

# Some remarks on the power product expansion of the $q$ -exponential series

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**Abstract.** We give an overview about the power product expansion of the exponential series and derive some  $q$ -analogs.

## 1. Introduction

Each formal power series  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  with  $a_0 = 1$  has a representation as an infinite product of the form

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n x^n), \quad (1)$$

a so-called power product expansion.

More precisely there are uniquely determined coefficients  $g_j$  such that  $\prod_{k=1}^n (1 + g_k x^k) = \sum_k b_{n,k} x^k$

with  $b_{n,k} = a_k$  for  $k \leq n$ . To see this observe that

$$\prod_{k=1}^n (1 + g_k x^k) = (1 + g_n x^n) \sum_k b_{n-1,k} x^k = \sum_{k=0}^{n-1} a_k x^k + (b_{n-1,n} + g_n) x^n + \dots$$

Therefore, there is a uniquely determined  $g_n$  such that  $b_{n-1,n} + g_n = a_n$ .

Such expansions have been studied in a number of papers, cf. [4] and [5] and the references cited there.

A simple example is

$$\sum_{n \geq 0} x^n = \prod_{n \geq 0} (1 + x^{2^n}). \quad (2)$$

Here both sides are convergent for  $|x| < 1$ . Since the zeros of the right-hand side are not in the domain of convergence there is no contradiction to the fact that the left-hand side has no zeros.

In this note I consider the case of the exponential series

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!} = \prod_{n \geq 1} (1 + e_n x^n) \quad (3)$$

and its  $q$ -analogs in some detail.

My interest in this problem has been aroused by the article “A dream of a (number-) sequence” by Gottfried Helms [6].

## 2. Some background information

Taking logarithms of (3) yields

$$x = \sum_{k \geq 1} \log(1 + e_k x^k) = \sum_{k \geq 1} \sum_{d \geq 1} (-1)^{d-1} \frac{1}{d} e_k^d x^{kd} = \sum_{n \geq 1} x^n \sum_{d|n} (-1)^{d-1} \frac{e_{n/d}^d}{d}. \quad (4)$$

Therefore, we get for  $n > 1$

$$\sum_{d|n} (-1)^{d-1} \frac{e_{n/d}^d}{d} = 0 \text{ which gives}$$

$$e_n = \sum_{d|n, d>1} (-1)^d \frac{e_{n/d}^d}{d}. \quad (5)$$

The first terms are

$$(e_n)_{n \geq 1} = \left(1, \frac{1}{2}, -\frac{1}{3}, \frac{3}{8}, -\frac{1}{5}, \frac{13}{72}, -\frac{1}{7}, \frac{27}{128}, \dots\right) \quad (6)$$

O. Kolberg [7] used this formula to show that  $e_p = -\frac{1}{p}$  for a prime number  $p \geq 3$  and that

$$(-1)^n e_n > 0 \text{ for } n > 1.$$

The first assertion follows immediately from (5). For the second assertion we consider

$$\exp(-x) = \prod_{n \geq 1} (1 + a_n x^n) \text{ and show that } 0 < a_n < \frac{2}{n} \text{ for } n > 1.$$

Let me reproduce his argument. The inequality is true for  $n < 20$  by direct computation. Let now

$n \geq 20$  and suppose that  $0 < a_m < \frac{2}{m}$  holds for  $m < n$ . Then

$$\left| a_n - \frac{1}{n} \right| < \left| \sum_{d|n, 1 < d < n} (-1)^d \frac{1}{d} a_{n/d}^d \right| < \frac{1}{2} \left( \frac{4}{n} \right)^2 + \frac{1}{3} \left( \frac{6}{n} \right)^3 + \frac{1}{4} \left( \frac{8}{n} \right)^4 + \sum_{d|n, 5 \leq d \leq \frac{n}{4}} \frac{1}{d} a_{n/d}^5 + \frac{2}{n} a_2^{\frac{n}{2}} + \frac{3}{n} a_3^{\frac{n}{3}}.$$

For  $d \leq \frac{n}{4}$  we have  $k = \frac{n}{d} \geq 4$  and  $a_k < \frac{2}{k}$ . This implies  $\sum_{d|n, 5 \leq d \leq \frac{n}{4}} \frac{1}{d} a_{n/d}^5 < \sum_{k \geq 4} \frac{k}{n} \left( \frac{2}{k} \right)^5$

For  $n \geq 20$  we get  $n \left( \frac{1}{2} \left( \frac{4}{n} \right)^2 + \frac{1}{3} \left( \frac{6}{n} \right)^3 + \frac{1}{4} \left( \frac{8}{n} \right)^4 + \frac{2}{n} a_2^{\frac{n}{2}} + \frac{3}{n} a_3^{\frac{n}{3}} \right) < 0.72$ .

This implies  $\left| a_n - \frac{1}{n} \right| < \frac{1}{n} \left( 0.72 + \sum_{k \geq 4} k \left( \frac{2}{k} \right)^5 \right) < \frac{1}{n} (0.72 + 0.24) < \frac{1}{n}$ .

It also follows that the expansion

$$\frac{\exp(-x)}{1-x} = \prod_{n \geq 2} (1 + a_n x^n) \quad (7)$$

converges for  $|x| < 1$ .

L. Carlitz [2] showed that  $e_n = \frac{c_n}{n!}$  with integers  $c_n$  and derived some arithmetic properties of  $c_n$ ,

for example that  $c_n \equiv 0 \pmod{p}$  if  $n > p$  is relatively prime to the prime number  $p$ . We shall give a different proof in Corollary 4.4.

The first terms of the sequence  $(c_n)$  are

$$(c_n)_{n \geq 1} = (1, 1, -2, 9, -24, 130, -720, 8505, \dots). \quad (8)$$

### Remark

The sequence  $(0, 1, 2, 9, 24, 130, 720, 8505, \dots)$  of coefficients of (7) also occurs as the dimensions of representations by Witt vectors (cf. [1] and OEIS [8], A006973).

Since  $a_p = \frac{1}{p}$  for a prime number  $p$  we get  $c_p = -(p-1)!$ .

Moreover J. Borwein and S. Lou [1] proved that  $|c_n| \leq (n-1)!$  for odd  $n$ , for example  $|c_9| = 35840 < 8! = 40320$ , and that  $c_n \geq (n-1)!$  for even  $n$ , for example  $c_4 = 9 > 3! = 6$  or  $c_6 = 130 > 5! = 120$ .

A look at OEIS [8] suggests the sequence OEIS [8], A067911 =  $\left( u_n = \prod_{k=1}^n \gcd(k, n) = \prod_{d|n} d^{\phi(\frac{n}{d})} \right)_{n \geq 1}$  as another choice for the numerators  $1, 2, 3, 8, 5, 72, 7, 128, \dots$ .

Putting  $e_n = \frac{r_n}{u_n}$  in (5) we get

$$r_n = \sum_{d|n, d>1} (-1)^d \frac{u_n}{du_{n/d}} r_{n/d}. \quad (9)$$

Now we see by induction that all  $r_n$  are integers if we can show that  $\frac{u_n}{du_{n/d}}$  is an integer for each divisor  $d$  of  $n$ .

In order to show this we consider the  $d$  intervals  $j\frac{n}{d}+1, \dots, (j+1)\frac{n}{d}$ ,  $0 \leq j < d$ , where in the last interval the number  $n$  is replaced by  $\frac{n}{d}$ . Since for each  $i$   $\gcd(i, n/d) | \gcd(i, n)$  we see that

$$u_{n/d}^d \mid \frac{u_n}{d}.$$

The first terms are

$$(r_n)_{n \geq 1} = (1, 1, -1, 3, -1, 13, -1, 27, \dots). \quad (10)$$

Note that  $r_n$  and  $u_n$  need not be relatively prime. For example,  $\gcd(u_{12}, r_{12}) = 3$ .

Let us mention some special cases of (9): For a prime  $p > 2$  we get  $r_p = -1$  and  $r_{p^2} = 1 - p^{p-1}$  and

for a product of two different primes  $n = pq$  we get  $u_{pq} = p^q q^p$  and

$$\begin{aligned} r_{pq} &= \sum_{\substack{d|n \\ d>1}} (-1)^d \frac{u_n}{du_{n/d}^d} r_{n/d}^d = -\frac{u_{pq}}{pu_p^p} r_q - \frac{u_{pq}}{qu_q^q} r_p - \frac{u_{pq}}{pq} = u_{pq} \left( \frac{1}{pq^p} + \frac{1}{qp^q} - \frac{1}{pq} \right) \\ &= p^q q^p \left( \frac{1}{pq^p} + \frac{1}{qp^q} - \frac{1}{pq} \right) = p^{q-1} + q^{p-1} - p^{q-1} q^{p-1}. \end{aligned}$$

### 3. Connections with Pascal's triangle.

**3.1.** Let  $P_n = \left( \binom{i}{j} \right)_{i,j=0}^{n-1}$  be the  $n \times n$ -Pascal matrix and let  $H_n = (h(i, j))_{i,j=0}^{n-1}$  with  $h(i, i-1) = i$

and  $h(i, j) = 0$  else, i.e.  $h(i, j) = i[i-j=1]$  by using Iverson's convention:  $[P]=1$  if property P is

true and  $[P]=0$  else. Then  $\frac{H_n^k}{k!} = H_{n,k} = \left( \binom{i}{k} [i-j=k] \right)_{i,j=0}^{n-1}$ .

Since  $H_n^k = 0$  for  $k \geq n$  we get

$$P_n = \sum_{k=0}^{n-1} H_{n,k} = \sum_{k=0}^{n-1} \frac{1}{k!} H_n^k = \sum_{k \geq 0} \frac{1}{k!} H_n^k = \exp(H_n). \quad (11)$$

From (3) we get

$$\begin{aligned} P_n &= \left( I_n + \frac{c_1}{1!} H_n^1 \right) \left( I_n + \frac{c_2}{2!} H_n^2 \right) \cdots \left( I_n + \frac{c_{n-1}}{(n-1)!} H_n^{n-1} \right) \\ &= (I_n + c_1 H_{n,1}) (I_n + c_2 H_{n,2}) \cdots (I_n + c_{n-1} H_{n,n-1}). \end{aligned} \quad (12)$$

For example

$$P_4 = (I_4 + H_{4,1})(I_4 + H_{4,2})(I_4 - 2H_{4,3}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}.$$

From this representation we see again that the numbers  $c_n$  are integers.

If we set  $(I_n + c_1 H_{n,1})(I_n + c_2 H_{n,2}) \cdots (I_n + c_k H_{n,k}) = (g_k(i, j))_{i,j=0}^{n-1}$  then  $g_k(i, 0) = 1$  for  $i \leq k$ .

If it is already proved that  $c_i$  is an integer for  $1 \leq i \leq k-1$ , then from

$(g_k(i, j))_{i,j=0}^{n-1} = (g_{k-1}(i, j))_{i,j=0}^{n-1} (I_n + c_k H_{n,k})$  we see that  $g_k(k, 0) = g_{k-1}(k, 0) + c_k$ . Choosing  $c_k = 1 - g_{k-1}(k, 0)$  we get  $g_k(k, 0) = 1$  and all  $g_k(i, j)$  are integers.

**3.2.** For later applications let us state a slight generalization. Let  $m$  be a positive integer and consider

the matrices  $H_{n,k}^{(m)} = (h_k^{(m)}(i, j))_{i,j=0}^{n-1} = \left( \binom{\lfloor \frac{i}{m} \rfloor}{k} [i - j = mk] \right)_{i,j=0}^{n-1}$  whose entries  $h_k^{(m)}(i, j)$  satisfy

$$h_k^{(m)}(i, i - mk) = \binom{\lfloor \frac{i}{m} \rfloor}{k} \text{ and } h_k^{(m)}(i, j) = 0 \text{ else.}$$

They satisfy  $H_{n,k}^{(m)} = k H_{n,k-1}^{(m)} H_{n,1}^{(m)}$  because

$$\begin{aligned} \sum_{\ell} h_{k-1}^{(m)}(i, \ell) h_1^{(m)}(\ell, i - mk) &= h_{k-1}^{(m)}(i, i - m(k-1)) h_1^{(m)}(i - m(k-1), i - mk) \\ &= \binom{\lfloor \frac{i}{m} \rfloor}{k-1} \binom{\lfloor \frac{i}{m} \rfloor - k + 1}{1} = k \binom{\lfloor \frac{i}{m} \rfloor}{k} \end{aligned}$$

and  $\sum_{\ell} h_{k-1}^{(m)}(i, \ell) h_1^{(m)}(\ell, j) = 0$  else. Therefore we get  $H_{n,k}^{(m)} = \frac{(H_n^{(m)})^k}{k!}$  for  $H_n = H_{n,1}$

and

$$P_n^{(m)} = \sum_{k \geq 0} H_{n,k}^{(m)} = \exp(H_n^{(m)}) = \prod_{k \geq 1} (I_n + c_k H_{n,k}^{(m)}). \quad (13)$$

For  $m = 2$  we get the ‘‘doubled’’ Pascal triangle (OEIS [8], A178112).

#### 4. Power product expansion of the $q$ – exponential series

Let us now consider the  $q$  – exponential series

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]!} \quad (14)$$

and its counterpart

$$\text{Exp}_q(x) = \sum_{n \geq 0} q^{\binom{n}{2}} \frac{x^n}{[n]!}. \quad (15)$$

As usual we set  $[n] = [n]_q = 1 + q + \dots + q^{n-1}$  and  $[n]! = [1][2] \dots [n]$ .

Depending on the context  $q$  is either a complex number or an indeterminate. The needed results about  $q$  – series may for example be found in [3].

##### Theorem 4.1

The coefficients  $e_n(q)$  of the power product expansion of

$$\exp_q(x) = \prod_{n \geq 1} (1 + e_n(q)x^n) \quad (16)$$

are given by

$$e_n(q) = \sum_{d|n, d>1} (-1)^d \frac{e_{n/d}(q)^d}{d} + \frac{(1-q)^{n-1}}{n[n]}. \quad (17)$$

The coefficients  $E_n(q)$  of the expansion

$$\text{Exp}_q(x) = \exp_{q^{-1}}(x) = \frac{1}{\exp_q(-x)} = \prod_{n \geq 1} (1 + E_n(q)x^n) \quad (18)$$

are given by

$$E_n(q) = \sum_{d|n, d>1} (-1)^d \frac{E_{n/d}(q)^d}{d} + \frac{(q-1)^{n-1}}{n[n]}. \quad (19)$$

They satisfy

$$e_{2n+1}(q) = E_{2n+1}(q) = e_{2n+1}\left(\frac{1}{q}\right). \quad (20)$$

##### Proof

For  $0 \leq q < 1$  we have  $\exp_q(x) = \prod_{k=0}^{\infty} (1 - q^k(1-q)x)^{-1}$  and therefore

$$\begin{aligned}
\log \exp_q(x) &= -\sum_{k=0}^{\infty} \log(1 - q^k(1-q)x) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(q^k(1-q)x)^n}{n} \\
&= \sum_{n=1}^{\infty} \frac{((1-q)x)^n}{n} \frac{1}{1-q^n} = \sum_{n=1}^{\infty} \frac{(1-q)^{n-1}}{n[n]} x^n.
\end{aligned} \tag{21}$$

This gives (17). In the same way we get (19). Comparing both formulas gives (20).

The first terms of the sequence  $(e_n(q))$  are

$$\begin{aligned}
e_1(q) &= 1, \quad e_2(q) = \frac{1}{1+q}, \quad e_3(q) = -\frac{q}{[3]}, \quad e_4(q) = \frac{(1+q^2+q^3)}{[2][4]}, \quad e_5(q) = -\frac{q(1-q+q^2)}{[5]}, \\
e_6(q) &= \frac{q^2(1+3q+2q^2+2q^3+2q^4+2q^5+q^6)}{[2]^2[3][6]}, \quad e_7(q) = -\frac{q(1-q+q^2)^2}{[7]}, \dots
\end{aligned}$$

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E_6(q) &= \frac{q(1+2q+2q^2+2q^3+2q^4+3q^5+q^6)}{[2]^2[3][6]}, \quad E_7(q) = -\frac{q(1-q+q^2)^2}{[7]}, \dots
\end{aligned}$$

Let now  $u_n(q) = \prod_{j=1}^n \gcd([j], [n])$  be the product of the polynomial greatest divisors of the polynomials  $[j] = 1 + q + \dots + q^{j-1}$  and  $[n] = 1 + q + \dots + q^{n-1}$ . Since  $(q^d - 1) \mid (q^n - 1)$  if and only if  $d \mid n$  we see that  $\lim_{q \rightarrow 1} u_n(q) = u_n$ .

Let us write  $e_n(q) = \frac{r_n(q)}{u_n(q)}$ . Then we get as above

$$r_n(q) = \sum_{d \mid n, d > 1} (-1)^d \frac{u_n(q)}{du_{n/d}^d(q)} r_{n/d}^d(q) + \frac{(1-q)^{n-1} u_n(q)}{n[n]}. \tag{22}$$

The first terms of  $(r_n(q))$  are

$$1, 1, -q, 1+q^2+q^3, -q(1-q+q^2), q^2(1+3q+2q^2+2q^3+2q^4+2q^5+q^6), -q(1-q+q^2)^2, \dots$$

**Remark**

For  $q = 0$  the series  $\exp_q(x)$  reduces to  $1 + x + x^2 + \dots = \frac{1}{1-x}$  and therefore (16) reduces to (2).

Comparing coefficients we get  $r_{2^n}(0) = 1$  and  $r_n(0) = 0$  else.

**Theorem 4.2**

The polynomials  $(-1)^n r_n(q)$  have integer coefficients and leading coefficient 1 for  $n > 1$ .

**Proof**

Let us first show that  $(-1)^n r_n(q)$  has leading coefficient 1 for  $n > 1$ .

The highest terms of  $q$  in  $(-1)^n r_n(q)$  occur in

$$\frac{u_n(q)}{n} - \frac{(q-1)^{n-1} u_n(q)}{n[n]} = \frac{u_n(q)}{n} \left( 1 - \frac{(q-1)^{n-1}}{[n]} \right) = \frac{u_n(q)}{[n]} \frac{[n] - (q-1)^{n-1}}{n}.$$

The coefficient of the leading term of  $\frac{u_n(q)}{[n]}$  is 1 and the leading term of  $\frac{[n] - (q-1)^{n-1}}{n}$  is  $q^{n-2}$ .

Next we show that  $r_n(q)$  is a polynomial in  $q$  with integer coefficients.

We show first that  $e_n(q) = \frac{c_n(q)}{[n]!}$  where  $c_n(q)$  is a polynomial in  $q$  with integer coefficients.

$$\text{Let } P_n(q) = \left( \left[ \begin{matrix} i \\ j \end{matrix} \right]_q \right)_{i,j=0}^{n-1} \text{ and let } H_n(q) = \left( \left[ \begin{matrix} i \\ 1 \end{matrix} \right]_q [i-j=1] \right)_{i,j=0}^{n-1}.$$

$$\text{Then } H_n^k(q) = \left( [k]! \left[ \begin{matrix} i \\ k \end{matrix} \right]_q [i-j=k] \right)_{i,j=0}^{n-1}.$$

This implies the well-known  $q$ - analog of (11)

$$\exp_q(H_n(q)) = \sum_{k=0}^{\infty} \frac{H_n^k(q)}{[k]!} = \left( \left[ \begin{matrix} i \\ j \end{matrix} \right]_q \right)_{i,j=0}^{n-1} = P_n(q). \quad (23)$$

If we write  $\frac{H_n^k(q)}{[k]!} = H_{n,k}(q)$  then

$$P_n(q) = \prod_{k \geq 1} (I_n + e_k(q) H_n^k(q)) = \prod_{k \geq 1} (I_n + c_k(q) H_{n,k}(q)). \quad (24)$$



We now show that each  $c_n(q)$  is a polynomial in  $q$  with integer coefficients. Since  $c_1(q) = 1$  this is

$$\text{true for } k = 1. \text{ Let } \prod_{j=0}^k (I_n + c_j(q)H_{n,j}(q)) = (g_{k,i,j}(q))_{i,j=0}^{n-1}.$$

Assume that all  $g_{k-1,i,j}(q)$  are polynomials with integer coefficients. We know that  $g_{k,0,k} = 1$  since for  $n = k$  we get  $P_k(q)$ . Then  $1 = g_{k,0,k} = g_{k-1,0,k} + g_{k-1,k,k}c_k(q) = g_{k-1,0,k} + c_k(q)$  shows that  $c_k(q)$  has integer coefficients and therefore that all  $g_{k,i,j}(q)$  are polynomials in  $q$  with integer coefficients.

Since  $\frac{1}{[k]!}$  is a formal power series with integer coefficients we see that

$$r_k(q) = \frac{u_k(q)c_k(q)}{[k]!} \in \mathbb{Z}[q] \text{ is also a polynomial with integer coefficients.}$$

The first terms of the sequence  $(r_n(q))_{n \geq 1}$  are  $1, 1, -q, 1+q^2+q^3, -q(1-q+q^2), q^2(1+3q+2q^2+2q^3+2q^4+2q^5+q^6), -q(1-q+q^2)^2, \dots$

For a prime number  $n = p$  we get  $u_p(q) = [p]$  and

$$\begin{aligned} r_p(q) &= -\frac{u_p(q)}{p} + \frac{(1-q)^{p-1}u_p(q)}{p[p]} = -\frac{[p]}{p} + \frac{(1-q)^{p-1}}{p} = \frac{1}{p(1-q)}(-1+q^p+(1-q)^p) \\ &= \frac{\sum_{j=1}^{p-1} \binom{p}{j} (-q)^j}{p(1-q)} = \frac{\sum_{j=1}^{p-1} \binom{p}{j} (-q)^j (1-q^{p-2j})}{p(1-q)} \in \mathbb{Z}[q]. \end{aligned}$$

For the sequence  $(c_n(q))_{n \geq 1}$  we get more information.

### Theorem 4.3

Let  $n \geq m$  be positive integers and  $\zeta_m = e^{\frac{2\pi i}{m}}$ . If  $n = mk$  we get  $c_{mk}(\zeta_m) = c_k$ , if  $n$  is not a multiple of  $m$  then  $c_n(\zeta_m) = 0$ .

### Proof

Since  $\frac{[km+r]_q!}{[km]_q!} = \frac{1-q^{km+1}}{1-q} \frac{1-q^{km+2}}{1-q} \dots \frac{1-q^{km+r}}{1-q}$  for  $q \rightarrow \zeta_m$  reduces to

$$\frac{1-\zeta_m}{1-\zeta_m} \frac{1-\zeta_m^2}{1-\zeta_m} \dots \frac{1-\zeta_m^r}{1-\zeta_m} = [r]_{\zeta_m}!$$

we see that for  $q \rightarrow \zeta_m$

$$H_{n,km+r}(\zeta_m) = \lim_{q \rightarrow \zeta_m} \frac{H_n^{km+r}(q)}{[km+r]!} = \lim_{q \rightarrow \zeta_m} \frac{H_n^{km}(q)}{[km]!} \frac{[km]!}{[km+r]!} H_n^r(q) = H_{n,km}(\zeta_m) \frac{H_n^r(\zeta_m)}{[r]_{\zeta_m}!}.$$

Therefore we get

$$P_n(\zeta_m) = \sum_{j=0}^{m-1} \frac{H_n^j(\zeta_m)}{[j]_{\zeta_m}!} \sum_{k \geq 0} H_{n,km}(\zeta_m). \quad (25)$$

Since

$$\begin{bmatrix} nm+r \\ km \end{bmatrix} = \frac{[nm+r]!}{[km]![(n-k)m+r]!} = \frac{[nm]!}{[km]![(n-k)m]!} \frac{[nm+r]}{[(n-k)m+r]} \frac{[nm+r-1]}{[(n-k)m+r-1]} \cdots \frac{[nm+1]}{[(n-k)m+1]}$$

and  $\frac{[nm+i]}{[(n-k)m+i]} = \frac{1-q^{nm+i}}{1-q^{(n-k)m+i}} = 1$  for  $q = \zeta_m$  and  $0 < i < m$  we see that

$$\lim_{q \rightarrow \zeta_m} \begin{bmatrix} nm+r \\ km \end{bmatrix}_q = \lim_{q \rightarrow \zeta_m} \begin{bmatrix} nm \\ km \end{bmatrix}_q = \lim_{q \rightarrow \zeta_m} \frac{1-q^{mn}}{1-q^{km}} \frac{1-q^{mn-1}}{1-q^{km-1}} \cdots \frac{1-q^{m(n-k)+1}}{1-q^1} = \lim_{q \rightarrow \zeta_m} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} = \lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

Therefore we get  $\begin{bmatrix} i \\ km \end{bmatrix}_{q=\zeta_m} = \begin{bmatrix} \frac{i}{m} \\ k \end{bmatrix}$  and

$$H_{n,km}(\zeta_m) = \left( \begin{bmatrix} i \\ km \end{bmatrix}_{\zeta_m} \begin{bmatrix} i-j \\ km \end{bmatrix}_{i,j=0} \right)^{n-1} = \left( \begin{bmatrix} \frac{i}{m} \\ k \end{bmatrix} \begin{bmatrix} i-j \\ km \end{bmatrix}_{i,j=0} \right)^{n-1} = H_{n,k}^{(m)} = \frac{(H_n^{(m)})^k}{k!}.$$

Thus (25) gives

$$\left( \sum_{j=0}^{m-1} \frac{H_n^j(\zeta_m)}{[j]_{\zeta_m}!} \right)^{-1} P_n(\zeta_m) = \exp(H_n^{(m)}) = \prod_{k \geq 1} (I_n + c_k H_{n,km}(\zeta_m)). \quad (26)$$

On the other hand we know that

$$\prod_{j=1}^{m-1} (I_n + c_j(q) H_{n,j}(q)) = \sum_{j=0}^{m-1} \frac{H_n^j(q)}{[j]!} + \sum_{j \geq m} b_j(q) H_n^j(q) \quad (27)$$

for some polynomials  $b_j(q) \in \mathbb{Z}[q]$ .

For  $q = \zeta_m$  this reduces to

$$\sum_{j=0}^{m-1} \frac{H_n^j(\zeta_m)}{[j]_{\zeta_m}!} = \prod_{j=1}^{m-1} (I_n + c_j(\zeta_m) H_{n,j}(\zeta_m)) \quad (28)$$

because  $H_n^m(\zeta_m) = H_{n,k}^{(m)}[m]_{\zeta_m}! = 0$ .

Therefore we get from (24)

$$\left( \sum_{j=0}^{m-1} \frac{H_n^j(\zeta_m)}{[j]_{\zeta_m}!} \right)^{-1} P_n(\zeta_m) = \prod_{j \geq m} (I_n + c_j(\zeta_m) H_{n,j}(\zeta_m)). \quad (29)$$

Comparing with (26) we see that  $c_j(\zeta_m) = 0$  if  $j \geq m$  is not a multiple of  $m$  and that  $c_{jm}(\zeta_m) = c_j$ .

**Corollary 4.4** (Carlitz [2])

Let  $p$  be a prime number. If  $n > p$  is relatively prime to  $p$  then  $c_n \equiv 0 \pmod{p}$ . If  $n = pm$  then  $c_n = c_{pm} \equiv c_m \pmod{p}$ .

In another direction we prove a slight extension of the fact that  $r_{2^n}(0) = 1$  and  $r_n(0) = 0$  else.

**Theorem 4.5**

The identity (16) reduces modulo  $q^2$  to

$$1 + x + (1-q)x^2 + (1-2q)x^3 + \cdots \equiv \prod_{n \geq 1} (1 + g_n(q)x^n) \pmod{q^2} \quad (30)$$

where  $g_1(q) = 1$ ,  $g_{2^n}(q) = 1 - 2^{n-1}q$  for  $n > 0$  and  $g_{2n}(q) = 0$ ,  $g_{2n+1}(q) = -q$  else.

**Proof**

For any commutative ring  $R$  with identity the infinite product  $\prod_{n \geq 1} (1 + g_n x^n)$  with  $g_n \in R$

can be expanded into a formal power series  $1 + a_1 x + a_2 x^2 + \cdots$ .

Let us choose  $R = \mathbb{Z}[q]/(q^2)$ . Its elements can be written as  $a + bq$  with integers  $a, b$  and  $q^2 = 0$ .

Since  $[n]! = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}) \equiv (1+q)^{n-1} \equiv 1 + (n-1)q \pmod{q^2}$  we see that

$$\frac{1}{[n]!} \equiv 1 - (n-1)q \pmod{q^2}. \text{ Therefore } \exp_q(x) \equiv 1 + x + (1-q)x^2 + (1-2q)x^3 + \cdots \pmod{q^2}.$$

On the other hand we have

$$e_n(q) = \sum_{d|n, d>1} (-1)^d \frac{e_{n/d}(q)^d}{d} + \frac{(1-q)^{n-1}}{n[n]}.$$

Since

$$\frac{(1-q)^{n-1}}{n[n]} \equiv \frac{1 - (n-1)q}{n(1+q)} \equiv \frac{(1 - (n-1)q)(1-q)}{n} \equiv \frac{1-nq}{n} = \frac{1}{n} - q \pmod{q^2}$$

it suffices to show that  $g_n(q)$  satisfies

$$\sum_{d|n} (-1)^d \frac{g_{n/d}^d(q)}{d} \equiv \left( q - \frac{1}{n} \right) \pmod{q^2}.$$

For  $d = 1$  we get  $-g_n(q)$ . For  $d = n$  we get  $\frac{(-1)^n}{n}$ .

For  $1 < d < n$  we get  $g_{n/d}^d = 0$  except if  $\frac{n}{d} = 2^k$  for some  $k$ .

Thus for odd  $n$  we get  $-g_n(q) - \frac{1}{n} = q - \frac{1}{n}$ .

Let now  $n = 2^k u$ . Here we need only consider  $d = 1$ ,  $d = n$  and  $d = 2^i u$  for  $0 \leq i < k$ .

$$\text{We get } (-1)^d \frac{g_{n/d}^d(q)}{d} = (-1)^{2^i u} \frac{g_{2^{k-i}}^{2^i u}(q)}{2^i u} = (-1)^{2^i u} \frac{(1 - 2^{k-i-1} q)^{2^i u}}{2^i u} = (-1)^{2^i u} \left( \frac{1}{2^i u} - 2^{k-i-1} q \right).$$

This gives summed up

$$-\left( \frac{1}{u} - 2^{k-1} q \right) + \left( \frac{1}{2u} - 2^{k-2} q \right) + \left( \frac{1}{4u} - 2^{k-3} q \right) + \cdots + \left( \frac{1}{2^{k-1} u} - q \right) = -\frac{1}{2^{k-1} u} + q$$

Together with  $-g_n(q) + \frac{1}{n} = \frac{1}{2^k u}$  we get  $q - \frac{1}{n}$ .

Let us verify this for the first terms of  $e_n(q)$ :

$$e_1(q) = 1 = g_1(q), \quad e_2(q) = \frac{1}{1+q} \equiv 1 - q = 1 - 2^0 q = g_2(q),$$

$$e_3(q) = -\frac{q}{1+q+q^2} \equiv -q(1-q) \equiv -q = g_3(q),$$

$$e_4(q) = \frac{1+q^2+q^3}{(1+q)(1+q+q^2+q^3)} \equiv (1-q)^2 \equiv 1 - 2q = g_4(q), \dots$$

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