On The Log-Concavity of Polygonal Figurate Number Sequences

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Abstract: This paper presents the log-concavity of the m-gonal figurate number sequences. The author gives and proves the recurrence formula for m-gonal figurate number sequences and its corresponding quotient sequences which are found to be bounded. Finally, the author also shows that for $m \geq 3$, the sequence $\{S_n(m)\}_{n\geq 1}$ of m-gonal figurate numbers is a log-concave.
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1 Introduction

Figurate numbers, as well as a majority of classes of special numbers, have long and rich history. They were introduced in Pythagorean school as an attempt to connect Geometry and Arithmetic [1]. A figurate number is a number that can be represented by regular and discrete geometric pattern of equally spaced points [2]. It may be, say, a polygonal, polyhedral or polytopic number if the arrangement form a regular polygon, a regular polyhedron or a regular polytope, respectively. In particular, polygonal numbers generalize numbers which can be arranged as a triangle(triangular numbers), or a square (square numbers), to an m-gon for any integer $m \geq 3$ [3].
Some scholars have been studied the log-concavity(or log-convexity) of different numbers sequences such as Fibonacci and Hyperfibonacci numbers, Lucas and Hyperlucas numbers, Bell numbers,Hyperpell numbers, Motzkin numbers, Fine numbers, Franel numbers of order 3 and 4, Apéry numbers, Large Schröder numbers, Central Delannoy numbers, Catalan-Larcombe-French numbers sequences, and so on. See for instance [4, 5, 6, 7, 8, 9, 10, 11, 12].
To the best of the author's knowledge, among all the aforementioned works on the log-concavity and log-convexity of numbers sequences, no one has studied the log-concavity(or log-convexity) of m-gonal figurate number sequences. Hence this paper presents the log-concavity behavior of m-gonal figurate number sequences.
The paper is structured as follows. Definitions and mathematical formulations of figurate numbers are provided in Section 2. Section 3 focuses on the log-concavity of figurate number sequences, and Section 4 is about the conclusion.

2 Definitions and Formulas of Figurate Numbers

In [1, 2, 3], some properties of figurate numbers are given. In this paper we continue discussing the properties of m-gonal figurate numbers. Now we recall some definitions involved in this paper.

Definition 2.1. *Let $\{s_n\}_{n\geq 0}$ be a sequence of positive numbers. If for all $j \geq 1$, $s_j^2 \geq s_{j-1}s_{j+1}(s_j^2 \leq s_{j-1}s_{j+1})$, the sequence $\{s_n\}_{n\geq 0}$ is called a log-concave(or a log-convex).*

Definition 2.2. *Let $\{s_n\}_{n\geq 0}$ be a sequence of positive numbers. The sequence $\{s_n\}_{n\geq 0}$ is log-concave(log-convex) if and only if its quotient sequence $\left\{\frac{s_{n+1}}{s_n}\right\}_{n\geq 0}$ is non-increasing(non-decreasing).*

Log-concavity and log-convexity are important properties of combinatorial sequences and they play a crucial role in many fields for instance economics, probability, mathematical biology, quantum physics and white noise theory [13, 4, 14, 15, 16, 17, 18, 19, 12].
Now we are going to consider the sets of points forming some geometrical figures on the plane. Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc.. Then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \cdots$, OEIS(Sloane's A000217), which are called *triangular numbers*, see [20, 21, 22, 23]. The n^{th} triangular number is given by the formula

$$S_n = \frac{n(n+1)}{2}, n \geq 1. \tag{1}$$

Following similar procedure, one can construct *square, pentagonal, hexagonal,heptagonal,octagonal, nonagonal,decagonal numbers, ..., m-gonal* numbers if the arrangement forms a regular m-gon [1]. The n^{th} term m-gonal number denoted by $S_n(m)$ is the sum of the first n elements of the arithmetic progression

$$1, 1+(m-2), 1+2(m-2), 1+3(m-2), \ldots, m \geq 3. \tag{2}$$

Lemma 2.3 ([1]). *For $m \geq 3$ and $n \geq 1$,the n^{th} term of m-gonal figurate number is given by*

$$S_n(m) = \frac{n}{2}\big[(m-2)n - m + 4\big]. \tag{3}$$

Proof. To prove (3),it suffices to find the sum of the first n elements of (2). Hence the first n elements of the arithmetic progression given in (2) is:

$$1, 1+(m-2), 1+2(m-2), 1+3(m-2), \ldots, 1+(n-1)(m-2), \forall m \geq 3.$$

Since the sum of the first n elements of an arithmetic progression $s_1, s_2, s_3, \ldots, s_n$ is equal to

$\frac{n}{2}\big[s_1 + s_n\big]$, it follows that

$$\begin{aligned} S_n(m) &= \frac{n}{2}\big[s_1 + s_n\big] \\ &= \frac{n}{2}\big[1 + (1 + (n-1)(m-2))\big] \\ &= \frac{n}{2}\big[2 + (m-2)n - m + 2\big] \\ &= \frac{n}{2}\big[(m-2)n - m + 4\big] \qquad \textit{or} \\ S_n(m) &= \left(\frac{m-2}{2}\right)\big[n^2 - n\big] + n \end{aligned}$$

This completes the proof. □

Lemma 2.4 ([1]). *For $m \geq 3$ and $n \geq 1$,the following recurrence formula for m-gonal numbers hold:*

$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n), S_1(m) = 1. \tag{4}$$

Proof. By definition, we have

$$S_n(m) = 1 + (1 + (m-2)) + (1 + 2(m-2)) + \cdots + (1 + (m-2)(n-2)) + (1 + (m-2)(n-1))$$

It follows that

$$S_{n+1}(m) = 1 + (1 + (m-2)) + (1 + 2(m-2)) + \cdots + (1 + (m-2)(n-1)) + (1 + (m-2)n)$$
$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n).$$

Thus,for $m \geq 3$ and $n \geq 1$,

$$S_{n+1}(m) = S_n(m) + (1 + (m-2)n), S_1(m) = 1.$$

□

3 Log-Concavity of m-gonal Figurate Number Sequences

In this section, we state and prove the main results of this paper.

Theorem 3.1. *For $m \geq 3$ and $n \geq 3$,the following recurrence formulas for m-gonal number sequences hold:*

$$S_n(m) = R(n)S_{n-1}(m) + T(n)S_{n-2}(m) \tag{5}$$

with the initial conditions $S_1(m) = 1, S_2(m) = m$ and the recurrence of its quotient sequence is given by

$$x_{n-1} = R(n) + \frac{T(n)}{x_{n-2}} \tag{6}$$

with the initial conditions $x_1 = m$, where

$$R(n) = \frac{m + 2(n-2)(m-2)}{1 + (n-2)(m-2)}$$

and

$$T(n) = -\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)}.$$

Proof. By Lemma 2.4, we have

$$S_{n+1}(m) = S_n(m) + (1+(m-2)n) \tag{7}$$

It follows that

$$S_{n+2}(m) = S_{n+1}(m) + (m-1+(m-2)n) \tag{8}$$

Rewriting (7) and (8) for $n \geq 3$, we have

$$S_{n-1}(m) = S_{n-2}(m) + (1+(m-2)(n-2)) \tag{9}$$
$$S_n(m) = S_{n-1}(m) + (m-1+(m-2)(n-2)) \tag{10}$$

Multiplying (9) by $m-1+(m-2)(n-2)$ and (10) by $1+(m-2)(n-2)$, and subtracting as to cancel the non homogeneous part, one can obtain the homogeneous second-order linear recurrence for $S_n(m)$:

$$S_n(m) = \left[\frac{m+2(n-2)(m-2)}{1+(n-2)(m-2)}\right] S_{n-1}(m) - \left[\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)}\right] S_{n-2}(m), \forall n, m \geq 3.$$

By denoting

$$\frac{m+2(n-2)(m-2)}{1+(n-2)(m-2)} = R(n)$$

and

$$-\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)} = T(n),$$

one can obtain

$$S_n(m) = R(n)S_{n-1}(m) + T(n)S_{n-2}(m), \forall n, m \geq 3 \tag{11}$$

with given initial conditions $S_1(m) = 1$ and $S_2(m) = m$.

By dividing (11) through by $S_{n-1}(m)$, one can also get the recurrence of its quotient sequence x_{n-1} as

$$x_{n-1} = R(n) + \frac{T(n)}{x_{n-2}}, n \geq 3 \tag{12}$$

with initial condition $x_1 = m$. □

Lemma 3.2. *For $m \geq 3$, the m-gonal figurate number sequence $\{S_n(m)\}_{n\geq 1}$, let $x_n = \dfrac{S_{n+1}(m)}{S_n(m)}$ for $n \geq 1$. Then we have $1 < x_n \leq m$ for $n \geq 1$.*

Proof. It is clear that

$$x_1 = m, x_2 = 3 - \frac{3}{m}, x_3 = 2 - \frac{2}{3(m-1)} > 1, \text{ for } m \geq 3.$$

Assume that $x_n > 1$ for all $n \geq 3$. It follows from (12) that

$$x_n = \frac{m + 2(n-1)(m-2)}{1 + (n-1)(m-2)} - \frac{m - 1 + (n-1)(m-2)}{(1 + (n-1)(m-2))x_{n-1}}, n \geq 2 \tag{13}$$

For $n \geq 3$, by (13), we have

$$\begin{aligned}
x_{n+1} - 1 &= \frac{m - 1 + n(m-2)}{1 + n(m-2)} - \frac{m - 1 + n(m-2)}{1 + n(m-2))x_n} && (14)\\
&= \frac{(m - 1 + n(m-2))x_n - (n(m-2) + m - 1))}{(1 + n(m-2))x_n} && (15)\\
&= \frac{(m - 1 + n(m-2))(x_n - 1)}{(1 + n(m-2))x_n} && (16)\\
&> 0 \text{ for } m \geq 3.
\end{aligned}$$

Hence $x_n > 1$ for $n \geq 1$ and $m \geq 3$.

Similarly, it is known that

$$x_1 = m, x_2 = 3 - \frac{3}{m}, x_3 = 2 - \frac{2}{3(m-1)} < m, \text{ for } m \geq 3. \tag{17}$$

Assume that $x_n \leq m$ for all $n \geq 3$. It follows from (12) that

$$x_n = \frac{m + 2(n-1)(m-2)}{1 + (n-1)(m-2)} - \frac{m - 1 + (n-1)(m-2)}{(1 + (n-1)(m-2))x_{n-1}}, n \geq 2 \tag{18}$$

For $n \geq 3$, by (18), we have

$$\begin{aligned}
x_{n+1} - m &= -\frac{n(m-2)^2}{1 + n(m-2)} - \frac{m - 1 + n(m-2)}{1 + n(m-2))x_n} && (19)\\
&= -\frac{n(m-2)^2 x_n + n(m-2) + m - 1)}{(1 + n(m-2))x_n} && (20)\\
&< -\frac{n(m-2)^2 + n(m-2) + m - 1)}{(1 + n(m-2))x_n} && (21)\\
&= -\frac{n(m-2)(2m-3)}{(1 + n(m-2))x_n} && (22)\\
&< 0 \text{ for } m \geq 3.
\end{aligned}$$

Hence $x_n \leq m$ for $n \geq 1$ and $m \geq 3$.

Thus, in general, from the above two cases it follows that $1 < x_n \leq m$ for $n \geq 1$ and $m \geq 3$. □

Lemma 3.3 ([15]). *Let $\{A_n\}_{n \geq 0}$ be a sequence of positive real numbers given by the recurrence*

$$A_n = R(n)A_{n-1} + T(n)A_{n-2}, n \geq 2$$

with given initial conditions A_0, A_1 and $\{x_n\}_{n \geq 1}$ its quotient sequence, given by

$$x_n = R(n) + \frac{T(n)}{x_{n-1}}, n \geq 2$$

with initial condition $x_1 = \dfrac{A_1}{A_0}$. *If there is* $n_0 \in \mathbb{N}$ *such that* $x_{n_0} \geq x_{n_0+1}$, $R(n) \geq 0, T(n) \leq 0$, *and*

$$\Delta R(n)x_{n-1} + \Delta T(n) \leq 0$$

for all $n \geq n_0$*, then the sequence* $\{A_n\}_{n \geq n_0}$ *is a log-concave.*

Theorem 3.4. *For all* $m \geq 3$*, the sequence* $\{S_n(m)\}_{n\geq 1}$ *of* m*-gonal figurate numbers is a log-concave.*

Proof. Let $\{S_n(m)\}_{n\geq 1}$ be a sequence of m-gonal figurate numbers given by the recurrence (5) and let $\{x_n\}_{n\geq 1}$ be its quotient sequence given by (6).
In order to prove the log-concavity of $\{S_n(m)\}_{n\geq 1}$ for all $m \geq 3$, by Lemma 3.3, we only need to show that $\{x_n\}_{n\geq 1}$ is non-increasing, $R(n) \geq 0$, $T(n) \leq 0$, and

$$\Delta R(n)x_{n-2} + \Delta T(n) \leq 0$$

for all $n \geq 3$.
By (11), since $R(n) \geq 0$ and $T(n) \leq 0$, for $m, n \geq 3$, assume , inductively that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{n-2} \geq x_{n-1}$.
Expressing x_n from (6) and taking in to account that $\dfrac{T(n+1)}{x_{n-1}} \leq \dfrac{T(n+1)}{x_{n-2}}$, one can obtain

$$x_n = R(n+1) + \frac{T(n+1)}{x_{n-1}} \leq R(n+1) + \frac{T(n+1)}{x_{n-2}} \tag{23}$$

Now, we need to show that $x_n \leq x_{n-1}$. To show this, consider

$$R(n+1) + \frac{T(n+1)}{x_{n-2}} \leq R(n) + \frac{T(n)}{x_{n-2}} = x_{n-1} \tag{24}$$

Hence from (23) and (24),we can conclude that the quotient sequence $\{x_n\}_{n\geq 1}$ is non-increasing.
It follows from (24) that

$$\big[R(n+1) - R(n)\big]x_{n-2} + T(n+1) - T(n) \leq 0 \tag{25}$$

By denoting $R(n+1) - R(n) = \Delta R(n)$ and $T(n+1) - T(n) = \Delta T(n)$, we get the compact expression for (25) as:

$$\Delta R(n)x_{n-2} + \Delta T(n) \leq 0, \forall n \geq 3.$$

Thus, by Lemma 3.3, the sequence $\{S_n(m)\}_{n\geq 1}$ of m-gonal figurate numbers is a log-concave for $m \geq 3$.
This completes the proof of the theorem. □

4 Conclusion

In this paper, we have discussed the log-behavior of m-gonal figurate number sequences. We have also proved that for $m \geq 3$, the sequence $\{S_n(m)\}_{n\geq 1}$ of m-gonal figurate numbers is a log-concave.

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