On 0012-avoiding inversion sequences and a Conjecture of Lin and Ma

Shane Chern

Abstract. The study of pattern avoidance in inversion sequences recently attracts extensive research interests. In particular, Zhicong Lin and Jun Ma conjectured a formula that counts the number of inversion sequences avoiding the pattern 0012. We will not only confirm this conjecture but also give a formula that enumerates the number of 0012-avoiding inversion sequences in which the last entry equals n - 1.

Keywords. Inversion sequence, pattern avoidance, generating function, kernel method. **2010MSC**. 05A05, 05A15.

1. Introduction

An inversion sequence of length n is a sequence $e = e_1 e_2 \cdots e_n$ such that $0 \le e_i \le i-1$ for each $1 \le i \le n$. We denote by \mathbf{I}_n the set of inversion sequences of length n. Given any word $w \in \{0, 1, \ldots, n-1\}^n$ of length n, we define its reduction by the word obtained via replacing the k-th smallest entries of e with k-1. For instance, the reduction of 0023252 is 0012131. We say that an inversion sequence e contains a given pattern p if there exists a subsequence of e such that its reduction is the same as p; otherwise, we say that e avoids the pattern p. For instance, 0023252 has a subsequence 022 whose reduction is 011 — hence, 0023252 contains the pattern 011. On the other hand, none of the length 3 subsequences of 0023252 have the reduction 110 — hence, 0023252 avoids the pattern 110.

Let p_1, p_2, \ldots, p_m be given patterns. We denote by $\mathbf{I}_n(p_1, p_2, \ldots, p_m)$ the set of inversion sequences of length n that avoid all of the patterns p_1, p_2, \ldots, p_m . Recently, the study of pattern avoidance in inversion sequences attracts extensive research interests. See [1-8, 10-15, 18, 19] for several instances of work on this topic. Among these work, one particular interesting problem is about the enumeration of inversion sequences that avoids fixed patterns. For example, in a pioneering work of Corteel, Martinez, Savage and Weselcouch [7], it was shown that

$$|\mathbf{I}_n(011)| = B_n$$
 and $|\mathbf{I}_n(021)| = S_n$

where B_n is the *n*-th Bell number (OEIS, [17, A000110]) and S_n is the *n*-th large Schröder number (OEIS, [17, A006318]).

In a recent paper [18], Yan and Lin proved a conjecture due to Martinez and Savage [15] that claims

$$|\mathbf{I}_n(021, 120)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}.$$
(1.1)

This sequence is registered as OEIS, [17, A279561]. Lin and Yan also showed that this sequence as well enumerates $|\mathbf{I}_n(102, 110)|$ and $|\mathbf{I}_n(102, 120)|$. This therefore

establishes the Wilf-equivalence

$$\mathbf{I}_n(021, 120) \sim \mathbf{I}_n(102, 110) \sim \mathbf{I}_n(102, 120).$$
 (1.2)

At the end of [18], a conjecture of Zhicong Lin and Jun Ma discovered in 2019 is recorded.

Conjecture 1.1 (Lin and Ma). For $n \ge 1$,

$$|\mathbf{I}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}.$$
(1.3)

In other words, it is possible to extend the Wilf-equivalence (1.2) as

$$\mathbf{I}_n(0012) \sim \mathbf{I}_n(021, 120) \sim \mathbf{I}_n(102, 110) \sim \mathbf{I}_n(102, 120).$$

In this paper, we will prove the above conjecture of Lin and Ma.

Theorem 1.1. Conjecture 1.1 is true.

Let us fix some notation. Given $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$, we define

 $\mathcal{R}(e) := \{ m : \exists i \neq j \text{ such that } e_i = e_j = m \}.$

In other words, $\mathcal{R}(e)$ is the set of letters that appear more than once in e. We further define

$$\operatorname{SRPT}(e) := \min \mathcal{R}(e),$$

that is, the smallest number in $\mathcal{R}(e)$. Notice that there is only one sequence $01 \cdots (n-1)$ in which none of the letters repeat. For this sequence, we assign that

$$\operatorname{SRPT}(01\cdots(n-1)) := n-1$$

Finally, we define

$$LAST(e) := e_n,$$

the last entry of e.

Apart from counting the number of inversion sequences that avoid the pattern 0012, we will also enumerate the number of sequences in $I_n(0012)$ in which the last entry equals n - 1.

Theorem 1.2. For $n \ge 1$,

$$|\{e \in \mathbf{I}_n(0012) : \text{LAST}(e) = n - 1\}| = \begin{cases} 1 & \text{if } n = 1, \\ 2^{n-2} & \text{if } n \ge 2. \end{cases}$$
(1.4)

2. Combinatorial observations

We collect some combinatorial observations about inversion sequences in $\mathbf{I}_n(0012)$.

Lemma 2.1. For $n \ge 1$ and $e \in \mathbf{I}_n(0012)$, if $\operatorname{SRPT}(e) = k$, then for $1 \le i \le k+1$, $e_i = i-1$.

Proof. If SRPT(e) = n - 1, then $e = 01 \cdots (n - 1)$ and hence the lemma is true. Let $\text{SRPT}(e) \neq n - 1$. If in this case the lemma is not true, then since $0 \leq e_i \leq i - 1$ for each *i*, there must exist some $k_1 < k = \text{SRPT}(e)$ that appears more than once among $e_1, e_2, \ldots, e_{k+1}$. This violates the assumption that SRPT(e) = k. \Box

Lemma 2.2. For $n \ge 2$ and $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$, let $\gamma(e) = e_1 e_2 \cdots e_{n-1}$. We further assume that $e \ne 01 \cdots (n-1)$. Then

(a). if $LAST(e) > SRPT(\gamma(e))$, then

$$\operatorname{SRPT}(e) = \operatorname{SRPT}(\gamma(e));$$

(b). if $LAST(e) \leq SRPT(\gamma(e))$, then

 $\operatorname{SRPT}(e) = \operatorname{LAST}(e).$

Proof. A simple observation is that $\gamma(e) \in \mathbf{I}_{n-1}(0012)$. Below let us assume that $\text{LAST}(e) = \ell$, SRPT(e) = k and $\text{SRPT}(\gamma(e)) = k'$.

First, if $\mathcal{R}(\gamma(e)) = \emptyset$, then for each $0 \leq i \leq n-1$, $e_i = i-1$. Since $e \neq 01 \cdots (n-1)$, we have $\text{LAST}(e) = \ell \leq n-2 = \text{SRPT}(\gamma(e))$. This fits into Case (b). Further, we find that $\mathcal{R}(e) = \{\ell\}$ and hence $\text{SRPT}(e) = \ell$. This implies that SRPT(e) = LAST(e).

Now we assume that $\mathcal{R}(\gamma(e)) \neq \emptyset$. Notice that Case (a) is trivial. For Case (b), we first deduce from $\mathcal{R}(\gamma(e)) \neq \emptyset$ that $k' \leq n-3$. By Lemma 2.1, we find that for $1 \leq i \leq k'+1$, $e_i = i-1$. If $\text{LAST}(e) = \ell \leq k'$, then we know that $e_{\ell+1} = \ell = e_n$. Also, we notice that the indices satisfy $\ell+1 \leq k'+1 \leq n-2 < n$. Hence, $\ell \in \mathcal{R}(e)$. Therefore, $\text{SRPT}(e) = \min\{\ell, k'\} = \ell = \text{LAST}(e)$.

Corollary 2.3. For $e \in \mathbf{I}_n(0012)$,

$$0 \leq \operatorname{SRPT}(e) \leq \operatorname{LAST}(e) \leq n-1.$$

Proof. If $e = 01 \cdots (n-1)$, the above inequalities are trivial since SRPT(e) = LAST(e) = n-1. If $e \neq 01 \cdots (n-1)$, the inequalities are direct consequences of Lemma 2.2 and the fact that $\text{SRPT}(e) \ge 0$ and $\text{LAST}(e) \le n-1$.

Lemma 2.4. For $n \ge 2$ and $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$, let e be such that $\operatorname{SRPT}(e) = \operatorname{LAST}(e) = k$ with $0 \le k \le n-2$. Then

(a). for $1 \le i \le k+1$,

 $e_i = i - 1;$

(b). if we denote $e' = e'_1 e'_2 \cdots e'_{n-k}$ by the sequence obtained via $e'_i = e_{k+i} - k$ for each $1 \le i \le n-k$, then $e' \in \mathbf{I}_{n-k}(0012)$ such that

$$\operatorname{SRPT}(e') = \operatorname{LAST}(e') = 0.$$

Proof. Part (a) simply comes from Lemma 2.1. Also, we know from Part (a) that for $k+1 \leq i \leq n$, it holds that $e_i \geq k$. On the other hand, $e_i \leq i-1$. Hence, e' is still an inversion sequence. Further, it is trivial to see that e' still avoids the pattern 0012. Finally, we have $e'_1 = e_{k+1}-k = k-k = 0$ and $\text{LAST}(e') = e'_{n-k} = e_n-k = k-k = 0$. Since $n-k \geq 2 > 1$, we have $0 \in \mathcal{R}(e')$ and hence SRPT(e') = 0.

3. Recurrences

Let

$$f_n(k,\ell) := \left\{ \begin{array}{l} \text{the number of sequences } e \in \mathbf{I}_n(0012) \text{ with} \\ \\ \text{SRPT}(e) = k \text{ and } \text{LAST}(e) = \ell \end{array} \right\}$$

We will establish the following recurrences.

Lemma 3.1. We have

(a). for $n \ge 1$,

$$f_n(n-1, n-1) = 1$$

(b). for $n \geq 2$,

$$f_n(n-2,n-1) = 0$$

(c). for $n \ge 2$ and $0 \le k \le n - 3$,

$$f_n(k, n-1) = \sum_{k'=k}^{n-2} f_{n-1}(k', n-2);$$

(d). for $n \ge 2$ and $0 \le \ell \le n-2$,

$$f_n(\ell, \ell) = \sum_{\ell'=\ell}^{n-2} \sum_{k'=\ell}^{\ell'} f_{n-1}(k', \ell');$$

(e). for $n \ge 2$ and $0 \le k < \ell \le n-2$,

$$f_n(k,\ell) = \sum_{k'=k}^{\ell} f_{n-1}(k',\ell) + \sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell').$$

Proof. Cases (a) and (b) are trivial. In particular, Case (a) enumerates the only inversion sequence $01 \cdots (n-1)$ in which none of the letters repeat. Below we always assume that $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n(0012)$. Let $\gamma(e)$ be as in Lemma 2.2.

For Case (c), let e be such that $\operatorname{SRPT}(e) = k \leq n-3$ and $\operatorname{LAST}(e) = n-1$. We first notice that $e_{n-1} = \operatorname{LAST}(\gamma(e)) \geq \operatorname{SRPT}(\gamma(e))$ by Corollary 2.3. Also, it is easy to see that $\operatorname{SRPT}(\gamma(e)) = \operatorname{SRPT}(e) = k$ since $\operatorname{LAST}(e) = n-1 > k$. Now we claim that $e_{n-1} = k$. Otherwise, namely, if $e_{n-1} > k$, we may find i < j < n-1 such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which contradicts the assumption that $e \in \mathbf{I}_n(0012)$. We therefore have a bijection

$$e = e_1 e_2 \cdots e_{n-2}(k)(n-1) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(n-2) = e'.$$

Notice that e' is still an inversion sequence avoiding the pattern 0012. Also, $\operatorname{SRPT}(e') \geq k$. Otherwise, there exists some k' < k that appears more than once among $e_1, e_2, \ldots, e_{n-2}$ and therefore $\operatorname{SRPT}(e) < k$, which leads to a contradiction. Finally, to prove Case (c), it suffices to show that e' could be any inversion sequence in $\mathbf{I}_{n-1}(0012)$ with $\operatorname{LAST}(e') = n - 2$ (which is of course true) and $\operatorname{SRPT}(e') \geq k$. Let e' be such a sequence and assume that $\operatorname{SRPT}(e') = k' \geq k$. By Lemma 2.1, we have $e_{k+1} = k$. Pulling back to e, we have $e_{k+1} = e_{n-1} = k$ with the indices $k + 1 \leq n - 2 < n - 1$. Therefore, for this e, we have $k \in \mathcal{R}(e)$ and hence $\operatorname{SRPT}(e) = \min\{k', k\} = k$.

For Case (d), let e be such that $\operatorname{SRPT}(e) = \operatorname{LAST}(e) = \ell$ with $0 \leq \ell \leq n-2$. We first find that $\operatorname{SRPT}(\gamma(e)) \geq \operatorname{SRPT}(e) = \ell$. On the other hand, let $e' = e'_1 e'_2 \cdots e'_{n-1} \in \mathbf{I}_{n-1}(0012)$ be such that $\operatorname{SRPT}(e') \geq \ell$. By Lemma 2.1, $e'_{\ell+1} = \ell$. Hence, by appending ℓ to the end of e', we obtain a sequence with both SRPT and LAST equal to ℓ . We therefore arrive at a bijection between e and e',

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e',$$

and the desired relation follows.

For Case (e), let e be such that $\operatorname{SRPT}(e) = k$ and $\operatorname{LAST}(e) = \ell$ with $0 \le k < \ell \le n-2$. Notice that $e_{n-1} \ge k$. Otherwise, we assume that $e_{n-1} = k' < k$. Then

by Lemma 2.1, $e_{k'+1} = k' = e_{n-1}$. However, k' + 1 < k + 1 < n - 1 and hence $k' \in \mathcal{R}(e)$. But this violates the fact that $k = \min \mathcal{R}(e)$. Now we have two cases.

▶ $e_{n-1} < e_n$. We claim that $e_{n-1} = k$. Otherwise, we may find i < j < n-1such that $e_i = e_j = k$. Hence, $e_i e_j e_{n-1} e_n$ has the reduction 0012, which violates the assumption that $e \in \mathbf{I}_n(0012)$. Now we have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\mathrm{SRPT}(e') \ge k$ and $\mathrm{LAST}(e') = \ell$ by

$$e = e_1 e_2 \cdots e_{n-2}(k)(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-2}(\ell) = e'.$$

The argument is similar to that for Case (c). This bijection leads to the first term in the right-hand side of the recurrence relation in Case (e).

▶ $e_{n-1} \ge e_n$. We have a bijection between e and $e' \in \mathbf{I}_{n-1}(0012)$ such that $\operatorname{SRPT}(e') = k$ and $\operatorname{LAST}(e') \ge \ell$ by

$$e = e_1 e_2 \cdots e_{n-1}(\ell) \longleftrightarrow e_1 e_2 \cdots e_{n-1} = e'.$$

The argument is similar to that for Case (d). This bijection leads to the second term in the right-hand side of the recurrence relation in Case (e).

The proof of the lemma is therefore complete.

We may therefore determine the support of $f_n(k, \ell)$.

Corollary 3.2. For $n \ge 1$, $f_n(k, \ell)$ is supported on

$$\{(k,\ell) \in \mathbb{N}^2 : 0 \le k \le \ell \le n-1\} \setminus \{(n-2, n-1)\}.$$

Proof. By Corollary 2.3, $f_n(k, \ell) = 0$ if

$$(k,\ell) \notin \{(k,\ell) \in \mathbb{N}^2 : 0 \le k \le \ell \le n-1\}.$$

Also, $f_n(n-2, n-1) = 0$ by Lemma 3.1(b). Finally, for the remaining (k, ℓ) , we have $f_n(k, \ell) \neq 0$ with the help of the recurrences in Lemma 3.1.

Finally, we have another recurrence.

Lemma 3.3. We have, for $n \ge 2$ and $0 \le k \le n-2$,

$$f_n(k,k) = f_{n-k}(0,0).$$

Proof. This is an immediate consequence of Lemma 2.4.

In the sequel, we require three auxiliary functions with q within a sufficiently small neighborhood of 0:

$$\mathcal{L}(x;q) := \sum_{n\geq 1} \left(\sum_{k=0}^{n-1} f_n(k,n-1)x^k \right) q^n,$$
$$\mathcal{D}(x;q) := \sum_{n\geq 1} \left(\sum_{\ell=0}^{n-2} f_n(\ell,\ell)x^\ell \right) q^n,$$
$$\mathcal{F}(x,y;q) := \sum_{n\geq 1} \left(\sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_n(k,\ell)x^k y^\ell \right) q^n.$$

In particular, we write, for $n \ge 1$,

$$L_n(x) := \sum_{k=0}^{n-1} f_n(k, n-1) x^k,$$

$$D_n(x) := \sum_{\ell=0}^{n-2} f_n(\ell, \ell) x^{\ell},$$
$$F_n(x, y) := \sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_n(k, \ell) x^k y^{\ell}.$$

~

Notice that $L_1(x) = 1$, $D_1(x) = 0$ and $F_1(x, y) = 1$. Also, since $f_n(n-1, n-1) = 1$, we have

$$\sum_{\ell=0}^{n-1} f_n(\ell,\ell) x^{\ell} = D_n(x) + x^{n-1}.$$

4. Proof of Theorem 1.2

Notice that Theorem 1.2 is equivalent to

$$\mathcal{L}(1;q) = \sum_{n \ge 1} \left(\sum_{k=0}^{n-1} f_n(k,n-1) \right) q^n$$

$$\stackrel{?}{=} q + q^2 + 2q^3 + 4q^4 + 8q^5 + 16q^6 + \cdots$$

$$= \frac{q(1-q)}{1-2q}.$$

We prove a strengthening of the above.

Theorem 4.1. We have

$$\mathcal{L}(x;q) = \frac{q(1-q)^2}{(1-2q)(1-xq)}.$$
(4.1)

Proof. For $n \ge 2$, it follows from (a), (b) and (c) of Lemma 3.1 that

$$\sum_{k=0}^{n-1} f_n(k, n-1)x^k = x^{n-1} + \sum_{k=0}^{n-3} \sum_{k'=k}^{n-2} f_{n-1}(k', n-2)x^k$$
$$= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k', n-2) \sum_{k=0}^{k'} x^k + f_{n-1}(n-2, n-2) \sum_{k=0}^{n-3} x^k$$
$$= x^{n-1} + \sum_{k'=0}^{n-3} f_{n-1}(k', n-2) \frac{1-x^{k'+1}}{1-x} + \frac{1-x^{n-2}}{1-x}.$$

Therefore,

$$L_n(x) = x^{n-1} + \frac{1}{1-x} \left(L_{n-1}(1) - xL_{n-1}(x) \right) - \frac{1-x^{n-1}}{1-x} + \frac{1-x^{n-2}}{1-x}$$

Multiplying the above by q^n and summing over $n\geq 2,$ we have

$$\mathcal{L}(x;q) - q = \frac{q}{1-x}\mathcal{L}(1;q) - \frac{xq}{1-x}\mathcal{L}(x;q) - \frac{q^2(1-x)}{1-xq},$$

or

$$(1 - xq)(1 - x + xq)\mathcal{L}(x;q) = q(1 - xq)\mathcal{L}(1;q) + q(1 - q)(1 - x).$$
(4.2)

Applying the kernel method (see [9, Exercise 4, §2.2.1, p. 243] or [16]) yields

$$\begin{cases} 1 - x + xq = 0, \\ q(1 - xq)\mathcal{L}(1;q) + q(1 - q)(1 - x) = 0. \end{cases}$$

Solving the first equation of the system for x gives

$$x = \frac{1}{1-q}.$$

Substituting the above into the second equation of the system, we have

$$\mathcal{L}(1;q) = \frac{q(1-q)}{1-2q}.$$

Substituting the above back to (4.2), we arrive at (4.1).

5. Proof of Theorem 1.1

We first establish two relations concerning $\mathcal{D}(x;q)$.

Lemma 5.1. We have

$$\mathcal{D}(x;q) = \frac{1}{1 - xq} \mathcal{D}(0;q) \tag{5.1}$$

$$=\frac{q}{1-xq}\mathcal{F}(1,1;q).$$
(5.2)

Proof. We know from Lemma 3.3 that

$$\begin{split} \sum_{n\geq 2} \sum_{k=0}^{n-2} f_n(k,k) x^k q^n &= \sum_{n\geq 2} \sum_{k=0}^{n-2} f_{n-k}(0,0) x^k q^n \\ \text{(with } n' = n-k) &= \sum_{n'\geq 2} \sum_{n\geq n'} f_{n'}(0,0) x^{n-n'} q^n \\ &= \sum_{n'\geq 2} f_{n'}(0,0) x^{-n'} \sum_{n\geq n'} (xq)^n \\ &= \frac{1}{1-xq} \sum_{n'\geq 2} f_{n'}(0,0) q^{n'}. \end{split}$$

Noticing that $D_1(x) = 0$, we have

$$\mathcal{D}(x;q) = \frac{1}{1-xq}\mathcal{D}(0;q),$$

which is the first part of the lemma. For the second part, we deduce from Lemma $3.1(\mathrm{d})$ that

$$\mathcal{D}(0;q) = \sum_{n \ge 2} f_n(0,0)q^n$$

= $\sum_{n \ge 2} \sum_{\ell'=0}^{n-2} \sum_{k'=0}^{\ell'} f_{n-1}(k',\ell')q^n$
= $q\mathcal{F}(1,1;q).$

Therefore, (5.2) follows.

Next, we show a relation between $\mathcal{F}(x, 1; q)$ and $\mathcal{F}(1, 1; q)$.

Lemma 5.2. We have

$$\mathcal{F}(x,1;q) = \frac{1-q}{1-xq} \mathcal{F}(1,1;q).$$
(5.3)

Proof. For $n \ge 2$, it follows from Lemma 3.1(d) that

$$D_n(x) = \sum_{\ell=0}^{n-2} f_n(\ell, \ell) x^{\ell}$$

= $\sum_{\ell=0}^{n-2} \sum_{\ell'=\ell}^{n-2} \sum_{k'=\ell}^{\ell'} f_{n-1}(k', \ell') x^{\ell}$
= $\sum_{\ell'=0}^{n-2} \sum_{k'=0}^{\ell'} f_{n-1}(k', \ell') \sum_{k=0}^{k'} x^{\ell}$
= $\sum_{\ell'=0}^{n-2} \sum_{k'=0}^{\ell'} f_{n-1}(k', \ell') \frac{1-x^{k'+1}}{1-x}$
= $\frac{1}{1-x} (F_{n-1}(1, 1) - xF_{n-1}(x, 1)).$

Therefore,

$$\mathcal{D}(x;q) = \frac{q}{1-x} \big(\mathcal{F}(1,1;q) - x \mathcal{F}(x,1;q) \big).$$

Substituting (5.2) into the above yields

$$\frac{q}{1-xq}\mathcal{F}(1,1;q) = \frac{q}{1-x} \big(\mathcal{F}(1,1;q) - x\mathcal{F}(x,1;q)\big),\,$$

from which (5.3) follows.

We then construct a functional equation for $\mathcal{F}(x, y; q)$.

$$\left(1 + \frac{xq}{1-x} + \frac{yq}{1-y}\right) \mathcal{F}(x,y;q) = \frac{q}{1-x} \mathcal{F}(1,y;q) + \frac{q(1-q)}{(1-y)(1-xyq)} \mathcal{F}(1,1;q) + \frac{q(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}.$$
(5.4)

Proof. We first observe that

$$\sum_{\ell=0}^{n-2} f_n(\ell,\ell) x^\ell y^\ell + \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_n(k,\ell) x^k y^\ell = F_n(x,y) - \sum_{k=0}^{n-1} f_n(k,n-1) x^k y^{n-1} = F_n(x,y) - y^{n-1} L_n(x).$$
(5.5)

Notice also that

$$\sum_{\ell=0}^{n-2} f_n(\ell,\ell) x^{\ell} y^{\ell} = D_n(xy).$$
(5.6)

Now, by Lemma 3.1(e), we may separate

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_n(k,\ell) x^k y^\ell = \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k'=k}^{\ell} f_{n-1}(k',\ell) x^k y^\ell + \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell') x^k y^\ell.$$

We further notice that the first term on the right-hand side can be separated as

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k'=k}^{\ell} f_{n-1}(k',\ell) x^k y^\ell = \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell) x^k y^\ell + \sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell) x^k y^\ell.$$

We have

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{k'=k}^{\ell-1} f_{n-1}(k',\ell) x^k y^\ell$$

= $\sum_{\ell=1}^{n-2} \sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell) y^\ell \sum_{k=0}^{k'} x^k$
= $\sum_{\ell=1}^{n-2} \sum_{k'=0}^{\ell-1} f_{n-1}(k',\ell) y^\ell \frac{1-x^{k'+1}}{1-x}$
= $\sum_{\ell=0}^{n-2} \sum_{k'=0}^{\ell} f_{n-1}(k',\ell) y^\ell \frac{1-x^{k'+1}}{1-x} - \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell) y^\ell \frac{1-x^{\ell+1}}{1-x}$
= $\frac{1}{1-x} (F_{n-1}(1,y) - xF_{n-1}(x,y))$
 $- \frac{1}{1-x} (D_{n-1}(y) + y^{n-2} - xD_{n-1}(xy) - x^{n-1}y^{n-2}).$

Also,

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_{n-1}(\ell,\ell) x^k y^\ell = \sum_{\ell=1}^{n-2} f_{n-1}(\ell,\ell) y^\ell \frac{1-x^\ell}{1-x}$$
$$= \sum_{\ell=0}^{n-2} f_{n-1}(\ell,\ell) y^\ell \frac{1-x^\ell}{1-x}$$
$$= \frac{1}{1-x} \left(D_{n-1}(y) + y^{n-2} - D_{n-1}(xy) - x^{n-2}y^{n-2} \right).$$

On the other hand,

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} \sum_{\ell'=\ell}^{n-2} f_{n-1}(k,\ell') x^k y^\ell = \sum_{\ell'=1}^{n-2} \sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell') x^k \sum_{\ell=k+1}^{\ell'} y^\ell$$
$$= \sum_{\ell'=1}^{n-2} \sum_{k=0}^{\ell'-1} f_{n-1}(k,\ell') x^k \frac{y^{k+1} - y^{\ell'+1}}{1 - y}$$
$$= \sum_{\ell'=0}^{n-2} \sum_{k=0}^{\ell'} f_{n-1}(k,\ell') x^k \frac{y^{k+1} - y^{\ell'+1}}{1 - y}$$

$$= \frac{y}{1-y} \big(F_{n-1}(xy,1) - F_{n-1}(x,y) \big).$$

Therefore,

$$\sum_{\ell=1}^{n-2} \sum_{k=0}^{\ell-1} f_n(k,\ell) x^k y^\ell$$

= $\frac{1}{1-x} (F_{n-1}(1,y) - xF_{n-1}(x,y)) + \frac{y}{1-y} (F_{n-1}(xy,1) - F_{n-1}(x,y))$
 $- D_{n-1}(xy) - x^{n-2}y^{n-2}.$ (5.7)

It follows from (5.5), (5.6) and (5.7) that

$$F_n(x,y) - y^{n-1}L_n(x)$$

= $D_n(xy) + \frac{1}{1-x} (F_{n-1}(1,y) - xF_{n-1}(x,y))$
+ $\frac{y}{1-y} (F_{n-1}(xy,1) - F_{n-1}(x,y)) - D_{n-1}(xy) - x^{n-2}y^{n-2}.$

Therefore,

$$\mathcal{F}(x, y; q) - y^{-1}\mathcal{L}(x; yq)$$

$$= \mathcal{D}(xy; q) + \frac{q}{1-x} \big(\mathcal{F}(1, y; q) - x\mathcal{F}(x, y; q) \big)$$

$$+ \frac{yq}{1-y} \big(\mathcal{F}(xy, 1; q) - \mathcal{F}(x, y; q) \big) - q\mathcal{D}(xy; q) - \frac{q^2}{1-xyq}.$$
(4.1), (5.2) and (5.3) gives the desired result.

Applying (4.1), (5.2) and (5.3) gives the desired result.

With the assistance of the kernel method, we may deduce a functional equation satisfied by
$$\mathcal{F}(1, y; q)$$
.

Lemma 5.4. We have

$$\mathcal{F}(1,y;q) = \frac{q}{1-y+y^2q} \mathcal{F}(1,1;q) + \frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)(1-y+y^2q)}.$$
 (5.8)

Proof. We multiply both sides of (5.4) by (1-x)(1-y). Then

$$((1 - y + yq) - x(1 - y - q + 2yq))\mathcal{F}(x, y; q)$$

= $q(1 - y)\mathcal{F}(1, y; q) + \frac{q(1 - q)(1 - x)}{1 - xyq}\mathcal{F}(1, 1; q)$
+ $\frac{q(1 - x)(1 - y)(1 - q - 2yq + 2yq^2 + y^2q^2)}{(1 - 2yq)(1 - xyq)}.$

We treat the kernel polynomial as a function in x and solve

$$(1 - y + yq) - x(1 - y - q + 2yq) = 0$$

so that

$$x = \frac{1 - y + yq}{1 - y - q + 2yq}.$$

Substituting the above into

$$0 = q(1-y)\mathcal{F}(1,y;q) + \frac{q(1-q)(1-x)}{1-xyq}\mathcal{F}(1,1;q)$$

$$+\frac{q(1-x)(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-2yq)(1-xyq)}$$

we arrive at (5.8) after simplification.

Finally, we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. It is known that (cf. [17, A279561])

$$1 + \sum_{n \ge 1} \left(1 + \sum_{i=1}^{n-1} \binom{2i}{i-1} \right) q^n = \frac{1 - 4q + (1 - 2q)\sqrt{1 - 4q}}{2(1 - q)(1 - 4q)}.$$
 (5.9)

We then rewrite (5.8) as

$$(1 - y + y^2 q)\mathcal{F}(1, y; q) = q\mathcal{F}(1, 1; q) + \frac{q(1 - y)(1 - q - 2yq + 2yq^2 + y^2 q^2)}{(1 - q)(1 - 2yq)}.$$

We treat the kernel polynomial as a function in y and solve

$$1 - y + y^2 q = 0.$$

Then

$$y_{1,2} = \frac{1 \pm \sqrt{1 - 4q}}{2q}$$

We choose the solution

$$y_1 = \frac{1 - \sqrt{1 - 4q}}{2q}$$

since $y_1 \to 0$ as $q \to 0$. Substituting $y = y_1$ into

$$0 = q\mathcal{F}(1,1;q) + \frac{q(1-y)(1-q-2yq+2yq^2+y^2q^2)}{(1-q)(1-2yq)}$$

we find that

$$\mathcal{F}(1,1;q) = \frac{-(1-2q)(1-4q) + (1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)}$$
$$= \frac{1-4q + (1-2q)\sqrt{1-4q}}{2(1-q)(1-4q)} - 1.$$
(5.10)

This implies that for $n \ge 1$,

$$1 + \sum_{i=1}^{n-1} \binom{2i}{i-1} = \sum_{\ell=0}^{n-1} \sum_{k=0}^{\ell} f_n(k,\ell) = |\mathbf{I}_n(0012)|.$$

Therefore, Conjecture 1.1 is true.

References

- 1. J. S. Auli, Pattern avoidance in inversion sequences, Thesis (Ph.D.)–Dartmouth College. 2020. 160 pp.
- J. S. Auli and S. Elizalde, Consecutive patterns in inversion sequences, *Discrete Math. Theor. Comput. Sci.* 21 (2019), no. 2, Paper No. 6, 22 pp.
- J. S. Auli and S. Elizalde, Consecutive patterns in inversion sequences II: Avoiding patterns of relations, J. Integer Seq. 22 (2019), no. 7, Art. 19.7.5, 37 pp.
- 4. J. S. Auli and S. Elizalde, Wilf equivalences between vincular patterns in inversion sequences, preprint. Available at arXiv:2003.11533.
- N. R. Beaton, M. Bouvel, V. Guerrini, and S. Rinaldi, Enumerating five families of patternavoiding inversion sequences; and introducing the powered Catalan numbers, *Theoret. Comput. Sci.* 777 (2019), 69–92.

- W. Cao, E. Y. Jin, and Z. Lin, Enumeration of inversion sequences avoiding triples of relations, Discrete Appl. Math. 260 (2019), 86–97.
- S. Corteel, M. Martinez, C. D. Savage, and M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci. 18 (2016), no. 2, Paper No. 2, 21 pp.
- D. Kim and Z. Lin, Refined restricted inversion sequences, Sém. Lothar. Combin. 78B (2017), Art. 52, 12 pp.
- D. E. Knuth, The Art of Computer Programming, Vol. 1: Fundamental Algorithms, Addison-Wesley, 1973, Third edition, 1997.
- Z. Lin, Restricted inversion sequences and enhanced 3-noncrossing partitions, *European J. Combin.* 70 (2018), 202–211.
- Z. Lin, Patterns of relation triples in inversion and ascent sequences, *Theoret. Comput. Sci.* 804 (2020), 115–125.
- Z. Lin and S. Fu, On <u>12</u>0-avoiding inversion and ascent sequences, preprint. Available at arXiv:2003.11813.
- Z. Lin and S. H. F. Yan, Vincular patterns in inversion sequences, Appl. Math. Comput. 364 (2020), 124672, 17 pp.
- T. Mansour and M. Shattuck, Pattern avoidance in inversion sequences, Pure Math. Appl. (PU.M.A.) 25 (2015), no. 2, 157–176.
- M. Martinez and C. D. Savage, Patterns in inversion sequences II: inversion sequences avoiding triples of relations, J. Integer Seq. 21 (2018), no. 2, Art. 18.2.2, 44 pp.
- H. Prodinger, The kernel method: a collection of examples, Sém. Lothar. Combin. 50 (2003/04), Art. B50f, 19 pp.
- 17. N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences; http://oeis.org.
- C. Yan and Z. Lin, Inversion sequences avoiding pairs of patterns, preprint. Available at arXiv:1912.03674.
- S. H. F. Yan, Bijections for inversion sequences, ascent sequences and 3-nonnesting set partitions, Appl. Math. Comput. 325 (2018), 24–30.

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA *E-mail address*: shanechern@psu.edu; chenxiaohang92@gmail.com