BETWEEN BROADWAY AND THE HUDSON: A BIJECTION OF CORRIDOR PATHS

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Canal street, running across Broadway to the Hudson, near the centre of the city,
is a spacious street, principally occupied by retail stores. . . .
The streets are generally well paved, with good side walks,
lighted at night with lamps, and some of them supplied with gas lights.
—*The Treasury of Knowledge, and Library of Reference* (1834)

Abstract. We present a substantial generalization of the equinumeracy of grand Dyck paths and Dyck-path prefixes, constrained within a band. The number of constrained paths starting at level i and ending in a window of size $2j+2$ is equal to the number starting at level j and ending in a window of size $2i+2$ centered around the same point. A new encoding of lattice paths provides a bijective proof.

1. Introduction

We are interested in enumerating lattice paths that remain within a band of height h, sometimes called *corridor paths* [1]. Sort of like walking in Manhattan, sticking west of Broadway (Figure 1).

Let $i \overset{n}{\underset{h}{\rightsquigarrow}} \ell$, or just $i \overset{n}{\rightsquigarrow} \ell$ (fixing h), denote the number of monotonic lattice paths from $\langle 0, i\rangle$ to $\langle n, \ell\rangle$ with n steps that stay within (but may touch) the boundaries $y = 0$ and $y = h$, for some given (maximum) height h.[1] Let $H = [0 : h]$ be the ordinate bounds within which steps are permissible. Steps are diagonal, NE (northeast, $\nearrow$), taking $\langle x, y\rangle \mapsto \langle x+1, y+1\rangle$, and SE (southeast, $\searrow$), taking $\langle x, y\rangle \mapsto \langle x+1, y-1\rangle$, both with the proviso that the new ordinate position $y \pm 1 \in H$, as the case may be. It is easy to see that one always has $n + i \equiv \ell \pmod 2$, or else there are zero n-step paths starting at level i and ending at ℓ. See Figure 2 for a sample path in $1 \overset{12}{\underset{4}{\rightsquigarrow}} 3$ (using the same notation for the set of paths as for its cardinality).

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[1] *Height* here is the maximum length of a unidirectional path (just NE or just SE). Some might prefer to say that the *width* of the corridor is $h+1$, since $h+1$ ordinate values are allowed.

FIGURE 1. Manhattan neighborhoods, with East-West streets and North-South avenues, bounded by Broadway on the East and the Hudson River on the West, with Union Square serving as origin. (Image © Hagstrom Map Company, Inc., in the public domain at https://www.maps-of-the-usa.com/usa/new-york/new-york/large-detailed-road-map-of-south-manhattan-nyc.)

The basic recurrence is

$$i \overset{n}{\rightsquigarrow} \ell = \begin{cases} 0 & \text{if } i \notin H \text{ or } \ell \notin H \\ [i = \ell] & \text{if } n = 0 \\ (i \overset{n-1}{\rightsquigarrow} \ell - 1) + (i \overset{n-1}{\rightsquigarrow} \ell + 1) & \text{otherwise} \end{cases}$$

where the bracketed condition $[i = \ell]$ is Iverson's notation for a characteristic function (1 when true; 0 when false), and the conditions are taken in order.

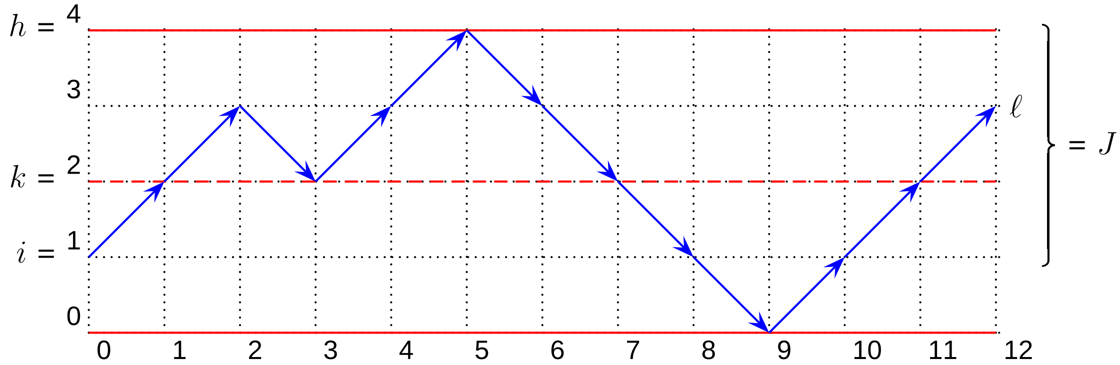

Figure 2. A diagonal path (counted by) $1 \overset{12}{\rightsquigarrow} 3$, which goes from $i = 1$ to $\ell = 3$ in a dozen steps, and consisting of 7 NE-steps and 5 SE-steps, with bound $h = 4$. The target region J is $[\![2 \pm 1]\!] = [1:4]$. Its center is $k = 2$ (the dashed red line), and $j = 1$ determines its width (1 feasible endpoint at or below k and 1 above).

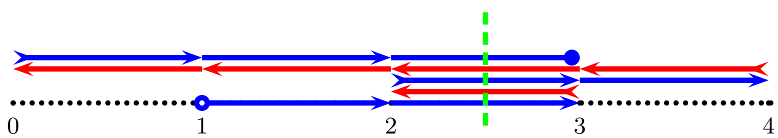

Figure 3. A right-left version of the constrained path $1 \underset{4}{\overset{12}{\rightsquigarrow}} 3$, consisting of 7 right (+) steps (colored blue) and 5 left (−) steps (red) along a 5-vertex point graph P_5 (labeled 0,1,2,3,4), starting at vertex $i = 1$. The path in this representation is ++−++−−−−+++, based at 1. It is an accordion fold of the blue path in Figure 2. The green vertical line serves as a "center of attraction" in Section 5 and Figure 5.

The ends of the paths we are interested in fall within a range, J, not just a single point ℓ. For example, the window $J = [5:10]$ has 6 possible landing spots, but only half of them are feasible, depending on whether $n + i$ is odd or even. Only those $\ell \in J$ with the same parity as $n + i$ are relevant. Our goal is to count

$$i \overset{n}{\rightsquigarrow} J = \sum_{\ell \in J} i \overset{n}{\rightsquigarrow} \ell = \sum_{\substack{\ell \in J \\ \ell \equiv n+i \pmod 2}} i \overset{n}{\rightsquigarrow} \ell$$

the number of paths constrained to any corridor $H = [0:h]$ and ending at any (feasible) ordinate in the window J.

These constrained lattice paths are equivalent to walks along a path graph, forward and backward. See Figure 3. When $i = 0$ (at the bottom) and $J = H$ (anywhere), walks for $h = 0, 1, 2, 3, 4, 5, 6, 7, 8$ are enumerated at A000007 (constant 0), A000012 (constant 1), A016116 ($2^{\lfloor n/2 \rfloor}$), A000045 (Fibonacci), A038754 ($\{1, 2\}3^n$), A028495, A030436, A061551,

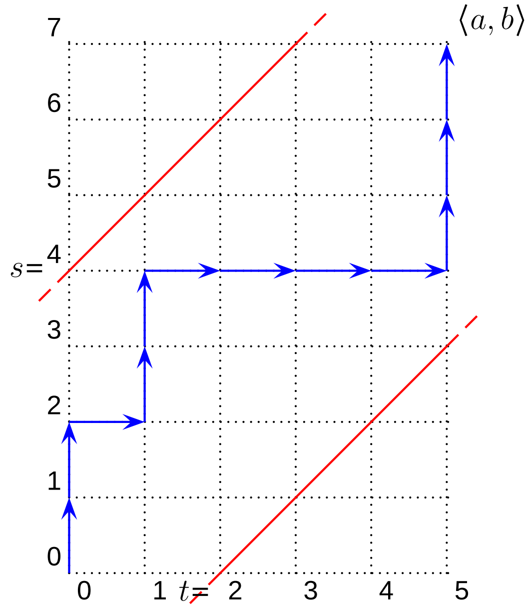

FIGURE 4. An orthogonal path, starting at the origin and ending at $\langle a, b\rangle = \langle 5, 7\rangle$, consisting of 7 N-steps and 5 E-steps, staying strictly within bounds $s = 4$ (below $y = x + 4$) and $t = 2$ (above $y = x - 2$). This path and its constraints are analogues of those in Figure 2; see Section 4.

A178381, respectively,[2] in Sloane's *Encyclopedia of Integer Sequences (OEIS)* [16]. The following sequence for $h = 9$, $n = 1, 2, 3, \ldots$, does not appear (yet!): 1, 2, 3, 6, 10, 20, 35, 70, 126, 251, 460, 911, 1690, 3327, 6225, 12190, 22950, 44744, 84626, 164407, ... But its odd elements are enumerated at A216710, while its even ones are A224514.

Such paths in a path graph having h edges can also be viewed as prefixes of Dyck paths of bounded height h, since they start at the bottom but may end anywhere above or on the bottom line. Their number is known to be equal to that of grand Dyck paths, of the same length, which start in the middle of the band, may go above or below that line – as long as they stay within bounds, and which we allow to end up either in the middle or just above [5].[3] So each of the above sequences also counts constrained grand Dyck paths.

More generally, walks can start anywhere in H ($0 \le i \le h$), with the position along the route always staying within the range $[0 : h]$. Table 1 lists values for the number of paths

[2]Compiled already by Jonathon Bryant [1].

[3]Usually, grand Dyck paths are defined to be of even length and to end up back on the starting line. To be more inclusive, we allow odd-length grand Dyck paths that terminate one line above – as in [6], for instance – adopting the same moniker in the odd case, too. Accordingly, we can say that the number of grand Dyck paths for $2n$ and even h (with the up-down symmetry of the corridor) is always twice that for $2n - 1$. See the middle case of Table 1. The term "grand Dyck" is used in [15], for example; these lattice paths are also referred to as "two-sided" or "bilateral" paths (e.g. [11]) on account of their shape, as "binomial" or "central binomial" paths (e.g. [13]) on account of their number, and as *free* Dyck paths (e.g. [3]); they are classified as "bridges" in [2].

through a corridor of height $h = 4$, with one subtable for each starting point ($i = 0, 1, 2, 3, 4$); Table 2 exhibits $h = 5$. These may be viewed as constrained versions of Pascal's triangle, with each entry the sum of two prior entries. (Cf. [1].)

In addition to pointing to the formula enumerating these more general sets of corridor paths ending in an arbitrary window, we explore a beautiful symmetry between such sets of paths, those starting at level i and ending in a window of size $2j + 2$ and those starting at level j and ending in a window of size $2i + 2$ centered around the same point. Three proofs are then provided: by induction, by counting, and by bijection. The final section describes prior work leading up to these results.

2. Main results

We use the notation $[\![k \pm j]\!]$ as shorthand for a range $[k - j : k + j + 1]$, which we make of even size, viz. $2j + 2$, by stretching the upper end one spot, to include $k + j + 1$. Thus, the window $[\![k \pm j]\!]$ covers $j + 1$ feasible endpoints – the odd ones or the even ones, as the case may be – centered about k.

Our main result is the following intriguing equivalence:

Theorem 1. *For all $n, h \in \mathbb{N}$, $k \in [0 : h]$, $i, j \in [0 : \min\{k + 1, h - k\}]$:*

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] \;=\; j \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm i]\!] \tag{1}$$

For example, $2 \overset{9}{\underset{4}{\rightsquigarrow}} [\![2 \pm 1]\!] = 162 = 1 \overset{9}{\underset{4}{\rightsquigarrow}} [\![2 \pm 2]\!]$; see Table 1. The bounds on i and j ensure that the starting points are in $H = [0 : h]$ and that the target windows $[\![k \pm j]\!]$ and $[\![k \pm i]\!]$ do not extend beyond one row above or below the corridor H.

Were i or j too big, $k \pm i$ or $k \pm j$ could extend too far beyond H, and the equality would not hold, as is the case for $2 \overset{9}{\underset{4}{\rightsquigarrow}} [\![3 \pm 1]\!] = 81 \neq 1 \overset{9}{\underset{4}{\rightsquigarrow}} [\![3 \pm 2]\!] = 121$. When $k \leq h/2$, the theorem holds as long as $i, j \leq k$.

The largest i and j can be (without being equal) is $i = \lfloor h/2 \rfloor$ and $j = \lceil h/2 \rceil$, which gives

$$\lfloor h/2 \rfloor \overset{n}{\rightsquigarrow} H \;=\; \lceil h/2 \rceil \overset{n}{\rightsquigarrow} H \tag{2}$$

and is no surprise.

This theorem also holds for the degenerate case $k = -1$ since the constraints impose $i = j = 0$, in which case the equivalence is true trivially.

By up-down symmetry:

Lemma 2. *For all $n, h \in \mathbb{N}$, $i, j, k \in [0 : h]$,*

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] \;=\; (h - i) \overset{n}{\underset{h}{\rightsquigarrow}} [\![h - k - 1 \pm j]\!] \tag{3}$$

So, for instances when $i > h \div 2$, we can combine this lemma with our theorem to obtain:

Corollary 3. *The equivalence*

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] \;=\; j \overset{n}{\underset{h}{\rightsquigarrow}} [\![h - k - 1 \pm h - i]\!]$$

holds for all $n, h \in \mathbb{N}$, $k \in [0 : h]$, $i \in [\max\{k, h - k - 1\} : h]$, $j \in [0 : \min\{k + 1, h - k\}]$.

Lastly, the closed-form formula for the paths of interest is as follows:

Theorem 4. *The number of corridor paths* $i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!]$ *is*

$$\sum_{z=\lfloor -n/4 \rfloor}^{\lfloor n/4 \rfloor} \sum_{\substack{s=0 \\ 0 \le k-j+2s \le h}}^{j} \left[\binom{n}{\lceil \frac{n+k-i-j}{2} \rceil + z(h+2) + s} - \binom{n}{\lceil \frac{n+k+i-j}{2} \rceil + z(h+2) + s + 1} \right]$$

for all $n, h, j, k \in \mathbb{N}$, $i \in [0 : h]$.

3. Inductive proof

One can prove Theorem 1, viz.

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] = j \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm i]\!]$$

by induction on the number of steps n, and with height h fixed throughout.

Recall that the bounds on i and j are

$$i, j \ge 0 \tag{4}$$
$$i, j \le k + 1 \tag{5}$$
$$i, j \le h - k \tag{6}$$

The cases where either is out of bounds are excluded from the theorem.

For $n = 0$, the starting and ending points must be the same. The two boundary conditions, viz.

$$\begin{aligned} i \overset{0}{\rightsquigarrow} [\![k \pm j]\!] &= [k - j \le i \le k + j + 1] \\ j \overset{0}{\rightsquigarrow} [\![k \pm i]\!] &= [k - i \le j \le k + i + 1] \end{aligned}$$

are equivalent since we are given that $0 \le i, j \le k + 1$.

In the general case ($n > 0$), we could argue inductively in the following fashion:

$$\begin{aligned} & i \overset{n}{\rightsquigarrow} [\![k \pm j]\!] \\ &= (i - 1 \overset{n-1}{\rightsquigarrow} [\![k \pm j]\!]) + (i + 1 \overset{n-1}{\rightsquigarrow} [\![k \pm j]\!]) && \text{basic recurrence} \\ &= (j \overset{n-1}{\rightsquigarrow} [\![k \pm i - 1]\!]) + (j \overset{n-1}{\rightsquigarrow} [\![k \pm i + 1]\!]) && \text{induction} \\ &= (j \overset{n-1}{\rightsquigarrow} [\![k - i - 1 \pm 0]\!]) + 2(j \overset{n-1}{\rightsquigarrow} [\![k \pm i - 1]\!]) + (j \overset{n-1}{\rightsquigarrow} [\![k + i + 1 \pm 0]\!]) && \text{definition} \\ &= (j \overset{n-1}{\rightsquigarrow} [\![k - 1 \pm i]\!]) + (j \overset{n-1}{\rightsquigarrow} [\![k + 1 \pm i]\!]) && \text{definition} \\ &= j \overset{n}{\rightsquigarrow} [\![k \pm i]\!] && \text{basic recurrence} \end{aligned}$$

But this only works if the two inductive cases also satisfy the theorem's constraints.

The problematic cases, when the inductive hypothesis cannot be applied, are three:

(a) $i = 0$, since then $i - 1 < 0$ violates (4) for the left inductive case $i - 1 \overset{n-1}{\rightsquigarrow} [\![k \pm j]\!]$;
(b) $i = k + 1$, since then $i + 1 > k + 1$ in violation of (5) for the right inductive case $i + 1 \overset{n-1}{\rightsquigarrow} [\![k \pm j]\!]$;
(c) $i = h - k$, violating (6) for the right case.

Fortuitously, the exact same argument may be applied in the opposite direction, with the rôles of i and j exchanged, to prove the identical equivalence:

$$\begin{aligned} i \overset{n}{\rightsquigarrow} [\![k \pm j]\!] &= (i \overset{n-1}{\rightsquigarrow} [\![k \pm j - 1]\!]) + (i \overset{n-1}{\rightsquigarrow} [\![k \pm j + 1]\!]) \\ &= (j - 1 \overset{n-1}{\rightsquigarrow} [\![k \pm i]\!]) + (j + 1 \overset{n-1}{\rightsquigarrow} [\![k \pm i]\!]) \\ &= j \overset{n}{\rightsquigarrow} [\![k \pm i]\!] \end{aligned} \tag{7}$$

The cases for which this version of the argument is problematic are analogous but different:

(a') $j = 0$;
(b') $j = k + 1$; or
(c') $j = h - k$.

For the first exception (a), when $i = 0$, all is well with just one induction:

$$0 \overset{n}{\rightsquigarrow} [\![k \pm j]\!] = 1 \overset{n-1}{\rightsquigarrow} [\![k \pm j]\!] = j \overset{n-1}{\rightsquigarrow} [\![k \pm 1]\!] = j \overset{n}{\rightsquigarrow} [\![k \pm 0]\!]$$

In the extreme case that $k = h$, and the induction is invalid, it must also be that $j = 0$, and the equivalence holds immediately, sans induction. By the same token, case (a') is also not an issue.

Furthermore, whenever $i = j$, the theorem holds trivially, so the two combined cases (b,b'), when $i = j = k + 1$, and (c,c'), when $i = j = h - k$, are fine, too.

So we only lack a proof for the following two combinations of the exceptions: (b,c'), when $i = k + 1$, $j = h - k$, and (c,b'), when $j = k + 1$ and $i = h - k$. These are symmetric, so let's delve just into the second. Taking constraints (5,6) into account, we find that $h = 2k + 1$, $i = k = \lfloor h/2 \rfloor$, and $j = \lceil h/2 \rceil$. So all we have to establish is the case $\lfloor h/2 \rfloor \overset{n}{\rightsquigarrow} H = \lceil h/2 \rceil \overset{n}{\rightsquigarrow} H$, which we've already seen (2).

4. Combinatorial proof

One can derive the enumeration of Theorem 4 using a standard result for bounded lattice paths. Our main theorem will then follow as a corollary.

The number $M(a, b, s, t)$ of "monotonic" paths from $\langle 0, 0\rangle$ to $\langle a, b\rangle$, taking a steps to the east (E, $\rightarrow$) and b steps to the north (N, $\uparrow$), while totally avoiding (not touching or crossing) the boundaries $y = x + s$ and $y = x - t$ ($s, t \in \mathbb{Z}^+$, $t < b - a < s$) is known (by a reflection argument) [8, 12, p. 6] to be

$$M(a, b, s, t) = \sum_{z \in \mathbb{Z}} \left[\binom{a + b}{b + z(s + t)} - \binom{a + b}{b + z(s + t) + t} \right] \tag{8}$$

with the (nonstandard) convention that $\binom{n}{m} = 0$ whenever $m \notin \mathbb{N}$. See Figure 4.

There is a straightforward relationship between these constrained N/E paths $\langle 0, 0\rangle \rightsquigarrow \langle a, b\rangle$ and those NE/SE paths $\langle 0, i\rangle \rightsquigarrow \langle n, \ell\rangle$ that we have set out to study (as illustrated in Figure 2):

$$n = a + b \qquad \ell - i = b - a$$
$$t = i + 1 \qquad s + t = h + 2$$

Plugging the solution

$$a = \frac{n+i-\ell}{2} \qquad\qquad b = \frac{n-i+\ell}{2}$$

$$s = h - i + 1 \qquad\qquad t = i + 1$$

into (8), we get (cf. [1]):

$$i \overset{n}{\underset{h}{\rightsquigarrow}} \ell = \sum_{z\in\mathbb{Z}} \left[\binom{n}{\frac{n-i+\ell}{2} + z(h+2)} - \binom{n}{\frac{n-i+\ell}{2} + z(h+2) + i + 1} \right] \tag{9}$$

as long as $0 \le i, \ell \le h$. For those ℓ for which $\frac{n-i+\ell}{2}$ is not a whole number, the binomial coefficients are all 0.

Letting ℓ move along the window from $k - j$ to $k + j + 1$, we get from (9) that

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] = \sum_{\ell=\max\{0,k-j\}}^{\min\{k+j+1,h\}} \sum_{z\in\mathbb{Z}} \left[\binom{n}{\frac{n-i+\ell}{2} + z(h+2)} - \binom{n}{\frac{n-i+\ell}{2} + z(h+2) + i + 1} \right]$$

The sum for z can be restricted to the range $\lfloor -n/4 \rfloor : \lfloor n/4 \rfloor$. Skipping over the impossible odd or even values (for which the denominators of the binomial coefficients are fractional), we arrive at the stated formula of Theorem 4:

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] = \sum_{z=\lfloor -n/4 \rfloor}^{\lfloor n/4 \rfloor} \sum_{\substack{s=0 \\ 0 \le k-j+2s \le h}}^{j} \left[\binom{n}{\left\lceil \frac{n-i+k-j}{2} \right\rceil + z(h+2) + s} - \binom{n}{\left\lceil \frac{n+i+k-j}{2} \right\rceil + z(h+2) + s + 1} \right] \tag{10}$$

Consider now only the cases considered in Theorem 1, which guarantee that $k - j \ge -1$ and that $k + j \le h + 1$, so s may run from 0 to j without exception – bearing in mind (as shown above) that any instances when $k - j + 2s = 0, h + 1$ have no impact on the sum. Reversing the order of the second sum in (10), replacing s with $j - s$, we get

$$i \overset{n}{\underset{h}{\rightsquigarrow}} [\![k \pm j]\!] = \sum_{z\in\mathbb{Z}} \sum_{s=0}^{j} \left[\binom{n}{\left\lceil \frac{n-i+k-j}{2} \right\rceil + z(h+2) + s} - \binom{n}{\left\lceil \frac{n+i+k+j}{2} \right\rceil + z(h+2) - s + 1} \right]$$

When $j > i$, the inner sums overlap (for $s > i$) and cancel each other. So the above sum is always equal to

$$\sum_{z\in\mathbb{Z}} \left[\sum_{s=0}^{\min\{i,j\}} \binom{n}{r + z(h+2) + s} - \binom{n}{r + z(h+2) + i + j - s + 1} \right]$$

where $r = \lceil (n + k - i - j)/2 \rceil$. This is symmetric in i and j; hence Theorem 1.

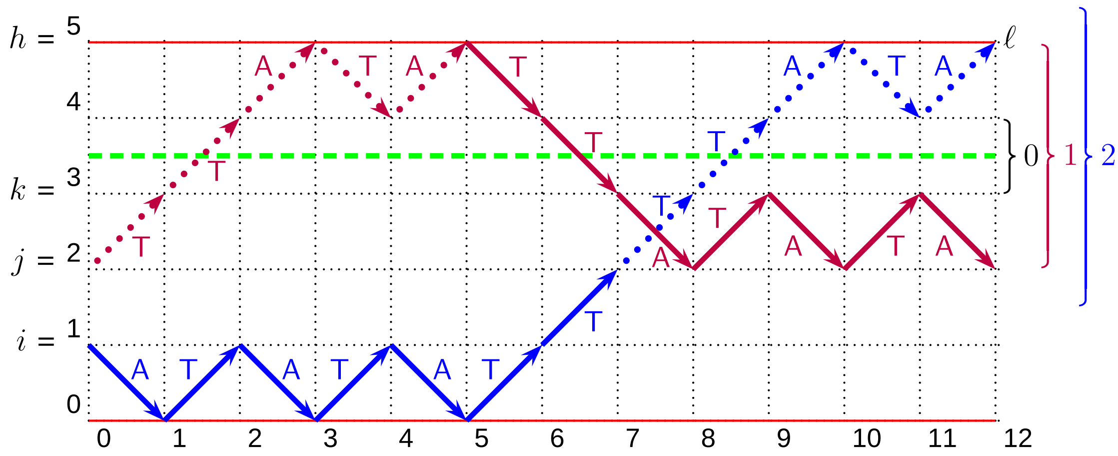

FIGURE 5. The 12-step blue path from level $i = 1$ to level $\ell = 5$ belongs to the class enumerated by $1 \overset{12}{\rightsquigarrow} [\![3 \pm 2]\!]$. Since $k = 3$, steps are labeled T when they start out towards the "attractor" $y = 3.5$ (the dashed green line), and A when they head away in the opposite direction. So the blue path is labeled ATATATTTTATA. The target window size is $j = 2$, so we are in the ($<$) case of the bijection. After seven (solid blue) steps ATATATT, the path touches $y = 2$, so the remaining 5 (dotted blue) steps, TTATA, are copied as is and placed with $\langle 0, 2\rangle$ as their initial point (dotted purple) , followed by the seven in reverse (solid purple), that is, TTATATA, to obtain the corresponding path. The result is TTATATTATATA, one of those counted by $2 \overset{12}{\rightsquigarrow} [\![3 \pm 1]\!]$, which all start from $j = 2$ and end in $[2 : 5]$. Because of the unusual encoding, the reversed (solid) path segments do not actually resemble each other visually. The counterpart of the latter (purple) path is again the former (in blue), and is obtained by proceeding from the end towards the beginning until the window size becomes 2, per case ($>$). See Section 5 for details.

5. BIJECTIVE PROOF

A bijection can be inferred from the inductive proof of Section 3 for the equivalence of the enumerations:

$$i \underset{h}{\overset{n}{\rightsquigarrow}} [\![k \pm j]\!] \;=\; j \underset{h}{\overset{n}{\rightsquigarrow}} [\![k \pm i]\!]$$

We use a novel representation for paths, which simplifies matters greatly.

Draw a line $y = k + 1/2$. Each step starting out towards that line is labeled `T`; each heading away is labeled `A`. From any given point, exactly one outgoing step ($\nearrow$ or $\searrow$) will be `T` and one `A`. We call this the `TA` representation of a lattice path (relative to k). See Figure 5.

Suppose the height of a point along the path is in the window $[k - j : k + j + 1]$. If we take an `A` step from there, then the next point is in the wider window $[k - j - 1 : k + j + 2]$; so j has been incremented. Conversely, a `T` step brings it into the narrower range $[k - j + 1 : k + j]$, with decremented j. Naturally, going backwards alongs the path has the opposite effect.

If we take this point of view and go through the cases of the inductive proof, we find that the correspondence simply reverses the order of steps, either moving the last step to the beginning or vice versa. When $i = j$, there is no need to do anything, since the two sides of the equivalence are identical. We are led to the following bijection between a path P starting at $y = i$ and ending in the range $[k - j : k + j + 1]$ and its counterpart path P^* starting at $y = j$ and ending in the range $[k - i : k + i + 1]$:

(=) If $i = j$, then $P^* = P$.

(<) If $i < j$, follow the path from the start at level i until it reaches j, if ever. At that point, we have $P = QR$, where $y = j$ first transpires at the end of prefix Q. Then $P^* = R\overleftarrow{Q}$, where $\overleftarrow{Q}$ is the reverse sequence of Q in its `TA` representation. If level j is never attained, then R is empty, and $P^* = \overleftarrow{Q}$.

(>) If $i > j$, follow the path from the end backwards, starting with a target window of size j, moving leftwards until it grows to be i, if ever. A `T` step enlarges the window, while `A` shrinks it. If R is shortest suffix such that the window size is i at its onset, so that we have $P = QR$, then we let $P^* = \overleftarrow{R}Q$.

It is not hard to verify that the transpositions involved keep the path within the bounded corridor, given that the original path satisfied $0 \le i, j \le \min\{k + 1, h - k\}$.

The path in Figure 2 is its own counterpart, as this is an instance of case (=) with $i = j = 1$. For a worked-out nontrivial example, see Figure 5.

The inductive proof allows for alternate bijections depending on the preferred order in which the different cases are to be considered.

6. Historical discussion

Theorem 1, our main result, is a significant generalization of the equality due to Johann Cigler [5], namely,

$$0 \overset{n}{\underset{h}{\rightsquigarrow}} [0 : h] \;=\; \hbar \overset{n}{\underset{h}{\rightsquigarrow}} [\hbar : \hbar + 1] \tag{11}$$

for all heights h, where $\hbar = h \div 2$ for short. Paths (counted by) $\hbar \overset{n}{\rightsquigarrow} [\hbar : \hbar + 1]$ start in the middle of the swath and end either in the middle – when the number of steps is even, or just above – when odd. As noted earlier, these are called "grand Dyck" paths. Dyck path prefixes $0 \overset{n}{\rightsquigarrow} [0 : h]$ start at the bottom and end anywhere within the swath. Cigler's (11) asserts the equality of cardinality of these two sets of paths. As such, it is a particular instance of our more general result (1) with $i = 0$ and $j = k = \hbar$. Phrased in our notation, Cigler proved:

$$0 \overset{n}{\underset{h}{\rightsquigarrow}} [\![\hbar \pm \hbar]\!] \;=\; \hbar \overset{n}{\underset{h}{\rightsquigarrow}} [\![\hbar \pm 0]\!]$$

Cigler solicited alternative proofs of his result. More specifically, he asked in [4] for a *bijective* proof of the height $h = 3$ case,

$$0 \overset{n}{\underset{3}{\rightsquigarrow}} [0 : 3] \;=\; 1 \overset{n}{\underset{3}{\rightsquigarrow}} [1 : 2]$$

which gives rise to the Fibonacci numbers. The wished-for bijective solution to this very particular case was discovered shortly thereafter by Thomas Prellberg [4, Answer], followed by another due to Helmut Prodinger [14]. Most recently, Nancy Gu and Prodinger [10] constructed a bijection for Cigler's full case (11) by extending the idea in [14]. When there are no upper and lower bounds on paths, there are long-standing well-known bijections between grand Dyck paths and Dyck path prefixes [7, 9].

The bijection of the previous section supplies an alternative proof of Cigler's (11). In that special case, the bijection amounts to simply reversing the order of steps in the `TA` representation. This works as is for even n in the grand Dyck ($i = k = \hbar$ and $j = 0$) to Dick-prefix ($i = 0$ and $j = k = \hbar$) case of Cigler, as this is the ($i > j$) case of the bijection and the window never grows too big (it may get to be $\hbar$, the maximum excess of `T` moves over `A` moves, but no larger) to continue all way the beginning. Unfortunately, it doesn't do the trick when n is odd and h is even (because proceeding only backwards can lead to a window wider than $\hbar = i$). For the odd n case, it is possible to modify the bijection by first reversing the grand Dyck path left to right (so it ends on $y = \hbar$ but begins at $i = \hbar + 1$) before converting to the `TA` representation and reversing. This now covers all cases of (11). The second bijection also works for even h and even n. For odd h, regardless of the parity of n, the first bijection actually succeeds for all i, j meeting the requirements of the theorem. So, when n and h have the same parity, both bijections work. In the more general cases, when $k \neq \hbar$, neither applies, and we resort to the slightly more complicated bijection of the previous section, wherein only part of the `TA` path is reversed.

We began our investigation seeking a bijective proof of (11). The simple bijection employing the `TA` path encoding didn't work in all cases. This led us to a sequence of generalizations, commencing from Cigler's (11):

$$
\begin{aligned}
0 \overset{n}{\rightsquigarrow} [0 : h] &= \hbar \overset{n}{\rightsquigarrow} [\hbar : \hbar + 1] \\
i \overset{n}{\rightsquigarrow} [0 : h] &= \hbar \overset{n}{\rightsquigarrow} [\hbar - i : \hbar + i + 1] \\
i \overset{n}{\rightsquigarrow} [\hbar - j : \hbar + j + 1] &= j \overset{n}{\rightsquigarrow} [\hbar - i : \hbar + i + 1] \\
i \overset{n}{\rightsquigarrow} [k - j : k + j + 1] &= j \overset{n}{\rightsquigarrow} [k - i : k + i + 1]
\end{aligned}
$$

First we let i be anywhere (not just 0), then we let j be any size (not just $\hbar$), and finally allowed it to be centered at any k (not just $\hbar$). Concurrently, we programmed various enumerations and potential bijections to lend support to – or refute – conjectures as they arose. Casting the equivalence in a fashion that highlights its symmetry also contributed to finding the generalizations and proofs.

All the above variants share the basic idea that, as the starting point of one set of paths moves from the edge of the corridor towards the middle, the target range of the corresponding equinumerous set of paths grows wider and wider. This behavior is what suggested the `TA` encoding in the first place.

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i	$n = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ℓ	OEIS
4	1		1		2		5		14		41		122		365		1094	4	A007051
3		1		2		5		14		41		122		365		1094		3	A007051
2			1		3		9		27		81		243		729		2187	2	A000244
1				1		4		13		40		121		364		1093		1	A003462
0					1		4		13		40		121		364		1093	0	A003462
4		1		2		5		14		41		122		365		1094		4	A007051
3	1		2		5		14		41		122		365		1094		3281	3	A007051
2		1		3		9		27		81		243		729		2187		2	A000244
1			1		4		13		40		121		364		1093		3280	1	A003462
0				1		4		13		40		121		364		1093		0	A003462
4			1		3		9		27		81		243		729		2187	4	A000244
3		1		3		9		27		81		243		729		2187		3	A000244
2	1		2		6		18		54		162		486		1458		4374	2	A025192
1		1		3		9		27		81		243		729		2187		1	A000244
0			1		3		9		27		81		243		729		2187	0	A000244
4				1		**4**		13		40		121		364		1093		4	A003462
3			**1**		**4**		**13**		40		121		**364**		1093		3280	3	A003462
2		**1**		**3**		9		**27**		81		**243**		729		2187		2	A000244
1	**1**		2		5		14		**41**		**122**		365		1094		3281	1	A007051
0		1		2		5		14		**41**		122		365		1094		0	A007051
4					1		4		13		40		121		364		1093	4	A003462
3				1		4		13		40		121		364		1093		3	A003462
2			1		3		9		27		81		243		729		2187	2	A000244
1		1		2		5		14		41		122		365		1094		1	A007051
0	1		1		2		5		14		41		122		365		1094	0	A007051

TABLE 1. The number of paths $i \overset{n}{\underset{4}{\rightsquigarrow}} \ell$ for $i, \ell \in [0:4]$, $n \in [0:16]$. For example, $2 \overset{16}{\rightsquigarrow} 2 = 0 \overset{16}{\rightsquigarrow} [0:4] = 4374$ and $3 \overset{16}{\rightsquigarrow} 3 = 4 \overset{16}{\rightsquigarrow} [2:4] = 3281$. Like for a bishop on a chessboard, half the squares are unreachable from any given starting point. The few squares that require backward steps are likewise inaccessible. The particular path of Figures 2–4 is highlighted in **blue boldface**. Sloane numbers of the sequences are provided in the last column.

i	$n=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ℓ	OEIS
5	1		1		2		5		14		42		131		417		1341	5	A080937
4		1		2		5		14		42		131		417		1341		4	A080937
3			1		3		9		28		89		286		924		2993	3	A094790
2				1		4		14		47		155		507		1652		2	A006053
1					1		5		19		66		221		728		2380	1	A005021
0						1		5		19		66		221		728		0	A005021
5		1		2		5		14		42		131		417		1341		5	A080937
4	1		2		5		14		42		131		417		1341		4334	4	A080937
3		1		3		9		28		89		286		924		2993		3	A094790
2			1		4		14		47		155		507		1652		5373	2	A006053
1				1		5		19		66		221		728		2380		1	A005021
0					1		5		19		66		221		728		2380	0	A005021
5			1		3		9		28		89		286		924		2993	5	A094790
4		1		3		9		28		89		286		924		2993		4	A094790
3	1		2		6		19		61		197		638		2069		6714	3	A052975
2		1		3		10		33		108		352		1145		3721		2	A060557
1			1		4		14		47		155		507		1652		5373	1	A006053
0				1		4		14		47		155		507		1652		0	A006053
5				1		4		14		47		155		507		1652		5	A006053
4			1		4		14		47		155		507		1652		5373	4	A006053
3		1		3		10		33		108		352		1145		3721		3	A060557
2	1		2		6		19		61		197		638		2069		6714	2	A052975
1		1		3		9		28		89		286		924		2993		1	A094790
0			1		3		9		28		89		286		924		2993	0	A094790
5					1		5		19		66		221		728		2380	5	A005021
4				1		5		19		66		221		728		2380		4	A005021
3			1		4		14		47		155		507		1652		5373	3	A006053
2		1		3		9		28		89		286		924		2993		2	A094790
1	1		2		5		14		42		131		417		1341		4334	1	A080937
0		1		2		5		14		42		131		417		1341		0	A080937
5						1		5		19		66		221		728		5	A005021
4					1		5		19		66		221		728		2380	4	A005021
3				1		4		14		47		155		507		1652		3	A006053
2			1		3		9		28		89		286		924		2993	2	A094790
1		1		2		5		14		42		131		417		1341		1	A080937
0	1		1		2		5		14		42		131		417		1341	0	A080937

TABLE 2. Paths $i \overset{n}{\underset{5}{\rightsquigarrow}} \ell$ constrained to height 5, $n \in [0:16]$. Sloane numbers of the sequences are provided in the last column.