Avoiding abelian powers cyclically

Jarkko Peltomäki*,[1,2,3] and Markus A. Whiteland[4]

[1]The Turku Collegium for Science and Medicine TCSM, University of Turku, Turku, Finland
[2]Turku Centre for Computer Science TUCS, Turku, Finland
[3]University of Turku, Department of Mathematics and Statistics, Turku, Finland
[4]Max Plank Institute for Software Systems, Saarbrücken, Germany

Abstract

We study a new notion of cyclic avoidance of abelian powers. A finite word w avoids abelian N-powers cyclically if for each abelian N-power of period m occurring in the infinite word w^ω, we have $m \geq |w|$. Let $\mathcal{A}(k)$ be the least integer N such that for all n there exists a word of length n over a k-letter alphabet that avoids abelian N-powers cyclically. Let $\mathcal{A}_\infty(k)$ be the least integer N such that there exist arbitrarily long words over a k-letter alphabet that avoid abelian N-powers cyclically.

We prove that $5 \leq \mathcal{A}(2) \leq 8$, $3 \leq \mathcal{A}(3) \leq 4$, $2 \leq \mathcal{A}(4) \leq 3$, and $\mathcal{A}(k) = 2$ for $k \geq 5$. Moreover, we show that $\mathcal{A}_\infty(2) = 4$, $\mathcal{A}_\infty(3) = 3$, and $\mathcal{A}_\infty(4) = 2$.

Keywords: abelian equivalence, abelian power, abelian power avoidance, cyclic abelian power avoidance, circular word

1 Introduction

Ever since the seminal work of A. Thue [36], repetitions or repetition avoidance in infinite words has been a central theme in the field of combinatorics on words. Thue showed that there exists a ternary word which avoids squares, in symbols xx, that is, two identical blocks occurring adjacently in the word. Further, he showed that there exists a binary word avoiding cubes, i.e., factors of the form xxx. These results are best possible concerning integral powers in terms of the size of the underlying alphabet. Thue's results have inspired numerous papers on avoiding powers culminating in the papers by Currie and Rampersad [12] and Rao [30] proving Dejean's conjecture on repetition thresholds.

An extremely prominent topic in combinatorics on words is the abelian equivalence of words. Two words u and v are *abelian equivalent*, in symbols $u \sim v$, if each letter of the underlying alphabet occurs equally many times in both words. This concept leads to that of an abelian power: an abelian N-power of period m is a word $u_0 u_1 \cdots u_{N-1}$ such that $u_0 \sim u_1 \sim \ldots \sim u_{N-1}$ and the words u_i have common length m. Thus avoidance of abelian squares, abelian cubes, etc. can be considered. Erdős suggested in 1957 the problem of whether abelian squares are avoidable on four letters [14]. Thus the problem of searching for an alphabet of minimal cardinality over which an infinite word avoiding abelian squares exists was initiated. Evdokimov [15] gave the first upper bound of 25. Later, Pleasants [28] lowered the bound to five, and finally in 1992 Keränen [21] answered Erdős's question in the positive by constructing an appropriate infinite

*Corresponding author.
E-mail addresses: r@turambar.org (J. Peltomäki), mawhit@mpi-sws.org (M. A. Whiteland).

word over four letters. Similar questions were considered for smaller alphabets but higher-order powers: Dekking [13] showed that there exist an infinite binary word which avoids abelian fourth powers and an infinite ternary word avoiding abelian cubes. Dekking's and Keränen's results are optimal: any binary word of length 10 contains an abelian cube, and any ternary word of length 8 contains an abelian square.

There are many variations of the study of avoidance of abelian powers. One direction is to consider avoidance in partial words [4, 5]. Another is to consider abelian powers occurring in words belonging to specific word classes; see, e.g., [16, 17]. Finally, the very notion of abelian equivalence can be generalized. See [20, 31, 37] for research on k-abelian equivalence and binomial equivalence. Of course, there is research on abelian equivalence beyond avoidance. We refer the reader to the recent survey [29]. Abelian equivalence has also been studied on graphs: the study of abelian 2-power-free graph colorings has been initiated under the term anagram-free colorings in [19, 38]. A coloring of a graph is anagram-free if no sequence of colors corresponding to a path in the graph is an abelian 2-power. We remark that anagram-free colorings of cycles correspond to circular avoidance of abelian 2-powers (see below).

A notion related to this paper is the notion of *circular avoidance*. A word w avoids N-powers *circularly* if no word in the conjugacy class of w contains an N-power as a factor. This is a more restrictive type of avoidance and more difficult to study because the language of words avoiding N-powers circularly is not closed under taking factors. This results in interesting phenomena. For example, Currie shows in [9] that there exists a ternary word of length n avoiding squares circularly for $n \geq 18$, but no such word of length 17 exists. For more on this notion, see, e.g., [11] and references therein. According to our knowledge no research on the abelian analogue of circular avoidance exists.

This paper introduces a stronger form of the circular avoidance called cyclic avoidance, and we mainly study it in the abelian setting. A word w avoids abelian N-powers *cyclically* if any abelian N-power occurring in $w^\omega = ww\cdots$ has period at least the length $|w|$ of w. The difference between circular and cyclic avoidance is that, in cyclic avoidance, periods up to length $|w| - 1$ are disallowed while in circular avoidance only periods up to $\lfloor |w|/N \rfloor$ are disallowed in w^ω. Cyclic avoidance of abelian powers was introduced in the recent paper [27] by the authors of this paper. There it served as a tool to construct infinite words with prescribed growth rate of abelian exponents. Due to the different focus, the abelian cyclic avoidance was only briefly studied in [27] and only to the extent that was necessary for the main result of that paper. The purpose of this paper is to extend this preliminary research by considering the question of what is the least number of letters required to avoid abelian N-powers cyclically.

Let $\mathcal{A}(k)$ be the least integer N such that for all n there exists a word of length n over a k-letter alphabet that avoids abelian N-powers cyclically. Similarly, let $\mathcal{A}_\infty(k)$ be the least integer N such that there exist arbitrarily long words over a k-letter alphabet that avoid abelian N-powers cyclically. The main results of this paper are as follows.

Theorem 1.1. *We have* $5 \leq \mathcal{A}(2) \leq 8$, $3 \leq \mathcal{A}(3) \leq 4$, $2 \leq \mathcal{A}(4) \leq 3$, *and* $\mathcal{A}(k) = 2$ *for* $k \geq 5$.

The lower bound for $\mathcal{A}(2)$ might be a bit surprising at first sight. However, it can be checked that no binary word of length 8 avoids abelian 4-powers cyclically. The bounds for $\mathcal{A}(3)$, $\mathcal{A}(4)$, and $\mathcal{A}(5)$ are quite straightforward from the results of Dekking and Keränen mentioned previously, but the upper bound for $\mathcal{A}(2)$ requires an explicit construction.

Extending the results of [27], we prove the following theorem.

Theorem 1.2. *We have* $\mathcal{A}_\infty(2) = 4$, $\mathcal{A}_\infty(3) = 3$, *and* $\mathcal{A}_\infty(4) = 2$.

The last result of the theorem can be seen as progress in resolving a conjecture appearing in [38, p. 17], which reads: for all but finitely many n, there exists a four-letter word of length n avoiding abelian 2-powers circularly.

We also extend previous results of Aberkane and Currie [1, 9] concerning the circular avoidance of powers to our cyclic setting. The results are as follows.

Theorem 1.3. *If $n \notin \{5, 7, 9, 10, 14, 17\}$ then there exists a word of length n over a 3-letter alphabet that avoids 2-powers cyclically.*

Theorem 1.4. *For each n, there exists a word of length n over a 2-letter alphabet that avoids $5/2^+$-powers cyclically.*

The paper is structured as follows. In Section 2, we introduce notation and the main notions. We develop preliminary properties of cyclic abelian repetitions, and recall relevant results from the literature. In Section 3 we prove Theorem 1.2. The binary and ternary cases were proved already in our previous work. The case of the four letter alphabet requires some technical developments. In Section 4, we prove Theorem 1.1. The nonbinary results follow quite straightforwardly from results in the literature when combined with our observations in Section 3. The upper bound for the binary case requires an involved construction, splitting into even and odd length words, and is the main technical part of the section. In Section 5, we extend known results on circular avoidance of ordinary powers to our cyclic setting. We then conclude with future directions of research in Section 6.

2 Preliminaries

We use standard terminology and notation of combinatorics on words; see [23, 24] for standard references. Let A be an *alphabet*, that is, a finite set of *letters*, or *symbols*. A *word* over the alphabet A is a sequence of letters of A obtained by concatenation. We denote the empty word by ε. The length of a word w is denoted by $|w|$, and the symbol $|w|_a$ stands for the number of occurrences of the letter a in w. If u and v are two words, then we denote their concatenation by uv. If $w = uzv$, then z is a *factor* of w. If $u = \varepsilon$ (resp. $v = \varepsilon$), then z is a *prefix* (resp. *suffix*) of w. A word z is a *proper prefix* (resp. *proper suffix*) of w if z is a prefix (resp. suffix) of w and $z \neq \varepsilon$ and $z \neq w$. If z is a factor of w, then we say that z *occurs* in w. If $w = uv$, then by $u^{-1}w$ and wv^{-1} we respectively mean the words v and u. If $w = uu \cdots u$ where u is repeated N times, we write $w = u^N$ and say that w is an (ordinary) N-*power* of period $|u|$. A *fractional power* with exponent R, $R > 1$, is a word of the form $x^N x'$, where x' is a prefix of x and $R = N + |x'|/|x|$. The set of all words over A is denoted by A^*. A *language* is a subset of A^*. A word w is *primitive* if $w = u^n$ only when $n = 1$. If there exist words x and y such that $u = xy$ and $v = yx$, then we say that u and v are *conjugate*. If $w = a_0 a_1 \cdots a_{n-1}$, $a_i \in A$, then the *reversal* of w is the word $a_{n-1} \cdots a_1 a_0$.

An infinite word $\mathbf{w}$ is a mapping from $\mathbb{N} \to A$ (we index words from 0). We refer to infinite words in boldface symbols. We denote the infinite repetition of a finite word u by u^ω.

Let us define the *Parikh mapping* $\psi \colon A^* \to \mathbb{N}^{|A|}$ by setting $\psi(w) = (|w|_a)_{a \in A}$. We refer to the vector $\psi(w)$ as the *Parikh vector* of w.

Definition 2.1. Let $u, v \in A^*$. We say that u and v are *abelian equivalent* if $\psi(u) = \psi(v)$.

The following definition generalizes N-powers.

Definition 2.2. Let $u_0, \ldots, u_{N-1}$, $N \geq 2$, be abelian equivalent and nonempty words of common length m. Then their concatenation $u_0 \cdots u_{N-1}$ is an *abelian N-power* of period m and exponent N. If a word (finite or infinite) w does not contain as factors abelian N-powers, then we say that w *avoids abelian N-powers* or that w is *abelian N-free*.

The next definition is central to this paper.

Definition 2.3. Let w be a word. Then w avoids abelian N-powers *cyclically* if for each abelian N-power of period m occurring in the infinite word w^ω, we have $m \geq |w|$.

Example 2.4. Let $w = 1000100$. Then both w and w^2 avoid abelian 5-powers. However, the word w^3 has the abelian 5-power $100 \cdot 010 \cdot 010 \cdot 001 \cdot 001$ of period 3 as a prefix. Therefore w does not avoid abelian 5-powers cyclically. Since w^4 contains an abelian 6-power of period 4 beginning from the second letter, the word w does not avoid abelian 6-powers cyclically either. By a straightforward inspection, it can be seen that it avoids abelian 7-powers cyclically. This fact is immediate from the example following the next lemma.

Notice that there might not be an integer N such that a word w avoids abelian N-powers cyclically. This happens when, e.g., w is conjugate to an abelian power. The following result characterizes this situation.

Lemma 2.5. *A word w avoids abelian $|w|$-powers cyclically if and only if for each $k < |w|$, w^k is not conjugate to an abelian power with period less than $|w|$. Further, if w does not avoid abelian $|w|$-powers cyclically, then it does not avoid abelian N-powers cyclically for any N.*

Proof. If w is such that w^k is conjugate to an abelian power of period m with $m < |w|$, then it is immediate that w does not avoid abelian N-powers cyclically for any N. Suppose that w is such that for each $k < |w|$, the word w^k is not conjugate to an abelian power with period less that $|w|$. Consider an abelian $|w|$-power $u_0 \cdots u_{|w|-1}$ of period m occurring in w^ω. By conjugating w if necessary, we may assume that $u_0 \cdots u_{|w|-1}$ is a prefix of w^ω. Let $\ell = m|w| / \gcd(m, |w|)$ so that $u_0 \cdots u_{\ell/m-1} = w^{\ell/|w|}$. The assumption then implies that $m \geq |w|$ or $\ell/|w| \geq |w|$. The latter also implies that $m \geq |w|$, so w avoids abelian $|w|$-powers cyclically. □

The previous example shows that the exponent $|w|$ in the above characterization is tight. We apply the above characterization to a subclass of words avoiding abelian $|w|$-powers cyclically.

Example 2.6. Let w be a word over A with $\gcd(\{|w|_a : a \in A\}) = 1$. This is satisfied, e.g., when w is not a power of a letter and $|w|$ is a prime number. We claim that w avoids abelian $|w|$-powers cyclically. If w^k is conjugate to an abelian N-power $u_0 \cdots u_{N-1}$ with $N > k$, then

$$k = \gcd(\{|w^k|_a : a \in A\}) = \gcd(\{N|u_0|_a : a \in A\}) = N \gcd(\{|u_0|_a : a \in A\}) \geq N > k$$

which is impossible. Thus if w^k is conjugate to an abelian power, the period of this abelian power must be at least $|w|$. The claim follows from Lemma 2.5.

The condition $\gcd(\{|w|_a : a \in A\}) = 1$ is not necessary: the word 001122 avoids abelian 3-powers cyclically and so avoids abelian 6-powers cyclically.

The following lemma is elementary, but it simplifies the arguments in the rest of the paper drastically.

Lemma 2.7. *Assume that x^ω contains an abelian N-power of period m with $\frac{1}{2}|x| \leq m < |x|$. Then it contains an abelian N-power with period $|x| - m$.*

Proof. There is nothing to prove when $m = \frac{1}{2}|x|$, so we may assume that $m > \frac{1}{2}|x|$. Without loss of generality, we may further assume that x^ω begins with an abelian N-power $u_0 \cdots u_{N-1}$. We show, by induction on N, that if x^ω begins with an abelian N-power with period m, then the word x^{N-1} ends with an abelian N-power $s_{N-1} \cdots s_0$ of period $|x| - m$.

Consider first the base case $N = 2$. Since m satisfies $\frac{1}{2}|x| < m < |x|$, we have $|u_0| < |x| < |u_0 u_1|$. We may write $x = u_0 s_0$ and $u_1 = s_0 p$, where s_0 is the length $|x| - m$ suffix of x and p is a prefix of x. Notice that $|p| < m$, so we have $u_0 = p s_1$ for the suffix s_1 of u_0 of length $|s_0|$. We find that

$$0 = \psi(u_0) - \psi(u_1) = \psi(p s_1) - \psi(s_0 p) = \psi(s_1) - \psi(s_0).$$

Thus s_1 is abelian equivalent to s_0, and x ends with the abelian 2-power $s_1 s_0$.

Let then $N > 2$. By proceeding as in the base case, we find that x ends with the abelian 2-power $s_1 s_0$ of period $|x| - m$. Consider the conjugate $z = s_0 u_0$ of x: the word z^ω begins with the abelian power $u_1 \cdots u_{N-1}$. By the induction hypothesis, z^{N-2} ends with the abelian power $s_{N-1} \cdots s_1$ of period $|x| - m$. To conclude the proof, we notice that $x^{N-1} = u_0 z^{N-2} s_0$. The claim follows. $\square$

3 Values of $\mathcal{A}_\infty(k)$

The aim of this section is to prove Theorem 1.2. Recall that $\mathcal{A}_\infty(k)$ is the least N such that there exist arbitrarily long words over a k-letter alphabet that avoid abelian N-powers cyclically. Our constructions for the main result of the section involve building arbitrarily long words with morphisms. Next we recall the definition of a morphism and related abelian avoidance results.

A *morphism* $\sigma \colon A^* \to B^*$ is a mapping such that $\sigma(uv) = \sigma(u)\sigma(v)$ for all words $u, v \in A^*$. The morphism σ is *prolongable* on a letter a if $\sigma(a)$ has prefix a and $\lim_{n\to\infty} |\sigma^n(a)| = \infty$. Thus iterating σ on the letter a produces an infinite word that is a fixed point of σ. We denote this fixed point by $\sigma^\omega(a)$. The set $\{w \colon w \text{ is a factor of } \sigma^n(a) \text{ for some } n \geq 0 \text{ and } a \in A\}$ is called the language of the morphism σ.

Definition 3.1. A morphism $\sigma \colon A^* \to B^*$ is *abelian N-free* if $\sigma(w)$ is abelian N-free for all abelian N-free words w in A^*.

Notice that $|\sigma(a)| \geq 1$ for each letter a when σ is a prolongable abelian N-free morphism. Indeed, if $\sigma(a) = \varepsilon$ and $\sigma(b) \neq \varepsilon$, then the abelian N-free word bab^{N-1} has an abelian N-power in its image.

There are several results in the literature concerning abelian N-free morphisms. For example, Dekking gave sufficient conditions for a morphism to be abelian N-free in [13]. Later, Carpi extended the results of Dekking by giving sharper sufficient conditions for a morphism to be abelian N-free in [6]. It is worth mentioning that Carpi's general conditions, for abelian N-freeness, are necessary and sufficient once the domain alphabet has cardinality at least 6 [6, Proposition 2]. It remains open to this day, whether the conditions characterize abelian N-free morphisms for smaller domain alphabets.

Let us recall some morphisms that are abelian N-free for small values of N. The first is the morphism σ_3, found in [10], that satisfies Dekking's conditions for the exponent 4. The morphism σ_3 is different from the morphism of [13, Thm. 1]. We use this different morphism to reduce the amount of computations required to prove Theorem 4.3. See [10, Example 2] for the proof of the following lemma.

Lemma 3.2. *The morphism $\sigma_3 \colon 0 \mapsto 0001, 1 \mapsto 101$ satisfies Dekking's conditions for the exponent 4. It is thus abelian 4-free.*

By [13, Thm. 2], the morphism $\sigma_4 \colon 0 \mapsto 0012, 1 \mapsto 112, 2 \mapsto 022$ satisfies Dekking's conditions for the exponent 3 and is thus abelian 3-free. The above two morphisms are prolongable on the letter 0. It thus follows that the infinite words $\sigma_3^\omega(0)$ and $\sigma_4^\omega(0)$ avoid abelian 4-powers and 3-powers respectively.

Let $\pi \colon 0 \mapsto 1, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0$. Consider the morphism $\phi \colon \{0,1,2,3\}^* \to \{0,1,2,3\}^*$ defined by setting

$$\begin{aligned}
\phi(0) &= 0120232123203231301020103101213121021232021 \cdot \\
& 0130102032123202312102123202321323031321 20, \\
\phi(1) &= \pi(\phi(0)), \\
\phi(2) &= \pi(\phi(1)), \\
\phi(3) &= \pi(\phi(2)).
\end{aligned}$$

Keränen proved in the breakthrough paper [21] that the fixed point $\phi^\omega(0)$ of the morphism ϕ is abelian 2-free. See also his more recent paper [22] for additional morphisms with this property. Carpi simplified Keränen's proof in [6] by showing that it satisfies Carpi's conditions for the exponent 2, and it is thus abelian 2-free. It is noteworthy that the morphism does not satisfy Dekking's conditions for the exponent 2.

Let us prove a general result related to abelian N-power cyclical avoidance and abelian N-free morphisms.

Proposition 3.3. *Let $\sigma\colon A^* \to B^*$ be an abelian N-free morphism, and assume that w in A^* is a word that avoids abelian N-powers cyclically. If $N > 2$, then $\sigma(w)$ avoids abelian N-powers cyclically. If $N = 2$ and $|w| \geq 2$, then $\sigma(w)$ avoids abelian 2-powers cyclically.*

Proof. Suppose for a contradiction that $\sigma(w)$ does not avoid abelian N-powers cyclically. Assume thus that $u_0 \cdots u_{N-1}$ is an abelian N-power occurring in $\sigma(w)^\omega$ with $|u_0| < |\sigma(w)|$. By Lemma 2.7, we may assume that $|u_0| \leq \lfloor |\sigma(w)|/2 \rfloor$, so

$$|u_0 \cdots u_{N-1}| \leq N\lfloor |\sigma(w)|/2 \rfloor \leq N|\sigma(w)|/2 \leq \lceil N/2 \rceil |\sigma(w)|.$$

We conclude that $u_0 \cdots u_{N-1}$ is a factor of $\sigma(w)^{\lceil N/2 \rceil + 1}$. Let $a_0 \cdots a_{\ell-1}$, $a_i \in A$, be a factor of $w^{\lceil N/2 \rceil + 1}$ of minimal length for which $\sigma(a_0 \cdots a_{\ell-1})$ contains $u_0 \cdots u_{N-1}$. We may write $\sigma(a_0 \cdots a_{\ell-1}) = p_0 u_0 \cdots u_{N-1} s_{\ell-1}$ with $\sigma(a_0) = p_0 s_0$ and $\sigma(a_{\ell-1}) = p_{\ell-1} s_{\ell-1}$. Since σ is abelian N-free, it follows that $a_0 \cdots a_{\ell-1}$ contains an abelian N-power $v_0 \cdots v_{N-1}$. As w avoids abelian N-powers cyclically, we have $|v_0| \geq |w|$. Therefore the word v_i has a conjugate of w as a factor, so $|\sigma(v_i)| \geq |\sigma(w)|$ for all i. Thus

$$N|\sigma(w)| \leq |\sigma(v_0 \cdots v_{N-1})| \leq |\sigma(a_0 \cdots a_{\ell-1})| \leq |\sigma(w)^{\lceil N/2 \rceil + 1}| = (\lceil N/2 \rceil + 1)|\sigma(w)|.$$

This inequality holds only when $N \leq 3$ in which case equality is forced. For $N \geq 4$, this contradiction suffices for the claim. For the remainder of the proof, we operate under the assumption $N \leq 3$. Observe that the above computation shows that $|\sigma(v_0 \cdots v_{N-1})| = |\sigma(a_0 \cdots a_{\ell-1})| = N|\sigma(w)|$. It follows that $v_i = w$ for all i and $w^N = a_0 \cdots a_{\ell-1}$.

We claim that either $|p_0 u_0| \geq |\sigma(w)|$ or $|u_{N-1} s_{\ell-1}| \geq |\sigma(w)|$. Indeed, this is clear if $N = 2$ and if $N = 3$ and $|p_0 u_0|, |u_2 s_{\ell-1}| < |\sigma(w)|$, then $|u_1| > |\sigma(w)|$ contrary to our assumptions. We assume that $|p_0 u_0| \geq |\sigma(w)|$; the other case is symmetric.

Next we claim that $|p_0 u_0 u_1| \leq 2|\sigma(w)|$. If not, then $|p_0| > |\sigma(w)| \geq |\sigma(a_0)|$ because $|u_0 u_1| \leq 2\lfloor |\sigma(w)/2| \rfloor \leq |\sigma(w)|$ by our assumption. Since p_0 is a prefix of $\sigma(a_0)$, this is impossible. We may thus write $\sigma(w) = p u_1 s$ in such a way that $p_0 u_0 = \sigma(w) p$.

Observe that $\psi(u_0) = \psi(\sigma(w)) - \psi(p_0) + \psi(p)$ and $\psi(u_1) = \psi(\sigma(w)) - \psi(p) - \psi(s)$. Since $\psi(u_0) = \psi(u_1)$, we conclude that $\psi(p_0) - \psi(p) = \psi(p) + \psi(s)$. Since the Parikh vector $\psi(p) + \psi(s)$ has nonnegative entries, we see that p is a prefix of p_0 (both words are prefixes of $\sigma(w)$). We conclude that the words sp and $p^{-1}p_0$ are abelian equivalent. Thus by writing $sp_0 = sp \cdot p^{-1}p_0$, we see that sp_0 is an abelian 2-power. Suppose now that $N = 2$. This implies that $s = s_{\ell-1}$, so $s_{\ell-1}p_0$ is an abelian 2-power. Since $s_{\ell-1}p_0$ is a factor of $\sigma(a_{\ell-1}a_0)$, it must be that $a_{\ell-1} = a_0$ as σ is abelian 2-free. Therefore w^ω contains the abelian 2-power $a_{\ell-1}a_0$ of period 1. Since w avoids abelian 2-powers cyclically, we infer that $|w| = 1$. This gives the latter claim.

Suppose finally that $N = 3$. Then $s\sigma(w) = u_2 s_{\ell-1}$. We have $\psi(u_2) = \psi(\sigma(w)) + \psi(s) - \psi(s_{\ell-1})$. Since $\psi(u_0) = \psi(u_1) = \psi(u_2)$, we get $\psi(p_0) - \psi(p) = \psi(p) + \psi(s) = \psi(s_{\ell-1}) - \psi(s)$. Since $\psi(p) + \psi(s)$ has nonnegative entries, we conclude that s is a suffix of $s_{\ell-1}$. We may now write $s_{\ell-1}p_0 = s_{\ell-1}s^{-1} \cdot sp \cdot p^{-1}p_0$ and conclude that $s_{\ell-1}p_0$ is an abelian 3-power. Now $s_{\ell-1}p_0$ is a factor of $\sigma(a_{\ell-1}a_0)$, so the image of the abelian 3-free word $a_{\ell-1}a_0$ contains an abelian 3-power. This contradicts the fact that σ is abelian 3-free. This proves the former claim. $\square$

Notice that for the case $N = 2$ in the above proposition, the assumption $|w| \geq 2$ cannot be omitted. Indeed, the word 0 avoids abelian 2-powers cyclically, but the word $\phi(0)$ does not (here ϕ is Keränen's morphism). This is evident from the fact that $\phi(0)$ begins and ends with the letter 0, so $\phi(0)^2$ contains the abelian 2-power 00.

The fact that $\mathcal{A}_\infty(2) = 4$ and $\mathcal{A}_\infty(3) = 3$ was already established in [27, Thm. 8]. Our following proof simplifies and unifies the arguments due to the above proposition. The main task here is to prove that $\mathcal{A}_\infty(4) = 2$, and we do this by iterating Keränen's morphism ϕ on suitable words.

Proof of Theorem 1.2. Recall the morphisms σ_3 and σ_4 defined above. They are abelian 4-free and abelian 3-free, respectively. Now the word 0 avoids abelian N-powers cyclically for all $N \geq 2$. Thus Proposition 3.3 implies that the words in the sequences $(\sigma_3^n(0))_n$ and $(\sigma_4^n(0))_n$ avoid abelian N-powers cyclically for $N = 4$ and $N = 3$, respectively. As the morphisms are prolongable on 0, this establishes that $\mathcal{A}_\infty(2) = 4$ and $\mathcal{A}_\infty(3) = 3$.

The word 01 avoids abelian 2-powers cyclically. Thus Proposition 3.3 implies that the words in the sequence $(\phi^n(01))_n$ avoid abelian 2-powers cyclically. Therefore $\mathcal{A}_\infty(4) = 2$ as ϕ is prolongable on 0. $\square$

4 Bounds for $\mathcal{A}(k)$

Recall that $\mathcal{A}(k)$ is the least N such that for all n there exists a word of length n over a k-letter alphabet that avoids abelian N-powers cyclically. This section is devoted to proving Theorem 1.1. When $k \geq 3$, the idea is simply to add a new letter to a word avoiding abelian N-powers cyclically. For $k = 2$, this idea does not work, and we provide an explicit construction of the required words.

Lemma 4.1. *Let w be a word that avoids abelian N-powers and # a letter that does not appear in w. Then the word w# avoids abelian N-powers cyclically.*

Proof. Set $\mathbf{w} = (w\#)^\omega$, and assume for a contradiction that an abelian N-power $u_0 \cdots u_{N-1}$ such that $|u_0| < |w\#|$ occurs in $\mathbf{w}$. By Lemma 2.7, we may assume that $|u_0| \leq \frac{1}{2}|w\#|$. Thus $|u_0 u_1| \leq |w\#|$ and # can occur in $u_0 u_1$ at most once. Thus # does not occur in u_0, and so $u_0 \cdots u_{N-1}$ must be a factor of w. This contradicts the assumption that w avoids abelian N-powers. $\square$

Theorem 4.2. *We have $3 \leq \mathcal{A}(3) \leq 4$, $2 \leq \mathcal{A}(4) \leq 3$, and $\mathcal{A}(k) = 2$ for $k \geq 5$.*

Proof. It is straightforward to verify that every ternary word of length 8 contains an abelian 2-power, so $\mathcal{A}(3) \geq 3$. Obviously $\mathcal{A}(k) \geq 2$ for $k \geq 4$.

Recall the abelian 4-free morphism σ_3 from Lemma 3.2. Taking w to be a factor of $\sigma_3^\omega(0)$ of length $n - 1$, we see by an application of Lemma 4.1 that the word w# of length n over the alphabet $\{0, 1, \#\}$ avoids abelian 4-powers cyclically. In addition, the word 0 avoids abelian 4-powers cyclically, so $\mathcal{A}(3) \leq 4$.

The morphisms σ_4 and ϕ, as defined in Section 3, are abelian 3-free and abelian 2-free, respectively. Similar to the previous paragraph, we see that $\mathcal{A}(4) \leq 3$ and $\mathcal{A}(5) \leq 2$. $\square$

Our next aim is to prove the following theorem.

Theorem 4.3. *We have $5 \leq \mathcal{A}(2) \leq 8$.*

We prove Theorem 4.3 by explicitly constructing the required words for each length. Our construction is inspired by the proof of [4, Thm. 4]. Consider the morphism $\sigma\colon 0 \mapsto 0001, 1 \mapsto 101$ of Lemma 3.2 and the prefix w of its fixed point $\sigma^\omega(0)$ of length n. Let $h\colon 0 \mapsto 1, 1 \mapsto 0$ and $\overline{w}$ be the reversal of $h(w)$. Set

$$f = \overline{w}\diamond w,$$
$$g_1 = \overline{w}w, \text{ and}$$
$$g_2 = \overline{w}^{\bullet}w,$$

where $\diamond \in \{0,1\}$ and $\overline{w}^{\bullet}$ is obtained from $\overline{w}$ by changing its final letter to 0. We further define $\mathbf{F} = f^{\omega}$, $\mathbf{G}_1 = g_1^{\omega}$, and $\mathbf{G}_2 = g_2^{\omega}$. Recall that the words w and $\overline{w}$ do not contain abelian 4-powers as factors. This follows from Lemma 3.2 and the discussion following it. Furthermore, $\overline{w}^{\bullet}$ avoids abelian 5-powers.

In Subsection 4.1, we prove that f avoids abelian 8-powers cyclically for all n. Subsection 4.2 establishes that g_1 avoids abelian 8-powers cyclically if n is odd and g_2 avoids abelian 8-powers cyclically when n is even. These results establish that $\mathcal{A}(2) \leq 8$. Theorem 4.3 follows from the observation that there does not exist a binary word of length 8 avoiding abelian 4-powers cyclically. However, such a word exists in the circular sense (see the introduction): 00010011.

The approach taken in Subsection 4.1 is identical to that of Subsection 4.2. Several of the structural lemmas carry over with very minor modifications. In particular, we encourage the reader to notice that the presence of the symbol $\diamond$ does not often play any role. We shall make use of the following notion.

Definition 4.4. Let u be a binary word over the alphabet $\{0,1\}$, and define $\Delta(u) = |u|_0 - |u|_1$. If $\Delta(u) > 0$ (resp. $\Delta(u) < 0$, $\Delta(u) = 0$), then u is *light* (resp. *heavy*, *neutral*).

Let us first establish some properties of the fixed point $\sigma^{\omega}(0)$ of σ. In particular, we consider properties of short factors of $\sigma^{\omega}(0)$, which can be verified with the help of a computer.

The word w below refers to the construction of the words f, g_1, and g_2.

Lemma 4.5. *If u is a factor of w such that $|u| \geq 29$, then u is light.*

Proof. It is straightforward to check that if u is a factor of the language of σ such that $29 \leq |u| \leq 2 \times 29 = 58$, then u is light. Any factor of length at least 58 can be written as a concatenation of words of length between 29 and 58, so it follows that all factors u with $|u| \geq 29$ are light. □

Lemma 4.6. *If u is a factor of w such that $|u| < 29$, then $\Delta(u) \geq -3$.*

Proof. This is a finite check. □

Lemma 4.7. *If u is a factor of w such that $|u| \geq 64$, then $\Delta(u) \geq 6$.*

Proof. Let u be a factor of w such that $|u| \geq 6 \times 29 = 174$ and factorize $u = u_0 \cdots u_5$ in such a way that $|u_i| \geq 29$ for all i. Since $|u_i| \geq 29$, we have $\Delta(u_i) > 0$ by Lemma 4.5. Consequently, we see that $\Delta(u) = \sum_{i=0}^{5} \Delta(u_i) \geq 6$. It can be verified with the help of a computer that if $64 \leq |u| < 174$, then $\Delta(u) \geq 6$. □

4.1 Odd Length Case

The aim of this subsection is to prove the following proposition.

Proposition 4.8. *The word f avoids abelian 8-powers cyclically.*

While the letter $\diamond$ can be freely chosen to be either 0 or 1, we use the symbol as a marker in the proofs that follow. Proposition 4.8 can be verified to be true when $|w| < 5 \times 29 = 145$. Thus in what follows, we implicitly assume that $|w| \geq 145$.

Let $u_0 \cdots u_{N-1}$ be an abelian N-power occurring in $\mathbf{F}$, and consider a word u_i for some i. We classify the word u_i as follows.

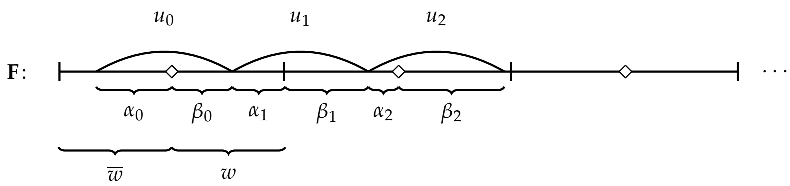

Figure 1: A depiction of the structure of $\mathbf{F}$. The words u_0 and u_2 are of type A and u_1 is of type B.

(A) $u_i = \alpha_i \diamond \beta_i$ for a suffix α_i of $\overline{w}$ and a prefix β_i of w;

(B) $u_i = \alpha_i \beta_i$ for a nonempty suffix α_i of w and a nonempty prefix β_i of $\overline{w}$.

In the proofs, we implicitly use the above factorizations using the words α_i and β_i. Notice that it is not necessary for u_i to have type A or B. See Figure 1 for clarification.

The following simple observation is very important in the subsequent proofs.

Lemma 4.9. *Suppose that u and v are words of common length such that $|u| \geq 29$. If u is a factor of w and v is a factor of $\overline{w}$, then u and v are not abelian equivalent.*

Proof. If u is a factor of w and $|u| \geq 29$, then u is light by Lemma 4.5. If v is a factor of $\overline{w}$, then $\overline{v}$ is a factor of w and must thus also be light. This means that v is heavy, so u and v cannot be abelian equivalent. □

Next we show that any abelian 8-power occurring in $\mathbf{F}$ must have a relatively large period.

Lemma 4.10. *If an abelian 8-power of period m occurs in $\mathbf{F}$, then $m > \frac{1}{2}|w|$.*

Proof. Assume for a contradiction that $\mathbf{F}$ contains an abelian 8-power $u_0 \cdots u_7$ such that $|u_0| \leq \frac{1}{2}|w|$. There exists u_i such that u_i is of type A or B because w and $\overline{w}$ avoid abelian 4-powers. We suppose that u_i is of type A; the case that it is of type B is analogous. Suppose first that $|u_0| < 29$. If $i \leq 3$, then $u_{i+1}u_{i+2}u_{i+3}u_{i+4}$ is a factor of w because $|w| \geq 5 \times 29 = 145$. This is impossible as w avoids abelian 4-powers. Thus $i \geq 4$, but then $u_{i-4}u_{i-3}u_{i-2}u_{i-1}$ is an abelian 4-power occurring in $\overline{w}$. We conclude that $|u_0| \geq 29$.

Assume that $1 \leq i \leq 6$, so that u_{i-1} and u_{i+1} exist. Since $|u_0| \leq \frac{1}{2}|w|$, the word u_{i+1} ends before the end of w and the word u_{i-1} begins after the beginning of $\overline{w}$. Therefore u_{i-1} is a factor of $\overline{w}$ and u_{i+1} is a factor of w. Lemma 4.9 shows that u_{i-1} and u_{i+1} cannot be abelian equivalent; a contradiction. Suppose then that $i = 0$. Then u_1 is a factor of w since $|u_0| \leq \frac{1}{2}|w|$. Since w avoids abelian 4-powers, the word $u_1u_2u_3u_4$ cannot be a factor of w. Thus u_2, u_3, or u_4 is of type B. Consequently, one of the words u_3, u_4, and u_5 must be a factor $\overline{w}$. This again contradicts Lemma 4.9. The case $i = 7$ is similar. □

The following two lemmas are technical lemmas that indicate what values $\Delta(u_i)$ may take for a u_i of type A or B depending on the lengths of the corresponding words α_i and β_i.

Lemma 4.11. *Suppose that the word $\mathbf{F}$ contains an abelian N-power $u_0 \cdots u_{N-1}$. Say u_i is of type B and write $u_i = \alpha_i \beta_i$.*

(i) If $|\alpha_i| \geq |\beta_i|$, then $\Delta(u_i) \geq -3$.

(ii) If $|\alpha_i| \leq |\beta_i|$, then $\Delta(u_i) \leq 3$.

Proof. Suppose that $|\alpha_i| \geq |\beta_i|$. Since β_i is a prefix of $\overline{w}$, the word $\overline{\beta_i}$ is a suffix of w. We may thus write $\alpha_i = z\overline{\beta_i}$ for some word z. Since $|\overline{\beta_i}\beta_i|_0 = |\overline{\beta_i}\beta_i|_1$, we have $\Delta(u_i) = \Delta(z)$. The word z is a factor of w, so if $\Delta(z) \leq 0$, then $|z| < 29$ by Lemma 4.5, and hence $\Delta(z) \geq -3$ by Lemma 4.6. Claim (i) follows. Claim (ii) is proved symmetrically. □

Lemma 4.12. *Suppose that the word* $\mathbf{F}$ *contains an abelian* N*-power* $u_0 \cdots u_{N-1}$*. Say* u_i *is of type A and write* $u_i = \alpha_i \diamond \beta_i$*.*

(i) If $|\alpha_i| \geq |\beta_i|$*, then* $\Delta(u_i) - \Delta(\diamond) \leq 3$*.*

(ii) If $|\alpha_i| \leq |\beta_i|$*, then* $\Delta(u_i) - \Delta(\diamond) \geq -3$*.*

Proof. This proof is similar to that of Lemma 4.11. Say $|\alpha_i| \geq |\beta_i|$. Then $\overline{\beta_i}$ is a suffix of $\overline{w}$, and we may write $\alpha_i = z\overline{\beta_i}$. Thus $\Delta(\alpha_i\beta_i) = \Delta(z)$. If $\Delta(z) \geq 0$, then $\Delta(z) \leq 3$ by Lemmas 4.5 and 4.6. It follows that $\Delta(u_i) = \Delta(\alpha_i\diamond\beta_i) = \Delta(z) + \Delta(\diamond) \leq 3 + \Delta(\diamond)$. Claim (ii) is analogous. □

We aim to combine Lemma 4.10 and the following observation. Together they imply that if an abelian 8-power $u_0 \cdots u_7$ occurs in $\mathbf{F}$, then each of the factors u_i has type A or type B.

Lemma 4.13. *Let* $u_0u_1u_2$ *be an abelian* 3*-power occurring in* $\mathbf{F}$*. If*

(i) u_0 *occurs in* w *or* $\overline{w}$ *or*

(ii) u_2 *occurs in* w *or* $\overline{w}$*,*

then $|u_0| \leq \frac{1}{2}|w|$*.*

Proof. Assume on the contrary that $|u_0| > \frac{1}{2}|w|$ and u_0 occurs in w. Now $|u_0| \geq 29$, so u_0 is light, and thus u_1 is also light. Since $|u_0| > \frac{1}{2}|w|$, the word u_1 is of type B. If $|\alpha_1| \leq |\beta_1|$, then $\Delta(u_0) = \Delta(u_1) \leq 3$ by Lemma 4.11, and this contradicts Lemma 4.7 (recall that we assume that $|w| \geq 145$, so $|u_0| > \frac{1}{2}|w| \geq 72$). Therefore $|\alpha_1| > |\beta_1|$. Since $u_0\alpha_1$ is a suffix of w, it follows that $|u_0\alpha_1| \leq |w|$. Consequently, we have $|\beta_1 u_2| \leq |w|$ which means that u_2 is a factor of $\overline{w}$. This contradicts Lemma 4.9 since u_0 is a factor of w.

The remaining cases are proved by applying the analogous Lemma 4.12. □

We next prove the main technical lemma of this part. The proof of Proposition 4.8 is almost immediate after this.

Lemma 4.14. *The word* $\mathbf{F}$ *does not contain abelian* 8*-powers of period* m *such that* $m \leq |w|$*.*

Proof. Assume for a contradiction that $\mathbf{F}$ contains an abelian 8-power $u_0 \cdots u_7$ such that $|u_0| \leq |w|$. By Lemma 4.10, we may assume that $|u_0| > \frac{1}{2}|w|$. By Lemma 4.13, the words $u_0, \ldots, u_7$ are not factors of w or $\overline{w}$. Therefore each u_i is of type A or B. In fact, the words u_0, u_2, u_4, and u_6 are of the same type, as are u_1, u_3, u_5, and u_7. Moreover, the word u_0 is of type A if and only if u_1 is of type B.

Notice that $v_0v_1v_2v_3$, with $v_i = u_{2i}u_{2i+1}$, is an abelian 4-power of period $2|u_0|$ occurring in $\mathbf{F}$. Let $M = |f| - 2|u_0|$. Since $|u_0| \leq |w| < |f|/2$, we have $M > 0$. By applying Lemma 2.7, we see that $\mathbf{F}$ contains an abelian 4-power $s_3s_2s_1s_0$ of period M. In fact, by inspecting the proof of the aforementioned lemma, the abelian 4-power $s_3 \cdots s_0$ ends where $v_0 \cdots v_3$ begins.

Assume that u_0 is of type A, the other case being symmetric. Let us write $\overline{w} = \beta_{-1}\alpha_0$ for a word β_{-1}. Since u_1 is of type B, we may write $v_0 = \alpha_0 \diamond w \beta_1$. Moreover, we have $\beta_{-1} = \beta_1 s_0'$ with $|s_0'| = M$. Since $s_3 \cdots s_0$ ends where $v_0 \cdots v_3$ begins, we see that $s_0' = s_0$. Repeating the argument for v_i, $i = 1, 2, 3$, in place of v_0, we see that $\beta_{2i-1} = \beta_{2i+1}s_i$. Hence $\beta_{-1} = \beta_7 s_3 s_2 s_1 s_0$. But now $\overline{w}$ contains the abelian 4-power $s_3 \cdots s_0$, which is absurd. □

Proof of Proposition 4.8. Suppose for a contradiction that $\mathbf{F}$ contains an abelian 8-power of period m such that $m < |f| = 2|w| + 1$. By Lemma 2.7, we may suppose that $m \leq |w|$. However, Lemma 4.14 indicates that no such abelian power exists. This is a contradiction. □

4.2 Even Length Case

In this section, we prove the following two propositions.

Proposition 4.15. *The word g_1 avoids abelian 8-powers cyclically if $|w|$ is odd.*

Proposition 4.16. *The word g_2 avoids abelian 8-powers cyclically if $|w|$ is even.*

As the reader might have observed, the letter $\diamond$ often did not play a particular role in the proofs of Subsection 4.1. This means that the previous lemmas transfer to the case of the words $\mathbf{G}_1$ and $\mathbf{G}_2$ mostly intact. Consequently, we omit repetitive details from the proofs of this section and indicate only what has changed.

Similar to Subsection 4.1, let $u_0 \cdots u_{N-1}$ be an abelian N-power occurring in $\mathbf{G}_1$ such that $|u_0| \leq |w|$, and consider a word u_i for some i. We classify the word u_i as follows.

(A) $u_i = \alpha_i \beta_i$ for a nonempty suffix α_i of $\overline{w}$ and a nonempty prefix β_i of w.

(B) $u_i = \alpha_i \beta_i$ for a nonempty suffix α_i of w and a nonempty prefix β_i of $\overline{w}$.

For an abelian N-power $u_0 \cdots u_{N-1}$ occurring in $\mathbf{G}_2$ such that $|u_0| \leq |w|$, we define the type of u_i as follows.

(A) $u_i = \alpha_i \beta_i$ for a nonempty suffix α_i of $\overline{w}^\bullet$ and a nonempty prefix β_i of w.

(B) $u_i = \alpha_i \beta_i$ for a nonempty suffix α_i of w and a nonempty prefix β_i of $\overline{w}^\bullet$.

Propositions 4.15 and 4.16 can be again verified when $|w| < 145$, so we assume that w has length at least 145 for the remainder of this section. In the following lemmas, we shall make no use of the parity of $|g_1|$ or $|g_2|$. In fact, the parity shall only play a role in the proofs of Proposition 4.15 and Proposition 4.16 at the end of this section.

Lemma 4.17. *If an abelian 8-power of period m occurs in $\mathbf{G}_1$ or $\mathbf{G}_2$, then $m > \frac{1}{2}|w|$.*

Proof. Assume for a contradiction that either of the words contains an abelian 8-power $u_0 \cdots u_7$ with period $m \leq \frac{1}{2}|w|$. We first show that u_i is of type A or type B for some i. Assume the contrary that no u_i is of type A or type B. Say the word u_0 occurs in w and that w is followed by w' where $w' \in \{\overline{w}, \overline{w}^\bullet\}$. Since w avoids abelian 4-powers, one of the words u_1, u_2, or u_3, say u_j, is a prefix of w' (since they do not have a type). Since $j \leq 3$, we see that u_{j+4} exists. There exists u_k such that u_k is a prefix of w and $w' = u_j u_{j+1} \cdots u_{k-1}$ for otherwise the abelian 5-power $u_j u_{j+1} \cdots u_{j+4}$ is a prefix of w', but neither $\overline{w}$ nor $\overline{w}^\bullet$ can have such a factor. It follows that either w' is an abelian N-power for some $N \leq 4$ or $w' = u_j$. In the former case, we have $|u_0| = |w'|/N \geq 145/4 \geq 36$, so u_0 is light by Lemma 4.5. However, the word u_j, a proper prefix of w', is heavy by Lemma 4.5. Therefore it must be that $w' = u_j$, but this contradicts the assumption that $|u_j| \leq \frac{1}{2}|w|$. The case that u_0 occurs in w' is symmetric.

To conclude the proof, we may now follow the proof of Lemma 4.10. Notice in particular that if u_i is of type A, then α_i is nonempty, and thus the change of the final letter of $\overline{w}$ does not affect u_{i-1}. $\square$

The following two lemmas are combinations of Lemmas 4.11 and 4.12 adjusted for the words $\mathbf{G}_1$ and $\mathbf{G}_2$.

Lemma 4.18. *Suppose that the word $\mathbf{G}_1$ contains an abelian N-power $u_0 \cdots u_{N-1}$. Suppose that u_i is of type A.*

(i) If $|\alpha_i| \geq |\beta_i|$, then $\Delta(u_i) \leq 3$.

(ii) If $|\alpha_i| \leq |\beta_i|$, then $\Delta(u_i) \geq -3$.

Suppose that u_i is of type B.

(i) If $|\alpha_i| \geq |\beta_i|$, then $\Delta(u_i) \geq -3$.

(ii) If $|\alpha_i| \leq |\beta_i|$, then $\Delta(u_i) \leq 3$.

Proof. Follow the proof of Lemma 4.11. □

Lemma 4.19. *Suppose that the word $\mathbf{G}_2$ contains an abelian N-power $u_0 \cdots u_{N-1}$. Suppose that u_i is of type A.*

(i) If $|\alpha_i| \geq |\beta_i|$, then $\Delta(u_i) \leq 5$.

(ii) If $|\alpha_i| \leq |\beta_i|$, then $\Delta(u_i) \geq -1$.

Suppose that u_i is of type B.

(i) If $|\alpha_i| \geq |\beta_i|$, then $\Delta(u_i) \geq -3$.

(ii) If $|\alpha_i| \leq |\beta_i|$, then $\Delta(u_i) \leq 3$.

Proof. We show how to handle the cases (i). Say u_i is of type A and $|\alpha_i| \geq |\beta_i|$. We may write $\alpha_i = z\overline{\beta_i}^{\bullet}$ for a word z (here we have $|\beta_i| > 0$ by definition). It follows that $\Delta(u_i) = \Delta(z) + 2$. Since z is a factor of $\overline{w}$, we have $\Delta(z) \leq 3$, so $\Delta(u_i) \leq 5$.

Suppose that u_i is of type B and $|\alpha_i| \geq |\beta_i|$. Since $|\alpha_i| > 0$ by its definition and $|u_i| \leq |w|$, we see that $|\beta_i| < |w|$. It follows that $\alpha_i = z\overline{\beta_i}$ for a word z. Thus $\Delta(u_i) = \Delta(z)$. Since z is a factor of w, we see that $\Delta(z) \geq -3$. □

Lemma 4.20. *Let $u_0u_1u_2$ be an abelian 3-power occurring in $\mathbf{G}_1$. If*

(i) u_0 occurs in w or $\overline{w}$ or

(ii) u_2 occurs in w or $\overline{w}$,

then $|u_0| \leq \frac{1}{2}|w|$.

Proof. Follow the proof of Lemma 4.13 and apply Lemma 4.18 appropriately. □

Lemma 4.21. *Let $u_0u_1u_2$ be an abelian 3-power occurring in $\mathbf{G}_2$. If*

(i) u_0 occurs in w or $\overline{w}^{\bullet}$ or

(ii) u_2 occurs in w or $\overline{w}^{\bullet}$,

then $|u_0| \leq \frac{1}{2}|w|$.

Proof. We show how to handle the case where u_0 occurs in $\overline{w}^{\bullet}$. Assume on the contrary that $|u_0| > \frac{1}{2}|w|$ and u_0 occurs in $\overline{w}^{\bullet}$. Suppose first that u_0 is a suffix of $\overline{w}^{\bullet}$. If $|u_0| = |w|$, then $u_1 = w$ and consequently $\overline{w}^{\bullet}$ and w are abelian equivalent. This means that $|w|_0 = |\overline{w}|_1 = |\overline{w}^{\bullet}|_1 + 1 = |w|_1 + 1$. Thus $\Delta(w) = -1$, and this contradicts Lemma 4.5. Therefore $|u_0| < |w|$ implying that u_1 is a factor of w. Thus $\Delta(u_1) \geq 6$ by Lemma 4.7. On the other hand, by taking into account the changed final letter of $\overline{w}^{\bullet}$, Lemma 4.6 implies that $\Delta(u_0) \leq -1$, so it is not possible that $\Delta(u_0) = \Delta(u_1)$.

We may thus assume that u_0 is not a suffix of $\overline{w}^{\bullet}$. Since $|u_0| > \frac{1}{2}|w|$, it follows that the word u_1 is of type A. If $|\alpha_1| \leq |\beta_1|$, then $\Delta(u_0) = \Delta(u_1) \geq -1$ by Lemma 4.19. This contradicts Lemma 4.7, so $|\alpha_1| > |\beta_1|$. Since $u_0\alpha_1$ is a suffix of $\overline{w}^{\bullet}$, it follows that $|u_0\alpha_1| \leq |w|$. Hence $|\beta_1 u_2| \leq |w|$ and u_2 is a factor of w. This contradicts Lemma 4.9. □

Lemma 4.22. *The word $\mathbf{G}_1$ does not contain abelian 8-powers of period m such that $m < |w|$.*

Proof. The proof of Lemma 4.14 works mostly as it is for the word $\mathbf{G}_1$. Indeed, Lemmas 4.17 and 4.20 guarantee that each u_i is of type A or B. Set $M = |g_1| - 2|u_0|$. Notice that we assume $|u_0| < |w| = |g_1|/2$, so $M > 0$. The remaining arguments are the same, only the $\diamond$ symbol is omitted. The conclusion is that $\overline{w}$ contains an abelian 4-power of period M ending at position $|w| - |\alpha_0|$. This is impossible. $\square$

Lemma 4.23. *The word $\mathbf{G}_2$ does not contain abelian 8-powers of period m such that $m < |w|$.*

Proof. We proceed as in the proof of Lemma 4.14. By Lemmas 4.17 and 4.21, we may suppose that each u_i is of type A or B. Set $M = |g_2| - 2|u_0|$. Again, $|u_0| < |w|$ is assumed so $M > 0$. Following the arguments of Lemma 4.14 (omitting $\diamond$), we find that $\overline{w}^\bullet$ contains an abelian 4-power ending at position $|w| - |\alpha_0|$. Observe that since the repetition is not a suffix of $\overline{w}^\bullet$ (as $|\alpha_0| > 0$), the same abelian 4-power occurs in $\overline{w}$. This is a contradiction. $\square$

Proof of Proposition 4.15. Say $|w|$ is odd, and suppose for a contradiction that $\mathbf{G}_1$ contains an abelian 8-power of period m such that $m < |g_1| = 2|w|$. By Lemma 2.7, we may suppose that $m \leq |w|$. Lemma 4.22 implies that $m = |w|$. By Lemma 4.20, it is not possible that $u_i = w$ or $u_i = \overline{w}$ for some i. Therefore all u_i are of type A or B. We handle the case that $|\alpha_0| \geq |\beta_0|$; the case $|\alpha_0| \leq |\beta_0|$ is symmetric. Write $\alpha_0 = z_0\overline{\beta_0}$ so that $\Delta(u_0) = \Delta(z_0)$. When u_0 is of type A, the word w has prefix $\beta_0\overline{z_0}$ and suffix $z_1\overline{\beta_1}$ (here $u_1 = \alpha_1\beta_1 = z_1\overline{\beta_1}\beta_1$). Since $m = |w|$, we have $|\beta_0| = |\beta_1|$. Since $m = 2|\beta_0| + |z_0| = 2|\beta_1| + |z_1|$, we conclude that $\overline{z_0} = z_1$. The same conclusion is reached if u_0 is type B. Since $\Delta(u_0) = \Delta(u_1) = \Delta(z_1)$, we have $\Delta(z_0) = \Delta(z_1) = \Delta(\overline{z_0})$, so $\Delta(z_0) = 0$. Therefore $|z_0|$ is even. Since $|w| = m = 2|\beta_0| + |z_0|$, it follows that $|w|$ is even. This is contrary to our hypothesis that $|w|$ is odd. $\square$

Proof of Proposition 4.16. Suppose that $|w|$ is even, and assume for a contradiction that $\mathbf{G}_2$ contains an abelian 8-power of period m with $m < 2|w|$. As in the proof of Proposition 4.15, we see that it must be that $m = |w|$. Moreover, the words u_i are of type A or B by Lemma 4.21. Suppose that u_0 is of type A and $|\alpha_0| \geq |\beta_0|$. The remaining cases are similar. Write $u_0 = \alpha_0^\bullet\beta_0 = z_0\overline{\beta_0}^\bullet\beta_0$ and $u_1 = \alpha_1\beta_1 = z_1\overline{\beta_1}\beta_1$ for some words z_0 and z_1 of the same length. Therefore $\Delta(u_0) = \Delta(z_0) + 2 = \Delta(u_1) = \Delta(z_1)$. Since $|\beta_0| > 0$, it is straightforward to see that $\overline{z_0} = z_1$. Thus $\Delta(z_0) + 2 = \Delta(\overline{z_0})$, that is, $|z_0|_0 - |z_0|_1 + 2 = |z_0|_1 - |z_0|_0$. It follows that $|z_0|_0 + 1 = |z_0|_1$, and so $|z_0| = |z_0|_0 + |z_0|_1 = 2|z_0|_0 + 1$. Therefore $|z_0|$ is odd, and consequently $|w| = 2|\beta_0| + |z_0|$ is odd. This is a contradiction. $\square$

Propositions 4.15 and 4.16 together with Proposition 4.8 imply Theorem 4.3.

5 Avoiding Ordinary Powers Cyclically

As mentioned in the introduction, previous research has considered the avoidance of ordinary powers in circular words. A circular word is simply a conjugacy class of words, that is, a word w avoids N-powers circularly if none of the conjugates of w contains an N-power as a factor. This constrains the periods to have length at most $\lfloor |w|/N \rfloor$ while our definition of cyclic avoidance disallows periods up to length $|w| - 1$. The purpose of this section is to generalize the known results on circular avoidance of powers to our cyclic setting.

Definition 5.1. Let w be a word. Then w avoids N-powers *cyclically* if for each N-power of period m occurring in the infinite word w^ω, we have $m \geq |w|$.

The following analogue of Lemma 2.7 is straightforward to prove.

Lemma 5.2. *Assume that x^ω contains an N-power of period m with $\frac{1}{2}|x| \leq m < |x|$. Then it contains an N-power with period $|x| - m$.*

This lemma implies that the concepts of avoiding 2-powers circularly and avoiding 2-powers cyclically are the same concept. When $N > 2$, this is not true. For example, the word 00 avoids abelian N-powers circularly for $N > 2$, but it never avoids abelian N-powers cyclically.

Currie proved in [9] that if $n \notin \{5, 7, 9, 10, 14, 17\}$ then there exists a word of length n over a 3-letter alphabet that avoids 2-powers circularly. By the preceding paragraph, we have the following result (notice that 2-powers cannot be avoided with just two letters).

Theorem 5.3. *If $n \notin \{5, 7, 9, 10, 14, 17\}$ then there exists a word of length n over a 3-letter alphabet that avoids 2-powers cyclically.*

Notice that for $n \in \{5, 7, 9, 10, 14, 17\}$ there exists a word of length n over a 4-letter alphabet avoiding 2-powers cyclically. Such words are, e.g., 01023, 0102013, 010201203, 0102010313, 01020103010213, and 01020103010212313. Notice in addition that for each n there exists a word of length n over a 3-letter alphabet that avoids 2^+-powers cyclically (see below for the definition). To see this, it is sufficient to observe that the words 00102, 0010012, 001001102, 0010011202, 00100112001002, and 00100112001001202 avoid 2^+-powers cyclically.

What is left is to determine the least exponent N such that for all n there exists a binary word w of length n such that w avoids N-powers cyclically. In the context of ordinary powers, it is natural to consider fractional exponents, and thus we give the following definition. We do not consider fractional abelian exponents in this paper; for discussion on this concept, see [7, 32].

Definition 5.4. Let w be a word and N be a rational number such that $N > 1$. Then w avoids N^+-powers *cyclically* if for each N^+-power of period m occurring in the infinite word w^ω, we have $m \geq |w|$. A word u is an N^+-power if u is an R-power for some $R > N$.

Let $\mathbf{t}$ be the fixed point $\sigma^\omega(0)$ of the morphism $\sigma\colon 0 \mapsto 01, 1 \mapsto 10$. The word $\mathbf{t}$ is the famous Thue-Morse word; see [3, Sect. 1.6]. Aberkane and Currie proved in [2] that the Thue-Morse word $\mathbf{t}$ contains a factor avoiding $5/2^+$-powers circularly for all lengths. We generalize this result to our cyclic setting. This result implies Theorem 1.4.

Theorem 5.5. *For each n, there exists a factor of length n of the Thue-Morse word avoiding $5/2^+$-powers cyclically.*

It can be shown that the exponent $5/2$ is optimal for binary words by inspecting all binary words of length 5.

In order to prove Theorem 5.5, we employ the automatic theorem-proving software Walnut [25]. Properties of automatic sequences [3] that are expressible in a certain first-order logic are decidable, and Walnut implements the decision procedure. The Thue-Morse word $\mathbf{t}$ is a 2-automatic word, so Walnut is applicable. We wish to keep the discussion on the decision procedure and usage of Walnut brief, so we merely describe the logical formulas necessary to encode our problem and refer the reader to [8] for a proof of Theorem 5.3 using Walnut. See also [34].

Let w be a factor of the Thue-Morse word. If w^ω contains an N-power of period m such that $m < |w|$ and $N > 3$, then w^ω contains a 3-power of period m. Therefore in order to show that w avoids $5/2^+$-powers cyclically, we only need to consider N-powers with $5/2 < N \leq 3$. Notice that such a power u is necessarily a factor of w^4. We first write a predicate $\mathrm{cRepK}(i, j, m, n, p)$, $K = 1, \ldots, 4$, that evaluates to true if and only if the factor w of length n beginning at the position i of the Thue-Morse word $\mathbf{t}$ is such that w^K has a factor u of length m beginning at position j, $i \leq j < i + n$, such that u has period p and u is not a factor of w^{K-1}. The predicate needs to be written somewhat awkwardly as w^K is not necessarily a factor of $\mathbf{t}$. For example, we have

$$\begin{aligned}\mathrm{cRep2}(i, j, m, n, p) = {} & (i \leq j < i + n) \wedge (i + n \leq j + m \leq i + 2n) \wedge \\ & (\forall k (j \leq k < i + n - p) \Longrightarrow \mathbf{t}[k] = \mathbf{t}[k + p]) \wedge \\ & (\forall k (i + n - p \leq k < i + n) \Longrightarrow \mathbf{t}[k] = \mathbf{t}[k + p - n]) \wedge \\ & (\forall k (i + n \leq k < j + m - p) \Longrightarrow \mathbf{t}[k - n] = \mathbf{t}[k + p - n]).\end{aligned}$$

We can then write a predicate $\mathrm{ncyc}(i, n)$ that evaluates to true if and only if the factor w of $\mathbf{t}$ of length n starting at position i is such that w^4 contains a factor that has period p with $5/2 < |u|/p \leq 3$. Its definition is:

$$\begin{aligned}\mathrm{ncyc}(i, n) = \exists j, m, p((0 < p < n) \wedge (5p < 2m \leq 6p) \wedge \\ (cRep1(i, j, m, n, p) \vee cRep2(i, j, m, n, p) \vee cRep3(i, j, m, n, p) \vee \\ cRep4(i, j, m, n, p))).\end{aligned}$$

Finally the following predicate evaluates to true if and only if Theorem 5.5 is true:

$$\forall n((n > 0) \implies (\exists i \neg \, \mathrm{ncyc}(i, n))).$$

Inputting the above predicates to Walnut produces an automaton accepting all inputs meaning that Theorem 5.5 is true.

6 Discussion on Future Research

Obviously the main question is what is the value of $\mathcal{A}(k)$ for $k = 2, 3, 4$. Theorem 1.2 seems to support the claims that $\mathcal{A}(2) = 4$, $\mathcal{A}(3) = 3$, and $\mathcal{A}(4) = 2$, but the first claim is false as there is no binary word of length 8 avoiding abelian 4-powers cyclically. This leads us to ask the following questions.

Question. *Is it the case that $\mathcal{A}(2) = 5$, $\mathcal{A}(3) = 3$, and $\mathcal{A}(4) = 2$?*

Question. *If $n \neq 8$, does there exist a word of length n over a 2-letter alphabet avoiding abelian 4-powers cyclically?*

Our computer experiments have not found a counterexample to the above questions among lengths less than 150. Notice that our question whether $\mathcal{A}(4) = 2$ is stronger than the conjecture of [38] mentioned in the preliminaries after Theorem 1.2. A positive answer to the latter question would imply that $\mathcal{A}(2) = 5$ as the word 00001011 of length 8 avoids abelian 5-powers cyclically.

We do not know how to approach these questions. The lowest hanging fruit is to improve the construction of Section 4 and lower the upper bound on $\mathcal{A}(2)$. We remark that the particular construction given here cannot be used to improve the upper bound 8 in Theorem 4.3 as some of the words constructed contain abelian 7-powers with short period. If two words that avoid abelian 4-powers are concatenated, then a priori abelian 7-powers could appear. An improved construction would need to take special care to concatenate the words in such a way that their respective abelian 3-powers of common period do not appear too close to each other. It seems that no precise information on the structure and location of abelian 3-powers in words that avoid abelian 4-powers is found in the literature. Even the sets of possible periods of abelian powers occurring in infinite words have been studied very little. The only papers in this direction are the papers [16, 26] concerning the abelian period sets of Sturmian words. This knowledge however is not helpful in this context as Sturmian words contain abelian powers of arbitrarily high exponent [16, Proposition 4.10]. It seems that making such concatenation arguments work for the alphabet sizes 3 and 4 is even more difficult especially because there is less room for improvement.

An alternative way to improve our results would be to find infinite words whose language contains the sought words. For example, Justin's morphism $0 \mapsto 00001$, $1 \mapsto 01111$ seems promising [18]. It has a factor of length n avoiding abelian 5-powers cyclically for $n = 1, \ldots, 400$. We do not know how to prove that such a factor exists for each length. Since the the fixed point of Justin's morphism is automatic, it might be possible to attack this problem via automatic theorem-proving as in Section 5. The problem in this plan is that this type of automatic theorem-proving requires the problem to be written in a certain restricted first order logic and generally abelian properties of words cannot be expressed in this logic [33, Sect. 5.2].

We have dealt in this paper only with the question of existence. A significantly harder problem would be to provide a lower bound, for example, for the number of binary words of length n that avoid abelian 4-powers cyclically. We have recorded this sequence as the sequence A334831 in Sloane's *On-Line Encyclopedia of Integer Sequences* [35]. The first values of the sequence are 2, 2, 6, 8, 10, 6, 28, 0, 36, 120, 132, 168, 364, 112.

Acknowledgements

We thank the reviewer for remarks that improved the quality of the paper. We also thank him/her for pointing out the conjecture in [38].

References

[1] A. Aberkane and J. D. Currie. There exist binary circular $5/2^+$ power free words of every length. *The Electronic Journal of Combinatorics* 11.1 (2004).
DOI: 10.37236/1763.

[2] A. Aberkane and J. D. Currie. The Thue-Morse word contains circular $5/2^+$ power free words of every length. *Theoretical Computer Science* 332.1–3 (2005), 573–581.
DOI: 10.1016/j.tcs.2004.12.024.

[3] J.-P. Allouche and J. Shallit. *Automatic Sequences. Theory, Applications, Generalizations*. Cambridge University Press, 2003.

[4] F. Blanchet-Sadri, S. Simmons, and D. Xu. Abelian repetitions in partial words. *Advances in Applied Mathematics* 48 (2012), 194–214.
DOI: 10.1016/j.aam.2011.06.006.

[5] F. Blanchet-Sadri et al. Avoiding abelian squares in partial words. *Journal of Combinatorial Theory, Series A* 119.1 (2012), 257–270.
DOI: 10.1016/j.jcta.2011.08.008.

[6] A. Carpi. On abelian power-free morphisms. *International Journal of Algebra and Computation* 3.2 (1993), 151–167.
DOI: 10.1142/S0218196793000123.

[7] J. Cassaigne and J. D. Currie. Words strongly avoiding fractional powers. *European Journal of Combinatorics* 20.8 (1999), 725–737.
DOI: 10.1006/eujc.1999.0329.

[8] T. Clokie, D. Gabric, and J. Shallit. Circularly squarefree words and unbordered conjugates: A new approach. *Combinatorics on Words. Proceedings of the 12th International Conference, WORDS 2019*. Lecture Notes in Computer Science 11682. Springer, 2019, pp. 133–144.
DOI: 10.1007/978-3-030-28796-2.

[9] J. D. Currie. There are ternary circular square-free words of length n for $n \geq 18$. *The Electronic Journal of Combinatorics* 9 (2002).
DOI: 10.37236/1671.

[10] J. D. Currie. The number of binary words avoiding abelian fourth powers grows exponentially. *Theoretical Computer Science* 319.1–3 (2004), 441–446.
DOI: 10.1016/j.tcs.2004.02.005.

[11] J. D. Currie, L. Mol, and N. Rampersad. Circular repetition thresholds on some small alphabets: Last cases of Gorbunova's conjecture. *The Electronic Journal of Combinatorics* 26.2 (2019).
DOI: 10.37236/7985.

[12] J. D. Currie and N. Rampersad. A proof of Dejean's conjecture. *Mathematics of Computation* 80.274 (2011), 1063–1070.
DOI: 10.1090/S0025-5718-2010-02407-X.

[13] F. M. Dekking. Strongly non-repetitive sequences and progression-free sets. *Journal of Combinatorial Theory, Series A* 27.2 (1979), 181–185.
DOI: 10.1016/0097-3165(79)90044-X.

[14] P. Erdős. Some unsolved problems. *The Michigan Mathematical Journal* 4.3 (1957), 291–300.

[15] A. A. Evdokimov. Strongly asymmetric sequences generated by a finite number of symbols. *Doklady Akademii Nauk SSSR* 179 (1968), 1268–1271.

[16] G. Fici et al. Abelian powers and repetitions in Sturmian words. *Theoretical Computer Science* 635 (2016), 16–34.
DOI: 10.1016/j.tcs.2016.04.039.

[17] Š. Holub. Abelian powers in paper-folding words. *Journal of Combinatorial Theory Series A* 120.4 (2013), 872–881.
DOI: 10.1016/j.jcta.2013.01.012.

[18] J. Justin. Characterization of the repetitive commutative semigroups. *Journal of Algebra* 21.1 (1973), 87–90.
DOI: 10.1016/0021-8693(72)90036-1.

[19] N. Kamčev, T. Łuczak, and B. Sudakov. Anagram-free colourings of graphs. *Combinatorics, Probability and Computing* 27 (2018), 623–642.
DOI: 10.1017/S096354831700027X.

[20] J. Karhumäki, A. Saarela, and L. Q. Zamboni. On a generalization of Abelian equivalence and complexity of infinite words. *Journal of Combinatorial Theory, Series A* 120 (2013), 2189–2206.
DOI: 10.1016/j.jcta.2013.08.008.

[21] V. Keränen. Abelian squares are avoidable on 4 letters. *Automata, Languages and Programming. 19th International Colloquium*. Lecture Notes in Computer Science 623. Springer, 1992, pp. 41–52.
DOI: 10.1007/3-540-55719-9.

[22] V. Keränen. A powerful abelian square-free substitution over 4 letters. *Theoretical Computer Science* 410.38 (2009), 3893–3900.
DOI: 10.1016/j.tcs.2009.05.027.

[23] M. Lothaire. *Combinatorics on Words*. Encyclopedia of Mathematics and Its Applications 17. Addison-Wesley, 1983.

[24] M. Lothaire. *Algebraic Combinatorics on Words*. Encyclopedia of Mathematics and Its Applications 90. Cambridge University Press, 2002.

[25] H. Mousavi. *Walnut Prover*. 2016.
URL: https://github.com/hamoonmousavi/Walnut.

[26] J. Peltomäki. Abelian periods of factors of Sturmian words. *Journal of Number Theory* 214 (2020), 251–285.
DOI: 10.1016/j.jnt.2020.04.007.

[27] J. Peltomäki and M. A. Whiteland. All growth rates of abelian exponents are attained by infinite binary words (2020). Preprint (to appear in Proceedings of MFCS 2020).

[28] P. A. B. Pleasants. Non-repetitive sequences. *Mathematical Proceedings of the Cambridge Philosophical Society* 68.2 (1970), 267–274.
DOI: 10.1017/S0305004100046077.

[29] S. Puzynina. Abelian properties of words. *Combinatorics on Words. Proceedings of the 12th International Conference, WORDS 2019*. Lecture Notes in Computer Science 11682. Springer, 2019, pp. 28–45.
DOI: 10.1007/978-3-030-28796-2.

[30] M. Rao. Last cases of Dejean's conjecture. *Theoretical Computer Science* 412.27 (2011), 3010–3018.
DOI: 10.1016/j.tcs.2010.06.020.

[31] M. Rigo and P. Salimov. Another generalization of abelian equivalence: Binomial complexity of infinite words. *Theoretical Computer Science* 601 (2015), 47–57.
DOI: 10.1016/j.tcs.2015.07.025.

[32] A. V. Samsonov and A. M. Shur. On abelian repetition threshold. *RAIRO - Theoretical Informatics and Applications* 46.1 (2012), 147–163.
DOI: 10.1051/ita/2011127.

[33] L. Schaeffer. *Deciding Properties of Automatic Sequences*. Master's thesis. University of Waterloo, 2013.
URL: http://hdl.handle.net/10012/7899.

[34] J. Shallit and R. Zarifi. Circular critical exponents for Thue-Morse factors. *RAIRO - Theoretical Informatics and Applications* 53.1–2 (2019), 37–49.
DOI: 10.1051/ita/2018008.

[35] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*.
URL: http://oeis.org.

[36] A. Thue. Über unendliche Zeichenreihen. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse* 7 (1906), 1–22.

[37] M. A. Whiteland. *On the k-Abelian Equivalence Relation of Finite Words*. Ph.D. dissertation. Turku, Finland: Turku Centre for Computer Science, University of Turku, 2019.
URL: http://urn.fi/URN:ISBN:978-952-12-3837-6.

[38] T. E. Wilson and D. R. Wood. Anagram-free graph colouring. *The Electronic Journal of Combinatorics* 25.2 (2018).
DOI: 10.37236/6267.