PATTERNS IN SHI TABLEAUX AND DYCK PATHS

Myrto Kallipoliti, Robin Sulzgruber and Eleni Tzanaki

Abstract. Shi tableaux are special binary fillings of certain Young diagrams which arise in the study of Shi hyperplane arrangements related to classical root systems. For type A, the set $\mathcal{T}$ of Shi tableaux naturally coincides with the set of Dyck paths, for which various notions of patterns have been introduced and studied over the years. In this paper we define a notion of pattern occurrence on $\mathcal{T}$ which, although it can be regarded as a pattern on Dyck paths, it is motivated by the underlying geometric structure of the tableaux. Our main goal in this work is to study the poset of Shi tableaux defined by pattern-containment. More precisely, we determine explicit formulas for upper and lower covers for each $T \in \mathcal{T}$, we consider pattern avoidance for the smallest non-trivial tableaux (size 2) and generalize these results to certain tableau of larger size. We conclude with open problems and possible future directions.

1. Introduction

The investigation of patterns in families of discrete objects is an active topic in Combinatorics, with connections to various areas in Mathematics, as well as other fields such as Physics, Biology, Sociology and Computer science [11, 12, 15]. Generally speaking, the notion of pattern occurrence or pattern avoidance can be described as the presence or, respectively, absence of a substructure inside a larger structure. Patterns were first considered for permutations: an occurrence of a pattern σ in a permutation π is a subword of π whose letters appear in the same relative order as those in σ. For instance, the permutation 132 occurs as a pattern in 32514 since the subsequence 254 (among others) is ordered in the same way as 132 (see Figure 1 for an illustration using matrix representations). The systematic investigation of

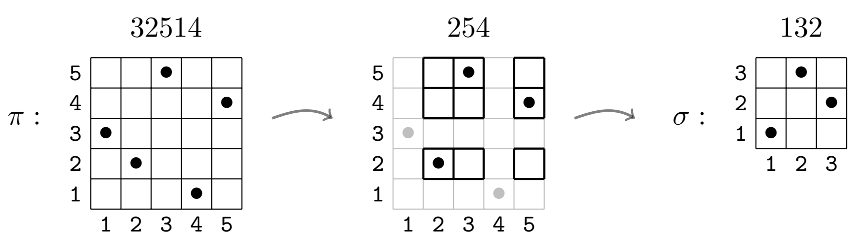


Figure 1. An occurrence of the pattern $\sigma = 132$ in $\pi = 32514$. σ is obtained by deleting the first and forth column, as well as the third and fifth row of π.

patterns in permutations and more generally in words began in the 70s with the work of Knuth on sorting permutations using data structures [13], and later with the work of Simion and Schmidt [18]. In the last decade the study of patterns in permutations and words has grown explosively (see [5, 12]) and has been extended in the context of various other structures such as set partitions, trees, lattice paths, fillings of Young diagrams, just to name a few. For instance, in [21] Spiridonov considered a notion of pattern occurrence for binary fillings of grid shapes, which naturally generalizes permutation pattern occurrence as follows: an occurrence of a pattern in a filling of a grid shape T is a filling of a sub-shape (or *minor shape*) S of T, obtained by removing some rows and columns of T (see Figure 2 for two such examples). In a totally different context, Bacher et al. in [4] considered a notion of pattern occurrence for Dyck paths, i.e., paths on the discrete plane from $(0, 0)$ to $(2n, 0)$, $n \in \mathbb{N}$, consisting of up-steps $U\colon(1, 1)$ and down-steps $D\colon(1, -1)$ never going below the x-axis. The pattern occurrence is defined by deleting U and

Key words and phrases. patterns, Dyck paths, Shi tableau.

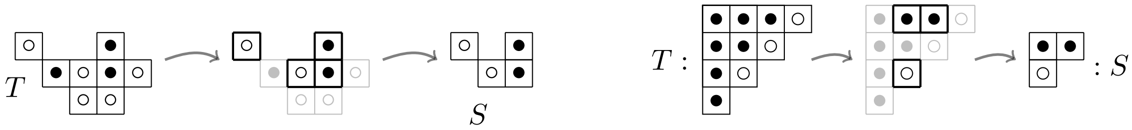


FIGURE 2. Two examples of pattern occurrence in binary fillings of grid shapes in the sense of [21]. In both cases, S is obtained by deleting two columns and a row of T.

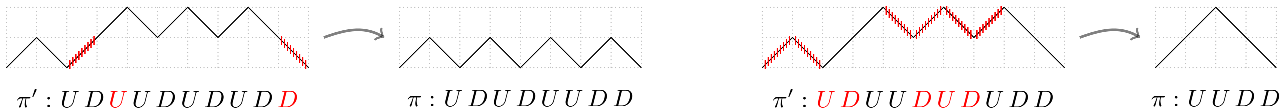


FIGURE 3. Two instances of pattern occurrence in the context of [4]. In both examples, π is a shorter Dyck path obtained from the original path π' by deleting the steps in red.

D-steps so that the resulting path is again a Dyck path (with fewer steps). See Figure 3 for an example of a Dyck path and a pattern occurrence.

Inspired by [4] and [21], we introduce a notion of patterns for the so called *Shi tableaux*, which are structures defined in [7] to encode dominant regions of the (m-extended) Shi arrangement. For $m = 1$ Shi tableaux can easily be described as binary fillings of certain Young diagrams. More precisely, a *Shi tableau* of size n is a binary filling with $\bullet$ or $\circ$ of a staircase Young diagram of shape $(n, n-1, \dots, 1)$, satisfying the property that if a cell contains $\bullet$ then all cells above and to its left contain $\bullet$. It is straightforward to see (cf. Figure 4) that Shi tableaux of size n biject to Dyck paths of semilength $n+1$.

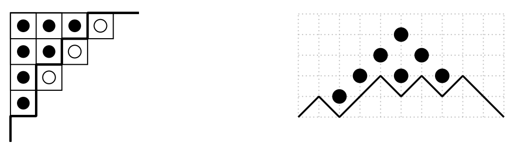


FIGURE 4. A Shi tableau $T \in \mathcal{T}_4$ and the corresponding Dyck path $\pi \in \mathcal{D}_5$.

In accordance with Spiridonov's definition [21], the notion of pattern occurrence in Shi tableaux can be defined by deleting columns and rows. However, in our pattern we impose a stronger condition; we allow deletions of rows and columns after which the underlying Young diagram is again a staircase Young diagram (see Section 2.2 for the precise definition). Part of the motivation of this work lies in [4, §6], where Bacher et al. ask whether it is possible to transport the pattern order on Dyck paths along some of the bijections between Dyck paths and other members of the Catalan family in order to obtain interesting order structures on different combinatorial objects. So far, our intuition to work on this subject has been affirmed by the enumerative results presented here, which hint at interesting connections to the theory of pattern avoidance for permutations. This is perhaps not surprising in view of the fact that one of the original problems of this area, the enumeration of 312-avoiding permutations, is also related to Dyck paths. Furthermore, there are ties to algebraic objects that arise in connection to crystallographic root systems. For example, *ad*-nilpotent ideals of a Borel subalgebra of the complex simple Lie algebra of type A with a bounded class of nilpotence, studied in [2], can be described using pattern avoidance for Shi tableaux.

This paper is organized as follows. In Section 2 the basic definitions and notation, including our definition of pattern occurrence on Shi tableaux, are provided. In Section 3 we give explicit formulas for the number of upper and lower covers for each Shi tableau, in the poset defined by pattern-containment. In Section 4 we give precise characterizations of pattern avoidance for each Shi tableaux of size 2 and generalize these results for certain tableaux of larger size. In the process we encounter *ad*-nilpotent ideals and the bijection on Dyck paths known as the zeta map. We complete the paper with a short discussion of open problems in Section 5 including possible connections to permutations avoiding a pair of patterns.

2. Preliminaries

2.1. **Dyck paths.** A *Dyck path* of *semilength* n is a path on the plane, from $(0,0)$ to $(2n,0)$, consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ that never go below the x-axis. It is well-known that the cardinality of the set $\mathcal{D}_n$ of Dyck paths of semilength n, is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Replacing each up-step of a Dyck path $\pi \in \mathcal{D}_n$ with the letter U and each down-step with the letter D, π can also be written as a word with $2n$ letters on the alphabet $\{U,D\}$. Clearly, a word $w \in \{U,D\}^n$ corresponds to a Dyck path if and only if each initial subword of w contains at least as many letters U as letters D. Another convenient way to encode paths in $\mathcal{D}_n$ is in terms of standard Young tableaux of size $2 \times n$, i.e., arrangements of the numbers $1,2,\ldots,2n$ in a $2 \times n$ rectangle so that each row and each column is increasing. The correspondence to Dyck paths is as follows: in the top (resp. bottom) row of the $2 \times n$ tableau we register in increasing order the positions of the U-steps (resp. D-steps) of the Dyck path (see Figure 5).

The *height* of a Dyck path is the highest y-coordinate attained in the path. A *return step* is a downstep that returns the path to the ground level. Let $\mathcal{H}(n,k)$ denote the set of Dyck paths of semilength n and height at most k. It is not hard to see that $|\mathcal{H}(n+1,k)| = \sum_{i=0}^{n} |\mathcal{H}(i,k)||\mathcal{H}(n-i,k-1)|$, with initial conditions $|\mathcal{H}(0,k)| = 1$ for all $k \geq 0$, and $|\mathcal{H}(n,0)| = 0$ for all $n > 0$ (see [6]).

The *bounce path* $b(\pi)$ of a Dyck path π is described by the following algorithm: starting at $(0,0)$ we travel along the up-steps of π until we encounter the beginning of a down-step. Then, we turn and travel down until we hit the x-axis. Then, we travel up until we again encounter the beginning of a down-step of π, we then turn down and travel to the x-axis, etc. We continue in this way until we arrive at $(0,2n)$. For instance, if $\pi = UDUUDUDUDD$, then $b(\pi) = UDUUDDUUDD$. A *peak at height* k of a Dyck path π is a point (x_0,k) of π which is immediately preceded by an up-step and immediately succeeded by a down-step. Similarly a *valley at height* k of π is a point $(x_0,k) \in \pi$ that is immediately preceded by a down-step and immediately followed by an up-step. Viewing the path π as a word, a peak is an occurrence of a UD and its height is the number of U's minus the number of D's that precede the peak. A valley is an occurrence of DU and its height is defined analogously.

2.2. **Shi tableaux and our poset structure.** In order to switch from Dyck paths of semilength $n+1$ to Shi tableaux of size n and vice versa, we make the convention that each Young diagram of shape $(n,\ldots,1)$ contains an additional empty row and empty column and we label rows from bottom to top and columns from left to right (so that, for $i = 1,\ldots,n+1$, the i-th row has $i-1$ boxes and the i-th column has $n+1-i$ boxes, as shown in Figure 5). In this way, the i-th D-step (U-step) of the Dyck path π is the horizontal (vertical) unit step on the i-th row (column). We denote by $\mathcal{T}_n$ the set of all Shi tableaux of size n and by $\mathcal{T}$ the set of all Shi tableaux of all sizes.

Our main goal in this paper is to introduce and study a new notion of *patterns* on the set $\mathcal{T}$ of Shi tableaux. The pattern containment is described in terms of two types of deletions on Shi tableaux which we call *bounce deletions*. For $2 \leq i \leq n+1$, we denote by $d_{i,i-1}$ the action of deleting the i-th row and the $(i-1)$-st column of T, and for $1 \leq i \leq n+1$ we denote by $d_{i,i}$ the action of deleting the i-th row and the i-th column of T (see Figure 5). The special cases $d_{i,i}$ with $i = 1$ or $n+1$, can be thought of as deleting the first column and top row respectively. Viewing the tableau T as a Dyck path π and indexing its U and D-steps from 1 to $n+1$, it is immediate to see that $d_{i,i-1}$ deletes U_i, D_{i-1}, whereas $d_{i,i}$ deletes U_i, D_i from π. Another way to define the actions $d_{i,i-1}, d_{i,i}$ is through the $2 \times (n+1)$ standard Young tableaux $S(T)$ of π. The standard Young tableau of $d_{i,i}(T)$ is obtained by deleting the i-th column of the standard $2 \times (n+1)$ tableau $S(T)$ of π and adjusting the larger entries. The standard Young tableau of $d_{i,i-1}(T)$ is obtained by deleting the i-th entry of the first row and $(i-1)$-st entry of the second row of $S(T)$ and adjusting the larger entries (see Figure 5).

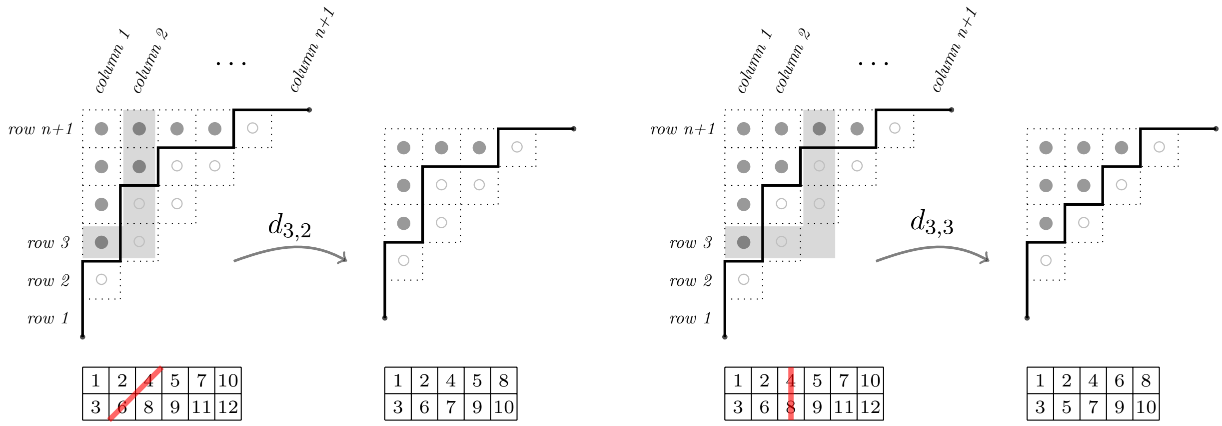


FIGURE 5. The bounce deletions $d_{3,2}$ and $d_{3,3}$ on the Dyck path $UUDUUDUDDUDD$ results in $UUDUUDDUDD$ and $UUDUDUDUDD$ respectively.

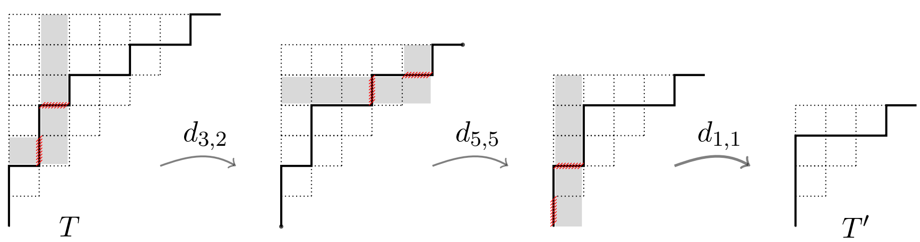


FIGURE 6

Using bounce deletions we can endow $\mathcal{T}$ with a poset structure. For any undefined terminology on posets we refer the reader to [24, Section 3].

Definition 2.1. *The set $\mathcal{T}$ of Shi tableaux becomes a poset by declaring that T covers T' if T' is obtained from T after a bounce deletion. We say that T'* occurs as a pattern *in T if $T' \preceq_{\mathcal{T}} T$, i.e., if T' can be obtained from T after an iteration of bounce deletions.*

See Figure 6 for an example of pattern occurrence in Shi tableaux. Notice that if the pattern T' occurs in T then, in terms of Dyck paths, T' occurs as a pattern in T in the sense of [21]. Indeed, the actions $d_{i,i-1}, d_{i,i}$ delete pairs of U, D so that the resulting path is a again a Dyck path (which is precisely the requirement of pattern occurrence in [21]). The reverse is not always true, since in our pattern definition the deleted pair U, D should obey stronger restrictions. For example, the path $\pi' = UUDD$ occurs as a pattern in $\pi = UDUDUD$ in the sense of [21] but not in the sense of Definition 2.1.

Remark 2.2. *Although we gave the definition of Dyck paths in terms of unit steps $(1, 1)$ and $(1, -1)$ and endpoints on the x-axis, in the remainder of the paper we rotate the setting by 45 degrees (see Figure 4). In other words, unless stated otherwise, we align all Dyck paths so that their unit steps are $(1, 0), (0, 1)$ and their endpoints lie on the main diagonal $x = y$.*

2.3. **Geometric interpretation of our poset structure.** In what follows, we discuss the poset structure $(\mathcal{T}, \preceq_{\mathcal{T}})$ and more precisely the cover relations from the viewpoint of Shi arrangements.

A hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong \mathbb{R}^n$. The *regions* of $\mathcal{A}$ are the connected components defined by the complement of the hyperplanes in $\mathcal{A}$. The *rank* $r(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A} \subseteq \mathbb{R}^n$ is the dimension of the space A spanned by the normals to the hyperplanes in $\mathcal{A}$. The *intersection poset* $\mathcal{L}_{\mathcal{A}}$ of $\mathcal{A}$ is the set of all intersections of subcollections of hyperplanes in $\mathcal{A}$, partially ordered by reverse inclusion. The poset $\mathcal{L}_{\mathcal{A}}$ is a lattice that captures all the

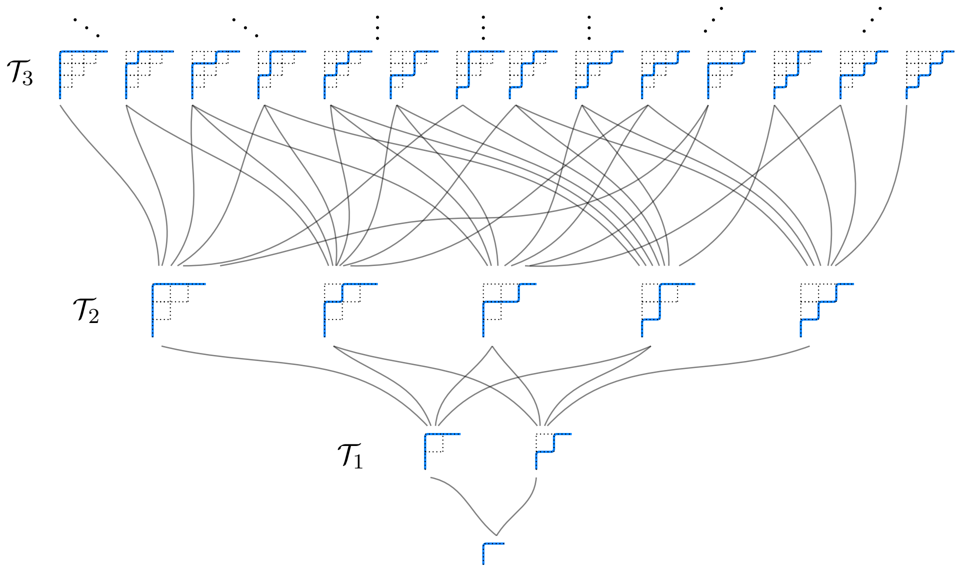


FIGURE 7. The first few levels of the poset $(\mathcal{T}, \preceq_{\mathcal{T}})$ of Shi tableau

combinatorial structure of $\mathcal{A}$. We say that two hyperplane arrangements are *combinatorially equivalent* if they have the same intersection lattice [23]. If the rank $r(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is smaller than the dimension of the ambient space $\mathbb{R}^n$, we can *essentialize* $\mathcal{A}$, i.e., intersect $\mathcal{A}$ with a subspace W of $\mathbb{R}^n$ without changing the combinatorial structure.

To do so, we chose a subspace Y complementary to A[1] and we define

$$W := Y^{\perp} = \{x \in \mathbb{R}^n : x \cdot y = 0 \text{ for all } y \in Y\},$$

where $x \cdot y$ denotes the standard Euclidean inner product. The set $\mathcal{A}_W = \{H \cap W : H \in \mathcal{A}\}$ is an *essentialization of* $\mathcal{A}$, i.e., a hyperplane arrangement in W combinatorially equivalent to $\mathcal{A}$ with $r(\mathcal{A}_W) = \dim(\mathcal{A}_W)$ [23, Section 1.1]. Notice that, if we do the above steps with A being a *proper* rather than the *whole* space spanned by the normals to the hyperplanes in $\mathcal{A}$, we again obtain a hyperplane arrangement $\mathcal{A}_W$ which is combinatorially equivalent to $\mathcal{A}$ but with $r(\mathcal{A}_W) < \dim(\mathcal{A}_W)$.

The *Shi arrangement* of type A_{n-1}, denoted by $\mathsf{Shi}(n)$, is the hyperplane arrangement in $\mathbb{R}^n$ consisting of the hyperplanes $x_i - x_j = 0, 1$ for all $1 \leq i < j \leq n$. It is well known that the regions of $\mathsf{Shi}(n)$ are in bijection with parking functions on $[n]$, thus counted by $(n+1)^{n-1}$ [22, Section 5]. Here, we focus on the set of *dominant* regions of $\mathsf{Shi}(n)$ which are those regions contained in the *dominant cone* $\mathcal{C}_n : x_1 > x_2 > \cdots > x_n$. The set of dominant regions is enumerated by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and can be encoded using Shi tableaux in $\mathcal{T}_{n-1}$ [17]. More precisely, each $T \in \mathcal{T}_{n-1}$ corresponds to the dominant region $\mathcal{R}(T)$ with defining inequalities

$$\begin{cases} 0 < x_i - x_j < 1 & \text{if the cell } (n-i+1, n-j+1) \text{ is empty} \\ 1 < x_i - x_j & \text{if the cell } (n-i+1, n-j+1) \text{ is full,} \end{cases} \tag{1}$$

where, for the position of each cell, we keep our earlier numbering on rows and columns (see Figure 8).

Shi arrangements form an *exponential sequence of arrangements (ESA)*, a family of hyperplane arrangements which posses a strong combinatorial symmetry [23, Section 5.3]. More precisely, $\mathsf{Shi}(n)$ has the property that, for each $S \subseteq [n]$, its subarrangement $\mathsf{Shi}(S)$ consisting of the hyperplanes $x_i - x_j = 0, 1$ for $i, j \in S$, is *combinatorially equivalent* to $\mathsf{Shi}(|S|)$.

Let us apply the properties of ESA to $\mathsf{Shi}(n)$. If $S_k := [n] \setminus \{k\}$ then $\mathsf{Shi}(S_k)$ is the subarrangement of $\mathsf{Shi}(n)$ from which we have deleted all hyperplanes whose equation involves x_k. It is immediate to see

[1] not necessarily the orthogonal complement

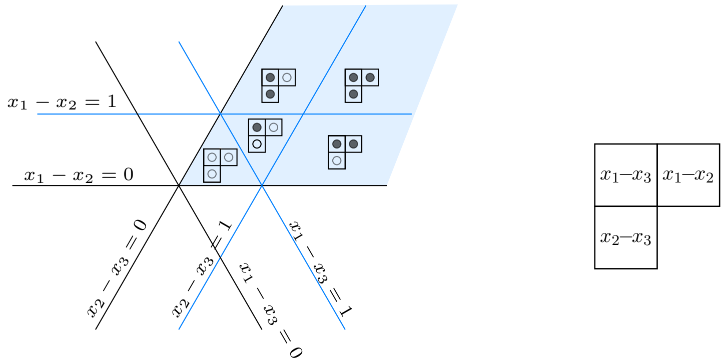
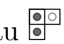


FIGURE 8. The dominant regions in $\mathsf{Shi}(3)$ and their correspondence to Shi tableaux. For example, the tableau  corresponds to the dominant region defined by the inequalities $x_1 - x_3 > 1$, $x_2 - x_3 > 1$ and $0 < x_1 - x_2 < 1$.

that the deleted hyperplanes are exactly the ones removed by the deletion $d_{i,i}$ with $i = n - k + 1$ (see Figure 9(a)). Therefore, the bounce deletions of type $d_{i,i}$ capture such instances of pattern occurrence.

The other type of bounce deletions do not constitute an instance of ESA, however they behave likewise if restricted to a certain hyperplane. To be more precise, let $\mathsf{Shi}(S'_k)$ be the subarrangement of $\mathsf{Shi}(n)$ from which we have deleted all hyperplanes $x_i - x_{k+1} = 0, 1$ $(i \leq k)$ or $x_k - x_i = 0, 1$ $(i > k)$.

It is immediate to see that the deleted hyperplanes are exactly the ones removed by the deletion $d_{i,i-1}$ with $i = n-k+1$ (see Figure 9(b)). Although $\mathsf{Shi}(S'_k)$ is not combinatorially equivalent to $\mathsf{Shi}(n-1)$, it can be shown that it becomes so if we intersect with the hyperplane $x_k - x_{k+1} = 0$. To see this denote this hyperplane by W and let $Y = \mathbb{R}(e_k - e_{k+1})$ be the span of its normal vector such that $W = Y^{\perp}$. Since Y is not contained in the span of the normals to the hyperplanes in $\mathsf{Shi}(S_k)$ it follows from the above discussion that $\mathsf{Shi}(S_k)_W$ and $\mathsf{Shi}(n-1)$ are combinatorially equivalent. But evidently $\mathsf{Shi}(S_k)_W$ is equal to the restriction $\{H \cap W : H \in \mathsf{Shi}(S'_k)\}$ of $\mathsf{Shi}(S'_k)$ to W.

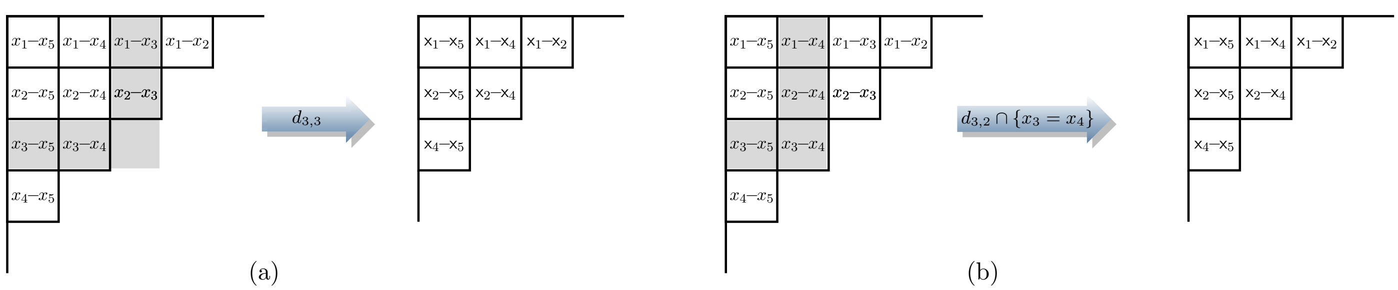


FIGURE 9

3. Cover relations

In this section we compute the number of upper and lower covers for each Shi tableau T in the poset $(\mathcal{T}, \preceq_{\mathcal{T}})$. To do so, we define irreducible and strongly irreducible Shi tableaux and we find closed/recursive formulas for their number of lower/upper covers. Then, we explain how each Shi tableau is decomposed into its irreducible components and how the lower/upper covers are computed in terms of the upper/lower covers of the decomposition.

We start with notation and definitions that will be used throughout the section. In this section, we find it more convenient to use the notation π for Dyck paths, rather than T for Shi tableaux, since most of the arguments use the realization of π as a word in $\{U, D\}^*$. However, the underlying tableau of π and more precisely its columns and rows is present in all the proofs.

Let T be a Shi tableau of size n and denote by $\pi \in \mathcal{D}_{n+1}$ the corresponding Dyck path. Let us denote by C_k the k-th column of the underlying tableau of $\pi \in \mathcal{D}_{n+1}$ and index the U-steps and D-steps of π from 1 to $n+1$. The portion of the path π contained in column C_k consists of all the steps that satisfy $k-1 \leq x \leq k$. In other words $\pi \cap C_k$ is the subpath between (and not including) D_{k-1} and D_{k+1}. An *ascent* of a Dyck path π is a maximal string of consecutive U-steps and a *descent* is a maximal string of consecutive D-steps of π. Each Dyck path π can be written as a concatenation of ascents and descents, i.e.,

$$\pi = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \cdots U^{a_\ell} D^{b_\ell} \quad \text{for some} \quad 1 \leq \ell \leq n+1 \quad \text{and} \quad a_i, b_i \geq 1. \tag{2}$$

For each ascent U^{a_i} of π we define a subpath $\overline{\pi}_i$ of π as follows:

$$\overline{\pi}_1 = \pi \cap \big(\bigcup_{r=1}^{a_1} C_r\big) \qquad \text{and} \qquad \overline{\pi}_i = \pi \cap \big(\bigcup_{r=0}^{a_i} C_{a_1+\cdots+a_{i-1}+r}\big) \quad \text{if} \quad i \geq 2 \tag{3}$$

(see Figure 11). As we will explain subsequently, the subpaths $\overline{\pi}_i$ are involved in the computation of the lower and uppers covers of π. For example, for the lower covers, $\overline{\pi}_i$ is the range of possible D-step deletions that correspond to the deletion of a U-step from the ascent U^{a_i} of π.

We say that $\pi \in \mathcal{D}_{n+1}$ is *irreducible* if it does not touch the the line $y = x$ except for the origin and the final point and *strongly irreducible* if it does not touch the the line $y = x + 1$ except for the first and last step. If π is irreducible then $b_1 + \cdots + b_i < a_1 + \cdots + a_i$ and if π is strongly irreducible then $b_1 + \cdots + b_i + 1 < a_1 + \cdots + a_i$ for all i with $1 \leq i < \ell$.

The situation where a Shi tableau T can be obtained from a tableau T' by two different bounce deletions is rather restrictive. The lemma below describes the conditions when this is possible. Pictorially, the following lemma implies that, if two bounce deletions produce the same lower cover, then they both delete a U and a D-step from the same ascent and descent respectively. The only exception is the case where π is symmetric, where deletion of first column or last row produce the same lower cover (see Figure 10). Given a word $w \in \{U, D\}^*$ let $\#U(w)$ denote the number of U-steps in w, and let $\#D(w)$ denote the number of D-steps in w.

Lemma 3.1. *Consider $i, j \in [n]$ with $i < j$ and let $k_i \in \{i-1, i\}$ and $k_j \in \{j-1, j\}$. Let π be a Dyck path with U and D-steps indexed by $U_1, \ldots, U_n, D_1, \ldots, D_n$, and let π' be a Dyck path obtained from π by deleting U_i and D_{k_i} and also by deleting U_j and D_{k_j}. That is, $\pi' = d_{i,k_i}(\pi) = d_{j,k_j}(\pi)$.*

- *(i) If D_{k_i} occurs after U_j, then U_i and U_j belong to the same ascent of π. Furthermore D_{k_i} and D_{k_j} belong to the same descent of π.*
- *(ii) If D_{k_i} occurs before U_j, then the segment of π connecting U_i and D_{k_j} is of the form $U^r(UD)^\ell D^t$ for some $r, \ell, t \in \mathbb{N}$ with $\ell > 0$. Furthermore this segment begins at height $i - k_i \in \{0, 1\}$ and ends at height $j - k_j \in \{0, 1\}$.*

Proof. First assume that we are in case (i). Then there exist unique (possibly empty) words $\alpha, x, y, z, \beta \in \{U, D\}^*$ such that

$$\pi' = \alpha x y z \beta \qquad \text{and} \qquad \pi = \alpha U x y D z \beta = \alpha x U y z D \beta$$

such that α contains $i-1$ U-steps, αx contains $j-1$ U-steps, $\alpha U x y$ contains $k_i - 1$ D-steps, and $\alpha x U y z$ contains $k_j - 1$ D-steps. In particular the two words

$$\begin{array}{cccccccccccc} U & x_1 & \cdots & x_{r-1} & x_r & y_1 & \cdots & y_s & D & z_1 & \cdots & z_{t-1} & z_t \\ x_1 & x_2 & \cdots & x_r & U & y_1 & \cdots & y_s & z_1 & z_2 & \cdots & z_t & D \end{array}$$

agree. Consequently $x = U^r$ and $z = D^t$ for some $r, t \in \mathbb{N}$.

(i) (ii)

FIGURE 10. The two situations described in Lemma 3.1

Next assume we are in case (ii). Then there exist unique (possibly empty) words $\alpha, x, y, z, \beta \in \{U, D\}^*$ such that

$$\pi' = \alpha xyz\beta \qquad \text{and} \qquad \pi = \alpha UxDyz\beta = \alpha xyUzD\beta$$

such that α contains $i-1$ U-steps, αUx contains $k_i - 1$ D-steps, αxy contains $j-1$ U-steps, and $\alpha xyUz$ contains $k_j - 1$ D-steps. In particular the two words

$$\begin{array}{cccccccccccccccc} U & x_1 & \cdots & x_{r-1} & x_r & D & y_1 & \cdots & y_{s-2} & y_{s-1} & y_s & z_1 & \cdots & z_{t-1} & z_t \\ x_1 & x_2 & \cdots & x_r & y_1 & y_2 & y_3 & \cdots & y_s & U & z_1 & z_2 & \cdots & z_t & D \end{array}$$

agree. It follows from

$$D = y_2 = y_4 = \dots$$

and

$$U = y_{s-1} = y_{s-3} = \dots$$

that s is even, say $s = 2\ell$ for some $\ell \in \mathbb{N}$, and $y = (UD)^\ell$. The assumption that the j-th D-step in π occurs after the k_i-th U-step implies $\ell > 0$. Moreover $x = U^r$ and $z = D^t$ for some $r, t \in \mathbb{N}$. We have

$$i - k_i = \#U(\alpha) - \#D(\alpha Ux) = \#U(\alpha) - \#D(\alpha).$$

Thus α ends at height $i - k_i$. Similarly

$$j - k_j = \#U(\alpha xy) - \#D(\alpha xyUz) = \#U(\alpha xyUzD) - \#D(\alpha xyUzD).$$

Thus β starts at height $j - k_j$. □

3.1. **Lower covers.** Our goal in this subsection is to describe and count the elements of the set $\mathcal{LC}(\pi)$ of lower covers of π. To this end, we write d_{s,k_s} where $k_s \in \{s, s-1\}$ and we group the lower covers of π according to the ascent from which we delete a U-step. More precisely, for each ascent U^{a_i} we define the set $\mathcal{LC}_i(\pi) \subseteq \mathcal{D}_n$ as follows:

$$\begin{aligned} \mathcal{LC}_1(\pi) &:= \{d_{s,k_s}(\pi) \text{ for } 1 \le s \le a_1\} && \text{if } i = 1 \\ \mathcal{LC}_i(\pi) &:= \{d_{s,k_s}(\pi) \text{ for } a_1 + \dots + a_{i-1} + 1 \le s \le a_1 + \dots + a_i\} && \text{if } 2 \le i \le \ell. \end{aligned} \tag{4}$$

Clearly, the lower covers of π are all paths in the union

$$\mathcal{LC}(\pi) = \bigcup_{i=1}^{\ell} \mathcal{LC}_i(\pi). \tag{5}$$

To describe the paths in each $\mathcal{LC}_i(\pi)$ with respect to the original path π, notice that the range of s for $d_{s,k_s}(\pi) \in \mathcal{LC}_i(\pi)$ is the index of all U-steps U_s in U^{a_i}. Since $d_{s,k_s}(\pi)$ acts by deleting the pair U_s, D_{k_s}

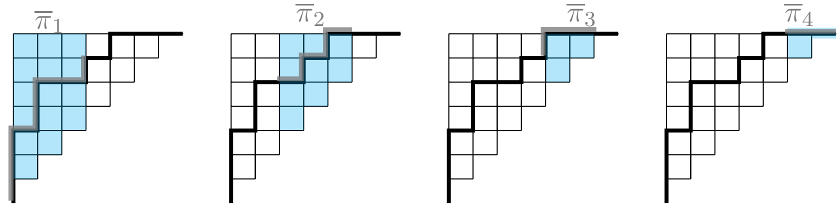


FIGURE 11. The path $\pi = U^3DU^2D^2UDUD^3$ and the subpaths $\overline{\pi}_i$ defined in (3).

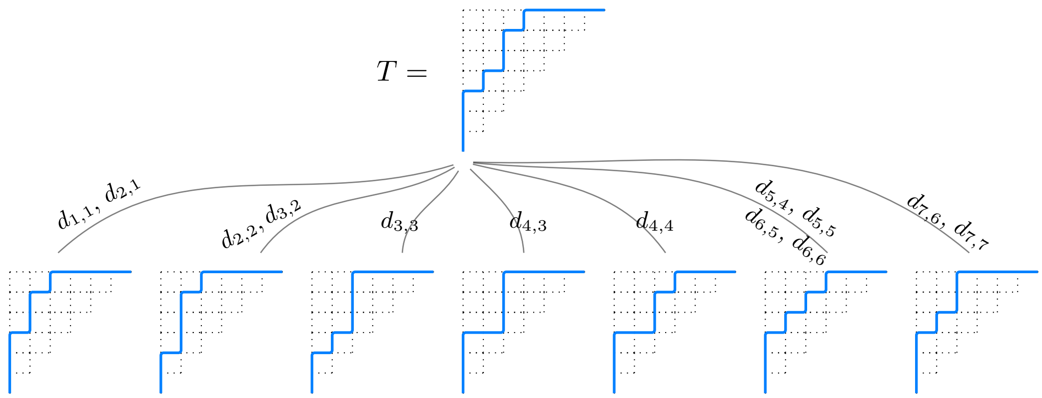


FIGURE 12. The tableau T corresponding to the path $\pi = UUUDUDUUDUDDDD$ and its lower covers. Since π is irreducible and has four ascents, its lower covers are divided into four pairwise disjoint sets $\mathcal{LC}_i(\pi)$ as follows. The first tree lower covers belong to $\mathcal{LC}_1(\pi)$, the next two in $\mathcal{LC}_2(\pi)$, the next in $\mathcal{LC}_3(\pi)$ and the last one in $\mathcal{LC}_4(\pi)$.

with $k_s \in \{s, s-1\}$, we deduce that all paths $\pi' \in \mathcal{LC}_i(\pi)$ are obtained from π by replacing U^{a_i} with $U^{a_i - 1}$ and deleting a D-step $D_{s'}$ where

$$\begin{aligned} 1 \le s' &\le a_1 && \text{if } \ i = 1 \quad \text{or} \\ a_1 + \cdots + a_{i-1} \le s' &\le a_1 + \cdots + a_i && \text{if } \ 2 \le i \le \ell. \end{aligned} \tag{6}$$

This means that the range of possible D-step deletions in $\mathcal{LC}_i(\pi)$ is precisely the set of D-steps of the subpath $\overline{\pi}_i$ described in (3). In other words, $\mathcal{LC}_i(\pi)$ is the set of paths obtained from π after deleting a U-step from the i-th ascent U^{a_i} and a D-step from $\overline{\pi}_i$.

Lemma 3.2. *Let $\pi = U^{a_1}D^{b_1}\dots U^{a_\ell}D^{b_\ell}$ be a Dyck path in $\mathcal{D}_{n+1}$.*

(i) If π is strongly irreducible with $\pi \neq U^a(DU)^{n+1-a}D^a$, the sets in the union (5) are pairwise disjoint.
(ii) If $\pi = U^a(DU)^{n+1-a}D^a$ with $2 < a < n+1$ then the sets in the union (5) are pairwise disjoint except for $\mathcal{LC}_1(\pi)$ and $\mathcal{LC}_\ell(\pi)$ which both contain the path $U^a(UD)^{n-a}D^a$.

Proof. Let $i < j$ and suppose that $\pi' \in \mathcal{LC}_i(\pi) \cap \mathcal{LC}_j(\pi)$. Then π' can be obtained from π by two bounce deletions d_{I,k_I} and d_{J,k_J} such that the I-th and J-th U-steps of π do not belong to the same acsent of π. By Lemma 3.1 this implies that π is of the form $\pi = \alpha U^r(DU)^\ell D^t\beta$ for some words $\alpha, \beta \in \{U, D\}^*$, and $r, \ell, t \in \mathbb{N}$ with $\ell > 0$, such that α ends at height $I - k_I \in \{0, 1\}$, and β starts at height $J - k_J \in \{0, 1\}$. If π is strongly irreducible then the only possibilities for the words α, β are $\alpha = U$ if $k_I = I - 1$ or α is empty if $k_I = I$. Similarly $\beta = D$ if $k_J = J - 1$ and β is empty if $k_J = J$. The claims follow. ☐

Using the above lemma, we can prove the following proposition, which counts the number of lower covers of all strongly irreducible Dyck paths.

Theorem 3.3.

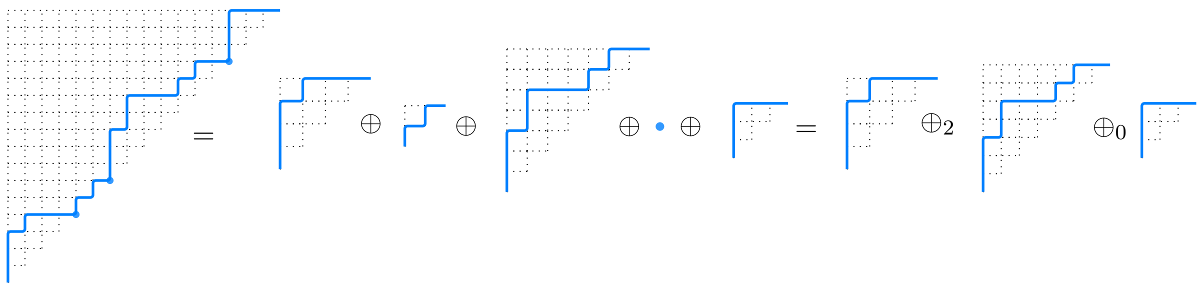


FIGURE 13. The irreducible decomposition of a path. The second and fourth components are connecting components while all others are irreducible components.

(i) If $\pi \in \mathcal{D}_{n+1}$ is strongly irreducible, the number of its lower covers is

$$|\mathcal{LC}(\pi)| = |peaks(\pi)| + |valleys(\pi)|$$

unless $\pi = U^a(DU)^{n+1-a}D^a$ for some $a > 2$ in which case

$$|\mathcal{LC}(\pi)| = |peaks(\pi)| + |valleys(\pi)| - 1 = 2(n+1-a).$$

(ii) If $\pi = (UD)^{n+1}$ or $\pi = U^{n+1}D^{n+1}$ then $|\mathcal{LC}(\pi)| = 1$.
(iii) If $\pi = U(UD)^nD$ and $n > 0$ then $|\mathcal{LC}(\pi)| = n$.

Proof. To prove (i), recall that all Dyck paths in $\mathcal{LC}_i(\pi)$ are obtained from π by replacing U^{a_i} by U^{a_i-1} and deleting a D-step from $\overline{\pi}_i$. The different ways to delete a D-step from $\overline{\pi}_i$ are as many as the ways to delete a D-step from a different descent of $\overline{\pi}_i$, which implies that $|\mathcal{LC}_i(\pi)| = |descents(\overline{\pi}_i)|$. Therefore, in view of Lemma 3.2, we have that $|\mathcal{LC}(\pi)| = \sum_{i=1}^{\ell} |\mathcal{LC}_i(\pi)| = \sum_{i=1}^{\ell} |descents(\overline{\pi}_i)|$. Next, notice that for all pairs i, j we have $\overline{\pi}_i \cap \overline{\pi}_j = \emptyset$ except for $j = i+1$ where $\overline{\pi}_i \cap \overline{\pi}_{i+1} = \pi \cap C_{a_1+\cdots+a_i}$. In other words, $\overline{\pi}_i, \overline{\pi}_{i+1}$ share a column and thus a descent. We can therefore rewrite the above sum as $|\mathcal{LC}(\pi)| = |descents(\pi)| + \ell - 1 = 2\ell - 1 = |peaks(\pi)| + |valleys(\pi)|$.

For (ii), it is immediate to see that the unique lower cover of $(UD)^{n+1}$ and $U^{n+1}D^{n+1}$ is $(UD)^n$ and U^nD^n respectively. For (iii), we leave it to the reader to check that the lower covers of $\pi = U(UD)^nD$ are the paths obtained from π by replacing a $UDUD$ by DU, or the path $U(UD)^{n-1}D$, which are altogether n. □

An *irreducible decomposition* $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ of a Dyck path π is the way to write π as a concatenation of Dyck paths π_i, where each π_i is either an irreducible Dyck path or a, possibly empty, sequence of peaks at height 1[2]. In the first case we say that π_i is an *irreducible component* and in the latter a *connecting component* of π. If π_1, π_3 are irreducible Dyck paths and π_2 is a sequence of k peaks at height 1, we abbreviate $\pi_1 \oplus \pi_2 \oplus \pi_3$ as $\pi_1 \oplus_k \pi_3$ (see Figure 13). Next, we introduce the symbol $\oplus'$ to denote concatenation of paths whose unique common point lies on $y = x + 1$. More precisely, a *strongly irreducible decomposition* of an *irreducible* Dyck path π is a way to write $\pi = U\pi_1 \oplus' \pi_2 \oplus' \cdots \oplus' \pi_k D$ as a concatenation of paths π_i, where each $U\pi_iD$ is either a strongly irreducible Dyck path or a non empty sequence of peaks at height 2 (see Figure 14). We separate the components of a strongly irreducible decomposition to *strongly irreducible* and *connecting* ones. We also abbreviate $U\pi_1 \oplus' \pi_2 \oplus' \pi_3 D$ to $U\pi_1 \oplus'_k \pi_3 D$ in the case where $U\pi_2D$ is a sequence of k peaks at height 2.

Theorem 3.4. *(i) If π is a Dyck path with irreducible decomposition $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ then*

$$|\mathcal{LC}(\pi)| = k' + \sum_{\substack{\pi_i \text{ irreducible} \\ \text{component}}} |\mathcal{LC}(\pi_i)|. \tag{7}$$

[2]i.e., we count the common endpoint between two consecutive irreducible components as a component itself. We refer to it as an *empty* component.

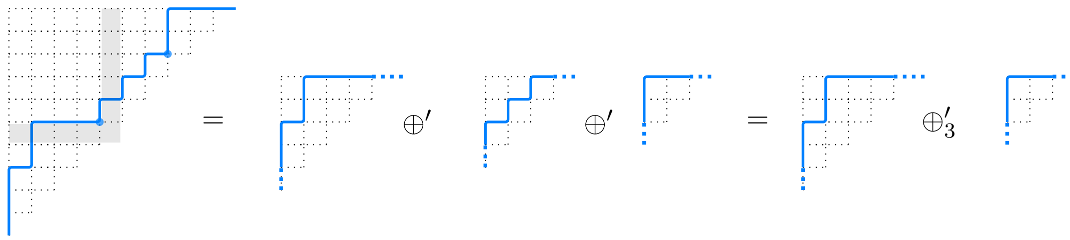


FIGURE 14. Decomposition of an irreducible path into its strongly irreducible components.

where k' is the number of connecting components of π, including the empty ones, i.e., those consisting of a simple point.

(ii) If π is an irreducible Dyck path with strongly irreducible decomposition $\pi = U\pi_1 \oplus' \cdots \oplus' \pi_k D$ then

$$|\mathcal{LC}(\pi)| = k - 1 + \sum_{i=1}^{k} |\mathcal{LC}(U\pi_i D)| \tag{8}$$

$$= k - 1 + \sum_{\substack{U\pi_i D \\ \text{strongly} \\ \text{irreducible}}} |\mathcal{LC}(U\pi_i D)| + \sum_{\substack{U\pi_i D= \\ U(UD)^{k_i} D}} k_i. \tag{9}$$

Proof. For (i), it is easy to see that the lower covers of π are obtained either by applying a bounce deletion on an irreducible component or on a nonempty connecting component. In other words, no bounce deletion deletes steps from different components. From the first we get $\sum_{i=1}^{k} |\mathcal{LC}(U\pi_i D)|$ possible lower covers and from the latter we get one lower cover for each connecting component. This leads to (7).

For (ii), the lower covers of π belong to one of the following categories: either they occur from bounce deletions on a single component $U\pi_i U$ (either strongly irreducible or of type $U(UD)^{k_i}D$) or they interact between two consecutive components by deleting the last U-step of $U\pi_i D$ and the first D-step of $U\pi_{i+1}D$ (as the deletion shown in Figure 14 on the left). More precisely, if $(a-1, a)$ is the common endpoint between the two consecutive components $\pi_i \oplus' \pi_{i+1}$ then $d_{a,a}(\pi)$ is such a lower cover. The first case contributes $\sum_{i=1}^{k} |\mathcal{LC}(U\pi_i D)|$ and the second contributes $k-1$, in the total number (8) of lower covers of $\mathcal{LC}(\pi)$. We can further refine (8) to (9) by distinguishing the type of each component π_i. □

3.2. **Upper covers.** Our next goal is the computation of the upper covers of a Dyck path $\pi \in \mathcal{D}_{n+1}$. To do so, we have to consider all the ways to insert a U and a D-step in π so that the resulting path π' is a Dyck path in $\mathcal{D}_{n+2}$ satisfying $d_{j,k_j}(\pi') = \pi$ for some $j = 1, \ldots, n+2$ and $k_j \in \{j-1, j\}$. In view of the definition of d_{j,k_j}, if the inserted U is the j-th U-step of the new path π' then the inserted D should be either the j-th or the $(j-1)$-st D-step of π'. Described more graphically, if $p = (x, j-1)$ is the integer point (other than $(0,0)$ or $(n+1, n+1)$) where we insert the new U-step, the range of possible insertions of the new D-step is the subpath $\pi \cap C_{j-1}$ contained in the $(j-1)$-th column of π. We can describe the above as a *bounce insertion*: we draw a horizontal line starting at the point p at which we want to insert U, until it hits the main diagonal and then we turn vertically up selecting the portion of the path contained in the column at which the horizontal line terminated. The selected subpath is the range of insertion of D, i.e., if we insert a U-step at p then we can insert a D-step at any integer point of the above subpath (see Figure 15). If $p = (i, i)$, $1 \le i \le n$, is on the main diagonal, then the bounce insertion described above reduces to selecting the portion of the path at column C_i, i.e., $\pi \cap C_i$. Trivially, if $p = (0, 0)$ or $(n+1, n+1)$ the only possibilities are to insert UD at p.

We next assume that π is strongly irreducible and we group all its upper covers according to the x-coordinate of the integer point where we insert the new U-step. This can be further partitioned into the case where we insert a new U-step at an ascent U^{a_i} or the case where we insert a new U-step between

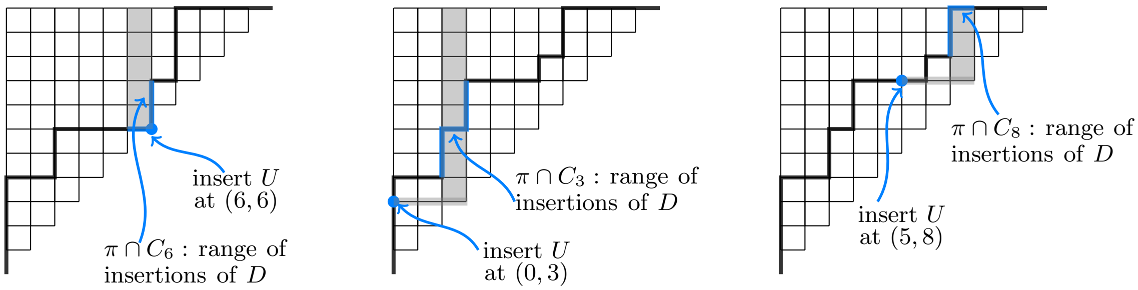


FIGURE 15. Bounce insertions: if we add a U-step at $(x, j-1) \in \pi$ and a D-step at any integer point of $\pi \cap C_{j-1}$, the resulting path π' satisfies $d_{j,k_j}(\pi') = \pi$ for some $k_j \in \{j-1, j\}$.

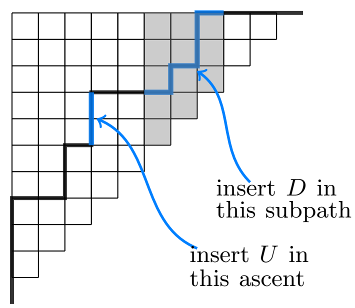


FIGURE 16. The set $\mathcal{U}C_3(\pi)$ contains all paths obtained by increasing the third ascent from U^2 to U^3 and inserting a D-step at any integer point on the indicated subpath.

two consecutive D-steps of a descent D^{b_i} of π. Altogether, we have the following possibilities for the upper covers of π:

(i) $\mathcal{U}C_1(\pi)$ is the set of upper covers of π obtained by inserting the new U-step at some integer point of U^{a_1}. In this case, the range of possible insertions of the new D-step is the set of all integer points of the subpath of π contained in the first a_1 columns, i.e., the subpath $\pi \cap (\cup_{s=1}^{a_i} C_s) = \overline{\pi}_1$.
(ii) $\mathcal{U}C_i(\pi)$, $2 \le i \le \ell$, is the set of upper covers of π obtained by inserting the new U-step at an integer point of U^{a_i}. Since the endpoints of U^{a_i} have coordinates $(b_1 + \cdots + b_{i-1}, a_1 + \cdots + a_{i-1})$, $(b_1 + \cdots + b_{i-1}, a_1 + \cdots + a_i)$ the range of possible insertions of the new D-step is the set of all integer points on the subpath $\pi \cap (\cup_{s=0}^{a_i} C_{a_1+\cdots+a_{i-1}+s}) = \overline{\pi}_i$.
(iii) $\overline{\mathcal{U}C}_i(\pi)$, $1 \le i \le \ell$, is the set of upper covers of π obtained by inserting a new U-step between a double descent DD of D^{b_i}. If D^{b_i} consists of a single descent, i.e., $b_i = 1$, then clearly $\overline{\mathcal{U}C}_i(\pi) = \emptyset$. If $b_i > 1$, let $(x_\circ, y_\circ)$ be the point between a double descent of D^{b_i}. Clearly $y_\circ = a_1 + \cdots + a_i$ which further implies that, when inserting a new U at $(x_\circ, y_\circ)$, the range of all possible insertions of the new D-step is the set of all integer points on the subpath $\pi \cap C_{a_1+\cdots+a_i}$ of π contained in the $(a_1 + \cdots + a_i)$-th column of the underlying tableau.
(iv) $\overline{\mathcal{U}C}_{\ell+1}(\pi)$ is the set containing the unique upper cover of π we obtain by inserting a U-step (and a D-step) at the end of the path π. In other words, $\overline{\mathcal{U}C}_{\ell+1}(\pi) = \{U^{a_1}D^{b_1} \ldots U^{a_\ell}D^{b_\ell}UD\}$.

Finally, since π is strongly irreducible it is not hard to see that all paths in the sets $\mathcal{U}C_i(\pi)$ and $\overline{\mathcal{U}C}_i(\pi)$ are again Dyck paths. We, therefore, conclude that the upper covers of a strongly irreducible Dyck path π are all paths in the union

$$\mathcal{U}C(\pi) = \bigcup_{i=1}^{\ell} \mathcal{U}C_i(\pi) \cup \bigcup_{i=1}^{\ell+1} \overline{\mathcal{U}C}_i(\pi). \tag{10}$$

The following lemma is the analogous of Lemma 3.2 in the case of the upper covers.

Lemma 3.5.

(i) If $\pi' \neq U^a(DU)^{n+1-a}D^a$ is strongly irreducible then the sets $\mathcal{U}C_i(\pi')$, $\overline{\mathcal{U}C}_j(\pi')$ are pairwise disjoint.

(ii) If $\pi' = U^a(DU)^{n+1-a}D^a$ with $a > 2$ then the sets $\mathcal{U}C_i(\pi')$, $\overline{\mathcal{U}C}_j(\pi')$ are pairwise disjoint except for $\mathcal{U}C_1(\pi')$ and $\mathcal{U}C_\ell(\pi')$ which both contain the path $U^a(DU)^{n+2-a}D^a$.

Proof. Suppose that π lies in the intersection of two distinct sets among sets $\mathcal{U}C_i(\pi')$ and $\overline{\mathcal{U}C}_j(\pi')$. Then π' can be obtained from π by two different bounce deletions d_{I,k_I} and d_{J,k_J} such that the I-th and J-th U-steps of π do not belong to the same ascent of π. By Lemma 3.1 this implies that π is of the form $\pi = \alpha U^r(DU)^\ell D^t\beta$ for some words $\alpha, \beta \in \{U, D\}^*$, and $r, \ell, t \in \mathbb{N}$ with $\ell > 0$, such that α ends at height $I - k_I$, and β starts at height $J - k_J$. Moreover $\pi' = \alpha U^r(DU)^{\ell-1}D^t\beta$. If π' is strongly irreducible we obtain that $\alpha = U$ if $k_I = I - 1$ and α is empty if $k_I = I$. Similarly $\beta = D$ if $k_J = J - 1$ and β is empty if $k_J = J$. The claims follow. ☐

Using the fact that the sets in the union (10) are disjoint if π is strongly irreducible, we arrive at a closed formula for $|\mathcal{U}C(\pi)|$. In what follows, we use the notation $\#U(\pi)$ for the number of U-steps of the path π.

Theorem 3.6. *Let $\pi = U^{a_1}D^{b_1}\cdots U^{a_\ell}D^{b_\ell}$ be a Dyck path in $\mathcal{D}_{n+1}$.*

(i) If $\pi \in \mathcal{D}_{n+1}$ is strongly irreducible with $\pi \neq U^a(DU)^{n+1-a}D^a$, the number of its upper covers is

$$|\mathcal{U}C(\pi)| = 2n + 3 + \sum_{i=1}^{\ell-1} b_i \cdot \big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big). \tag{11}$$

(ii) If $\pi = U^aDU\cdots DUD^a$ with $a \geq 2$, then $|\mathcal{U}C(\pi)|$ is given by (11) reduced by 1.

(iii) If $\pi = (UD)^{n+1}$ or $\pi = U^{n+1}D^{n+1}$ then $|\mathcal{U}C(\pi)| = 2(n+1)$.

(iv) If $\pi = U(UD)^nD$ then $|\mathcal{U}C(\pi)| = 4n - 5$.

Proof. If $\pi \neq U^aDU\cdots DUD^a$ is strongly irreducible then the sets in (10) are disjoint, hence we have

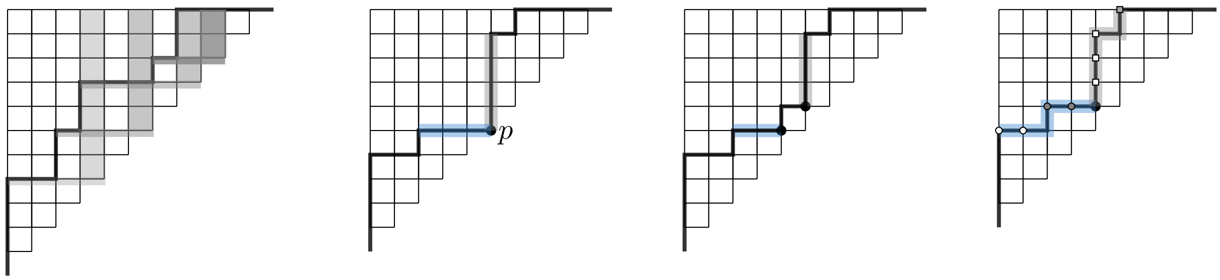
$$|\mathcal{U}C(\pi)| = \sum_{i=1}^{\ell}|\mathcal{U}C_i(\pi)| + \sum_{i=1}^{\ell+1}|\overline{\mathcal{U}C}_i(\pi)| = 1 + \sum_{i=1}^{\ell}|\mathcal{U}C_i(\pi)| + \sum_{i=1}^{\ell}|\overline{\mathcal{U}C}_i(\pi)|. \tag{12}$$

Each set $\mathcal{U}C_i(\pi)$ has as many elements as there are ways to insert a D-step in $\overline{\pi}_i$. The different ways to insert a D in any path are, either to insert a D in the beginning of the path or to insert a D immediately after a U-step. This implies that $|\mathcal{U}C_i(\pi)| = \#U(\overline{\pi}_i) + 1$. Likewise, by the definition of $\overline{\mathcal{U}C}_i(\pi)$ (see (**??**)) since for each double descent DD in D^{b_i} the range of possible ways to insert a new D-step is the subpath $\pi \cap C_{a_1+\cdots+a_i}$, we conclude that the sum in (12) becomes:

$$\begin{aligned}
|\mathcal{U}C(\pi)| &= 1 + \sum_{i=1}^{\ell}\big(\#U(\overline{\pi}_i) + 1\big) + \sum_{i=1}^{\ell}(b_i - 1)\big(\#U(\pi \cap C_{a_1+\cdots+a_i}) + 1\big)\\
&= 1 + \sum_{i=1}^{\ell} b_i + \sum_{i=1}^{\ell}\#U(\overline{\pi}_i) + \sum_{i=1}^{\ell}(b_i - 1)\big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big)\\
&= 1 + (n+1) + \sum_{i=1}^{\ell}\#U(\overline{\pi}_i) + \sum_{i=1}^{\ell}(b_i - 1)\big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big).
\end{aligned} \tag{13}$$

Counting the U-steps of each $\overline{\pi}_i$ on each vertical line $x = s$ and taking into account that the U-steps in each $\overline{\pi}_i \cap \overline{\pi}_{i+1} = \pi \cap C_{a_1+\cdots+a_i}$ are counted twice, we have

$$\sum_{i=1}^{\ell}\#U(\overline{\pi}_i) = \sum_{s=0}^{n+1}\Big(\#U(\pi \cap \{x = s\})\Big) + \sum_{i=1}^{\ell-1}\big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big).$$



(a) (b) $\pi = \pi_1 \oplus \pi_2$ (c) $\pi = U\pi_1 \oplus_1' \pi_2 D$ (d) $\pi = U\pi_1 \oplus' \pi_2 D$

FIGURE 17. (a) In view of Remark 3.7, the number of upper covers of the path on the left is $|\mathcal{U}C(\pi)| = (2 \cdot 10 + 3) + (2 \cdot 2 + 1 \cdot 1 + 3 \cdot 2 + 1 \cdot 0)$.

Thus, we continue from (13) as

$$n + 2 + \sum_{s=0}^{n+1} \Big(\#U(\pi \cap \{x = s\})\Big) + \sum_{i=1}^{\ell-1} \big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big) + \sum_{i=1}^{\ell} (b_i - 1)\big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big) \tag{14}$$

$$= 2n + 3 + \sum_{i=1}^{\ell-1} b_i \big(\#U(\pi \cap C_{a_1+\cdots+a_i})\big), \tag{15}$$

where, to go from (14) to (15), we used the fact that the total number of U-steps in the first sum of (14) is $n+1$, and that the last column has no U-steps (i.e., $\#U(\pi \cap C_{a_1+\cdots+a_\ell}) = 0$).

The claim in *(ii)* is straightforward from Lemma3.5(ii). For *(iii)*, it is not hard to see that all the upper covers of $(UD)^{n+1}$ are obtained either by replacing a UD by U^2D^2, or a $(UD)^2$ by $U(UD)^2D$, or $(UD)^{n+2}$. In the case where $\pi = U^{n+1}D^{n+1}$, the upper covers are either $U^kDU^{n+2-k}D^{n+1}$ for $1 \le k \le n+2$, or $U^{n+1}D^kUD^{n+2-k}$ for $2 \le k \le n+1$.

Finally, for *(iv)*, the upper covers $\pi' \in \mathcal{U}C(\pi)$ of $\pi = U(UD)^nD$ fall in one of the following categories:

(i) $\pi' = U(UD)^{n+1}D$,
(ii) π' is obtained from π by increasing a DU to D^2U^2,
(iii) π' is obtained from π by increasing a UD to U^2D^2,
(iv) π' is obtained from π by increasing a $UDUD$ to U^2DUD^2,
(v) π' is obtained from π by increasing a $DUDU$ to U^2UDU^2.

All the above, sum up to $4n - 5$ possible upper coves. ☐

Remark 3.7. *The sum in* (11) *can be easily computed as follows. Extend each descent* U^{b_i} *of* π *until it hits the line* $x = y$ *and turn vertically up selecting the subpath in the column at which the extended descent terminated. This is the subpath* $\pi \cap C_{a_1+\cdots+a_i}$ *whose number of* U*-steps is multiplied by* b_i *in* (11) *(see, for example, 17(a)).*

The computation of the number of upper covers of an arbitrary Dyck path π is recursive. Unlike the case with the lower covers, we cannot have an explicit formula of $|\mathcal{U}C(\pi)|$ in terms of the upper covers of the irreducible components of π, since there exist upper covers that occur by inserting a U-step in a component of π and a D-step in the subsequent one. In the following proposition we use the notation $\oplus, \oplus_1'$ without requiring that the components are irreducible. In other words, $\pi_1 \oplus \pi_2$ is the concatenation of any two Dyck paths π_1, π_2. If $U\pi_1D, U\pi_2D$ are Dyck paths then $U\pi_1 \oplus' \pi_2D$ is the concatenation of $U\pi_1, \pi_2D$, whereas $U\pi_1 \oplus_1' \pi_2D$ is the concatenation of $U\pi_1U$ and $D\pi_2D$.

Theorem 3.8.

(i) If π_1, π_2 *are Dyck paths and* $\pi = \pi_1 \oplus \pi_2$ *then*

$$|\mathcal{U}C(\pi)| = |\mathcal{U}C(\pi_1)| + |\mathcal{U}C(\pi_2)| - 1 + b \cdot a \tag{16}$$

where D^b is the last descent of π_1 and U^a is the first ascent of π_2.

(ii) *If $\pi = U\pi_1 \oplus_1' \pi_2 D$ then*

$$|\mathcal{UC}(\pi)| = |\mathcal{UC}(U\pi_1 D)| + |\mathcal{UC}(U\pi_2 D)| + (b+1)(a+1) \tag{17}$$

where D^{b+1} is the last descent of $U\pi_1 D$ and U^{a+1} is the first ascent of $U\pi_2 D$.

(iii) *If $\pi = U\pi_1 \oplus' \pi_2 D$ with $U\pi_1 D, U\pi_2 D$ strongly irreducible Dyck paths then*

$$|\mathcal{UC}(\pi)| = |\mathcal{UC}(U\pi_1 D)| + |\mathcal{UC}(U\pi_2 D)| - 2 + a'b + ab + ab' \tag{18}$$

where $D^{b'}UD^b$ is the subpath in the last row of $U\pi_1 D$ and $U^a DD^{a'}$ is the subpath in the first column of $U\pi_2 D$.

Proof. For (i), first notice that all Dyck paths of the form $\pi' \oplus \pi_2$ with $\pi' \in \mathcal{UC}(\pi_1)$ or the form $\pi_1 \oplus \pi'$ with $\pi' \in \mathcal{UC}(\pi_2)$ are upper covers of π. Since the upper cover $\pi_1 UD\pi_2$ occurs in both cases, we have $|\mathcal{UC}(\pi_1)| + |\mathcal{UC}(\pi_2)| - 1$ such paths. We next need to count the upper covers that occur by inserting a U-step in π_1 and a D-step in π_2 and are not encountered in the previous cases. Assume that $p = (k, k)$ is the intersection point of π_1, π_2 (see Figure 17(b)). Viewing these upper covers in terms of bounce insertions, one can see that these are the paths we get by inserting a U-step at any integer point, other than p, of the last descent of π_1 (light blue, Figure 17(b)) and a D-step at any integer point, other than p, of the first ascent of π_2 (light gray, Figure 17(b)). This contributes the term $b \cdot a$ in (16).

For (ii), notice that all Dyck paths of the form $U\pi' \oplus_1' \pi_2 D$ where $U\pi' D \in \mathcal{UC}(U\pi_1 D)$ or of the form $\pi_1 \oplus_1' \pi'$ where $U\pi' D \in \mathcal{UC}(U\pi_2 D)$ are upper covers of π. This contributes $|\mathcal{UC}(\pi_1)| + |\mathcal{UC}(\pi_2)|$ in (17). As before, we also have to count the upper covers that occur by inserting a U-step in π_1 and a D-step in π_2. Viewing these upper covers in terms of bounce insertions, these are all paths we get by inserting a U-step at any integer point of the last descent of π_1 (light blue, Figure 17(c)) and a D-step at any integer point of the first ascent of π_2 (light gray, Figure 17(c)). This contributes $(b+1)(a+1)$ in (17).

For (iii), in order to compute $|\mathcal{UC}(\pi)|$, notice that all Dyck paths of the form $U\pi' \oplus' \pi_2 D$ with $U\pi' D \in \mathcal{UC}(U\pi_1 D)$ or of the form $U\pi_1 \oplus' \pi' D$ with $U\pi' D \in \mathcal{UC}(U\pi_2 D)$ are upper covers of π. Since the upper covers $U\pi_1 UD\pi_2 D$ and $U\pi_1 DU\pi_2 D$ occur in both instances, we have a total of $|\mathcal{UC}(\pi_1)| + |\mathcal{UC}(\pi_2)| - 2$ contributed in (18). We next count all upper covers that occur by inserting a U-step in π_1 and a D-step in π_2. Viewing these upper covers in terms of bounce insertions, we have two categories:
a) we insert a U-step in one of the first b' integer points of the last but one descent $D^{b'}$ of π_1 (i.e., points ○ in Figure 17(c)) and insert a D-step in one of the last a integer points of the first ascent U^a of π_2 (i.e., points □ in Figure 17(c)). This contributes $a \cdot b'$ upper covers in (18), or
b) we insert a U-step in one of the first b integer points of the last descent D^b of π_1 (i.e., points ● in Figure 17(c)) and insert a D-step in one of the last a integer points of U^a or the last a' integer points of $U^{a'}$ (i.e., points □ or ■ in Figure 17(c)). This contributes $b \cdot (a + a')$ upper covers in (18). □

Example 3.9. In Figure 17

(b) we depict the path $\pi = \pi_1 \oplus \pi_2$ for $\pi_1 = U^4D^2UD^3$ and $\pi_2 = U^4DUD^4$. Using Theorem 3.6 we compute $|\mathcal{UC}(\pi_1)| = 11$ and $|\mathcal{UC}(\pi_2)| = 10$. In view of Theorem 3.8(i), we have $|\mathcal{UC}(\pi)| = 11 + 10 + 3 \cdot 4 - 1 = 32$.

(c) we depict the path $\pi = U\pi_1 \oplus_1' \pi_2 D$ for $U\pi_1 D = U^4D^2UD^3$ and $U\pi_2 D = U^4DUD^4$. In view of Theorem 3.8(ii) we have $|\mathcal{UC}(\pi)| = 11 + 10 + 3 \cdot 4 = 33$

(d) we depict the path $\pi = U\pi_1 \oplus' \pi_2 D$ for $U\pi_1 D = U^4D^2UD^3$ and $U\pi_2 D = U^4DUD^4$. In view of Theorem 3.8(iii) we have $|\mathcal{UC}(\pi)| = 11 + 10 - 2 + 2 \cdot 3 + 2 \cdot 3 + 2 \cdot 1 = 33$.

4. Pattern avoidance in Shi tableaux

Recall from our definition in Section 2.2 that T' occurs as a pattern in T if T' can be obtained from T after an iteration of bounce deletions. Otherwise, we say that T *avoids* T'. We denote by $Av(T')$ the subset of Shi tableaux that avoid T', and by $Av_n(T')$ the subset of Shi tableaux in $\mathcal{T}_n$ that avoid T'. If

$|Av_n(T')| = |Av_n(T)|$ for all $n \in \mathbb{N}$, we say that T and T' are *Wilf-equivalent.* A fundamental problem on patterns is related to Wilf-equivalence and more precisely to finding classes of Wilf equivalent objects.

We begin this section with an analysis of small patterns. More precisely, we present closed formulas for $|Av_n(T)|$, where T is any Shi tableau of size 2. We continue with analogous results for certain tableau of size k, which can be considered as natural generalizations of the ones of size 2. Finally, we compare the results we obtain with known results in the theory of permutation-patterns.

4.1. **Shi tableaux of size 2.** There are five Shi tableaux of size two

$$T_{\circ\circ\circ} = U^3D^3, T_{\bullet\circ\circ} = U^2DUD^2, T_{\bullet\bullet\circ} = U^2D^2UD, T_{\circ\circ\bullet} = UDU^2D^2 \text{ and } T_{\bullet\bullet\bullet} = (UD)^3, \tag{19}$$

which are divided into two Wilf-equivalence classes, as shown in the following proposition.

Proposition 4.1. *For every $n \geq 2$ we have*

(i) $|Av_n(T_{\circ\circ\circ})| = |Av_n(T_{\bullet\bullet\bullet})| = 2^n$, and
(ii) $|Av_n(T_{\bullet\circ\circ})| = |Av_n(T_{\bullet\bullet\circ})| = |Av_n(T_{\circ\circ\bullet})| = \binom{n+1}{2} + 1$.

To prove Proposition 4.1 we need the following lemma which provides us with precise characterizations of pattern avoidance for each of the five tableau of size 2.

Lemma 4.2. *(i) $T \in \mathcal{T}_n$ avoids $T_{\bullet\bullet\bullet}$ if and only if its bounce path $b(T)$ has at most two return points.*
(ii) $T \in \mathcal{T}_n$ avoids $T_{\circ\circ\circ}$ if and only if its height is at most 2.
(iii) $T \in \mathcal{T}_n$ avoids $T_{\bullet\circ\circ}$ if and only if it has at most one peak at height ≥ 2.
(iv) $T \in \mathcal{T}_n$ avoids $T_{\circ\circ\bullet}$ if and only if is $T = U^{n+1}D^{n+1}$ or $T = U^rD\cdots D(UD)^{n-r}$ with $1 \leq r \leq n$. Since $T \in \mathcal{T}_n$, the subword between $D \cdots D$ is a permutation of $r-1$ D's and a single U.
(iv′) $T \in \mathcal{T}_n$ avoids $T_{\bullet\bullet\circ}$ if and only if is $T = U^{n+1}D^{n+1}$ or $T = (UD)^{n-r}U\cdots UD^r$ with $1 \leq r \leq n$. Since $T \in \mathcal{T}_n$, the subword between $U \cdots U$ is a permutation of $r-1$ U's and a single D.

Proof. For (i), notice that if the bounce path of T has at most two return points, say (k,k) with $1 \leq k \leq n+1$ and $(n+1, n+1)$, then the columns $C_{k+1}, \ldots, C_{n+1}$ are empty (see Figure 18). Thus, if we want to arrive at the pattern $T_{\bullet\bullet\bullet}$ we have to delete $k-1$ among the k first rows of T. This eliminates $k-1$ among the first k columns of T, resulting in a tableau with at most one non-empty column which, clearly, cannot contain the pattern $T_{\bullet\bullet\bullet}$. For the reverse, we leave the reader to check that if T has more than two return points, there exists an iteration of bounce deletions which leads to $T_{\bullet\bullet\bullet}$.

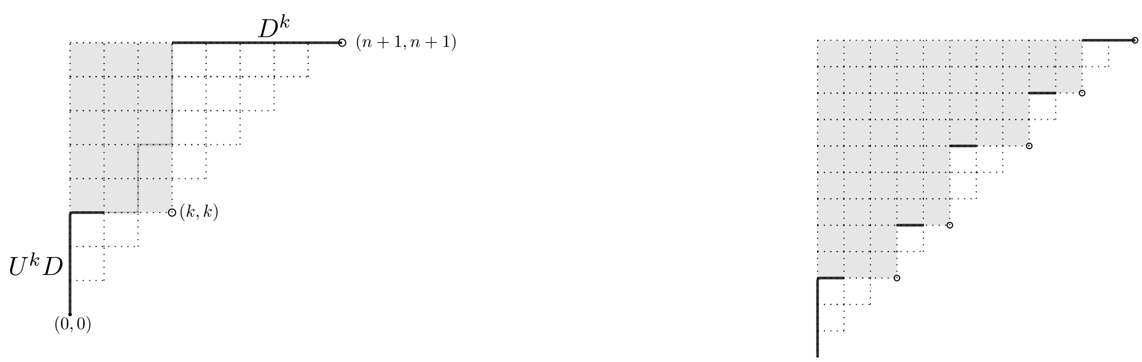
For (ii), it is immediate to see that T has height at most 2 if all boxes, except possibly the last, in each row are full. To prove the claim in (ii), notice that if T has height more than 2, then it has a row ending with at least two empty boxes, from which we can obtain the pattern $T_{\circ\circ\circ}$. Reversely, if no row has more than one empty boxes then T can be written as a concatenation of UD's and U^2D^2's, i.e., $T = (UD)^{a_1}U^2D^2(UD)^{a_2}U^2D^2\cdots$. Given such a path, each bounce deletion either deletes a peak at height 1 or 2 (i.e., a consecutive UD) or a valley at height 0 (i.e., a consecutive DU). In both cases the resulting path will again be of a similar form, i.e., a concatenation of UD's and U^2D^2's. Thus, it will never contain the pattern $T_{\circ\circ\circ} = U^3D^3$.

For (iii) notice that if T has at most one peak at height ≥ 2, then it can be written as $T = (UD)^{a_1}U^{a_2}D^{a_2}(UD)^{a_3}$ with $a_1 + a_2 + a_3 = n+1$ (see Figure 19(a)). Arguing as in (ii), iteration of any bounce deletions leads to a tableau of a similar form, i.e., $(UD)^{a'_1}U^{a'_2}D^{a'_2}(UD)^{a'_3}$ with $a'_i \leq a_i$. This implies that we can never obtain the pattern $T_{\bullet\circ\circ} = U^2DUD^2$. For the reverse, we leave the reader to check that if T has more than one peaks at height ≥ 2, then T contains the pattern $T_{\bullet\circ\circ}$.

The claim in (iv) is obvious for $T = U^{n+1}D^{n+1}$. We next show that if $T = U^rD\cdots D(UD)^{n-r}$ as described in the statement then it avoids the pattern $T_{\circ\circ\bullet} = UDU^2D^2$ (see Figure 20(a)). Indeed, in order to obtain the pattern UDU^2D^2 we have to delete at least $r-1$ among the first r U-steps. This deletes at least $r-1$ among the first r columns of T, resulting to one of the following tableaux: $U^2D^2(UD)^{n-r}$, $UDUD(UD)^{n-r}$ or $UD(UD)^{n-r}$, none of which contains the pattern $T_{\circ\circ\bullet} = UDU^2D^2$. For the reverse, we leave the reader to check that if there exists $r+2 \leq i \leq n+1$ such that the i-th row has an empty box, then there is a sequence of bounce deletions leading to UDU^2D^2.

The statement in (iv$'$) is completely analogous by symmetry. □

Proof of Proposition 4.1. For (i) we need to enumerate the tableaux described in Lemma 4.2(i),(ii). We first count the tableaux having at most two return points. Since the second return point is always the endpoint $(n+1, n+1)$, let us assume that the first return point of the bounce path is (k,k) with $1 \leq k \leq n+1$. To count the tableau whose bounce path has first return point (k,k) it suffices to count the paths within the rectangle defined by $(1,k),(k,k),(k,n),(1,n)$ (see Figure 18a). The latter are $\binom{n}{k-1}$, from which we deduce that $|Av_n(T_{\therefore})| = \sum_{k=1}^{n+1}\binom{n}{k-1} = 2^n$. Next, notice that the tableaux with height at most 2 are just those all whose boxes are full, except possibly the last box of each row. This immediately implies that $|Av_n(T_{\therefore})| = 2^n$.



(a) Size 2: a tableau avoids $T_{\therefore}$ iff its bounce path has at most two return points.

(b) Generalization: a tableau avoids $T_{\therefore}^5$ iff its bounce path has at most 5 return points.

FIGURE 18

For (ii), we need to enumerate the tableaux described in Lemma 4.2(iii),(iv). Recall from the proof of Lemma 4.2(iii) that if a tableau T has at most one peak at height ≥ 2, then it can be written as $T = (UD)^{a_1}U^{a_2}D^{a_2}(UD)^{a_3}$ with $a_1+a_2+a_3 = n+1$ and $a_2 \geq 2$ (see Figure 19(a)). Thus, we have to count the non-negative integer solutions of $a_1+a_2+a_3 = n+1$ with $a_2 \geq 2$. These are $\binom{n+1}{2}$. Including also the tableau $T_{\therefore}^n$, all whose peaks are at height 1, we conclude that $|Av_n(T_{\therefore})| = \binom{n+1}{2}+1$. Finally, in view of the characterisation in Lemma 4.2(iv), for each $1 \leq r \leq n$ we count the words with $r-1$ D's and a single U. Including $T = T_{\therefore}^n$, we have that $|Av_n(T_{\therefore})| = 1 + \sum_{r=1}^{n} r = 1 + \binom{n+1}{2}$. □

4.2. **Generalizations.** In this subsection we generalize the five cases in (19) to the following Shi tableaux of size $k \geq 3$:

$$T_{\therefore}^k = U^{k+1}D^{k+1},\ T_{\therefore}^k = U^kDUD^k,\ T_{\therefore}^k = U^kD^kUD,\ T_{\therefore}^k = UDU^kD^k \text{ and } T_{\therefore}^k = (UD)^{k+1}. \tag{20}$$

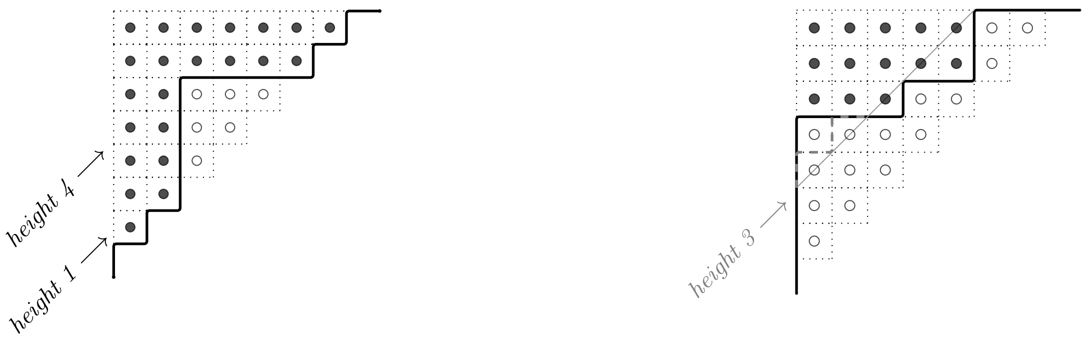
It turns out that for fixed $k \geq 3$ the tableaux in (20) fall into two Wilf-equivalence classes.

Proposition 4.3. *For every $3 \leq k \leq n$ we have*

(i) $|Av_n(T_{\therefore}^k)| = |Av_n(T_{\therefore}^k)| = |Av_n(T_{\therefore}^k)| = |\mathcal{H}(n+1,k)|$,

(ii) $|Av_n(T_{\therefore}^k)| = |Av_n(T_{\therefore}^k)| = \sum_{\ell=0}^{k-1}\frac{n-\ell+1}{n+1}\binom{n+\ell}{\ell} + |\mathcal{F}(n,n,k-1)| + \sum_{\ell=k}^{n-1}\sum_{h=0}^{k-1}\binom{n-\ell+h-1}{h}|\mathcal{F}(\ell-h,\ell,k-1)|$,

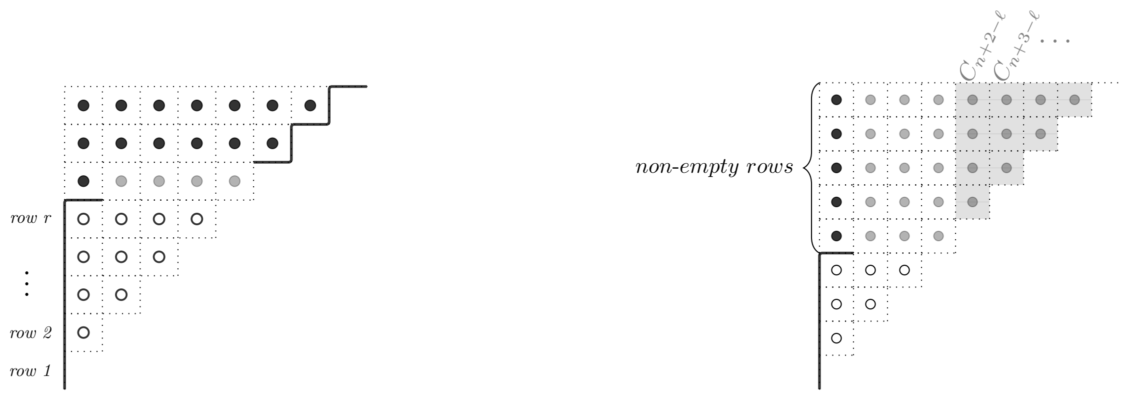
where $|\mathcal{H}(n,k)|$ is the number of Dyck paths in $\mathcal{D}_n$ with height at most k and $|\mathcal{F}(m,n,k)|$ is the number of lattice paths from $(0,0)$ to (n,m) with steps $(1,0)$ and $(0,1)$ such that all points (x,y) visited by the path satisfy $x \leq y \leq x+k$.



(a) Size 2: a tableau avoids $T_{\bullet\circ\circ}$ iff it has at most one peak at height ≥ 2.

(b) Generalization: a tableau avoids $T^4_{\bullet\circ\circ}$ iff it has no valley at height ≥ 3.

FIGURE 19



(a) Size 2: Tableaux avoiding $T_{\bullet\bullet\circ}$. Gray bullets can be either full or empty boxes.

(b) Generalization: If T has ℓ non-empty rows then, T avoids $T^k_{\bullet\bullet\circ}$ iff the tableau in colums $C_{n+2-\ell}, \ldots, C_{n+1}$ (shaded) avoids $T^{k-1}_{\circ\circ\circ}$. Gray bullets can be either full or empty boxes.

FIGURE 20

Remark 4.4. *The numbers $|\mathcal{F}(m,n,k)|$ can be computed efficiently using the following formula, which is a consequence of the iterated reflection principle* [16, Chap. 1.3, Thm. 2].

$$|\mathcal{F}(m,n,k)| = \sum_{i=0}^{\infty} \tfrac{n-m+2i(k+2)+1}{n+i(k+2)+1}\tbinom{m+n}{m-i(k+2)} + \sum_{i=1}^{\infty} \tfrac{n-m-2i(k+2)+1}{m+i(k+2)}\tbinom{m+n}{m+i(k+2)}.$$

To prove Proposition 4.3 we need the following lemma.

Lemma 4.5. *Let $k \geq 3$ and $T \in \mathcal{T}_n$. Then*

(i) T avoids $T^k_{\bullet\bullet\bullet}$ if and only if its bounce path has at most k return points,
(ii) T avoids $T^k_{\circ\circ\circ}$ if and only if its height is at most k,
(iii) T avoids $T^k_{\bullet\circ\circ}$ if and only if it has no valley at height $\geq k-1$.
(iv) Let ℓ denote the number of the non-empty rows of T. Then $T \in Av_n(T^k_{\bullet\bullet\circ})$ if and only if the tableau $T' \in \mathcal{T}_{\ell-1}$ obtained by deleting the first $n+1-\ell$ columns of T avoids $T^{k-1}_{\circ\circ\circ}$.

Proof. To prove (i) we use induction on k, the base case $k=2$ being Lemma 4.2(i). Thus assume that the claim holds for k and consider the case $k+1$. Suppose on the contrary that there exists some $T \in \mathcal{T}_n$ containing $T^{k+1}_{\bullet\bullet\bullet}$ and whose bounce path has $k+1$ return points. Then $n > k+1$ and there there exist a non-empty sequence $d_{i_1,*}, \ldots, d_{i_\ell,*}$ of bounce deletions such that $d_{i_1,*} \cdots d_{i_\ell,*}(T) = T^{k+1}_{\bullet\bullet\bullet}$. Applying one more bounce deletion we have $d_{i_0,*}\, d_{i_1,*} \cdots d_{i_\ell,*}(T) = T^k_{\bullet\bullet\bullet}$. In particular the tableau $d_{i_\ell,*}(T)$ contains $T^k_{\bullet\bullet\bullet}$. However, the deletion $d_{i_\ell,*}$ can only possibly reduce the number of return points of the bounce path by one. Thus the bounce path of $d_{i_\ell,*}(T)$ has at least k return points and, by the induction hypothesis, $d_{i_\ell,*}(T)$ avoids $T^k_{\bullet\bullet\bullet}$. This is a contradiction.

For the other direction, we show that any T whose bounce path has at least $k+1$ return points must contain $T^k_{\bullet\bullet\bullet}$. Indeed, this can be done by applying $d_{i,i}$ on each row $i > 0$ that does not contain a bounce step (see Figure 18b).

For (ii), we use the fact that T has height at most k if and only if no row has more than $k-1$ empty boxes. In view of this, it is immediate to see that if a row of T has more than $k-1$ empty boxes, then T contains $T^k_{\circ\circ\circ}$. Reversely, if all rows of T have at most $k-2$ empty boxes, then no iteration of bounce deletions can lead to a tableau having a row with k or more empty boxes, and thus to $T^k_{\circ\circ\circ}$.

To prove (iii), we use the fact that T has no valley at height $\geq k-1$ if and only if each initial subword of the form wUD satisfies $\#U(wDU) - \#D(wDU) \leq k$. Next, notice that each bounce deletion either deletes a U, D or a single U(the corresponding D being on a subsequent position) from each such initial subword wDU. Thus, the above inequality still holds after any bounce deletion, which futher implies that T cannot contain the pattern $T^k_{\bullet\circ\circ}$. Reversely, if T has a valley at height $\geq k-1$ then, deleting all rows and columns above and to the left of this valley we get a tableau $T^\kappa_{\bullet\circ\circ}$ with $\kappa \geq k$ which, clearly, contains $T^k_{\bullet\circ\circ}$.

Finally, notice that (iv) is trivially true for $\ell < k$ since both assumptions that T avoids $T^k_{\bullet\circ\bullet}$ and T' avoids $T^{k-1}_{\circ\circ\circ}$ are satisfied (by size restrictions). If $\ell \geq k$, assume first that T' contains $T^{k-1}_{\circ\circ\circ}$. Then, it is easy to see that there exist bounce deletions on T leading to a Shi tableau T'' which is T' with a full column adjoint on its left. Since T' contains $T^{k-1}_{\circ\circ\circ}$ we conclude that T'' contains $T^k_{\bullet\circ\bullet}$ and consequently that T contains $T^k_{\bullet\circ\bullet}$. Next, assume that $\ell \geq k$ and T' avoids $T^{k-1}_{\circ\circ\circ}$. In order to answer the existence (or not) of the pattern $T^k_{\bullet\circ\bullet} = UDU^kD^k$ in T and since T begins with $U^{n-\ell+1}D$, we have to delete $n-\ell$ among the first $n-\ell+1$ U-steps of T in a way that the resulting tableaux T'' has a full first column, i.e., begins with UD. This will delete $n-\ell$ among the first $n-\ell+1$ columns of T so that T'' is T' with a full column adjoint on its left. Then, it is clear that since the pattern $T^{k-1}_{\circ\circ\circ}$ does not occur in T', the pattern $T^k_{\bullet\circ\bullet} = UDU^kD^k$ does not occur in T'' and hence in T.

□

Now we are ready to prove Proposition 4.3

Proof of Proposition 4.3. (i) The fact that $T^k_{\bullet\bullet\bullet}, T^k_{\circ\circ\circ}$ are Wilf-equivalent is a consequence of the well-known *zeta map* [2, 8]. Consider a Dyck path $\pi \in \mathcal{D}_{n+1}$ with area vector $a(\pi) = (a_1, a_2, \ldots, a_{n+1})$. That is, a_i denotes the number of empty boxes in row i of the corresponding Shi tableau. For each $j = 0, \ldots, n$ we define w_j as a word in the alphabet $\{U, D\}$ obtained as follows: reading $a(\pi)$ from left to right, we draw a down-step whenever we encounter an entry $a_i = j-1$ and a up-step whenever we encounter an entry $a_i = j$. Finally, we set $\zeta(\pi) = w_0 w_1 \cdots w_n$. It is easy to see that $\zeta(\pi) \in \mathcal{D}_{n+1}$ since every entry of $a(\pi)$ contributes twice: an up-step first and a down-step later. The map $\zeta : \mathcal{D}_{n+1} \to \mathcal{D}_{n+1}$ is a bijection with surprising properties. The following are equivalent:

(a) The area vector $a(\pi)$ satisfies $a_i \leq k-1$ for all i.
(b) The word w_j is empty for all $j > k$.
(c) The bounce path of $\zeta(\pi)$ has at most k return points.

Notice that a Dyck path has height at most k if and only if all entries of its area vector are less than k. The equality $|Av_n(T^k_{\circ\circ\circ})| = |Av_n(T^k_{\bullet\bullet\bullet})| = |\mathcal{H}(n+1, k)|$ therefore follows from the characterization of the Shi tableaux in $Av_n(T^k_{\circ\circ\circ})$ and $Av_n(T^k_{\bullet\bullet\bullet})$ in Lemma 4.5 (i) and (ii) and the equivalence of (a) and (c) above.

It remains to prove that $|Av_n(T^k_{\bullet\circ\circ})| = |\mathcal{H}(n+1, k)|$. In view of Lemma 4.5 (iii) suppose that π is a Dyck path with no valleys at height greater or equal to $k-1$. Then the steps of π above the line $y = x + k - 1$ form a (possibly empty) set of disconnected peaks. Replacing each such peak of the form $U^\ell D^\ell$ by a sequence $(UD)^\ell$ we obtain a Dyck path of height at most k. This yields a bijection between $T^k_{\substack{\bullet\circ\\\circ}}$-avoiding and $T^k_{\substack{\circ\circ\\\circ}}$-avoiding Shi tableaux.

To prove (ii), let $T \in Av_n(T^k_{\substack{\bullet\circ\\\bullet}})$ with ℓ non-empty rows. By Lemma 4.5 (iv) the tableau T falls in one of three categories: (a) $\ell < k$, (b) $\ell = n$ and the tableau $T' \in \mathcal{T}_{n-1}$ obtained by deleting the first column of T avoids $T^{k-1}_{\substack{\circ\circ\\\circ}}$, or (c) $k \le \ell < n$ and the tableau $T' \in \mathcal{T}_{\ell-1}$ obtained by deleting the first $n+1-\ell$ columns of T avoids $T^{k-1}_{\substack{\circ\circ\\\circ}}$. The set of Shi tableaux that satisfy $\ell < k$ corresponds naturally to ballot paths with n U-steps and ℓ D-steps, that is, lattice paths from $(0,0)$ to (ℓ, n) with steps $(1,0)$ and $(0,1)$ that never go below the line $x = y$. Such ballot paths are known to be counted by $\frac{n-\ell+1}{n+1}\binom{n+\ell}{\ell}$. The second category contains $|\mathcal{H}(n, k-1)| = |\mathcal{F}(n, n, k-1)|$ elements by part (i) of the proposition. It remains to count the tableaux that fall into the third category. To this end write $T = U^{n+1-\ell} D\pi D\pi'$, where π contains $n-\ell-1$ D-steps, and let h denote the number of U-steps of π. Thus the tableau T' corresponds to the path $U^h\pi'$. In order for T' to avoid $T^{k-1}_{\substack{\circ\circ\\\circ}}$ we must have $h \le k-1$ and π' must correspond to a ballot path from $(0,0)$ to $(\ell-h, \ell)$ of height at most $k-1$. There are $|\mathcal{F}(\ell-h, \ell, k-1)|$ such ballot paths. The fact that T' avoids $T^{k-1}_{\substack{\circ\circ\\\circ}}$ imposes no restriction on the path π. Thus there are $\binom{n-\ell+h-1}{h}$ possible choices for π. $\square$

Remark 4.6. *The zeta map and the bounce path of a Dyck path were first defined by Andrews et al. in* [2] *in order to enumerate ad-nilpotent ideals in a borel subalgebra of the complex simple Lie algebra of type A with class of nilpotence less than k. These ideals are in bijection with certain Dyck paths (*[2, Thm. 4.1]*), which correspond precisely to the Shi tableaux that avoid* $T^k_{\substack{\bullet\circ\\\bullet}}$. *The zeta map provides a bijection to Dyck paths of height at most k (see* [2, §5]*), which of course coincide with* $T^k_{\substack{\circ\circ\\\circ}}$*-avoiding tableaux. Thus our use of the zeta map in the proof of 4.3 (i) really agrees with its original purpose. The combinatorial description of ad-nilpotent ideals with bounded class of nilpotence in terms of pattern avoidance is, in the opinion of the authors, quite intriguing.*

We further remark that the zeta map also ties to different problems in algebraic combinatorics such as the Hilbert series of diagonal harmonics [8,9]. *The zeta map and the bounce path have also previously been considered in conjuction with the Shi arrangement, for example, in* [3].

5. Open Problems

Once a notion of pattern avoidance for a class of combinatorial objects has been defined there are plenty of interesting questions which have been studied for permutation patterns and can be transferred to the new setting. We mention here only two such (potentially difficult) problems and conclude the paper with two more possible further directions that are more particular to the case of Shi tableaux.

Problem 5.1. *Does the poset* $\mathcal{T}$ *of Shi tableaux contain infinite antichains?*

The analogous question can be answered affirmatively for permutations ordered by pattern containment [20]. On the other hand the poset of finite words over a finite alphabet is an example of a poset with no infinite antichains [10].

Problem 5.2. *Let* $T \in \mathcal{T}$ *be a Shi tableau (or a collection of Shi tableaux). What can be said about the generating function* $\sum_{n\in\mathbb{N}} Av_n(T)x^n$*? Are all formal power series obtained in this way rational, algebraic, D-finite,…?*

We have no strong intuition as to what the correct answer should be. Pattern avoidance in permutations gives rise to (conjecturally) very complicated generating functions [1]. However, there is at least a chance that the poset of Shi tableaux is sufficiently simpler such some results can be obtained.

5.1. **Connections to pattern-avoiding permutations.** Considering our first enumerative results on pattern avoidance in Shi tableaux, we observe that the sequences in Proposition 4.1 and Proposition 4.3 (i) have already appeared in the literature in the context of permutation-patterns. The following table summarizes these results.

Shi tableaux of type A	Sequence	OEIS [19]	Pairs of permutations
$\lvert Av_n(T_{\substack{\circ\circ\\ \circ}})\rvert, \lvert Av_n(T_{\substack{\bullet\bullet\\ \bullet}})\rvert$	2^n	A000079	$\lvert Av_{n+1}(132,123)\rvert$
$\lvert Av_n(T_{\substack{\bullet\circ\\ \circ}})\rvert, \lvert Av_n(T_{\substack{\bullet\circ\\ \bullet}})\rvert, \lvert Av_n(T_{\substack{\bullet\bullet\\ \circ}})\rvert$	$\binom{n}{2}+n+1$	A000124	$\lvert Av_{n+1}(132,321)\rvert$
$\lvert Av_n(T^k_{\substack{\circ\circ\\ \circ}})\rvert, \lvert Av_n(T^k_{\substack{\bullet\bullet\\ \bullet}})\rvert, \lvert Av_n(T^k_{\substack{\bullet\circ\\ \circ}})\rvert$	$\lvert\mathcal{H}(n+1,k)\rvert$	A080934	$\lvert Av_{n+1}(132,12\dots k)\rvert$

Using Proposition 4.3 (ii) we compute the first few values of $|Av_n(T^k_{\substack{\bullet\circ\\ \bullet}})|$. For $k=2$ we find the sequence of $\{132,321\}$-avoiding permutations [19, A000124]. For $k=3$ the numbers seem to match the sequence of $\{123,3241\}$-avoiding permutations [19, A116702]. For $k=4$ we apparently obtain the sequence of $\{123,51432\}$-avoiding permutations [19, A116847]. The first few values for $k=5$ and $k=6$ are as follows:

$$k=5: \qquad 1,2,5,14,42,131,413,1294,4007,12272,37277,112622,339152,1019457,\dots$$
$$k=6: \qquad 1,2,5,14,42,132,428,1411,4675,15463,50928,166999,545682,1778631,\dots,$$

and are not part of the oeis yet. The above data might lead to the guess that there should exist also patterns σ and τ of lengths three and $k+1$ respectively such that $|Av_n(T^k_{\substack{\bullet\circ\\ \bullet}})|$ equals the number of $\{\sigma,\tau\}$-avoiding permutations.

Problem 5.3. *Find an explanation for (or quantify) the above phenomenon linking pattern avoidance in Shi tableaux of type A to permutations avoiding a pair of patterns.*

Considering the fact that Shi tableaux of are counted by Catalan numbers and are hence in bijection with σ-avoiding permutations for any pattern σ of length three, one might hope to explain this enumerative parallel by finding a bijection that translates the bounce deletions into pattern containment on the permutation side. Unfortunately such an attempt is bound to fail: There is no choice of pattern σ of length three for which the poset of Shi tableaux is isomorphic to the poset of σ-avoiding permutations. This can be seen quickly for example from the number of cover relations in Figure 7.

5.2. ad**-nilpotent ideals.** A Shi arrangement can be attached to any crystallographic root system. In each case the dominant regions are indexed by Shi tableaux – binary fillings whose shape is determined by the root poset. In types B and C this shape is a doubled staircase as opposed to the staircase shape in type A. In type D the root poset is no longer planar but with some adjustments the combinatorics can still be made to work.

Problem 5.4. *Find and investigate the cover relations that define patterns in Shi tableaux for other classical types.*

There are two facts that should serve as guidelines as well as encouragement in this endeavor. First, the Shi arrangement is an exponential sequence of arrangements for the classical types. Hence there is geometric motivation for (at least some of) the bounce deletions.

Secondly, there are enumerative results on ad-nilpotent ideals in a Borel subalgebra of the complex simple Lie algebra of classical type with bounded class of nilpotence [14]. In type C the ideals of class of nilpotence k are counted by formulas depending on the parity of k. At least in one of the two cases there seems to be an immediate connection between such ideals and the Shi tableaux avoiding a full Shi tableau.

References

[1] M. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter, *Generating permutations with restricted containers*, Journal of Combinatorial Theory, Series A **157** (2018), 205–232.

[2] G. Andrews, C. Krattenthaler, L. Orsina, and P. Papi, *ad-nilpotent $\mathfrak{b}$-ideals in $sl(n)$ having fixed class of nilpotence: combinatorics and enumeration*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 3835–3853.

[3] D. Armstrong, *Hyperplane arrangements and diagonal harmonics*, J. Comb. **4** (2002), no. 2, 157–190.

[4] A. Bacher, A. Bernini, L. Ferrari, B. Gunby, R. Pinzani, and J. West, *The Dyck pattern poset*, Discrete Math. **321** (2014), 12–23.

[5] M. Bóna, *Combinatorics of permutations*, Second edition, Discrete Mathematics and Its Applications, Chapman & Hall/CRC, 2004.

[6] N. G. de Bruijn, D. E. Knuth, and S. O. Rise, *The average height of planted plane trees*, Graph Theory and Computing (1972), 15–22.

[7] S. Fishel, M. Vazirani, and E. Tzanaki, *Counting shi regions with a fixed separating wall*, Annals of Combinatorics **17** (2013), no. 4, 671–693.

[8] J. Haglund, *The q,t-catalan numbers and the space of diagonal harmonics*, Univ. Lecture Ser. **41** (2008), 319–334.

[9] J. Haglund, *Conjectured statistics for the q,t-catalan numbers*, Adv. Math. **175** (2013), 319–334.

[10] G. Higman, *Ordering by divisibility in abstract algebras*, Proceedings of the London Mathematical Society (3) **2** (1952), 326–336.

[11] M. E. Karim, L. Parida, and A. Lakhotia, *Using permutation patterns for content-based phylogeny*, Pattern Recognition in Bioinformatics (2006), 115–125.

[12] S. Kitaev, *Patterns in permutations and words*, Monographs in Theoretical Computer Science, Springer, 2011.

[13] D. E. Knuth, *The art of computer programming, i: Fundamental algorithms*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[14] C. Krattenthaler, L. Orsina, and P. Papi, *Enumeration of ad-nilpotent $\mathfrak{b}$-ideals for simple Lie algebras*, Advances in Applied Mathematics **28** (2002), 478–522.

[15] M.-L. Lackner and M. Lackner, *On the likelihood of single-peaked preferences*, Soc. Choice Welf. **48** (2017), no. 4, 717–745.

[16] S. G. Mohanty, *Lattice path counting and applications*, Probability and Mathematical Statistics, Academic Press, New York, 1979. ISBN 0-12-504050-4.

[17] J-Y. Shi, *Alcoves corresponding to an affive Weyl group*, J. London Math. Soc. **35** (1987), 42–55.

[18] R. Simion and F. W. Schmidt, *Restricted permutations*, European J. Combin. **6 (4)** (1985), 383–406.

[19] N. J. A. Sloane, *The on-line encyclopedia of integer sequences* (2017). `http://oeis.org`oeis.org.

[20] D. A. Spielman and M. Bóna, *An infinite antichain of permutations*, Electron. J. Combin **7** (2000), #N2.

[21] A. Spiridonov, *Pattern-avoidance in binary fillings of grid shapes*, Discrete Math. Theor. Comput. Sci. Proc. (2008), 677–690.

[22] R. P. Stanley, *Hyperplane arrangements, interval orders, and trees*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), no. 6, 2620–2625.

[23] R. P. Stanley, *An introduction to hyperplane arrangements*, Lecture notes, ias/park city mathematics institute, 2004.

[24] R. P. Stanley, *Enumerative combinatorics: Volume 1*, 2nd ed., Cambridge University Press, New York, NY, USA, 2011.

Faculty of Mathematics, University of Vienna, Vienna, Austria
E-mail address: `myrto.kallipoliti@univie.ac.at`

Deptartment of Mathematics and Statistics, York University, Toronto, Canada
E-mail address: `rsulzg@yorku.ca`

Department of Mathematics and Applied Mathematics, University of Crete, Heraklion, Greece
E-mail address: `etzanaki@uoc.gr`