Stuttering Conway Sequences Are Still Conway Sequences

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Abstract. A look-and-say sequence is obtained iteratively by reading off the digits of the current value, grouping identical digits together: starting with 1, the sequence reads: 1, 11, 21, 1211, 111221, 312211, etc. (OEIS A005150). Starting with any digit $d \neq 1$ gives Conway's sequence: d, 1d, 111d, 311d, 13211d, etc. (OEIS A006715). Conway popularised these sequences and studied some of their properties [Con87].

In this paper we consider a variant subbed "*look-and-say again*" where digits are repeated twice. We prove that the "*look-and-say again*" sequence contains only the digits 1,2,4,6,*d*, where *d* represents the starting digit. Such sequences decompose and the ratio of successive lengths converges to Conway's constant.

In fact, these properties result from a commuting diagram between look-and-say again sequences and "classical" look-and-say sequences. Similar results apply to the "look-and-say three times" sequence.

1 Introduction

The look-and-say (LS) sequence [CG12], also known as the *Morris* or the *Conway sequence* [Con87, Hil96, EZ97] is a recreational integer sequence having very intriguing properties.

A LS sequence is obtained iteratively by reading off the digits of the current value, and counting the number of digits in groups of the identical digit.

Starting with 1, the sequence reads (OEIS A005150): 1, 11, 21, 1211, 111221, 312211, etc. Starting with any digit $d \neq 1$ gives Conway's sequence (OEIS A006715): d, 1d, 111d, 311d, 13211d, etc. Conway popularised these sequences and studied some of their properties. For example, an LS sequence contains only the digits 1, 2, 3, and satisfies a so-called *cosmological decay* [EZ97], if L_n denotes the number of digits of the *n*th term of the sequence, then

$$\lim_{n \to \infty} \frac{L_{n+1}}{L_n} = \lambda \simeq 1.303577$$

where λ is the only real root of a degree-71 polynomial [Fin03, §6.12]. Conway showed that $\exists N \in \mathbb{N}^*$ such that every term of the LS term decays in at most N rounds to a compound of "common" and "transuranic" terms.

Following Conway's work, numerous variants of LS sequences were proposed and studied. For instance, Pea Pattern sequences [Mul12], Sloane's sequences [Slo09] or Kolakoski sequences [Kol66,Ü66]. In this paper we consider a new LS sequence and study some of its properties. The concerned variant, called "*look-and-say again*" sequence, consists in repeating each LS digit twice. We prove that the such sequences contain only the digits 1, 2, 4, 6, d, where d is the starting digit.

2 Notations and definitions

In this paper we assume that numbers are written in base 10. Any integer *T* can thus be written $T = t_1 t_2 \cdots t_k$ with $t_1, \ldots, t_k \in \{0, 1, \ldots, 9\}$. To avoid any ambiguity, *ab* will denote the *concatenation* of the numbers *a* and *b*; accordingly a^b indicates that a digit *a* is repeated *b* times. If we want to emphasise concatenation we use a || b instead of *ab*.

Definition 1 (**Run-length representation**). *Let* $T \in \mathbb{N}^*$ *, we can write*

$$T = \underbrace{a_1 \dots a_1}_{n_1} \underbrace{a_2 \dots a_2}_{n_2} \dots \underbrace{a_k \dots a_k}_{n_k}$$

with $a_1 \neq a_2, a_2 \neq a_3, \ldots, a_{k-1} \neq a_k$. The run-length representation of T is the sequence $\text{RunL}(T) = a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$. Conversely, any finite sequence of couples $(a_i, b_i)_i$ where $a \in \mathbb{N}^*$ and $0 \leq b_i \leq 9$ is such that $b_{i-1} \neq b_i \neq b_{i+1}$, corresponds to an integer with run-length representation $(a_i^{b_i})_i$.

Note that the run-length representation of an integer is unique.

Definition 2 (Pieces). If $N = (a_i^{b_i})$ is a run-length encoded integer, we call each $a_i^{b_i}$ a piece of N.

Definition 3 (Look-and-say-again sequence). Let T_0 be a decimal digit, and for each T_n define

 $T_{n+1}=n_1n_1a_1a_1n_2n_2a_2a_2\cdots n_kn_ka_ka_k.$

where $(a_i^{n_i})_i = \text{RunL}(T_n)$. We call the sequence $(T_k)_{k \in \mathbb{N}}$ the look-and-say-again sequence of seed T_0 , and denote it by $\text{LSA}(T_0)$.

Example 1.

$$\begin{split} \mathsf{LSA}(1) &= \mathsf{1} \to \mathsf{1111} \to \mathsf{4411} \to \mathsf{22442211} \to \mathsf{2222224422222211} \to \mathsf{6622224466222211} \to \ldots \\ \mathsf{LSA}(2) &= \mathsf{2} \to \mathsf{1122} \to \mathsf{22112222} \to \mathsf{222222114422} \to \mathsf{6622221122442222} \to \mathsf{226644222211222222444422} \to \ldots \end{split}$$

3 The look-and-say-again sequence

Theorem 1 (Digits of LSA). Only the digits 1, 2, 3, 4, 6, d appear in LSA(d).

Proof. Let $n \in \mathbb{N}$ and $(a_i^{n_i})_i = \operatorname{RunL}(n)$, we write

$$P(n) := \{ \forall i, a_i \in \{1, 2, 4, 6\} \text{ and } n_i \in \{2, 4, 6\} \}$$

(the matter of *d* will be settled further down). Assume that $P(T_n)$ is true, and let $a_i^{n_i} \in \text{RunL}(T_n)$. We have four situations:

Case 1: $\text{RunL}(T_n) = a_i^{n_i}$, in other terms $T_n = a_i a_i \cdots a_i$ and there is no other digit. Then the next term in the sequence is $T_{n+1} = n_i n_i a_i a_i$, which clearly satisfies $P(T_{n+1})$ since $a_i \in \{1, 2, 4, 6\}$ and $n_i \in \{2, 4, 6\}$.

Case 2: i = 1, i.e., T_n starts with the repeated digits a_i . In this case

$$T_{n+1} = n_1 n_1 a_1 a_1 n_2 n_2 a_2 a_2 \cdots$$

with $a_k \neq a_{k+1}$ for all k. It is clear that no a_k can be contained in a piece that also contains a_{k+1} , therefore the possible pieces that n_1 and a_1 can take part to are either $n_1n_1a_1a_1n_2n_2$ (in the case $n_1 = a_1 = n_2$), or $n_1n_1a_1a_1$ (in the case $n_1 = a_1 \neq n_2$), or n_1n_1 and $a_1a_1 / a_1a_1n_2n_2$ (respectively in the cases $n_1 \neq a_1 \neq n_2$ and $n_1 \neq a_1 = n_2$).

In conclusion a_1 and n_1 generate pieces made of digits in $\{1, 2, 4, 6\}$ with multiplicity either 2, 4, or 6.

- Case 3: T_n ends with a_i ; this is analogous to case 2 above.
- Case 4: The piece $a_i^{n_i}$ is neither at the end nor at the beginning of T_n . The next term in the sequence is:

$$T_{n+1} = \cdots a_{i-2}n_{i-1}n_{i-1}a_{i-1}a_{i-1}n_in_ia_ia_in_{i+1}n_{i+1}a_{i+1}a_{i+1}a_{i+1}\cdots$$

By definition of the run-length representation, $a_{i-2}, a_{i-1}, a_i, a_{i+1}$ contains no consecutive values, hence the possible pieces resulting for n_i and a_i are:

(a) $n_{i-1}n_{i-1}a_{i-1}a_{i-1}n_in_i$, and either a_ia_i or $a_ia_in_{i+1}n_{i+1}$

- (b) $a_{i-1}a_{i-1}n_in_i$, and either a_ia_i or $a_ia_in_{i+1}n_{i+1}$
- (c) $n_i n_i$, and either $a_i a_i$ or $a_i a_i n_{i+1} n_{i+1}$
- (d) $n_i n_i a_i a_i$ or $n_i n_i a_i a_i n_{i+1} n_{i+1}$
- (e) $n_i n_i$ and $a_i a_i$

In each case, since $\forall k, a_k \in \{1, 2, 4, 6\}$ and $n_k \in \{2, 4, 6\}$, we see that n_i and a_i can appear only in pieces that are of multiplicity 2, 4, or 6, and which contain numbers $\in \{1, 2, 4, 6\}$.

If $T_0 = d \in \{1, 2, 4, 6\}$, then $P(T_0)$ holds and by the above case exhaustion argument $P(T_n)$ hold for all *n*. It remains to discuss the case $T_0 = d \notin \{1, 2, 4, 6\}$. Writing the first few terms of the resulting sequence

shows that this is easily dealt with:

 $T_0 = d$ $T_1 = 11 dd$ $T_2 = 221122 dd$ $T_3 = 22222211222222 dd$...

Indeed, save for the first term, the digit d only appears as dd at the end of T_n . The rest of T_n satisfies $P(T_n)$ discussed previously.

To prove this, assume that $T_n = k || dd$, which means that T_n starts with an integer k and ends with the two digits dd, and further assume that P(k) is true and k's last piece is 2^k , with $k \in \{2,4,6\}$. Then the next term in the sequence is $T_{n+1} = k' || 22dd$, where k' is an integer that ends with the digit 2 and such that P(k') is true (thanks to what we have proved in the first part of the theorem). Let S = k' || 22. Consider three cases, as a function of the last piece of k, denoted 2^{ω} :

$$\omega = 2 \Rightarrow k'$$
 ends with 2222 $\Rightarrow S$ ends with 22222 $\Rightarrow P(S)$ holds
 $\omega = 4 \Rightarrow k'$ ends with 4422 $\Rightarrow S$ ends with 2222 $\Rightarrow P(S)$ holds
 $\omega = 6 \Rightarrow k'$ ends with 6622 $\Rightarrow S$ ends with 2222 $\Rightarrow P(S)$ holds

Since P(k') holds, the only problem in *S* was at the interface between the ending 2's of k' and the couple 22 at the end of *S*. With the exhaustion argument above we have shown that in each possible case P(S) holds. Finally the number S || dd is such that *d* only appears as a couple *dd* at the end, whereas *S* is made of digits belonging to $\{1, 2, 4, 6\}$ with multiplicities in $\{2, 4, 6\}$.

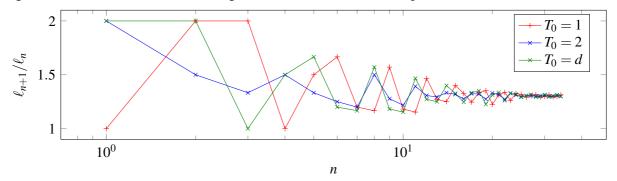
 T_0 and T_1 contain only digits $\in \{1, 2, 4, 6, d\}$. Since T_2 is of the form k || dd with P(k) true and k's last digit being 2, the above argument shows that all subsequent terms in the sequence can be written in this way, and the proof is completed.

Corollary 1. If $T_0 = d \neq 1, 2$ then LSA (T_0) gives the same sequence, save for the two last digits of each term which are dd.

Remark 1. The length sequences for $T_0 = 1, 2, d$ are respectively:

 $1,4,4,8,16,16,24,40,48,56,88,104,120,176,224,280,392,520,648,864,1168\\1,4,8,12,16,24,32,40,48,72,92,112,156,204,264,352,464,592,784,1036,1320\\1,4,8,16,16,24,40,48,56,88,104,120,176,224,280,392,520,648,864,1168,1432$

Remark 2. For all seeds *s*, LSA(*d*) grows to infinity, namely the ratio of lengths $\lambda_n = \ell_{n+1}/\ell_n$ for two consecutive terms of sequence (which is between 1 and 2) tends towards Conway's constant $\lambda \simeq 1.303577$, regardless of the seed T_0 . The following numerical simulation backs up this intuition:



This is in fact a consequence of the following result:

Theorem 2. Consider the following operations on pieces:

$$C: a^b \mapsto ba$$
 $L: a^b \mapsto b^2 a^2$ $\eta: a^b \mapsto \kappa(a)^{2l}$

with $\kappa : a \mapsto 2a$ for $a \in \{1, 2, 3\}$ and $\kappa(a) = 1$ otherwise. Then $L \circ \eta = \eta \circ C$.

Proof. Let a^b be a piece,

$$(\boldsymbol{\eta} \circ C)(a^b) = \boldsymbol{\eta}(b^1 a^1) = \boldsymbol{\eta}(b^1)\boldsymbol{\eta}(a^1) = \boldsymbol{\kappa}(b)^2 \boldsymbol{\kappa}(a)^2$$
$$(L \circ \boldsymbol{\eta})(a^b) = L(\boldsymbol{\kappa}(a)^{\boldsymbol{\kappa}(b)}) = \boldsymbol{\kappa}(b)^2 \boldsymbol{\kappa}(a)^2.$$

A result of this theorem is that LSA is equivalent to Conway's sequence, where κ and η allow us to translate from one to the other. Then LSA inherits many of the properties that are known of Conway's sequence: decomposition into "elements", convergence to λ , and so forth.

Remark 3. This would also work with $L(a^b) = b^3 a^3$ and $\kappa(a) = 3a$ for $a \in \{1, 2, 3\}$ (i.e. an LSA variant where elements are repeated three times instead of two). However the argument breaks down for a "look-and-say four times" sequence. We leave this sequence and the study of its properties open for further research.

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