$uuuddudd$



See [RSA14, pp.966-967] or [KT08, Lemma 5.33] for details.

Proposition 5.3. *The rank of* $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_m(a)} \mathbb{1}$ *is equal to the number of Dyck paths of length* $2n$ *whose first peak occurs at height* m *or greater.*

Proof. Proposition 2.4 shows that $\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}$ has basis given by the n-planar diagrams with black box of size m, i.e. the diagrams that have no cups among dots $1, \dots, m$ on the right. These are precisely the diagrams which have no arcs that start and end among dots $1, \dots, m$ on the right. Therefore, under the bijection between planar diagrams and Dyck paths, these diagrams correspond exactly to the paths that start with m up steps, i.e. the paths whose first peak has height m or greater. ☐

We are now in a position to prove Theorem A, which states that the Euler characteristic of $W(n)$ is $(-1)^{n-1}F_n$, where F_n is the n-th Fine number.

Proof of Theorem A. Let us fix n and define B_m to be the number of Dyck paths of length $2n$ whose first peak occurs at height m or greater, so that $\mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_m} \mathbb{1}) = B_m$. And let us write A_m for the number of Dyck paths of length $2n$ whose first peak occurs at height exactly m. Then

$$\begin{aligned}
\chi(W(n)) &= -\mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_n} \mathbb{1}) + \mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-1}} \mathbb{1}) - \mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-2}} \mathbb{1}) + \cdots \\
&\quad \cdots + (-1)^{n-2}\,\mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_1} \mathbb{1}) + (-1)^{n-1}\,\mathrm{rank}(\mathrm{TL}_n \otimes_{\mathrm{TL}_0} \mathbb{1}) \\
&= -B_n + B_{n-1} - B_{n-2} + \cdots + (-1)^{n-1}B_0 \\
&= (-1)^{n-1}[(B_0 - B_1) + (B_2 - B_3) + \cdots]
\end{aligned}$$

with final term in the bracket either B_n if n is even, or $(B_{n-1} - B_n)$ if n is odd. But this is precisely $(-1)^{n-1}[A_0 + A_2 + A_4 + \cdots + A_n]$ if n is even, and $(-1)^{n-1}[A_0 + A_2 + A_4 + \cdots + A_{n-1}]$ if n is odd. In either case, we obtain $(-1)^{n-1}F_n$. ☐

Combined with Proposition 4.1, the proof above gives us Equation (1):

$$\begin{aligned}
F_n &= (-1)^{n-1}\chi(W(n)) \\
&= B_0 - B_1 + \cdots + (-1)^n B_n \\
&= \frac{1}{n+1}\left[\binom{2n}{n} - 2\binom{2n-1}{n} + 3\binom{2n-2}{n} - \cdots + (-1)^n(n+1)\binom{n}{n}\right]
\end{aligned}$$

We also obtain the representation-theoretic analogue Corollary B. Indeed, for a bounded chain complex C of finitely generated modules over any ring or algebra A, the relation

$$\sum_m (-1)^m [C_m] = \sum_m (-1)^m [H_m(C)] \tag{2}$$

holds in $K_0(A)$. In particular, if A is semisimple then $K_0(A)$ coincides with the representation ring of A, and the same relation holds there. We therefore obtain:

$$\begin{aligned}
[\mathcal{F}_n] &= (-1)^{n-1} \sum_{d=-1}^{n-1} (-1)^d [H_d(W(n))] \\
&= (-1)^{n-1} \sum_{d=-1}^{n-1} (-1)^d [W(n)_d] \\
&= (-1)^{n-1} \sum_{d=-1}^{n-1} (-1)^d [\mathbb{1} \uparrow_{\mathrm{TL}_{n-d-1}}^{\mathrm{TL}_n}] \\
&= \sum_{m=0}^{n} (-1)^m [\mathbb{1} \uparrow_{\mathrm{TL}_m}^{\mathrm{TL}_n}].
\end{aligned}$$

Here the first equation is a consequence of Theorem 3.4, the second is an instance of (2), and the third follows from the definition

$$W(n)_d = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-d-1}} \mathbb{1} = \mathbb{1} \uparrow_{\mathrm{TL}_{n-d-1}}^{\mathrm{TL}_n} .$$

6. Young tableaux

In this section we will describe the top-dimensional homology $\mathcal{F}_n = H_{n-1}(W(n))$ as a module over TL_n when our ground ring R is the complex numbers and the algebra TL_n is semisimple. In this case the irreducible representations of TL_n are indexed by certain Young diagrams, and we are able to identify the multiplicity of each irreducible in $\mathcal{F}_n$. A nice account of the theory used here is given in chapters 4 and 5 of [KT08], and see also the brief account in section 11 of Jones' paper [Jon87]. In particular we will use the language of partitions, Young diagrams and Young tableaux, for which one can refer to Sections 5.1 and 5.2 of [KT08]. More detailed references are recalled in Remark 6.1.

For this section we will fix $n \geqslant 1$ and assume that our ground ring R is the field of complex numbers $\mathbb{C}$, that v and $a = v + v^{-1}$ are non-zero complex numbers, and that $q = v^2$ is not a d-th root of unity for $2 \leqslant d \leqslant n$. The latter condition guarantees that $\mathrm{TL}_n(a)$ is semisimple. We also assume that $(\lambda, \mu) = (-1, v)$ in order to accord with the conventions of [KT08].

Under these assumptions, the Temperley-Lieb algebra $\mathrm{TL}_p(a)$ is semisimple for each $0 \leqslant p \leqslant n$, with one irreducible representation V_λ for each partition $\lambda \vdash p$ whose Young diagram has at most two columns. The representation corresponding to the partition $1^p = (1, \dots, 1) \vdash p$, whose Young diagram is a single column of p boxes, is

$$V_{1^p} = \mathbb{1}.$$

The operations of restriction and induction on the modules V_λ are now determined by the rules

$$V_\lambda \downarrow^{\mathrm{TL}_p}_{\mathrm{TL}_{p-1}} \cong \bigoplus_{\mu \hookrightarrow \lambda} V_\mu, \qquad \lambda \vdash p$$

$$V_\lambda \uparrow^{\mathrm{TL}_p}_{\mathrm{TL}_{p-1}} \cong \bigoplus_{\lambda \hookrightarrow \mu} V_\mu, \qquad \lambda \vdash (p-1)$$

where all λ and μ are assumed to have diagrams with at most two columns. Recall that the notation $\mu \hookrightarrow \lambda$ means that the diagram of μ is obtained from that of λ by deleting a single corner box, and that $\lambda \hookrightarrow \mu$ means that the diagram of μ is obtained from that of λ by adding a single corner box. See Remark 6.1 below for references.

Proof of Theorem C. For this proof, all partitions are assumed to have diagrams with at most two columns.

To prove the claim it suffices to identify the isomorphism class $[\mathcal{F}_n]$ within the ring of isomorphism classes of TL_n-modules. We have

$$(-1)^{n-1}[\mathcal{F}_n] = \sum_{j=-1}^{n-1} (-1)^j [W(n)_j].$$

(This is analogous to the usual relationship between the Euler characteristic of a chain complex and the Euler characteristic of its homology, and is proved in the same way, using the fact that because TL_n is semisimple, any short exact sequence of TL_n-modules splits.) Observe that

$$W(n)_j = \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-j-1}} \mathbb{1} = V_{1^{n-j-1}} \uparrow^{\mathrm{TL}_n}_{\mathrm{TL}_{n-j-1}}$$

so that altogether we have

$$[\mathcal{F}_n] = (-1)^{n-1} \sum_{j=-1}^{n-1} (-1)^j [V_{1^{n-j-1}} \uparrow^{\mathrm{TL}_n}_{\mathrm{TL}_{n-j-1}}].$$

We now identify the induced modules appearing above. Given $\lambda \vdash n$ and $p \leqslant n$, let $N_{\lambda,p}$ denote the number of SYT for which the labels in the first column begin, starting from the top, with $1, \dots, p$. Observe that if i is in the range $0 \leqslant i \leqslant n$, then $N_{\lambda,i-1} - N_{\lambda,i}$ is precisely the number of SYT of shape λ whose second column has top entry i, and whose first column necessarily has top entries $1, \dots, i-1$.

We claim:

$$V_{1^p} \uparrow^{\mathrm{TL}_n}_{\mathrm{TL}_p} \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus N_{\lambda,p}}$$

To see this, we induce V_{1^p} up to TL_{p+1}, then TL_{p+2}, and so on. At each stage, the module will be a direct sum of modules V_λ for various λ, and we will consider each irreducible summand V_λ to be labelled by an SYT of the relevant shape λ, according to the following rules. At the initial step, the single summand V_{1^p} is labelled by the unique SYT of shape 1^p, which is a single column with labels $1, \ldots, p$. And as one passes from one step to the next, we interpret the rule

$$V_\lambda \uparrow_{\mathrm{TL}_m}^{\mathrm{TL}_{m+1}} \cong \bigoplus_{\lambda \hookrightarrow \mu} V_\mu$$

as saying that when we induce up the module labelled by an SYT Q, we obtain the sum of the two modules labelled by the SYT obtained from Q by adding a single box containing $(m+1)$. Beginning with the unique SYT of shape 1^p, and adding single boxes labelled $p+1, p+2, \ldots, n$ so that one has an SYT at each stage, produces precisely one copy of each SYT with n boxes whose first column starts $1, \ldots, p$. Compare with Exercise 5 on p.93 of [Ful97]. This proves the claim.

The claim above gives us

$$\begin{aligned}
[\mathcal{F}_n] &= (-1)^{n-1} \sum_{j=-1}^{n-1} (-1)^j [V_{1^{n-j-1}} \uparrow_{\mathrm{TL}_{n-j-1}}^{\mathrm{TL}_n}] \\
&= (-1)^{n-1} \sum_{j=-1}^{n-1} (-1)^j \sum_{\lambda \vdash n} N_{\lambda, n-j-1} [V_\lambda] \\
&= \sum_{\lambda \vdash n} \left[\sum_{k=0}^{n} (-1)^k N_{\lambda,k} \right] [V_\lambda].
\end{aligned}$$

Thus the multiplicity of V_λ in $\mathcal{F}_n$ is $\sum_{k=0}^{n} (-1)^k N_{\lambda,k}$. If $\lambda \neq 1^n$, then this multiplicity is precisely

$$\sum_{i \text{ odd},\ i \leqslant n} (N_{\lambda,i-1} - N_{\lambda,i}) = |\{\text{SYT of shape } \lambda \text{ with top entry of second column odd}\}|$$

as required. (The assumption $\lambda \neq 1^n$ guarantees that $N_{\lambda,n} = 0$, so that a potential final term in the case of n even does not make a difference.) And if $\lambda = 1^n$ then $N_{\lambda,i} = 1$ for all i, so that the multiplicity is 0 if n is odd and 1 if n is even, which agrees with the special convention outlined in the statement. This completes the proof. □

Remark 6.1 (References for the representation theory of Temperley-Lieb algebras)**.** With the assumptions from the start of the section, Theorem 2.2 of [Wen88] shows that the Iwahori-Hecke algebra $\mathcal{H}_n(q)$ is semisimple. Theorem 5.18 of [KT08] shows that the distinct irreducible modules of $\mathcal{H}_n(q)$ are the modules V_λ, one for each partition $\lambda \vdash n$, with no restriction on the shape. Section 5.7.3 of [KT08] then shows that $\mathrm{TL}_n(a)$ is semisimple, with one irreducible representation V_λ for each partition $\lambda \vdash n$ of n whose Young diagram has at most two columns, and that these V_λ pull back to the representations of $\mathcal{H}_n(q)$ with the same names. The fact that $V_{1^n} = \mathbb{1}$ can be seen by comparing Examples 5.12(b) and Theorem 5.29 of [KT08]. The rules for induction and restriction of the representations V_λ of $\mathcal{H}_n(q)$ are

$$V_\lambda \downarrow^{\mathcal{H}_n(q)}_{\mathcal{H}_{n-1}(q)} \cong \bigoplus_{\mu \hookrightarrow \lambda} V_\mu, \qquad \lambda \vdash n,$$

$$V_\lambda \uparrow^{\mathcal{H}_n(q)}_{\mathcal{H}_{n-1}(q)} \cong \bigoplus_{\lambda \hookrightarrow \mu} V_\mu, \qquad \lambda \vdash (n-1),$$

again with no restriction on the shape of λ and μ. The first of these rules is Proposition 5.13 of [KT08], and the second follows by Frobenius reciprocity. From the first of these we can immediately deduce the stated rule for restriction in the Temperley-Lieb case, and the rule for induction then follows, again by Frobenius reciprocity. The assumptions on n stated at the start of this section imply the analogous assumptions for all p in the range $0 \leqslant p \leqslant n$, so that we can replace n with any such p throughout this remark.

7. Jacobsthal numbers and the boundary maps of $W(n)$

In this section we give a combinatorial description of the boundary maps in $W(n)$ and relate them to the *Jacobsthal numbers*. We start by recalling the boundary maps, and the Jacobsthal numbers.

Recall from Definition 3.2 that for $i \geqslant 0$ the boundary map has the following description

$$\begin{array}{rcl} d^i\colon W(n)_i & \to & W(n)_{i-1} \\ d^i\colon \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i-1}} \mathbb{1} & \to & \mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i}} \mathbb{1} \\ x \otimes r & \mapsto & \displaystyle\sum_{j=0}^{i} (-1)^j d^i_j(x \otimes r) \\ & = & \displaystyle\sum_{j=0}^{i} (-1)^j (x \cdot s_{n-i+j-1} \cdots s_{n-i}) \otimes \lambda^{-j} r \\ & = & \displaystyle\sum_{j=0}^{i} (-1)^j \lambda^{-j} (x \cdot (\lambda - \mu U_{n-i+j-1}) \cdots (\lambda - \mu U_{n-i})) \otimes r. \end{array}$$

Here there are two possibilities for λ and μ, namely $(\lambda, \mu) = (-1, v)$ or $(\lambda, \mu) = (v^2, -v)$, and note for future reference that $\frac{\mu}{\lambda} = -v$ and $\frac{\mu}{\lambda} = -v^{-1}$ respectively.

The n-th *Jacobsthal number* J_n [Slo] is (among other things) the number of compositions of n that end with an odd number. So for example, taking $n = 4$ the relevant compositions are 31, 13, 211, 121, 1111. The Jacobsthal number J_n can also be described as the number of sequences $n > a_1 > a_2 > \cdots > a_r > 0$

whose initial term has the opposite parity to n. These sequences are in one-to-one correspondence with the compositions. If you label the 'gaps' in $1, \ldots, n$ by $1, \ldots, n-1$ the sequence dictates at which gaps one should divide up $1, \ldots, n$ to obtain the composition. For the above examples, when $n = 4$, the relevant sequences are 3, 1, $3 > 2$, $3 > 1$ and $3 > 2 > 1$. (We allow the empty sequence, and say that by convention its initial term is $a_1 = 0$, and $r = 0$. Of course this only occurs when n is odd.) More precisely, the correspondence between compositions and sequences is as follows: Given a composition $c_1 c_2 \cdots c_r$, the corresponding sequence is $n > a_1 > \cdots > a_{r-1} > 0$ where $a_j = n - (c_r + c_{r-1} + \cdots + c_{r-j+1})$. Observe that the initial term is $a_1 = n - c_r$, so that since c_r is odd, a_1 has the opposite parity to n.

The Jacobsthal numbers are determined by the recursion $J_n = J_{n-1} + 2J_{n-2}$ for $n \geqslant 2$, and also satisfy the closed form $J_n = \frac{2^n - (-1)^n}{3}$. Thus, the compositions and sequences counted by the Jacobsthal number are about one-third of the total possible sequences and compositions.

Definition 7.1. Let $a = v + v^{-1}$ where $v \in R^\times$ is a unit. For every $0 \leqslant l \leqslant n$, we define the l^{th} *Jacobsthal* element in $\mathrm{TL}_n(a)$ as follows:

$$\mathcal{J}_l^n = \sum_{\substack{l > a_1 > \cdots > a_r > 0 \\ l - a_1 \text{ odd}}} (-1)^{(r-1)+l} \left(\frac{\mu}{\lambda}\right)^r U_{a_1+n-l} \cdots U_{a_r+n-l}$$

The indices of the U_j which occur vary from $(n-(l-1))$ to $(n-1)$ and hence are non-trivial in $\mathrm{TL}_n(a) \otimes_{\mathrm{TL}_{n-(l-1)}(a)} \mathbb{1}$. Recall that we allow the empty sequence ($a_1 = 0$ and $r = 0$) when l is odd. This corresponds to a constant summand 1 in $\mathcal{J}_l^n$ for odd l. Note that the number of irreducible terms in $\mathcal{J}_l^n$ is J_l. When $l = n$, $m_n = 0$ and the formula simplifies. We call $\mathcal{J}_n^n$ the *Jacobsthal element*, and denote it $\mathcal{J}_n$.

Proof of Theorem D. Firstly we note that the terms appearing in $\mathcal{J}_{i+1}^n$ are non-zero in $\mathrm{TL}_n \otimes_{\mathrm{TL}_{n-i}} \mathbb{1}$: the target of d^i. We consider the cases i odd and i even for clarity. For ease of notation, let $p = n - i - 1$. When i is odd, then d^i is a sum over an even number of terms, and acts by right multiplication on the left factor of $x \otimes r$ by the following element:

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^j\lambda^{-j}(\lambda-\mu U_{p+j})\cdots(\lambda-\mu U_{p+1})\\
=&\sum_{j=0}^{(i-1)/2}\Big(\lambda^{-2j}(\lambda-\mu U_{p+2j})\cdots(\lambda-\mu U_{p+1})\\
&-\lambda^{-2j-1}(\lambda-\mu U_{p+(2j+1)})(\lambda-\mu U_{p+(2j+1)-1})\cdots(\lambda-\mu U_{p+1})\Big)\\
=&\sum_{j=0}^{(i-1)/2}\Big(\big[\lambda^{-2j}(\lambda-\mu U_{p+2j})\cdots(\lambda-\mu U_{p+1})\\
&-\lambda^{-2j-1}(\lambda)(\lambda-\mu U_{p+(2j+1)-1})\cdots(\lambda-\mu U_{p+1})\big]\\
&+\lambda^{-2j-1}(\mu U_{p+(2j+1)})(\lambda-\mu U_{p+(2j+1)-1})\cdots(\lambda-\mu U_{p+1})\Big)\\
=&\sum_{j=0}^{(i-1)/2}\lambda^{-2j-1}\mu U_{p+(2j+1)}(\lambda-\mu U_{p+(2j+1)-1})\cdots(\lambda-\mu U_{p+1})
\end{aligned}$$

Here the final equality is given by noting that the terms in the square bracket cancel out. Substituting $k = 2j + 1$ gives that d^i is multiplication by

$$\sum_{\substack{0<k<i+1\\ k \text{ odd}}}\lambda^{-k}\mu U_{p+k}[(\lambda-\mu U_{p+(k-1)})\cdots(\lambda-\mu U_{p+1})]$$

and multiplying out the terms in the square bracket above gives:

$$\sum_{\substack{0<k<i+1\\ k \text{ odd}}}\lambda^{-k}\mu U_{p+k}\Big[\lambda^{k-1}+\sum_{k>a_2>\ldots>a_r>0}\lambda^{k-1-r}(-1)^{r-1}\mu^{r-1}U_{p+a_2}\ldots U_{p+a_r}\Big].$$

Let $k = a_1$, and note that since i is odd, then k being odd equates to $(i + 1) - a_1$ being odd. Putting the two sums in the previous equation together corresponds to the sequences which enumerate the Jacobsthal compositions. Recall that $p = n-i-1 = n-(i+1)$ and since i is odd then multiplication by $(-1)^{(i+1)}$ does not change the sign. It follows that d^i is right multiplication on the left of the tensor product by

$$\mathcal{J}^n_{i+1}=\sum_{\substack{i+1>a_1>\cdots>a_r>0\\ (i+1)-a_1 \text{ odd}}}(-1)^{(r-1)+(i+1)}\left(\frac{\mu}{\lambda}\right)^r U_{a_1+n-(i+1)}\cdots U_{a_r+n-(i+1)}.$$

When i is even, d^i is a sum over an odd number of terms, and the element which we left multiply by can be written in a similar fashion to the odd case as follows (once again fixing $p = n - i - 1$).

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^j\lambda^{-j}(\lambda-\mu U_{p+j})\cdots(\lambda-\mu U_{p+1})\\
=\;& 1+\sum_{j=1}^{i}(-1)^j\lambda^{-j}(\lambda-\mu U_{p+j})\cdots(\lambda-\mu U_{p+1})\\
=\;& 1-\sum_{j=1}^{i/2}\lambda^{-2j}\mu U_{p+2j}(\lambda-\mu U_{p+(2j-1)})\cdots(\lambda-\mu U_{p+1})).
\end{aligned}$$

Substituting $k = 2j$ gives

$$1-\sum_{\substack{0<k<i+1\\ k\text{ even}}}\lambda^{-k}\mu U_{p+k}(\lambda-\mu U_{p+(k-1)})\cdots(\lambda-\mu U_{p+1}))$$

and we use the computation for i odd to identify this with $\mathcal{J}^n_{i+1}$. We note that setting $k = a_1$ gives $(i+1)-a_1$ odd, since both i and k are even, and the negative coefficient of the sum is a consequence of the factor $(-1)^{(i+1)}$. The constant term 1 corresponds to the empty partition, for which we set $r = 0$ and $a_1 = 0$. □

We now restrict ourselves to a study of the top differential in this setting. Recall when $l = n$ we call $\mathcal{J}^n_n$ the *Jacobsthal element*, and denote it $\mathcal{J}_n$. It is a sum of J_n terms.

Since $\mathcal{F}_n$ is the homology of $W(n)$ in the top degree, it is simply the kernel of the top differential $d^{n-1}\colon W(n)_{n-1} \to W(n)_{n-2}$. There are identifications $W(n)_{n-1} = \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_0(a)} \mathbb{1} \cong \mathrm{TL}_n(a)$ and $W(n)_{n-2} \cong \mathrm{TL}_n(a) \otimes_{\mathrm{TL}_1(a)} \mathbb{1} \cong \mathrm{TL}_n(a)$.

Proposition 7.2. *Under the above identifications, the top differential of $W(n)$ is right-multiplication by $\mathcal{J}_n$. In particular, there is an exact sequence*

$$0 \longrightarrow \mathcal{F}_n(a) \longrightarrow \mathrm{TL}_n(a) \xrightarrow{-\cdot\mathcal{J}_n} \mathrm{TL}_n(a).$$

Proof. This is an application of Theorem D for the case $i = n-1$, which shows $d^{n-1}(x\otimes r) = x\cdot\mathcal{J}_n \otimes r$. The identifications above send $x \otimes r$ to $x \cdot r$ and so under these the map d^{n-1} is left multiplication by $\mathcal{J}_n$ as described. □

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Max Planck Institute for Mathematics, Bonn
E-mail address: rachaelboyd@mpim-bonn.mpg.de
URL: https://guests.mpim-bonn.mpg.de/rachaelboyd/

Institute of Mathematics, University of Aberdeen
E-mail address: r.hepworth@abdn.ac.uk
URL: http://homepages.abdn.ac.uk/r.hepworth/pages/