# On 0012-avoiding inversion sequences and a Conjecture of Lin and Ma 

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#### Abstract

The study of pattern avoidance in inversion sequences recently attracts extensive research interests. In particular, Zhicong Lin and Jun Ma conjectured a formula that counts the number of inversion sequences avoiding the pattern 0012. We will not only confirm this conjecture but also give a formula that enumerates the number of 0012-avoiding inversion sequences in which the last entry equals $n-1$.


Keywords. Inversion sequence, pattern avoidance, generating function, kernel method.
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## 1. Introduction

An inversion sequence of length $n$ is a sequence $e=e_{1} e_{2} \cdots e_{n}$ such that $0 \leq e_{i} \leq$ $i-1$ for each $1 \leq i \leq n$. We denote by $\mathbf{I}_{n}$ the set of inversion sequences of length $n$. Given any word $w \in\{0,1, \ldots, n-1\}^{n}$ of length $n$, we define its reduction by the word obtained via replacing the $k$-th smallest entries of $e$ with $k-1$. For instance, the reduction of 0023252 is 0012131 . We say that an inversion sequence $e$ contains a given pattern $p$ if there exists a subsequence of $e$ such that its reduction is the same as $p$; otherwise, we say that $e$ avoids the pattern $p$. For instance, 0023252 has a subsequence 022 whose reduction is 011 - hence, 0023252 contains the pattern 011. On the other hand, none of the length 3 subsequences of 0023252 have the reduction 110 - hence, 0023252 avoids the pattern 110.

Let $p_{1}, p_{2}, \ldots, p_{m}$ be given patterns. We denote by $\mathbf{I}_{n}\left(p_{1}, p_{2}, \ldots, p_{m}\right)$ the set of inversion sequences of length $n$ that avoid all of the patterns $p_{1}, p_{2}, \ldots, p_{m}$. Recently, the study of pattern avoidance in inversion sequences attracts extensive research interests. See $[1-8,10-15,18,19]$ for several instances of work on this topic. Among these work, one particular interesting problem is about the enumeration of inversion sequences that avoids fixed patterns. For example, in a pioneering work of Corteel, Martinez, Savage and Weselcouch [7], it was shown that

$$
\left|\mathbf{I}_{n}(011)\right|=B_{n} \quad \text { and } \quad\left|\mathbf{I}_{n}(021)\right|=S_{n}
$$

where $B_{n}$ is the $n$-th Bell number (OEIS, [17, A000110]) and $S_{n}$ is the $n$-th large Schröder number (OEIS, [17, A006318]).

In a recent paper [18], Yan and Lin proved a conjecture due to Martinez and Savage [15] that claims

$$
\begin{equation*}
\left|\mathbf{I}_{n}(021,120)\right|=1+\sum_{i=1}^{n-1}\binom{2 i}{i-1} \tag{1.1}
\end{equation*}
$$

This sequence is registered as OEIS, [17, A279561]. Lin and Yan also showed that this sequence as well enumerates $\left|\mathbf{I}_{n}(102,110)\right|$ and $\left|\mathbf{I}_{n}(102,120)\right|$. This therefore
establishes the Wilf-equivalence

$$
\begin{equation*}
\mathbf{I}_{n}(021,120) \sim \mathbf{I}_{n}(102,110) \sim \mathbf{I}_{n}(102,120) \tag{1.2}
\end{equation*}
$$

At the end of [18], a conjecture of Zhicong Lin and Jun Ma discovered in 2019 is recorded.

Conjecture 1.1 (Lin and Ma). For $n \geq 1$,

$$
\begin{equation*}
\left|\mathbf{I}_{n}(0012)\right|=1+\sum_{i=1}^{n-1}\binom{2 i}{i-1} \tag{1.3}
\end{equation*}
$$

In other words, it is possible to extend the Wilf-equivalence (1.2) as

$$
\mathbf{I}_{n}(0012) \sim \mathbf{I}_{n}(021,120) \sim \mathbf{I}_{n}(102,110) \sim \mathbf{I}_{n}(102,120)
$$

In this paper, we will prove the above conjecture of Lin and Ma.
Theorem 1.1. Conjecture 1.1 is true.
Let us fix some notation. Given $e=e_{1} e_{2} \cdots e_{n} \in \mathbf{I}_{n}$ (0012), we define

$$
\mathcal{R}(e):=\left\{m: \exists i \neq j \text { such that } e_{i}=e_{j}=m\right\}
$$

In other words, $\mathcal{R}(e)$ is the set of letters that appear more than once in $e$. We further define

$$
\operatorname{SRPT}(e):=\min \mathcal{R}(e),
$$

that is, the smallest number in $\mathcal{R}(e)$. Notice that there is only one sequence $01 \cdots(n-1)$ in which none of the letters repeat. For this sequence, we assign that

$$
\operatorname{SRPT}(01 \cdots(n-1)):=n-1
$$

Finally, we define

$$
\operatorname{LAST}(e):=e_{n},
$$

the last entry of $e$.
Apart from counting the number of inversion sequences that avoid the pattern 0012, we will also enumerate the number of sequences in $\mathbf{I}_{n}(0012)$ in which the last entry equals $n-1$.

Theorem 1.2. For $n \geq 1$,

$$
\left|\left\{e \in \mathbf{I}_{n}(0012): \operatorname{LAST}(e)=n-1\right\}\right|= \begin{cases}1 & \text { if } n=1  \tag{1.4}\\ 2^{n-2} & \text { if } n \geq 2\end{cases}
$$

## 2. Combinatorial observations

We collect some combinatorial observations about inversion sequences in $\mathbf{I}_{n}(0012)$.
Lemma 2.1. For $n \geq 1$ and $e \in \mathbf{I}_{n}(0012)$, if $\operatorname{SRPT}(e)=k$, then for $1 \leq i \leq k+1$,

$$
e_{i}=i-1
$$

Proof. If $\operatorname{sRPt}(e)=n-1$, then $e=01 \cdots(n-1)$ and hence the lemma is true. Let $\operatorname{SRPT}(e) \neq n-1$. If in this case the lemma is not true, then since $0 \leq e_{i} \leq i-1$ for each $i$, there must exist some $k_{1}<k=\operatorname{SRPT}(e)$ that appears more than once among $e_{1}, e_{2}, \ldots, e_{k+1}$. This violates the assumption that $\operatorname{SRPT}(e)=k$.

Lemma 2.2. For $n \geq 2$ and $e=e_{1} e_{2} \cdots e_{n} \in \mathbf{I}_{n}(0012)$, let $\gamma(e)=e_{1} e_{2} \cdots e_{n-1}$. We further assume that $e \neq 01 \cdots(n-1)$. Then
(a). if $\operatorname{LAST}(e)>\operatorname{SRPT}(\gamma(e))$, then

$$
\operatorname{SRPT}(e)=\operatorname{SRPT}(\gamma(e)) ;
$$

(b). if $\operatorname{LAST}(e) \leq \operatorname{sRPT}(\gamma(e))$, then

$$
\operatorname{SRPT}(e)=\operatorname{LAST}(e) .
$$

Proof. A simple observation is that $\gamma(e) \in \mathbf{I}_{n-1}(0012)$. Below let us assume that $\operatorname{LAST}(e)=\ell, \operatorname{SRPt}(e)=k$ and $\operatorname{SRPT}(\gamma(e))=k^{\prime}$.

First, if $\mathcal{R}(\gamma(e))=\emptyset$, then for each $0 \leq i \leq n-1, e_{i}=i-1$. Since $e \neq$ $01 \cdots(n-1)$, we have $\operatorname{LAST}(e)=\ell \leq n-2=\operatorname{SRPT}(\gamma(e))$. This fits into Case (b). Further, we find that $\mathcal{R}(e)=\{\ell\}$ and hence $\operatorname{srpt}(e)=\ell$. This implies that $\operatorname{SRPT}(e)=\operatorname{LAST}(e)$.

Now we assume that $\mathcal{R}(\gamma(e)) \neq \emptyset$. Notice that Case (a) is trivial. For Case (b), we first deduce from $\mathcal{R}(\gamma(e)) \neq \emptyset$ that $k^{\prime} \leq n-3$. By Lemma 2.1, we find that for $1 \leq i \leq k^{\prime}+1, e_{i}=i-1$. If $\operatorname{LAST}(e)=\ell \leq k^{\prime}$, then we know that $e_{\ell+1}=\ell=e_{n}$. Also, we notice that the indices satisfy $\ell+1 \leq k^{\prime}+1 \leq n-2<n$. Hence, $\ell \in \mathcal{R}(e)$. Therefore, $\operatorname{SRPT}(e)=\min \left\{\ell, k^{\prime}\right\}=\ell=\operatorname{LAST}(e)$.

Corollary 2.3. For $e \in \mathbf{I}_{n}(0012)$,

$$
0 \leq \operatorname{SRPT}(e) \leq \operatorname{LAST}(e) \leq n-1 .
$$

Proof. If $e=01 \cdots(n-1)$, the above inequalities are trivial since $\operatorname{SRPT}(e)=$ $\operatorname{LAST}(e)=n-1$. If $e \neq 01 \cdots(n-1)$, the inequalities are direct consequences of Lemma 2.2 and the fact that $\operatorname{SRPT}(e) \geq 0$ and $\operatorname{LAST}(e) \leq n-1$.
Lemma 2.4. For $n \geq 2$ and $e=e_{1} e_{2} \cdots e_{n} \in \mathbf{I}_{n}(0012)$, lete be such that $\operatorname{SRPT}(e)=$ LAST $(e)=k$ with $0 \leq k \leq n-2$. Then
(a). for $1 \leq i \leq k+1$,

$$
e_{i}=i-1 ;
$$

(b). if we denote $e^{\prime}=e_{1}^{\prime} e_{2}^{\prime} \cdots e_{n-k}^{\prime}$ by the sequence obtained via $e_{i}^{\prime}=e_{k+i}-k$ for each $1 \leq i \leq n-k$, then $e^{\prime} \in \mathbf{I}_{n-k}(0012)$ such that

$$
\operatorname{SRPT}\left(e^{\prime}\right)=\operatorname{LAST}\left(e^{\prime}\right)=0 .
$$

Proof. Part (a) simply comes from Lemma 2.1. Also, we know from Part (a) that for $k+1 \leq i \leq n$, it holds that $e_{i} \geq k$. On the other hand, $e_{i} \leq i-1$. Hence, $e^{\prime}$ is still an inversion sequence. Further, it is trivial to see that $e^{\prime}$ still avoids the pattern 0012. Finally, we have $e_{1}^{\prime}=e_{k+1}-k=k-k=0$ and $\operatorname{LAST}\left(e^{\prime}\right)=e_{n-k}^{\prime}=e_{n}-k=k-k=0$. Since $n-k \geq 2>1$, we have $0 \in \mathcal{R}\left(e^{\prime}\right)$ and hence $\operatorname{SRPT}\left(e^{\prime}\right)=0$.

## 3. Recurrences

Let

$$
f_{n}(k, \ell):=\left\{\begin{array}{c}
\text { the number of sequences } e \in \mathbf{I}_{n}(0012) \text { with } \\
\operatorname{SRPT}(e)=k \text { and } \operatorname{LAST}(e)=\ell
\end{array}\right\} .
$$

We will establish the following recurrences.
Lemma 3.1. We have
(a). for $n \geq 1$,

$$
f_{n}(n-1, n-1)=1
$$

(b). for $n \geq 2$,

$$
f_{n}(n-2, n-1)=0
$$

(c). for $n \geq 2$ and $0 \leq k \leq n-3$,

$$
f_{n}(k, n-1)=\sum_{k^{\prime}=k}^{n-2} f_{n-1}\left(k^{\prime}, n-2\right)
$$

(d). for $n \geq 2$ and $0 \leq \ell \leq n-2$,

$$
f_{n}(\ell, \ell)=\sum_{\ell^{\prime}=\ell}^{n-2} \sum_{k^{\prime}=\ell}^{\ell^{\prime}} f_{n-1}\left(k^{\prime}, \ell^{\prime}\right)
$$

(e). for $n \geq 2$ and $0 \leq k<\ell \leq n-2$,

$$
f_{n}(k, \ell)=\sum_{k^{\prime}=k}^{\ell} f_{n-1}\left(k^{\prime}, \ell\right)+\sum_{\ell^{\prime}=\ell}^{n-2} f_{n-1}\left(k, \ell^{\prime}\right) .
$$

Proof. Cases (a) and (b) are trivial. In particular, Case (a) enumerates the only inversion sequence $01 \cdots(n-1)$ in which none of the letters repeat. Below we always assume that $e=e_{1} e_{2} \cdots e_{n} \in \mathbf{I}_{n}(0012)$. Let $\gamma(e)$ be as in Lemma 2.2.

For Case (c), let $e$ be such that $\operatorname{SRPT}(e)=k \leq n-3$ and $\operatorname{LASt}(e)=n-1$. We first notice that $e_{n-1}=\operatorname{LAST}(\gamma(e)) \geq \operatorname{SRPT}(\gamma(e))$ by Corollary 2.3. Also, it is easy to see that $\operatorname{SRPT}(\gamma(e))=\operatorname{SRPT}(e)=k$ since $\operatorname{LAST}(e)=n-1>k$. Now we claim that $e_{n-1}=k$. Otherwise, namely, if $e_{n-1}>k$, we may find $i<j<n-1$ such that $e_{i}=e_{j}=k$. Hence, $e_{i} e_{j} e_{n-1} e_{n}$ has the reduction 0012, which contradicts the assumption that $e \in \mathbf{I}_{n}(0012)$. We therefore have a bijection

$$
e=e_{1} e_{2} \cdots e_{n-2}(k)(n-1) \longleftrightarrow e_{1} e_{2} \cdots e_{n-2}(n-2)=e^{\prime}
$$

Notice that $e^{\prime}$ is still an inversion sequence avoiding the pattern 0012. Also, $\operatorname{SRPT}\left(e^{\prime}\right) \geq k$. Otherwise, there exists some $k^{\prime}<k$ that appears more than once among $e_{1}, e_{2}, \ldots, e_{n-2}$ and therefore $\operatorname{SRPT}(e)<k$, which leads to a contradiction. Finally, to prove Case (c), it suffices to show that $e^{\prime}$ could be any inversion sequence in $\mathbf{I}_{n-1}(0012)$ with $\operatorname{LAST}\left(e^{\prime}\right)=n-2$ (which is of course true) and $\operatorname{SRPT}\left(e^{\prime}\right) \geq k$. Let $e^{\prime}$ be such a sequence and assume that $\operatorname{SRPT}\left(e^{\prime}\right)=k^{\prime} \geq k$. By Lemma 2.1, we have $e_{k+1}=k$. Pulling back to $e$, we have $e_{k+1}=e_{n-1}=k$ with the indices $k+1 \leq n-2<n-1$. Therefore, for this $e$, we have $k \in \mathcal{R}(e)$ and hence $\operatorname{SRPT}(e)=\min \left\{k^{\prime}, k\right\}=k$.

For Case (d), let $e$ be such that $\operatorname{SRPT}(e)=\operatorname{LAST}(e)=\ell$ with $0 \leq \ell \leq n-2$. We first find that $\operatorname{SRPT}(\gamma(e)) \geq \operatorname{SRPT}(e)=\ell$. On the other hand, let $e^{\prime}=e_{1}^{\prime} e_{2}^{\prime} \cdots e_{n-1}^{\prime} \in$ $\mathbf{I}_{n-1}(0012)$ be such that $\operatorname{SRPT}\left(e^{\prime}\right) \geq \ell$. By Lemma 2.1, $e_{\ell+1}^{\prime}=\ell$. Hence, by appending $\ell$ to the end of $e^{\prime}$, we obtain a sequence with both SRPT and LAST equal to $\ell$. We therefore arrive at a bijection between $e$ and $e^{\prime}$,

$$
e=e_{1} e_{2} \cdots e_{n-1}(\ell) \longleftrightarrow e_{1} e_{2} \cdots e_{n-1}=e^{\prime}
$$

and the desired relation follows.
For Case (e), let $e$ be such that $\operatorname{SRPT}(e)=k$ and $\operatorname{LAST}(e)=\ell$ with $0 \leq k<\ell \leq$ $n-2$. Notice that $e_{n-1} \geq k$. Otherwise, we assume that $e_{n-1}=k^{\prime}<k$. Then
by Lemma 2.1, $e_{k^{\prime}+1}=k^{\prime}=e_{n-1}$. However, $k^{\prime}+1<k+1<n-1$ and hence $k^{\prime} \in \mathcal{R}(e)$. But this violates the fact that $k=\min \mathcal{R}(e)$. Now we have two cases.

- $e_{n-1}<e_{n}$. We claim that $e_{n-1}=k$. Otherwise, we may find $i<j<n-1$ such that $e_{i}=e_{j}=k$. Hence, $e_{i} e_{j} e_{n-1} e_{n}$ has the reduction 0012, which violates the assumption that $e \in \mathbf{I}_{n}(0012)$. Now we have a bijection between $e$ and $e^{\prime} \in \mathbf{I}_{n-1}(0012)$ such that $\operatorname{SRPT}\left(e^{\prime}\right) \geq k$ and $\operatorname{LAST}\left(e^{\prime}\right)=\ell$ by

$$
e=e_{1} e_{2} \cdots e_{n-2}(k)(\ell) \longleftrightarrow e_{1} e_{2} \cdots e_{n-2}(\ell)=e^{\prime}
$$

The argument is similar to that for Case (c). This bijection leads to the first term in the right-hand side of the recurrence relation in Case (e).

- $e_{n-1} \geq e_{n}$. We have a bijection between $e$ and $e^{\prime} \in \mathbf{I}_{n-1}(0012)$ such that $\operatorname{SRPT}\left(e^{\prime}\right)=k$ and $\operatorname{LAST}\left(e^{\prime}\right) \geq \ell$ by

$$
e=e_{1} e_{2} \cdots e_{n-1}(\ell) \longleftrightarrow e_{1} e_{2} \cdots e_{n-1}=e^{\prime}
$$

The argument is similar to that for Case (d). This bijection leads to the second term in the right-hand side of the recurrence relation in Case (e).
The proof of the lemma is therefore complete.
We may therefore determine the support of $f_{n}(k, \ell)$.
Corollary 3.2. For $n \geq 1, f_{n}(k, \ell)$ is supported on

$$
\left\{(k, \ell) \in \mathbb{N}^{2}: 0 \leq k \leq \ell \leq n-1\right\} \backslash\{(n-2, n-1)\}
$$

Proof. By Corollary 2.3, $f_{n}(k, \ell)=0$ if

$$
(k, \ell) \notin\left\{(k, \ell) \in \mathbb{N}^{2}: 0 \leq k \leq \ell \leq n-1\right\} .
$$

Also, $f_{n}(n-2, n-1)=0$ by Lemma 3.1(b). Finally, for the remaining $(k, \ell)$, we have $f_{n}(k, \ell) \neq 0$ with the help of the recurrences in Lemma 3.1.

Finally, we have another recurrence.
Lemma 3.3. We have, for $n \geq 2$ and $0 \leq k \leq n-2$,

$$
f_{n}(k, k)=f_{n-k}(0,0)
$$

Proof. This is an immediate consequence of Lemma 2.4.
In the sequel, we require three auxiliary functions with $q$ within a sufficiently small neighborhood of 0 :

$$
\begin{aligned}
\mathcal{L}(x ; q) & :=\sum_{n \geq 1}\left(\sum_{k=0}^{n-1} f_{n}(k, n-1) x^{k}\right) q^{n}, \\
\mathcal{D}(x ; q) & :=\sum_{n \geq 1}\left(\sum_{\ell=0}^{n-2} f_{n}(\ell, \ell) x^{\ell}\right) q^{n}, \\
\mathcal{F}(x, y ; q) & :=\sum_{n \geq 1}\left(\sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_{n}(k, \ell) x^{k} y^{\ell}\right) q^{n} .
\end{aligned}
$$

In particular, we write, for $n \geq 1$,

$$
L_{n}(x):=\sum_{k=0}^{n-1} f_{n}(k, n-1) x^{k}
$$

$$
\begin{aligned}
D_{n}(x) & :=\sum_{\ell=0}^{n-2} f_{n}(\ell, \ell) x^{\ell}, \\
F_{n}(x, y) & :=\sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_{n}(k, \ell) x^{k} y^{\ell} .
\end{aligned}
$$

Notice that $L_{1}(x)=1, D_{1}(x)=0$ and $F_{1}(x, y)=1$. Also, since $f_{n}(n-1, n-1)=1$, we have

$$
\sum_{\ell=0}^{n-1} f_{n}(\ell, \ell) x^{\ell}=D_{n}(x)+x^{n-1} .
$$

## 4. Proof of Theorem 1.2

Notice that Theorem 1.2 is equivalent to

$$
\begin{aligned}
\mathcal{L}(1 ; q) & =\sum_{n \geq 1}\left(\sum_{k=0}^{n-1} f_{n}(k, n-1)\right) q^{n} \\
& \stackrel{?}{=} q+q^{2}+2 q^{3}+4 q^{4}+8 q^{5}+16 q^{6}+\cdots \\
& =\frac{q(1-q)}{1-2 q} .
\end{aligned}
$$

We prove a strengthening of the above.
Theorem 4.1. We have

$$
\begin{equation*}
\mathcal{L}(x ; q)=\frac{q(1-q)^{2}}{(1-2 q)(1-x q)} . \tag{4.1}
\end{equation*}
$$

Proof. For $n \geq 2$, it follows from (a), (b) and (c) of Lemma 3.1 that

$$
\begin{aligned}
\sum_{k=0}^{n-1} f_{n}(k, n-1) x^{k} & =x^{n-1}+\sum_{k=0}^{n-3} \sum_{k^{\prime}=k}^{n-2} f_{n-1}\left(k^{\prime}, n-2\right) x^{k} \\
& =x^{n-1}+\sum_{k^{\prime}=0}^{n-3} f_{n-1}\left(k^{\prime}, n-2\right) \sum_{k=0}^{k^{\prime}} x^{k}+f_{n-1}(n-2, n-2) \sum_{k=0}^{n-3} x^{k} \\
& =x^{n-1}+\sum_{k^{\prime}=0}^{n-3} f_{n-1}\left(k^{\prime}, n-2\right) \frac{1-x^{k^{\prime}+1}}{1-x}+\frac{1-x^{n-2}}{1-x} .
\end{aligned}
$$

Therefore,

$$
L_{n}(x)=x^{n-1}+\frac{1}{1-x}\left(L_{n-1}(1)-x L_{n-1}(x)\right)-\frac{1-x^{n-1}}{1-x}+\frac{1-x^{n-2}}{1-x} .
$$

Multiplying the above by $q^{n}$ and summing over $n \geq 2$, we have

$$
\mathcal{L}(x ; q)-q=\frac{q}{1-x} \mathcal{L}(1 ; q)-\frac{x q}{1-x} \mathcal{L}(x ; q)-\frac{q^{2}(1-x)}{1-x q},
$$

or

$$
\begin{equation*}
(1-x q)(1-x+x q) \mathcal{L}(x ; q)=q(1-x q) \mathcal{L}(1 ; q)+q(1-q)(1-x) . \tag{4.2}
\end{equation*}
$$

Applying the kernel method (see [9, Exercise 4, §2.2.1, p. 243] or [16]) yields

$$
\left\{\begin{array}{l}
1-x+x q=0 \\
q(1-x q) \mathcal{L}(1 ; q)+q(1-q)(1-x)=0
\end{array}\right.
$$

Solving the first equation of the system for $x$ gives

$$
x=\frac{1}{1-q} .
$$

Substituting the above into the second equation of the system, we have

$$
\mathcal{L}(1 ; q)=\frac{q(1-q)}{1-2 q}
$$

Substituting the above back to (4.2), we arrive at (4.1).

## 5. Proof of Theorem 1.1

We first establish two relations concerning $\mathcal{D}(x ; q)$.
Lemma 5.1. We have

$$
\begin{align*}
\mathcal{D}(x ; q) & =\frac{1}{1-x q} \mathcal{D}(0 ; q)  \tag{5.1}\\
& =\frac{q}{1-x q} \mathcal{F}(1,1 ; q) \tag{5.2}
\end{align*}
$$

Proof. We know from Lemma 3.3 that

$$
\begin{aligned}
\sum_{n \geq 2} \sum_{k=0}^{n-2} f_{n}(k, k) x^{k} q^{n} & =\sum_{n \geq 2} \sum_{k=0}^{n-2} f_{n-k}(0,0) x^{k} q^{n} \\
\left(\text { with } n^{\prime}=n-k\right) & =\sum_{n^{\prime} \geq 2} \sum_{n \geq n^{\prime}} f_{n^{\prime}}(0,0) x^{n-n^{\prime}} q^{n} \\
& =\sum_{n^{\prime} \geq 2} f_{n^{\prime}}(0,0) x^{-n^{\prime}} \sum_{n \geq n^{\prime}}(x q)^{n} \\
& =\frac{1}{1-x q} \sum_{n^{\prime} \geq 2} f_{n^{\prime}}(0,0) q^{n^{\prime}}
\end{aligned}
$$

Noticing that $D_{1}(x)=0$, we have

$$
\mathcal{D}(x ; q)=\frac{1}{1-x q} \mathcal{D}(0 ; q)
$$

which is the first part of the lemma. For the second part, we deduce from Lemma 3.1(d) that

$$
\begin{aligned}
\mathcal{D}(0 ; q) & =\sum_{n \geq 2} f_{n}(0,0) q^{n} \\
& =\sum_{n \geq 2} \sum_{\ell^{\prime}=0}^{n-2} \sum_{k^{\prime}=0}^{\ell^{\prime}} f_{n-1}\left(k^{\prime}, \ell^{\prime}\right) q^{n} \\
& =q \mathcal{F}(1,1 ; q)
\end{aligned}
$$

Therefore, (5.2) follows.

Next, we show a relation between $\mathcal{F}(x, 1 ; q)$ and $\mathcal{F}(1,1 ; q)$.
Lemma 5.2. We have

$$
\begin{equation*}
\mathcal{F}(x, 1 ; q)=\frac{1-q}{1-x q} \mathcal{F}(1,1 ; q) \tag{5.3}
\end{equation*}
$$

Proof. For $n \geq 2$, it follows from Lemma 3.1(d) that

$$
\begin{aligned}
D_{n}(x) & =\sum_{\ell=0}^{n-2} f_{n}(\ell, \ell) x^{\ell} \\
& =\sum_{\ell=0}^{n-2} \sum_{\ell^{\prime}=\ell}^{n-2} \sum_{k^{\prime}=\ell}^{\ell^{\prime}} f_{n-1}\left(k^{\prime}, \ell^{\prime}\right) x^{\ell} \\
& =\sum_{\ell^{\prime}=0}^{n-2} \sum_{k^{\prime}=0}^{\ell^{\prime}} f_{n-1}\left(k^{\prime}, \ell^{\prime}\right) \sum_{k=0}^{k^{\prime}} x^{\ell} \\
& =\sum_{\ell^{\prime}=0}^{n-2} \sum_{k^{\prime}=0}^{\ell^{\prime}} f_{n-1}\left(k^{\prime}, \ell^{\prime}\right) \frac{1-x^{k^{\prime}+1}}{1-x} \\
& =\frac{1}{1-x}\left(F_{n-1}(1,1)-x F_{n-1}(x, 1)\right) .
\end{aligned}
$$

Therefore,

$$
\mathcal{D}(x ; q)=\frac{q}{1-x}(\mathcal{F}(1,1 ; q)-x \mathcal{F}(x, 1 ; q))
$$

Substituting (5.2) into the above yields

$$
\frac{q}{1-x q} \mathcal{F}(1,1 ; q)=\frac{q}{1-x}(\mathcal{F}(1,1 ; q)-x \mathcal{F}(x, 1 ; q))
$$

from which (5.3) follows.
We then construct a functional equation for $\mathcal{F}(x, y ; q)$.
Lemma 5.3. We have

$$
\begin{align*}
& \left(1+\frac{x q}{1-x}+\frac{y q}{1-y}\right) \mathcal{F}(x, y ; q) \\
& \quad=\frac{q}{1-x} \mathcal{F}(1, y ; q)+\frac{q(1-q)}{(1-y)(1-x y q)} \mathcal{F}(1,1 ; q)+\frac{q\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-2 y q)(1-x y q)} \tag{5.4}
\end{align*}
$$

Proof. We first observe that

$$
\begin{align*}
\sum_{\ell=0}^{n-2} f_{n}(\ell, \ell) x^{\ell} y^{\ell}+\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n}(k, \ell) x^{k} y^{\ell} & =F_{n}(x, y)-\sum_{k=0}^{n-1} f_{n}(k, n-1) x^{k} y^{n-1} \\
& =F_{n}(x, y)-y^{n-1} L_{n}(x) \tag{5.5}
\end{align*}
$$

Notice also that

$$
\begin{equation*}
\sum_{\ell=0}^{n-2} f_{n}(\ell, \ell) x^{\ell} y^{\ell}=D_{n}(x y) \tag{5.6}
\end{equation*}
$$

Now, by Lemma 3.1(e), we may separate

$$
\begin{aligned}
\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n}(k, \ell) x^{k} y^{\ell}= & \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k^{\prime}=k}^{\ell} f_{n-1}\left(k^{\prime}, \ell\right) x^{k} y^{\ell} \\
& +\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{\ell^{\prime}=\ell}^{n-2} f_{n-1}\left(k, \ell^{\prime}\right) x^{k} y^{\ell}
\end{aligned}
$$

We further notice that the first term on the right-hand side can be separated as

$$
\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k^{\prime}=k}^{\ell} f_{n-1}\left(k^{\prime}, \ell\right) x^{k} y^{\ell}=\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k^{\prime}=k}^{\ell-1} f_{n-1}\left(k^{\prime}, \ell\right) x^{k} y^{\ell}+\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n-1}(\ell, \ell) x^{k} y^{\ell}
$$

We have

$$
\begin{aligned}
\sum_{\ell=1}^{n-2} & \sum_{k=0}^{\ell-1} \sum_{k^{\prime}=k}^{\ell-1} f_{n-1}\left(k^{\prime}, \ell\right) x^{k} y^{\ell} \\
= & \sum_{\ell=1}^{n-2} \sum_{k^{\prime}=0}^{\ell-1} f_{n-1}\left(k^{\prime}, \ell\right) y^{\ell} \sum_{k=0}^{k^{\prime}} x^{k} \\
= & \sum_{\ell=1}^{n-2} \sum_{k^{\prime}=0}^{\ell-1} f_{n-1}\left(k^{\prime}, \ell\right) y^{\ell} \frac{1-x^{k^{\prime}+1}}{1-x} \\
= & \sum_{\ell=0}^{n-2} \sum_{k^{\prime}=0}^{\ell} f_{n-1}\left(k^{\prime}, \ell\right) y^{\ell} \frac{1-x^{k^{\prime}+1}}{1-x}-\sum_{\ell=0}^{n-2} f_{n-1}(\ell, \ell) y^{\ell} \frac{1-x^{\ell+1}}{1-x} \\
= & \frac{1}{1-x}\left(F_{n-1}(1, y)-x F_{n-1}(x, y)\right) \\
& -\frac{1}{1-x}\left(D_{n-1}(y)+y^{n-2}-x D_{n-1}(x y)-x^{n-1} y^{n-2}\right)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n-1}(\ell, \ell) x^{k} y^{\ell} & =\sum_{\ell=1}^{n-2} f_{n-1}(\ell, \ell) y^{\ell} \frac{1-x^{\ell}}{1-x} \\
& =\sum_{\ell=0}^{n-2} f_{n-1}(\ell, \ell) y^{\ell} \frac{1-x^{\ell}}{1-x} \\
& =\frac{1}{1-x}\left(D_{n-1}(y)+y^{n-2}-D_{n-1}(x y)-x^{n-2} y^{n-2}\right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{\ell^{\prime}=\ell}^{n-2} f_{n-1}\left(k, \ell^{\prime}\right) x^{k} y^{\ell} & =\sum_{\ell^{\prime}=1}^{n-2} \sum_{k=0}^{\ell^{\prime}-1} f_{n-1}\left(k, \ell^{\prime}\right) x^{k} \sum_{\ell=k+1}^{\ell^{\prime}} y^{\ell} \\
& =\sum_{\ell^{\prime}=1}^{n-2} \sum_{k=0}^{\ell^{\prime}-1} f_{n-1}\left(k, \ell^{\prime}\right) x^{k} \frac{y^{k+1}-y^{\ell^{\prime}+1}}{1-y} \\
& =\sum_{\ell^{\prime}=0}^{n-2} \sum_{k=0}^{\ell^{\prime}} f_{n-1}\left(k, \ell^{\prime}\right) x^{k} \frac{y^{k+1}-y^{\ell^{\prime}+1}}{1-y}
\end{aligned}
$$

$$
=\frac{y}{1-y}\left(F_{n-1}(x y, 1)-F_{n-1}(x, y)\right)
$$

Therefore,

$$
\begin{align*}
& \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n}(k, \ell) x^{k} y^{\ell} \\
& \quad=\frac{1}{1-x}\left(F_{n-1}(1, y)-x F_{n-1}(x, y)\right)+\frac{y}{1-y}\left(F_{n-1}(x y, 1)-F_{n-1}(x, y)\right) \\
& \quad-D_{n-1}(x y)-x^{n-2} y^{n-2} \tag{5.7}
\end{align*}
$$

It follows from (5.5), (5.6) and (5.7) that

$$
\begin{aligned}
& F_{n}(x, y)-y^{n-1} L_{n}(x) \\
& \qquad D_{n}(x y)+\frac{1}{1-x}\left(F_{n-1}(1, y)-x F_{n-1}(x, y)\right) \\
&+\frac{y}{1-y}\left(F_{n-1}(x y, 1)-F_{n-1}(x, y)\right)-D_{n-1}(x y)-x^{n-2} y^{n-2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathcal{F}(x, y ; q)-y^{-1} \mathcal{L}(x ; y q) \\
& \quad= \mathcal{D}(x y ; q)+\frac{q}{1-x}(\mathcal{F}(1, y ; q)-x \mathcal{F}(x, y ; q)) \\
& \quad+\frac{y q}{1-y}(\mathcal{F}(x y, 1 ; q)-\mathcal{F}(x, y ; q))-q \mathcal{D}(x y ; q)-\frac{q^{2}}{1-x y q}
\end{aligned}
$$

Applying (4.1), (5.2) and (5.3) gives the desired result.
With the assistance of the kernel method, we may deduce a functional equation satisfied by $\mathcal{F}(1, y ; q)$.
Lemma 5.4. We have

$$
\begin{equation*}
\mathcal{F}(1, y ; q)=\frac{q}{1-y+y^{2} q} \mathcal{F}(1,1 ; q)+\frac{q(1-y)\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-q)(1-2 y q)\left(1-y+y^{2} q\right)} \tag{5.8}
\end{equation*}
$$

Proof. We multiply both sides of $(5.4)$ by $(1-x)(1-y)$. Then

$$
\begin{aligned}
&((1-y+y q)-x(1-y-q+2 y q)) \mathcal{F}(x, y ; q) \\
& \quad q(1-y) \mathcal{F}(1, y ; q)+\frac{q(1-q)(1-x)}{1-x y q} \mathcal{F}(1,1 ; q) \\
&+\frac{q(1-x)(1-y)\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-2 y q)(1-x y q)} .
\end{aligned}
$$

We treat the kernel polynomial as a function in $x$ and solve

$$
(1-y+y q)-x(1-y-q+2 y q)=0
$$

so that

$$
x=\frac{1-y+y q}{1-y-q+2 y q}
$$

Substituting the above into

$$
0=q(1-y) \mathcal{F}(1, y ; q)+\frac{q(1-q)(1-x)}{1-x y q} \mathcal{F}(1,1 ; q)
$$

$$
+\frac{q(1-x)(1-y)\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-2 y q)(1-x y q)},
$$

we arrive at (5.8) after simplification.
Finally, we are ready to complete the proof of Theorem 1.1.
Proof of Theorem 1.1. It is known that (cf. [17, A279561])

$$
\begin{equation*}
1+\sum_{n \geq 1}\left(1+\sum_{i=1}^{n-1}\binom{2 i}{i-1}\right) q^{n}=\frac{1-4 q+(1-2 q) \sqrt{1-4 q}}{2(1-q)(1-4 q)} \tag{5.9}
\end{equation*}
$$

We then rewrite (5.8) as

$$
\left(1-y+y^{2} q\right) \mathcal{F}(1, y ; q)=q \mathcal{F}(1,1 ; q)+\frac{q(1-y)\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-q)(1-2 y q)}
$$

We treat the kernel polynomial as a function in $y$ and solve

$$
1-y+y^{2} q=0
$$

Then

$$
y_{1,2}=\frac{1 \mp \sqrt{1-4 q}}{2 q}
$$

We choose the solution

$$
y_{1}=\frac{1-\sqrt{1-4 q}}{2 q}
$$

since $y_{1} \rightarrow 0$ as $q \rightarrow 0$. Substituting $y=y_{1}$ into

$$
0=q \mathcal{F}(1,1 ; q)+\frac{q(1-y)\left(1-q-2 y q+2 y q^{2}+y^{2} q^{2}\right)}{(1-q)(1-2 y q)}
$$

we find that

$$
\begin{align*}
\mathcal{F}(1,1 ; q) & =\frac{-(1-2 q)(1-4 q)+(1-2 q) \sqrt{1-4 q}}{2(1-q)(1-4 q)} \\
& =\frac{1-4 q+(1-2 q) \sqrt{1-4 q}}{2(1-q)(1-4 q)}-1 \tag{5.10}
\end{align*}
$$

This implies that for $n \geq 1$,

$$
1+\sum_{i=1}^{n-1}\binom{2 i}{i-1}=\sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_{n}(k, \ell)=\left|\mathbf{I}_{n}(0012)\right|
$$

Therefore, Conjecture 1.1 is true.

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