# Stuttering Conway Sequences Are Still Conway Sequences 

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#### Abstract

A look-and-say sequence is obtained iteratively by reading off the digits of the current value, grouping identical digits together: starting with 1 , the sequence reads: $1,11,21,1211,111221,312211$, etc. (OEIS A005150). Starting with any digit $d \neq 1$ gives Conway's sequence: $d, 1 d, 111 d, 311 d, 13211 d$, etc. (OEIS A006715). Conway popularised these sequences and studied some of their properties [Con87]. In this paper we consider a variant subbed "look-and-say again" where digits are repeated twice. We prove that the "look-and-say again" sequence contains only the digits $1,2,4,6, d$, where $d$ represents the starting digit. Such sequences decompose and the ratio of successive lengths converges to Conway's constant. In fact, these properties result from a commuting diagram between look-and-say again sequences and "classical" look-and-say sequences. Similar results apply to the "look-and-say three times" sequence.


## 1 Introduction

The look-and-say (LS) sequence [CG12], also known as the Morris or the Conway sequence [Con87, Hil96, EZ97] is a recreational integer sequence having very intriguing properties.

A LS sequence is obtained iteratively by reading off the digits of the current value, and counting the number of digits in groups of the identical digit.

Starting with 1 , the sequence reads (OEIS A005150): 1, 11, 21, 1211, 111221, 312211, etc. Starting with any digit $d \neq 1$ gives Conway's sequence (OEIS A006715): $d, 1 d, 111 d, 311 d, 13211 d$, etc. Conway popularised these sequences and studied some of their properties. For example, an LS sequence contains only the digits $1,2,3$, and satisfies a so-called cosmological decay [EZ97], if $L_{n}$ denotes the number of digits of the $n^{\text {th }}$ term of the sequence, then

$$
\lim _{n \rightarrow \infty} \frac{L_{n+1}}{L_{n}}=\lambda \simeq 1.303577
$$

where $\lambda$ is the only real root of a degree-71 polynomial [Fin03, §6.12]. Conway showed that $\exists N \in \mathbb{N}^{*}$ such that every term of the LS term decays in at most $N$ rounds to a compound of "common" and "transuranic" terms.

Following Conway's work, numerous variants of LS sequences were proposed and studied. For instance, Pea Pattern sequences [Mul12], Sloane's sequences [Slo09] or Kolakoski sequences [Kol66,Ü66]. In this paper we consider a new LS sequence and study some of its properties. The concerned variant, called "look-and-say again" sequence, consists in repeating each LS digit twice. We prove that the such sequences contain only the digits $1,2,4,6, d$, where $d$ is the starting digit.

## 2 Notations and definitions

In this paper we assume that numbers are written in base 10 . Any integer $T$ can thus be written $T=$ $t_{1} t_{2} \cdots t_{k}$ with $t_{1}, \ldots, t_{k} \in\{0,1, \ldots, 9\}$. To avoid any ambiguity, $a b$ will denote the concatenation of the numbers $a$ and $b$; accordingly $a^{b}$ indicates that a digit $a$ is repeated $b$ times. If we want to emphasise concatenation we use $a \| b$ instead of $a b$.

Definition 1 (Run-length representation). Let $T \in \mathbb{N}^{*}$, we can write

$$
T=\underbrace{a_{1} \ldots a_{1}}_{n_{1}} \underbrace{a_{2} \ldots a_{2}}_{n_{2}} \ldots \underbrace{a_{k} \ldots a_{k}}_{n_{k}}
$$

with $a_{1} \neq a_{2}, a_{2} \neq a_{3}, \ldots, a_{k-1} \neq a_{k}$. The run-length representation of $T$ is the sequence $\operatorname{RunL}(T)=$ $a_{1}^{n_{1}} a_{2}^{n_{2}} \cdots a_{k}^{n_{k}}$. Conversely, any finite sequence of couples $\left(a_{i}, b_{i}\right)_{i}$ where $a \in \mathbb{N}^{*}$ and $0 \leq b_{i} \leq 9$ is such that $b_{i-1} \neq b_{i} \neq b_{i+1}$, corresponds to an integer with run-length representation $\left(a_{i}^{b_{i}}\right)_{i}$.

Note that the run-length representation of an integer is unique.
Definition 2 (Pieces). If $N=\left(a_{i}^{b_{i}}\right)$ is a run-length encoded integer, we call each $a_{i}^{b_{i}}$ a piece of $N$.
Definition 3 (Look-and-say-again sequence). Let $T_{0}$ be a decimal digit, and for each $T_{n}$ define

$$
T_{n+1}=n_{1} n_{1} a_{1} a_{1} n_{2} n_{2} a_{2} a_{2} \cdots n_{k} n_{k} a_{k} a_{k} .
$$

where $\left(a_{i}^{n_{i}}\right)_{i}=\operatorname{RunL}\left(T_{n}\right)$. We call the sequence $\left(T_{k}\right)_{k \in \mathbb{N}}$ the look-and-say-again sequence of seed $T_{0}$, and denote it by $\mathrm{LSA}\left(T_{0}\right)$.

## Example 1.

LSA $(1)=1 \rightarrow 1111 \rightarrow 4411 \rightarrow 22442211 \rightarrow 2222224422222211 \rightarrow 6622224466222211 \rightarrow \ldots$
LSA $(2)=2 \rightarrow 1122 \rightarrow 22112222 \rightarrow 222222114422 \rightarrow 6622221122442222 \rightarrow 226644222211222222444422 \rightarrow \ldots$

## 3 The look-and-say-again sequence

Theorem 1 (Digits of LSA). Only the digits $1,2,3,4,6$, d appear in $\operatorname{LSA}(d)$.
Proof. Let $n \in \mathbb{N}$ and $\left(a_{i}^{n_{i}}\right)_{i}=\operatorname{RunL}(n)$, we write

$$
P(n):=\left\{\forall i, a_{i} \in\{1,2,4,6\} \text { and } n_{i} \in\{2,4,6\}\right\}
$$

(the matter of $d$ will be settled further down). Assume that $P\left(T_{n}\right)$ is true, and let $a_{i}^{n_{i}} \in \operatorname{RunL}\left(T_{n}\right)$. We have four situations:

Case 1: $\operatorname{RunL}\left(T_{n}\right)=a_{i}^{n_{i}}$, in other terms $T_{n}=a_{i} a_{i} \cdots a_{i}$ and there is no other digit. Then the next term in the sequence is $T_{n+1}=n_{i} n_{i} a_{i} a_{i}$, which clearly satisfies $P\left(T_{n+1}\right)$ since $a_{i} \in\{1,2,4,6\}$ and $n_{i} \in\{2,4,6\}$.
Case 2: $i=1$, i.e., $T_{n}$ starts with the repeated digits $a_{i}$. In this case

$$
T_{n+1}=n_{1} n_{1} a_{1} a_{1} n_{2} n_{2} a_{2} a_{2} \cdots
$$

with $a_{k} \neq a_{k+1}$ for all $k$. It is clear that no $a_{k}$ can be contained in a piece that also contains $a_{k+1}$, therefore the possible pieces that $n_{1}$ and $a_{1}$ can take part to are either $n_{1} n_{1} a_{1} a_{1} n_{2} n_{2}$ (in the case $n_{1}=a_{1}=n_{2}$ ), or $n_{1} n_{1} a_{1} a_{1}$ (in the case $n_{1}=a_{1} \neq n_{2}$ ), or $n_{1} n_{1}$ and $a_{1} a_{1} / a_{1} a_{1} n_{2} n_{2}$ (respectively in the cases $n_{1} \neq a_{1} \neq n_{2}$ and $n_{1} \neq a_{1}=n_{2}$ ).
In conclusion $a_{1}$ and $n_{1}$ generate pieces made of digits in $\{1,2,4,6\}$ with multiplicity either 2 , 4 , or 6 .
Case 3: $T_{n}$ ends with $a_{i}$; this is analogous to case 2 above.
Case 4: The piece $a_{i}^{n_{i}}$ is neither at the end nor at the beginning of $T_{n}$. The next term in the sequence is:

$$
T_{n+1}=\cdots a_{i-2} n_{i-1} n_{i-1} a_{i-1} a_{i-1} n_{i} n_{i} a_{i} a_{i} n_{i+1} n_{i+1} a_{i+1} a_{i+1} \cdots
$$

By definition of the run-length representation, $a_{i-2}, a_{i-1}, a_{i}, a_{i+1}$ contains no consecutive values, hence the possible pieces resulting for $n_{i}$ and $a_{i}$ are:
(a) $n_{i-1} n_{i-1} a_{i-1} a_{i-1} n_{i} n_{i}$, and either $a_{i} a_{i}$ or $a_{i} a_{i} n_{i+1} n_{i+1}$
(b) $a_{i-1} a_{i-1} n_{i} n_{i}$, and either $a_{i} a_{i}$ or $a_{i} a_{i} n_{i+1} n_{i+1}$
(c) $n_{i} n_{i}$, and either $a_{i} a_{i}$ or $a_{i} a_{i} n_{i+1} n_{i+1}$
(d) $n_{i} n_{i} a_{i} a_{i}$ or $n_{i} n_{i} a_{i} a_{i} n_{i+1} n_{i+1}$
(e) $n_{i} n_{i}$ and $a_{i} a_{i}$

In each case, since $\forall k, a_{k} \in\{1,2,4,6\}$ and $n_{k} \in\{2,4,6\}$, we see that $n_{i}$ and $a_{i}$ can appear only in pieces that are of multiplicity 2,4 , or 6 , and which contain numbers $\in\{1,2,4,6\}$.

If $T_{0}=d \in\{1,2,4,6\}$, then $P\left(T_{0}\right)$ holds and by the above case exhaustion argument $P\left(T_{n}\right)$ hold for all $n$.
It remains to discuss the case $T_{0}=d \notin\{1,2,4,6\}$. Writing the first few terms of the resulting sequence shows that this is easily dealt with:

$$
T_{0}=d \quad T_{1}=11 d d \quad T_{2}=221122 d d \quad T_{3}=22222211222222 d d \quad \ldots
$$

Indeed, save for the first term, the digit $d$ only appears as $d d$ at the end of $T_{n}$. The rest of $T_{n}$ satisfies $P\left(T_{n}\right)$ discussed previously.

To prove this, assume that $T_{n}=k \| d d$, which means that $T_{n}$ starts with an integer $k$ and ends with the two digits $d d$, and further assume that $P(k)$ is true and $k$ 's last piece is $2^{k}$, with $k \in\{2,4,6\}$. Then the next term in the sequence is $T_{n+1}=k^{\prime} \| 22 d d$, where $k^{\prime}$ is an integer that ends with the digit 2 and such that $P\left(k^{\prime}\right)$ is true (thanks to what we have proved in the first part of the theorem). Let $S=k^{\prime} \| 22$. Consider three cases, as a function of the last piece of $k$, denoted $2^{\omega}$ :

$$
\begin{aligned}
& \omega=2 \Rightarrow k^{\prime} \text { ends with } 2222 \Rightarrow S \text { ends with } 222222 \Rightarrow P(S) \text { holds } \\
& \omega=4 \Rightarrow k^{\prime} \text { ends with } 4422 \Rightarrow S \text { ends with } 2222 \Rightarrow P(S) \text { holds } \\
& \omega=6 \Rightarrow k^{\prime} \text { ends with } 6622 \Rightarrow S \text { ends with } 2222 \Rightarrow P(S) \text { holds }
\end{aligned}
$$

Since $P\left(k^{\prime}\right)$ holds, the only problem in $S$ was at the interface between the ending 2's of $k^{\prime}$ and the couple 22 at the end of $S$. With the exhaustion argument above we have shown that in each possible case $P(S)$ holds. Finally the number $S \| d d$ is such that $d$ only appears as a couple $d d$ at the end, whereas $S$ is made of digits belonging to $\{1,2,4,6\}$ with multiplicities in $\{2,4,6\}$.
$T_{0}$ and $T_{1}$ contain only digits $\in\{1,2,4,6, d\}$. Since $T_{2}$ is of the form $k \| d d$ with $P(k)$ true and $k$ 's last digit being 2 , the above argument shows that all subsequent terms in the sequence can be written in this way, and the proof is completed.

Corollary 1. If $T_{0}=d \neq 1,2$ then $\operatorname{LSA}\left(T_{0}\right)$ gives the same sequence, save for the two last digits of each term which are dd.

Remark 1. The length sequences for $T_{0}=1,2, d$ are respectively:

$$
\begin{aligned}
& 1,4,4,8,16,16,24,40,48,56,88,104,120,176,224,280,392,520,648,864,1168 \\
& 1,4,8,12,16,24,32,40,48,72,92,112,156,204,264,352,464,592,784,1036,1320 \\
& 1,4,8,16,16,24,40,48,56,88,104,120,176,224,280,392,520,648,864,1168,1432
\end{aligned}
$$

Remark 2. For all seeds $s, \operatorname{LSA}(d)$ grows to infinity, namely the ratio of lengths $\lambda_{n}=\ell_{n+1} / \ell_{n}$ for two consecutive terms of sequence (which is between 1 and 2 ) tends towards Conway's constant $\lambda \simeq 1.303577$, regardless of the seed $T_{0}$. The following numerical simulation backs up this intuition:


This is in fact a consequence of the following result:
Theorem 2. Consider the following operations on pieces:

$$
C: a^{b} \mapsto b a \quad L: a^{b} \mapsto b^{2} a^{2} \quad \eta: a^{b} \mapsto \kappa(a)^{2 b}
$$

with $\kappa: a \mapsto 2 a$ for $a \in\{1,2,3\}$ and $\kappa(a)=1$ otherwise. Then $L \circ \eta=\eta \circ C$.
Proof. Let $a^{b}$ be a piece,

$$
\begin{aligned}
& (\eta \circ C)\left(a^{b}\right)=\eta\left(b^{1} a^{1}\right)=\eta\left(b^{1}\right) \eta\left(a^{1}\right)=\kappa(b)^{2} \kappa(a)^{2} \\
& (L \circ \eta)\left(a^{b}\right)=L\left(\kappa(a)^{\kappa(b)}\right)=\kappa(b)^{2} \kappa(a)^{2} .
\end{aligned}
$$

A result of this theorem is that LSA is equivalent to Conway's sequence, where $\kappa$ and $\eta$ allow us to translate from one to the other. Then LSA inherits many of the properties that are known of Conway's sequence: decomposition into "elements", convergence to $\lambda$, and so forth.

Remark 3. This would also work with $L\left(a^{b}\right)=b^{3} a^{3}$ and $\kappa(a)=3 a$ for $a \in\{1,2,3\}$ (i.e. an LSA variant where elements are repeated three times instead of two). However the argument breaks down for a "look-and-say four times" sequence. We leave this sequence and the study of its properties open for further research.

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