Traces On Diagram Algebras I: Free Partition Quantum Groups, Random Lattice Paths And Random Walks On Trees

by Jonas Wahl[1]

Abstract

We classify extremal traces on the seven direct limit algebras of noncrossing partitions arising from the classification of free partition quantum groups of Banica-Speicher [BS09] and Weber [We13]. For the infinite-dimensional Temperley-Lieb-algebra (corresponding to the quantum group O_N^+) and the Motzkin algebra (B_N^+), the classification of extremal traces implies a classification result for well-known types of central random lattice paths. For the 2-Fuss-Catalan algebra (H_N^+) we solve the classification problem by computing the *minimal or exit boundary* (also known as the *absolute*) for central random walks on the Fibonacci tree, thereby solving a probabilistic problem of independent interest, and to our knowledge the first such result for a nonhomogeneous tree. In the course of this article, we also discuss the branching graphs for all seven examples of free partition quantum groups, compute those that were not already known, and provide new formulas for the dimensions of their irreducible representations.

1 Introduction

The classification problem for extremal traces on direct limits of finite-dimensional C^*-algebras has a decennia long history in mathematics that is intertwined with many different fields such as probability theory, symmetric functions, K-theory or representation theory [BO16] just to name a few. For instance, Thoma's classification of extremal traces on $\mathbb{C}[S_\infty]$ [Th64], the group algebra of the infinite symmetric group, has spurned a vast literature of beautiful results ranging from applications to determinantal point processes [BO98] [O03] to free probability [Bn98] to solutions of the Yang-Baxter equation [LPW19].

In this article, we are interested in a family of direct limit algebras that arises from the theory of partition (a.k.a. easy) quantum groups initiated by Banica and Speicher in their seminal article [BS09]. More precisely, Banica and Speicher introduced the notion of *categories of partitions* $\mathcal{C} = (\mathcal{C}(k,l))_{k,l\geq 0}$ which model the representation theory of the partition quantum groups. The elements of $\mathcal{C}(k,k),\ k \geq 0$ have a simple diagrammatical representation and serve as a basis for an inductive sequence of finite-dimensional algebras $\ldots \subset A_{(\mathcal{C},\delta)}(k) \subset A_{(\mathcal{C},\delta)}(k+1) \subset \ldots$ depending on an additional loop parameter $\delta > 0$. If the loop parameter δ is chosen generically, the algebras become semisimple and thus admit a limit object $A_{(\mathcal{C},\delta)}(\infty)$. Banica and Speicher further divided the categories of partitions in several subfamilies that are amenable to classification. In this article, we are concerned with the limit algebras of categories of *noncrossing* partitions of which there are exactly seven (corresponding to the compact quantum groups $\mathbb{G} = O_N^+, S_N^+, B_N^+, H_N^+, S_N^{'+} B_N^{'+}, B_N^{\#+}$ when $\delta = N$) as shown by Weber [We13].

The most prominent limit algebras appearing in this setting are the infinite Temperley-Lieb algebra $\mathrm{TL}(\infty,\delta)$ (in the three cases $\mathbb{G} = O_N^+, S_N^+, S_N^{'+}$) and the 2-Fuss-Catalan-algebra $\mathrm{FC}_2(\infty,\delta)$ (when $\mathbb{G} = H_N^+$) introduced by Bisch and Jones [BiJo95] in their analysis of intermediate subfactors, see also [La01]. The extremal traces of the infinite Temperley-Lieb algebra have been classified in the PhD thesis of A. Wassermann [Was81] with each extremal trace corresponding

[1]Hausdorff Center for Mathematics, Bonn (Germany).
E-mail: wahl@iam.uni-bonn.de.

to a parameter value $\lambda \in [1/2, 1]$. Wassermann also pointed out that this classification problem is equivalent to the computation of the *minimal boundary* in the sense of Vershik and Kerov of the *semi-Pascal graph* (drawn in Figure 2) which, loosely speaking, describes how the irreducible components of the algebras $\mathrm{TL}(0,\delta) \subset \mathrm{TL}(1,\delta) \subset \mathrm{TL}(2,\delta) \subset \dots$ are nested inside each other. Our first order of business in this article is to rephrase Wassermann's result as a classification result for *central random ballot paths* and to make his result much more explicit in this probabilistically natural context. A ballot path (the terminology originates from Bertrand's famous ballot problem) is a lattice path on $\mathbb{N}^2$ starting at $(0,0)$ that is allowed to take steps $(1,1)$ and $(1,-1)$. Consequently an infinite random ballot path is a probability measure μ on the space of infinite ballot paths which we require to satisfy the following memory-loss condition (called centrality): given the event that the random path passes through a point (n,k) on the lattice, every path from $(0,0)$ to (n,k) is chosen with the same probability. Wassermann's result then implies that every central random ballot path is the mixing of Markov chains $M_\lambda,\ \lambda \in [1/2, 1]$ (Theorem 5.1) for which we provide explicit transition probabilities in Section 5.1.

In Section 5.2, we address the trace classification problem for the direct limit algebra $A_{(\mathcal{B}^+,\delta)}(\infty)$ associated to the free bistochastic groups B_N^+ and $B_N^{\prime +}$. Again, the problem admits an equivalent formulation in terms of random lattice paths and the type of lattice paths appearing is known in the combinatorial literature as *Motzkin paths* which resemble ballot paths but allow for additional even level steps $(1,0)$. We establish this connection by a thorough analysis of the representations of the the algebras $A_{(\mathcal{B}^+,\delta)}(k),\ k \geq 0$ using the methods developed in [FrWe16]. In particular, this yields a description of the branching graph of the sequence $A_{(\mathcal{B}^+,\delta)}(0) \subset A_{(\mathcal{B}^+,\delta)}(1) \subset \dots$ (see also [BH14] for another approach to this). Combining our reinterpretation of this branching graph in terms of Motzkin paths with path counting results from the algebraic combinatorics literature (our main reference here is [Kra15]), we obtain explicit formulas for the dimensions of all irreducible representations of $A_{(\mathcal{B}^+,\delta)}(k)$ and more generally, for multiplicities in decompositions of restricted and induced representations. Next, using Wassermann's approach and the ergodic method of Vershik and Kerov [VK81], we prove in Theorem 5.10 that every central random Motzkin path is a mixture of Markov chains $M_{(\lambda_1,\lambda_2)}$ indexed by parameters $\lambda_1 \leq \lambda_2$ with $0 \leq \lambda_1 + \lambda_2 \leq 1$. Although we do not have explicit formulas for the transition probabilities of the chain $M_{(\lambda_1,\lambda_2)}$ as in the ballot path case, we do provide formulas (in terms of λ_1, λ_2) for the probability to return to the root after a given number of steps and a recursion that determines the transition probabilities uniquely.

In Section 5.3, we analyse the representation theory of the direct limit algebra corresponding to the freely modified bistochastic quantum group $B_N^{\#+}$ and determine its branching graph. It is convenient to interpret this branching graph as one arising from a smaller graph, the *derooted Fibonacci tree* through a process called *pascalization* in [VN06]. This technique (although not the terminology) is also well-known in subfactor theory, where one would describe the derooted Fibonacci tree as the *principal graph* of the bigger branching graph. The branching graph associated to $B_N^{\#+}$ closely resembles the one of our last missing case, the direct limit algebra of the free hyperoctahedral group H_N^+. This algebra is also known as the infinite 2-*Fuss-Catalan algebra* $\mathrm{FC}_2(\infty,\delta)$) for which the principal graph is the (full) Fibonacci tree [BiJo95]. Therefore, we will discuss dimension formulas and the trace classification problem for both $B_N^{\#+}$ and H_N^+ together in Section 5.4.

In both cases, the description of the branching graph in terms of the Fibonacci tree translates the trace classification problem into an equivalent probabilistic problem of independent interest, the computation of the *exit boundary for random walks* on the Fibonacci tree (to follow the terminology of [VM15]). In [VM15], this problem was solved for homogeneous trees by making clever use of their symmetric nature. For the less symmetric Fibonacci tree, the combinatorics to deal with become more involved and as a result the computation of the exit boundary becomes more subtle. For a discussion on exit boundaries in a different setting, see also [VM18], where the exit boundary is called the *absolute*.

In essence, we prove that every element of the exit boundary, that is to say every ergodic central Markov chain on the pascalization $\mathcal{P}(\mathbb{FT})$ of the Fibonacci tree is the lift of a transient random walk $S_{(t,\eta)}$ on $\mathbb{FT}$. This random walk is uniquely determined by a *structure constant* $\eta \in [0, 4/27]$, which describes the probability to cross edges once in both directions, and an *end* $t \in \partial\mathbb{FT}$ of the tree $\mathbb{FT}$ to which $S_{(t,\eta)}$ converges almost surely. If the reader is unfamiliar with the terminology, an end of a tree is an infinite path on the tree starting at the root. In the case of the Fibonacci tree $\mathbb{FT}$, an end can alternatively be understood as an infinite Fibonacci word, i.e. a word in letters 1 and 2 such that the letter 1 is always followed by the letter 2. To summarize we have:

Theorem A (see also Theorem 6.5)**.** *The central ergodic Markov measures on $\mathcal{P}(\mathbb{FT})$ are the lifts of random walks $S_{(t,\eta)}$ with*

$$(t, \eta) \in \partial\mathbb{FT} \times [0, 4/27].$$

The random walks $S_{(t,\eta)}$ and their lifts to the path space of $\mathcal{P}(\mathbb{FT})$ are defined in Section 6. Their transition probabilities admit nice formulas on all edges that do not lie on the specified t and these formulas involve the generating function $G(z) = \sum_{n=0}^{\infty} C_n^2 z^n$ of the *Fuss-Catalan numbers* $(C_n^2)_{n\geq 0}$ evaluated in η. This also explains the upper bound $4/27$ for the parameter η which is exactly the radius of convergence of the power series $G(z)$. More generally, we define these random walks for general s-Fuss-Catalan trees with $s \geq 2$ in which case the critical value for the structure constant becomes $\frac{s^s}{(s+1)^{s+1}}$.

In order to prove Theorem A, we show first that the random walk $S_{(t,\eta)}$ in fact converges to the end $t = (t_k)_{k\geq 0}$ (Corollary 6.13). Another important ingredient in the proof of Theorem A and an interesting result on its own, is the following law of large numbers for the exit times N_k at the vertex t_k (the number of steps after which $S_{(t,\eta)}$ leaves t_k forever), see Definition 6.14.

Theorem B (see also Theorem 6.15)**.** *Let $t = (t_k)_{k\geq 0} \in \partial\mathbb{FT}$ an end and let $\eta \in [0, 4/27]$. Denote by N_k the exit time at t_k for the random walk $S_{(t,\eta)}$ and by $r(t_k) = \sum_{i=1}^{k} l(t_i)$ the sum of the number of sons $l(t_i)$ of the vertices $\emptyset = t_0, \ldots, t_k$. Then the limit*

$$\lim_{k\to\infty} \frac{N_k - k}{r(t_k)} \in [0, \infty]$$

exists (and is almost surely constant). It is finite if and only $\eta < 4/27$.

The proof of Theorem A also works almost verbatim for the derooted Fibonacci trees and the ergodic central measures are exactly the pullbacks of random walks $\tilde{S}_{(t,\eta)}$ with the same transition probabilities as $\tilde{S}_{(t,\eta)}$ outside the end t, this time starting on the first level of $\mathbb{FT}$. Together with the results on random lattice paths mentioned above, this settles the trace classification problem for diagram algebras associated to free partition quantum groups.

Corollary. *The extremal traces on the infinite dimensional diagram algebras associated to the seven examples of free partition quantum groups are fully classified by Theorem 5.1 (O_N^+), Theorem 5.10 (B_N^+), Remark 5.2 ($S_N^+, S_N'^+, B_N'^+$) and Corollary 6.6 ($H_N^+, B_N^{\#+}$).*

There are several other classes of infinite diagram algebras appearing in the work of Banica and Speicher [BS09], Weber [We13] and others, e.g. [RaWe16] [FlP18] [VV19], for which the trace classification problem is still open and for which in many cases even the branching graphs are unknown. We expect that for these algebras, the trace classification problem admits interesting probabilistic reformulations as well, a point that we will further emphasize in the forthcoming article [Wa20]. In that article, we will classify the extremal traces on the next prominent class of infinite diagram algebras appearing in [BS09], namely those containing the crossing partition. Originally, these algebras arose in the context of Schur-Weyl duality for compact groups [W46], and once more, it will be advantageous to think about the involved branching graphs as pascalizations of smaller graphs. In contrast to the results of this article, however, these graphs will not be trees but variations of the famous Young graph, the branching graph associated to the infinite symmetric group S_∞.

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2 Preliminaries on lattice paths

We will now introduce the lattice path models whose random behaviour we would like to study in the first part of this article. The literature on the combinatorics of lattice paths is vast and we will mostly refer to the survey [Kra15].

Generally speaking, a lattice path in $\mathbb{Z}^2$ is a (finite or infinite) sequence of points in the lattice $\mathbb{Z}^2$ that is only allowed to change according to an a priori specified set of steps (a step is the difference between two consecutive elements of the sequence). The lattice paths we will focus on in this article, will not be allowed to cross the x- and y-axes and thus will be restricted to the upper right quadrant $\mathbb{N} \times \mathbb{N}$, where we include 0 in $\mathbb{N}$ by convention. We will call the tuples $(1,1),(1,-1),(1,0)$ an *up-step, down-step* and *level-step* respectively.

Definition 2.1. (1) A *ballot path* from $(a,b) \in \mathbb{N} \times \mathbb{N}$ to $(c,d) \in \mathbb{N} \times \mathbb{N}$ is a lattice path starting at (a,b) and ending at (c,d) that takes only up- or down-steps. We will also allow for infinite ballot paths, that is to say ballot paths with a specified starting point performing an infinite number of steps.

(2) A *Motzkin path* from $(a,b) \in \mathbb{N} \times \mathbb{N}$ to $(c,d) \in \mathbb{N} \times \mathbb{N}$ is a lattice path starting at (a,b) and ending at (c,d) that is allowed to take up-steps, down-steps and level-steps. Again Motzkin paths are allowed to be infinite.

Remark 2.2. (1) Ballot paths that start at the root $(0,0)$ and end in a point on the x-axis are commonly known as *Dyck paths*.

(2) It is also common to define ballot paths somewhat differently as lattice paths that take steps $(0,1)$ and $(1,0)$, start above the diagonal $x=y$ and do not cross it.

(3) Ballot paths can also be identified with walks on the one sided line (i.e. the graph with vertex set $\mathbb{N}$ and edges between neighboring non-negative integers). Similarly, Motzkin paths correspond to walks on the graph obtained from the previous one by adding a loop on every vertex.

Let us denote the set of ballot paths (resp. Motzkin paths) from (a,b) to (c,d) by $\mathrm{Ba}((a,b),(c,d))$ (resp. $\mathrm{Mo}((a,b),(c,d))$). For the following result, see [Kra15, Theorem 10.3.1]. Note that there, this theorem is formulated in the setting mentioned above in Remark 2.2.

Theorem 2.3. *The number of ballot paths from (a,b) to (c,d) is*

$$|\operatorname{Ba}((a,b),(c,d))| = \binom{c-a}{(c-a-d+b)/2} - \binom{c-a}{(c-a+d+b+2)/2}$$

where by convention a binomial coefficient is 0 *if its bottom component is not an integer.*

The analogous result for Motzkin paths is the following (see [Kra15, Theorem 10.6.1]).

Theorem 2.4. *The number of Motzkin paths from (a,b) to (c,d) is*

$$|\operatorname{Mo}((a,b),(c,d))| = \sum_{k=0}^{c-a} \binom{c-a}{k}\left(\binom{c-a-k}{(c-a-k+d-b)/2} - \binom{c-a-k}{(c-a-k+b+d+2)/2}\right).$$

Remark 2.5. 1. The numbers

$$m_n := |\operatorname{Mo}((0,0),(n,0))| = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2l)!(l+1)!l!}$$

are known in the literature as the *Motzkin numbers* (sequence A001006 in the OEIS) and they satisfy the identity $m_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k$, where $C_k = \frac{1}{k+1}\binom{2k}{k}$ are the Catalan numbers.

3 Preliminaries on branching graphs

A *branching graph* or *Bratelli diagram* $\Gamma = (V,E)$ is a bipartite locally finite $\mathbb{N}$-graded rooted graph, that is to say, a graph whose set of vertices V can be subdivided into levels $V = \sqcup_{n \in N} V_n$ with $V_0 = \{\emptyset\}$ containing the only the root $\emptyset$, and whose edges can only connect vertices of adjacent levels. All concrete examples of branching graphs in this article will have finite level sets V_n.

Branching graphs are typically used to encode the induction/restriction rules of inductive sequences $G_0 = \{e\} \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \dots$ of finite groups of more generally of inductive sequences $A_0 = \mathbb{C} \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$ of semisimple $*$-algebras. To recall how this graph is obtained, let us fix the following notation. If $A \subset B$ is an inclusion of semisimple algebras and M is a left B-module, we denote by $M^{\downarrow}$ the left A-module obtained by restricting the action of B to A.

Definition 3.1. Let $A_0 = \mathbb{C} \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$ be an inductive sequence of semisimple $*$-algebras. The *induction/restriction graph* of $(A_n)_{n \geq 0}$ is the branching graph $\Gamma = (V,E)$ defined in the following way.

- The n-th level vertex set V_n is the set of (equivalence classes of) simple modules of A_n.
- There are exactly $\operatorname{mult}(v, w^{\downarrow})$ edges between the simple A_n-module $v \in V_n$ and the simple A_{n+1}-module $w \in V_{n+1}$ where $\operatorname{mult}(v, w^{\downarrow})$ denotes the multiplicity of v in the decomposition of w into simple A_n-modules.

In most of the examples that we will consider in this article, the branching graphs in question are generated by smaller ones through a process dubbed *pascalization* in [VN06].

Definition 3.2. Let $\Gamma = (V,E)$ be a branching graph. The *pascalized graph* $\mathcal{P}(\Gamma) = (\mathcal{P}(V), \mathcal{P}(E))$ of Γ is defined in the following way.

- The vertex set of level n is $\mathcal{P}(V)_n = \{(n, v) \ ; \ v \in V_k, \ k \leq n, \ k \equiv n \mod 2\}$. We will denote the natural projection $\mathcal{P}(V) \to V, \ (n, v) \mapsto v$ by π.

- In $\mathcal{P}(\Gamma)$, the number of edges between $(n, v) \in \mathcal{P}(V)_n$ and $(n+1, w) \in \mathcal{P}(V)_{n+1}$ is the number of (undirected) edges between $v \in V_k$ and $w \in V_l$ in the original graph Γ. In particular, this number can only be non-zero if v and w are on neighbouring levels of Γ, that is $|k - l| = 1$.

Example 3.3. (1) If we consider the set of integers $\mathbb{Z}$ as a branching graph with $\mathbb{N}$-grading $n \mapsto |n|$ and edges connecting neighboring integers, then $\mathcal{P}(\mathbb{Z})$ is the *Pascal graph*, whence the name pascalization.

(2) Similarly if we consider the set of non-negative integers $\mathbb{N}$ as a branching graph, its pascalization is the semi-Pascal graph $\mathcal{P}(\mathbb{N})$. This branching graph is the Bratelli diagram of the inductive family of Temperley-Lieb algebras at a generic parameter, see e.g [Jo83] [GHJ89]. We will discuss this example in more detail in Section 5.

(3) More generally, the procedure of pascalization will be familiar to readers with a background in *subfactor theory* although the name pascalization is not commonly used there. It is exactly the method by which one constructs the Bratteli diagram of irreducible bimodules of a subfactor $N \subset M$ from its principal graph.

3.1 The minimal boundary of a branching graph

In this section, we will recall the fundamentals on the minimal boundary of branching graphs. The methods explained here go mostly back to the work of Vershik and Kerov [VK81] [VK82] and they are heavily used in asymptotic representation theory of inductive limit groups such as $S(\infty)$ or $U(\infty)$, see e.g. the book [BO16].

For a Bratteli diagram $\Gamma = (V, E)$, let us denote by $(\Omega, \mathcal{F}) := (\Omega_\Gamma, \mathcal{F}_\Gamma)$ the space $\Omega \subset \prod_{n \geq 0} V_n$ of infinite paths on Γ equipped with the product σ-algebra $\mathcal{F}$. The n-th coordinate projection will be denoted by $X_n : \Omega \to V_n, \ X_n((v_k)_{\geq 1}) = v_n$. We will say that two infinite paths $x, y \in \Omega$ are *tail equivalent* ($x \sim y$) if they coincide up to finitely many steps. The corresponding equivalence relation, the *tail relation*, is Borel and will be referred to by $\mathcal{T}$. We will also make use of the common notation $\dim(v, w)$ for the number of paths leading from $v \in V_n$ to $w \in V_m, \ m > n$ and we will shorten $\dim(*, v)$ to $\dim(v)$.

Definition 3.4. A probability measure ν on $(\Omega, \mathcal{F})$ will be called *central* if it is invariant under $\mathcal{T}$ or equivalently if for all $k \geq 0$, $v \in V_k$, and every path $v_0 = \emptyset, v_1, \ldots, x_k = v$ from the root to v, we have

$$\nu(\{x \in \Omega \ ; \ x_1 = v_1, \ldots, x_k = v\}) = 1/\dim v.$$

The central measure ν will be called *ergodic* if it is ergodic w.r.t. $\mathcal{T}$, that is to say if the only $\mathcal{T}$-invariant subsets of Ω are those that have ν-measure 0 or 1.

The topological space of ergodic central measures on $(\Omega, \mathcal{F})$ (equipped with the weak topology) is known as the *minimal* or *exit boundary* of Γ and we will denote it by $\partial\Gamma$. Note that an arbitrary central probability ν' can always be decomposed into its ergodic components by Choquet's theorem, that is to say, there exists a (unique up to nullsets) probability measure μ on $\partial\Gamma$ such that $\nu'(A) = \int_{\partial\Gamma} \nu(A) \, d\mu(\mathbb{P})$. In other words, the set $\mathcal{M}_c(\Gamma)$ of central probability measures on $(\Omega, \mathcal{F})$ forms a Choquet simplex (w.r.t. the weak topology) whose extremal points are given by the boundary. The following theorem is well-known, see e.g [BO16].

Theorem 3.5. *Let $\mathbb{C} = A_0 \subset A_1 \subset \dots$ a sequence of finite-dimensional semisimple $*$-algebras with inductive limit algebra A_∞. Further, let Γ be the induction/restriction branching graph of this sequence, so that the vertices $v \in V_n$ correspond to the irreducible summands of A_n. Then, the Choquet simplex of tracial states on the envelopping C^*-algebra $C^*(A_\infty)$ is homeomorphic to $\mathcal{M}_c(\Gamma)$. More precisely, this homeomorphism takes the measure $\mathbb{P}$ to the tracial state $\tau_\mathbb{P}$ determined by*

$$\tau_\mathbb{P}(x) = \sum_{i=1}^m \mathbb{P}(X_n = v_i)\frac{\tau_i(x)}{\dim(v_i)} \qquad (x \in A_n),$$

where τ_i is the extremal trace on A_n belonging to the irreducible summand v_i. Under this homeomorphism, the pure tracial states correspond to the elements of the boundary $\partial\Gamma$.

Thanks to the works of Vershik and Kerov [VK81] [VK82], there is a well-developed machinery available to compute the minimal boundary of a branching graph Γ. For an ergodic central measure $\nu \in \partial\Gamma$ and connected vertices $v_n \in V_n,\ v_{n+1} \in V_{n+1}$, we will denote by $p_\nu(v_n, v_{n+1})$ the transition probability $\nu(X_{n+1} = v_{n+1}|X_n = v_n)$. The following theorem is known as the *(Vershik-Kerov-)ergodic method*, see e.g. [VM15].

Theorem 3.6. *Let $\nu \in \partial\Gamma$ be an ergodic central measure on $(\Omega_\Gamma, \mathcal{F}_\Gamma)$, let $(v_0 = \emptyset, v_2, \dots, v_n)$ be a finite path on Γ and let (v, w) be an edge on Γ.*

(i) The set S of paths $(\omega_i)_{i\geq 0} \in \Omega$ for which the sequence $\frac{\dim(v_n,\omega_i)}{\dim(\omega_i)}$ has a limit has full ν-measure $\nu(S) = 1$. Moreover, for any such path,

$$\lim_{i\to\infty} \frac{\dim(v_n, \omega_i)}{\dim(\omega_i)} = \frac{\nu(X_n = v_n)}{\dim(v_n)} = \prod_{l=0}^{n-1} p_\nu(v_l, v_{l+1}).$$

(ii) Similarly, the set of paths $(\omega_i)_{i\geq 0} \in \Omega$ for which the sequence $\frac{\dim(w,\omega_i)}{\dim(v,\omega_i)}$ has a limit has full ν-measure and for any such path,

$$\lim_{i\to\infty} \frac{\dim(w, \omega_i)}{\dim(v, \omega_i)} = p_\nu(v, w).$$

The above theorem identifies the minimal boundary as a subset of another space of central measures, the so-called *Martin boundary*. Roughly speaking, the Martin boundary is build from all central measures that appear as limits in the above theorem. However, the ergodic method does not make any statements on whether or not the measures obtained from it are in fact ergodic (they need not be) and one typically has to check this 'by hand' for the example in question.

3.2 Random walk interpretation of the boundary of a pascalized graph

If one is interested in the boundary $\partial\mathcal{P}(\Gamma)$ of the pascalization of a branching graph Γ, it can be difficult to target this problem directly as the pascalized graph can be significantly larger than the original one. However, it is possible to reduce the study of the boundary $\partial\mathcal{P}(\Gamma)$ to a problem on Γ if one notes the following, see e.g. [VM15].

By a *walk* on Γ we mean a sequence of vertices $(v_n)_{n\geq 0}$ such that v_n and v_{n+1} are connected by an edge for all $n \geq 0$. We turn the space $\mathcal{W}_\Gamma$ of walks on Γ that start at $\emptyset$ into a Borel space by equipping it with the product σ-algebra $\mathcal{E}_\Gamma$ inherited from the inclusion $\mathcal{W}_\Gamma \subset \prod_{n\geq 1} V_{\leq n}$, $V_{\leq n} = \cup_{l\leq n} V_l$. The proof of the following result is straighforward.

Lemma 3.7. *The projection* $\pi : ((l, w_n))_{n\geq 0} \mapsto (w_n)_{n\geq 0}$ *yields a one-to-one correspondence between infinite paths on* $\mathcal{P}(\Gamma)$ *and rooted walks on* Γ*. More strongly, this map is an isomorphism of Borel spaces* $(\Omega_{\mathcal{P}(\Gamma)}, \mathcal{F}_{\mathcal{P}(\Gamma)}) \cong (\mathcal{W}_\Gamma, \mathcal{E}_\Gamma)$*.*

Lemma 3.7 asserts in particular that every measure on $(\Omega_{\mathcal{P}(\Gamma)}, \mathcal{F}_{\mathcal{P}(\Gamma)})$ can be interpreted as a (possible time-nonhomogenous) random walk on Γ. We will therefore call a measure on $(\mathcal{W}_\Gamma, \mathcal{E}_\Gamma)$ central (resp. ergodic) iff its corresponding boundary measure on $(\Omega_{\mathcal{P}(\Gamma)}, \mathcal{F}_{\mathcal{P}(\Gamma)})$ is central (resp. ergodic).

4 Preliminaries on diagram algebras

The branching graphs to be discussed in this article will always be derived from towers of finite-dimensional algebras whose basis will be given by *partition diagrams* and whose operations can be encoded diagrammatically. These algebras originate from Schur-Weyl duality of compact groups and have been used in [BS09] by Banica and Speicher to introduce a class of compact quantum groups which they called *easy*. These easy (or partition) quantum groups have also been extensively studied in [We13] and three important subclasses have been classified: the *free* easy quantum groups, the *half-liberated* easy quantum groups and the easy (classical) groups. The main focus of this paper are the free easy quantum groups. The branching graphs associated to easy groups will be addressed in the forthcoming article [Wa20].

Let us recall that by a partition with k upper and l lower points, we mean a decomposition of the set $\{1, 2, \ldots, k, 1', \ldots, l'\}$ into disjoint subsets (the *blocks* of the partition). A diagrammatic depiction of a partition is to be found below in Figure 1. The set of all partitions with k upper and l lower points will be denoted by $\mathrm{Part}(k, l)$ and the set of non-crossing partitions with k upper and l lower points will be denoted by $\mathrm{NC}(k, l)$. See [BS09] or [We13, Definition 1.4] for more details on the operations in the following definition.

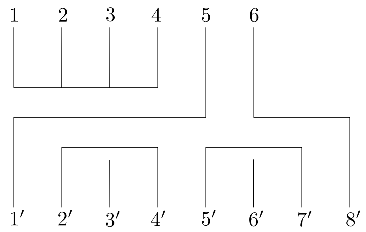

Figure 1: A partition in $\mathrm{NC}(6, 8)$.

Definition 4.1. A *category of partitions* $\mathcal{C}$ is a collection $(\mathcal{C}(k, l))_{k,l\in\mathbb{N}}$ of subsets $\mathcal{C}(k, l) \subset \mathrm{Part}(k, l)$ such that

- $\mathcal{C}(1, 1)$ contains the identity partition that connects the upper and the lower point.
- the family is invariant under the category operations tensor product (i.e. vertical concatenation of diagrams), rotation, involution (i.e. reflecting a diagram along a horizontal line in the middle) and composition (i.e. vertical concatenation of compatible partitions $p_1 \in \mathcal{C}(k, l),\ p_2 \in \mathcal{C}(l, m)$).

The k-th diagram algebra $A_{(\mathcal{C},\delta)}(k)$ of the category of partitions $(\mathcal{C}, \delta)$ with loop parameter δ will be the free vector space with basis $\mathcal{C}(k, k)$. Multiplication of two basis vectors e_{p_1}, e_{p_2} associated to partitions p_1 and p_2 will be implemented by drawing p_2 atop p_1 and connecting blocks of p_1 and p_2 that meet in the middle. We are allowed to erase every block appearing in the middle of the picture that is not connected to any upper or lower points at the cost of multiplying with the loop parameter δ for each such block. In other words, if the diagram obtained after erasing loops is p_3, then $e_{p_1} \cdot e_{p_2} = \delta^{\#erased\ loops} e_{p_3}$. As mentioned above, the involution p^* of a diagram p is given by reflecting p along a horizontal line in the middle, whence we get an involution $e_p^* = e_{p^*}$ on $A_{(\mathcal{C},\delta)}(k)$.

Recall that a finite-dimensional algebra A is semisimple if and only if it possesses a *positive* involution, i.e. one for which $x^*x = 0$ implies $x = 0$, see e.g. [GHJ89, Appendix II]. In addition, if there is a positive involution on A, there is also a unique C^*-norm on A turning it into a C^*-algebra. Lastly, if $A \subset B$ is an inclusion of finite-dimensional semisimple algebras, for any positive involution on A, there is a positive involution on B extending it. Therefore, when given an inductive sequence of finite-dimensional semisimple algebras, we can always consider its inductive limit in the category of C^*-algebras which is the viewpoint we want to take here.

Definition 4.2. Let $(\mathcal{C}, \delta)$ be a category of partitions at loop parameter δ and assume that for all $k \geq 1$, $A_{(\mathcal{C},\delta)}(k)$ is semisimple. Let $\alpha_k : A_{(\mathcal{C},\delta)}(k) \to A_{(\mathcal{C},\delta)}(k+1)$ be the embedding obtained by mapping $e_p \to e_{p'}$, where p' is obtained from p by adding a through-string on the right of the diagram. The inductive limit (in the category of C^*-algebras) w.r.t. these embeddings will be denoted $A_{(\mathcal{C},\delta)}(\infty)$.

Note that the set $\mathcal{C}(\infty) := \bigcup_{k\geq 1} \mathcal{C}(k,k)$ yields a natural basis for $A_{(\mathcal{C},\delta)}(\infty)$. Also we remark that by [HR05, Theorem 5.13], the set of generic parameters ((that is parameter values for which $A_{(\mathcal{C},\delta)}(k)$ is semisimple for all $k \geq 0$)) is always co-countable. For more precise results on genericity of the loop parameter, see [FM20].

4.1 The classification of free partition quantum groups

We will now recall the classification of categories of *non-crossing* set partitions due to Banica, Speicher [BS09] and Weber [We13]. These categories describe the representation theory of the *free* easy quantum groups in the setting of Banica and Speicher and there are exactly seven of them.

Theorem 4.3 ([BS09],[We13])**.** *There are exactly seven categories of non-crossing partitions, namely*

1. *The category $\mathcal{C}_{S^+} = \mathrm{NC}$ of all non-crossing partitions, corresponding to the quantum permutation groups S_N^+;*
2. *The category $\mathcal{C}_{O^+} = \mathrm{NC}_2$ of all non-crossing pair partitions, corresponding to the free orthogonal groups O_N^+;*
3. *The category $\mathcal{C}_{B^+}$ of all non-crossing partitions with blocks of size one or two, corresponding to the bistochastic quantum groups B_N^+;*
4. *The category $\mathcal{C}_{H^+}$ of all non-crossing partitions with blocks of even size, corresponding to the hyperoctahedral quantum groups H_N^+;*
5. *The category $\mathcal{C}_{S'^+}$ of all non-crossing partitions with an even number of blocks of odd size, corresponding to the modified quantum permutation groups $S_N'^+$;*
6. *The category $\mathcal{C}_{B'^+}$ of all non-crossing partitions with an even number of blocks of size one and an arbitrary number of blocks of size two, corresponding to the modified bistochastic quantum groups $B_N'^+$;*

7. *The category $\mathcal{C}_{B\#+}$ of all non-crossing partitions with boundary points labelled by alternating symbols a, b with an even number of blocks of size one and an arbitrary number of blocks of size two connecting one a to one b. This category corresponds to the freely modified bistochastic quantum groups $B_N^{\#+}$.*

5 Traces on intertwiner algebras of free easy quantum groups

We will now approach the new results of this article, namely the description of the traces on the algebras $A_{(\mathcal{C},\delta)}(\infty)$ at the generic parameter. We will begin with the diagram algebra associated to the category $\mathcal{C}_{O^+} = \mathrm{NC}_2$ of all non-crossing pair partitions, which is better known as the *infinite Temperley-Lieb algebra* $\mathrm{TL}(\infty, \delta)$. The main result of Subsection 5.1 is Theorem 5.1, which rephrases the classification of extremal traces on $\mathrm{TL}(\infty, \delta)$ as a classification result for random ballot paths. As explained in Remark 5.2, Theorem 5.1 will also settle the trace classification problem for the examples $\mathcal{C}_{S^+}$ and $\mathcal{C}_{S'^+}$. The trace classification for the diagram algebras of the categories of partitions $\mathcal{C}_{B^+}$ and $\mathcal{C}_{B'^+}$ or equivalently, the classification of central random Motzkin paths, will be addressed in Subsection 5.2. Before stating the classification result (Theorem 5.10), we will first carry out the algebraic legwork of computing the relevant branching graph which yields new dimension formulas for irreducible representations of the quantum groups B_N^+ as a side product. In Subsection 5.3 , we compute the branching graph associated to the category $\mathcal{C}_{B\#+}$ and in Subsection 5.4, we review known results on the branching graph of $\mathcal{C}_{H^+}$, which will also deal with more general *Fuss-Catalan-algebras*. In Subsection 5.4, we will also deduce new dimension formulas for irreducible representations of the quantum groups $B_N^{\#+}$. The trace classification for the last two examples $\mathcal{C}_{B\#+}$ and $\mathcal{C}_{H^+}$ is in our eyes the most interesting and it will follow from the stochastic results on the boundary of the Fibonacci tree that we will obtain in Section 6.

5.1 Traces on the infinite-dimensional Temperley-Lieb algebra and random ballot paths

The algebra spanned by noncrossing pair partitions in $\mathrm{NC}_2(k, k)$ between k upper and k lower points is the *Temperley-Lieb algebra* $\mathrm{TL}(k, \delta)$ and it is typically defined as the associative unital algebra generated by elements $e_i,\ i = 1, \dots, k-1$ satisfying the relations

$$\begin{aligned} e_i^2 &= e_i & i &= 1, \dots k-1 \\ e_i e_j &= e_j e_i & |i-j| &> 1 \\ \delta e_i e_j e_i &= e_i & |i-j| &= 1. \end{aligned}$$

The set of generic parameter values $\delta \in \mathbb{C}$ for the Temperley-Lieb algebras was determined by Jones [Jo83] and contains in particular the interval $[2, \infty)$. If $\delta \in [2, \infty)$, the declaration $e_i^* = e_i,\ i = 1, \dots, k-1$ defines a positive involution on $\mathrm{TL}(k, \delta)$ for all k, thus turning the generating idempotents into projections. The k-th Temperley-Lieb algebra $\mathrm{TL}(k, \delta)$ embeds naturally into the $\mathrm{TL}(k+1, \delta)$ by mapping $e_i \in \mathrm{TL}(k, \delta)$ to $e_i \in \mathrm{TL}(k+1, \delta)$ or equivalently by adding a through string on the right of every diagram in the diagrammatical interpretation. The inductive limit algebra under these embeddings will be denoted $\mathrm{TL}(\infty, \delta)$. The images of $\mathrm{TL}(k, \delta)$ at parameter value $\delta = N = 2, 3 \dots$ under the Banica-Speicher representation in [BS09], yield the endomorphism spaces the fundamental representations of the easy quantum groups O_N^+.

In the generic case, the Bratelli diagram of $\mathrm{TL}(\infty, \delta)$ does not depend on the choice of the parameter δ (as long as it in fact is generic) and its boundary is the semi-Pascal graph, see

Example 3.3. Let us label the vertices of $\mathcal{P}(\mathbb{N})_m$ by $(m, s),\ s = 0, 2, \ldots, m$ if m is even and by $(m, s),\ s = 1, 3, \ldots m$, if m is odd such that we have an edge between (m, s) and $(m + 1, t)$ if and only if $t \in \{s - 1, s + 1\}$. A picture of the first few levels of the semi-Pascal graph $\mathcal{P}(\mathbb{N})$ is drawn in Figure 2 below.

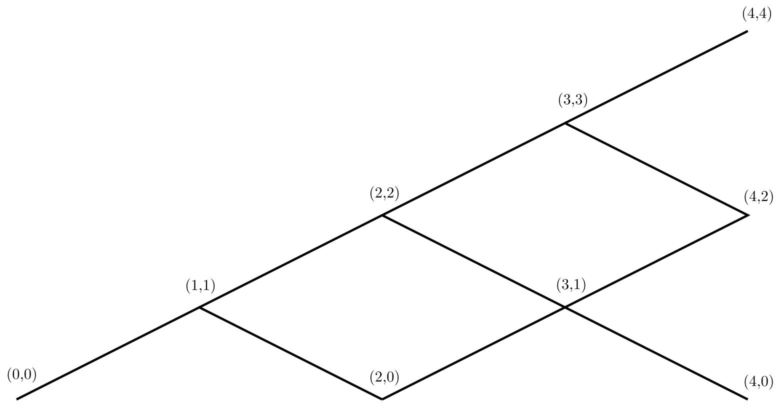

Figure 2: The first five levels of the semi-Pascal graph.

From the description of the Bratteli diagram as the semi-Pascal graph, it is immediate that its infinite rooted paths exactly conincide with infinite ballot paths starting at $(0, 0)$. Central measures on the semi-Pascal graph therefore translate into random ballot paths such that the conditional probability measure given that the random path passes through (n, k) is the uniform distribution on $\mathrm{Ba}((0, 0), (n, k))$.

The boundary of the semi-Pascal graph (and thus the trace simplex on $\mathrm{TL}(\infty, \delta)$) has been studied in the PhD thesis of Wassermann [Was81], where it is shown that the extremal trace simplex is homeomorphic to the half interval $[1/2, 1]$. An algebraic explanation of this goes as follows. $\mathrm{TL}(\infty, \delta)$ is a quotient of the infinite Hecke algebra $H(\infty, q),\ q + q^{-1} = \delta$, a q-deformation of the symmetric group algebra. The extremal traces/ II_1-factor representations of $H(\infty, q)$ coincide with those of S_∞ as $C^*(H_q(\infty)) \cong C^*(S_\infty)$. The only II_1-factor representations that factor through the quotient map onto the Temperley-Lieb algebra are the ones corresponding to the Thoma parameters satisfying $\alpha_1 + \alpha_2 = 1$. The extremal trace simplex is thus homeomorphic to the half interval $[1/2, 1]$.

We will now recall and further refine Wassermann's results in view of the interpretation of central measures as random ballot paths.

Let $\lambda \in (1/2, 1]$. Define the Markov chain M^λ on the space $(\Omega_{\mathcal{P}(\mathbb{N})}, \mathcal{F}_{\mathcal{P}(\mathbb{N})})$ of infinite rooted ballot paths with transition probabilities

$$\begin{aligned} p_\lambda((m, s), (m + 1, s + 1)) &= \frac{(1 - \lambda)^{s+2} - \lambda^{s+2}}{(1 - \lambda)^{s+1} - \lambda^{s+1}}, \\ p_\lambda((m, s), (m + 1, s - 1)) &= 1 - \frac{(1 - \lambda)^{s+2} - \lambda^{s+2}}{(1 - \lambda)^{s+1} - \lambda^{s+1}}. \end{aligned}$$

For $\lambda = 1/2$, define the transition probabilities of $M_{1/2}$ by

$$\begin{aligned} p_{1/2}((m, s), (m + 1, s + 1)) &= \frac{1}{2} \cdot \frac{s + 2}{s + 1}, \\ p_{1/2}((m, s), (m + 1, s - 1)) &= 1 - \frac{1}{2} \cdot \frac{s + 2}{s + 1}. \end{aligned}$$

Note that the Markov chains M^λ are time-homogeneous in the sense that the transition probabilities do not depend on m. Denote the law of M^λ by ν_λ. The following theorem can be interpreted as a one-sided de Finetti theorem for discrete stochastic processes conditioned to stay non-negative.

Theorem 5.1. *Every central rooted random ballot path M with law ν is a mixing of the Markov chains M^λ, $\lambda \in [1/2, 1]$, that is to say there is a probability measure μ on $[1/2, 1]$ such that*

$$\nu_M = \int_{1/2}^{1} \nu_\lambda \, d\mu(\lambda).$$

Proof. We need to show that the family ν_λ, $\lambda \in [1/2, 1]$ exhausts the ergodic central measures on $(\Omega_{\mathcal{P}(\mathbb{N})}, \mathcal{F}_{\mathcal{P}(\mathbb{N})})$. The theorem then follows from the ergodic decomposition theorem.
Thus, let ν be an ergodic central measure on $(\Omega_{\mathcal{P}(\mathbb{N})}, \mathcal{F}_{\mathcal{P}(\mathbb{N})})$. We invoke the ergodic method of Theorem 3.6 to compute the transition probabilities $p_\nu((m,s),(m+1,s+1))$. Set $k = (m-s)/2$, let (N,S) be another vertex of $\mathcal{P}(\mathbb{N})$ and set $K = (N-S)/2$. By Theorem 2.3, we have

$$\frac{\dim_{\mathcal{P}(\mathbb{N}_0)}((m+1,s+1),(N,S))}{\dim_{\mathcal{P}(\mathbb{N}_0)}((m,s),(N,S))} = \frac{\binom{N-m-1}{K-k} - \binom{N-m-1}{N-K-k+1}}{\binom{N-m}{K-k} - \binom{N-m}{N-K-k+1}}.$$

A quick computation shows that for large values of N this expression is asymptotically equal to

$$\frac{N-K}{K} \cdot \frac{1-\left(\frac{K}{N-K}\right)^{m-2k+2}}{1-\left(\frac{K}{N-K}\right)^{m-2k+1}}.$$

This expression converges to the desired value

$$\frac{(1-\lambda)^{s+2} - \lambda^{s+2}}{(1-\lambda)^{s+1} - \lambda^{s+1}},$$

if and only if $\frac{K}{N} \to \lambda \in (1/2, 1]$ and to $\frac{1}{2} \cdot \frac{s+2}{s+1}$ if and only if $\frac{K}{N} \to \frac{1}{2}$.
On the other hand, it is easily checked that the measures defined by the Markov chains M_λ are indeed central. Their ergodicity is proven in [Was81, p. 119-122] by showing that the corresponding trace τ_λ on $\mathrm{TL}(\infty, \delta)$ is the restriction of the product state $\bigotimes_{n\geq 0} \mathrm{Tr}(\,\cdot\,\mathrm{diag}(\lambda, 1-\lambda))$ on $\bigotimes_{n\geq 0} M_2(\mathbb{C})$ to $\mathrm{TL}(\infty,\delta)$ and are therefore extremal. We will not present the details of this argument here, but we will invoke a similar argument in the classification of central ergodic Motzkin paths in the next section.

□

Remark 5.2.
1. For the category $\mathcal{C}_{S^+} = \mathrm{NC}$ of all noncrossing partitions, it has been observed in the literature that $A_{(\mathcal{C}_{S^+},\delta)}(k) \cong A_{(\mathcal{C}_{O^+},\sqrt{\delta})}(2k) = \mathrm{TL}(2k, \sqrt{\delta})$ through the so-called fattening isomorphism, see e.g [LT16]. Since this isomorphism respects the inclusion $A_{(\mathcal{C}_{S^+},\delta)}(k) \subset A_{(\mathcal{C}_{S^+},\delta)}(k+1)$ to $\mathrm{TL}(2k,\sqrt{\delta}) \subset \mathrm{TL}(2k+2,\sqrt{\delta})$, one readily obtains the branching graph of $A_{(\mathcal{C}_{S^+},\delta)}(\infty)$ as the graph of even levels $\mathcal{P}(\mathbb{N}_0)_{2n}$ of the semi-Pascal graph. In this graph, there is an edge from v to w for every two step path from v to w in $\mathcal{P}(\mathbb{N}_0)$. The minimal boundary is therefore also homeomorphic to $[1/2, 1]$ and the transition probabilities of the ergodic central Markov process indexed by λ are the two step transition probabilities of M_λ above.

2. Since the difference between the categories of partitions $\mathcal{C}_{S^{+\prime}}$ and $\mathcal{C}_{S^+}$ is not visible in $\mathcal{C}(k,k)$, $k \geq 0$, the branching graphs associated to these categories are the same. By the same argument, the categories $\mathcal{C}_{B^+}$ and $\mathcal{C}_{B^{+\prime}}$ yield the same branching graph, which we will compute in the next section.

5.2 Traces on the infinite-dimensional Mozkin algebra and random Motzkin paths

The next direct limit algebra that we would like to study is the one associated to the category $\mathcal{B} = \mathcal{C}_{B+}$ which describes the representation theory of the free bistochastic quantum group in the setting of Banica and Speicher. Recall from above that $\mathcal{C}_{B+}$ is formed by the non-crossing partitions with blocks of size one and two. In the next section we will compute the Bratelli diagram of $A_{(\mathcal{B},\delta)}(\infty)$. We learned after these computations were carried out that this branching graph had already been computed in [BH14], where the algebras $A_{(\mathcal{B},\delta)}(n)$ appear under the name *Motzkin algebras*. We decided to nevertheless include our computations in the present article since they are a nice application of the recipe to compute the representation theory of free easy quantum group presented in [FrWe16].

5.2.1 Determining the branching graph of $A_{(\mathcal{B},\delta)}(\infty)$

We will describe the representations of the finite-dimensional algebras $A_{(\mathcal{B},\delta)}(n),\ n \geq 1$ in a way that is similar to the standard description of Temperley-Lieb representations through link states.

Definition 5.3. A *n-link state* for $\mathcal{B}$ is a partition of n points into pairs and singletons of two types called *proper singletons* and *defects*. A defect is not allowed to be braced by a pair, i.e. if $i < j < k$ and i and k are paired, then j must be a proper singleton. A (n, d)-link state for $\mathcal{B}$ is a n-link state with d defects.

n-Link states for $\mathcal{B}$ arise as upper halves of projective partitions in $\mathcal{B}(n, n)$ (as defined in [FrWe16]) when we cut them across a horizontal line in the middle of the diagram. A diagramatic example in which defects are marked by arrows is depicted in Figure 3 below.

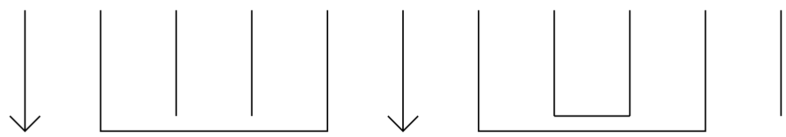

Figure 3: A $(11, 2)$-link state.

Let us denote the set of (n, d)-link states by $\mathcal{L}_{(n,d)}$.

Analogous to the multiplication of partition diagrams, we can define an action $v \mapsto p \cdot v$ of a partition $p \in \mathcal{B}(n, n)$ on an n-link state v by drawing p above v, connecting lines and deleting any lines that do not connect to points on the upper boundary. For any closed block (that is any line without defect) deleted in this way, we pay by multiplying the resulting diagram with the loop parameter δ. Any line with a defect that is not connected to upper points may however be deleted right away (that is without multiplying by a scalar). Let M_n denote the complex span of all n-link states. We can linearly extend the action of partitions on link states to a linear representation $A_{(\mathcal{B},\delta)}(n) \to \mathrm{End}(M_n)$, turning M_n into a left $A_{(\mathcal{B},\delta)}(n)$-module.

Definition 5.4. We denote by $M_{(n,d)} \subset M_n$ the complex span of all (n, d')-link states with $d' \leq d$.

It is clear that $M_{(n,d)}$ is a submodule of M_n. Moreover, by definition we have $M_{(n,d)} \subset M_{(n,d+1)}$ for all $d = 0, \dots, n-1$. Let us denote the quotient modules by

$$V_{(n,0)} := M_{(n,0)}, \qquad V_{(n,d)} := {}^{M_{(n,d)}}\!/\!_{M_{(n,d-1)}},\ d = 1, \dots n.$$

Proposition 5.5. *$\{V_{(n,0)}, \dots V_{(n,n)}\}$ is a full set of inequivalent irreducibles modules for $A_{(\mathcal{B},\delta)}(n)$, that is to say we have a decomposition*

$$A_{(\mathcal{B},\delta)}(n) \cong \bigoplus_{d=0}^{n} \operatorname{End}(V_{(n,d)}).$$

Moreover, the set $B_{(n,d)} = \{v + M_{(n,d-1)};\ v \in \mathcal{L}_{(n,d)}\}$ is a basis of $V_{(n,d)}$ for $d = 1, \dots, n$.

Proof. The claim about the basis follows directly from the definition of our modules $V_{(n,d)}$. We recall from [FrWe16] that the minimal projections in $A_{(\mathcal{B},\delta)}(n)$ are precisely the projective partitions (up to a scalar multiple) and that two projective partitions are equivalent if and only if the number of through strings coincide. The projective partition are in one-to-one correspondence with n-link state as every link state arises by cutting a projective partition along a horizontal line in the middle and every projective partition is uniquely determined by its upper half. One checks that the projective partition p_s associated to a (n,d)-link state s acts on $V_{(n,d)}$ as the projection onto $\langle s\rangle$. Hence, if $V \subset V_{(n,d)}$ is a nontrivial submodule containing a vector w, it must at least contain one element v of the basis $B_{(n,d)}$. Since the projections $p_s, s \in \mathcal{L}_{(n,d)}$ are all equivalent, it follows that $V = V_{(n,d)}$ and $V_{(n,d)}$ is irreducible. Finally, since the equivalence class of a minimal projection is determined by its number of through strings, it follows that $A_{(\mathcal{B},\delta)}(n) \cong \bigoplus_{d=0}^{n-1} \operatorname{End}(V_{(n,d)})$. □

To determine the branching graph of $A_{(\mathcal{B},\delta)}(\infty)$, we need to determine the decomposition of the modules $V_{(n,0)}, \dots V_{(n,n-1)}$ of $A_{(\mathcal{B},\delta)}(n)$, when considered as modules of $A_{(\mathcal{B},\delta)}(n-1)$. Here, as always, we work with the natural embedding $A_{(\mathcal{B},\delta)}(n-1) \to A_{(\mathcal{B},\delta)}(n)$ that adds a through string to any partition diagram. When considered as an $A_{(\mathcal{B},\delta)}(n-1)$, we write $V_{(n,d)}$ as $V^{\downarrow}_{(n,0)}$.

Proposition 5.6. *The $A_{(\mathcal{B},\delta)}(n-1)$-module $V^{\downarrow}_{(n,d)}$ decomposes as*

$$V^{\downarrow}_{(n,d)} \cong V_{(n-1,d-1)} \oplus V_{(n-1,d)} \oplus V_{(n-1,d+1)}$$

for $d = 1, \dots, n-2$. In the remaining cases $d = 0, n-1, n$, we have

$$V^{\downarrow}_{(n,0)} \cong V_{(n-1,0)} \oplus V_{(n-1,1)}, \quad V^{\downarrow}_{(n,n-1)} \cong V_{(n-1,n-2)} \oplus V_{(n,n-1)}, \quad V^{\downarrow}_{(n,n)} \cong V_{(n-1,n-1)}.$$

Proof. For a $(n-1,d)$-link state v, let us define $\dot{v}$ to be the (n,d)-link state obtained by adding a proper singleton on the right. Similarly, we define $\vec{v}$ as the $(n,d+1)$-link state obtained from v by adding a defect on the right. Lastly, if $d > 0$, we define $\breve{v}$ to be the $(n,d-1)$-link state that we obtain from v by adding a point on the right and pairing it with the rightmost defect of v. For $d = 1, \dots, n-2$, we define the map

$$V_{(n-1,d-1)} \oplus V_{(n-1,d)} \oplus V_{(n-1,d+1)} \to V^{\downarrow}_{(n,d)},\ (u, v, w) \mapsto \vec{u} + \dot{v} + \breve{w},$$

where $u \in B_{(n-1,d-1)}, v \in B_{(n-1,d)}, w \in B_{(n-1,d+1)}$ and where we ignored the equivalence classes in the quotient to lighten the notation (note that this is well-defined). It is now easy to check that this map indeed extends to an isomorphism $V^{\downarrow}_{(n,d)} \cong V_{(n-1,d-1)} \oplus V_{(n-1,d)} \oplus V_{(n-1,d+1)}$ of $A_{(\mathcal{B},\delta)}(n-1)$-modules. The cases $d = 0, n-1, n$ are analogous. □

A pictorial depiction of the first levels of the branching graph $\Gamma_{\mathcal{B}^+}$ associated to $A_{(\mathcal{B},\delta)}(\infty)$ is shown in Figure 4.

The following observation is now immediate.

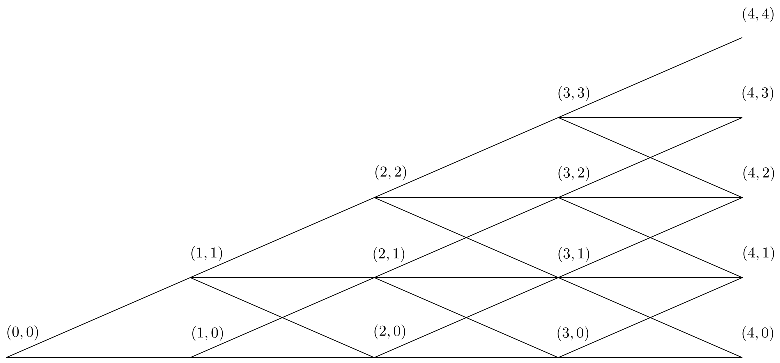

Figure 4: The first five levels of the branching graph $\Gamma_{\mathcal{B}^+}$.

Proposition 5.7. *There is a natural bijection between paths on the branching graph $\Gamma_{\mathcal{B}^+}$ starting at vertex (n,d) and Motzkin paths starting at (n,d).*

Proof. For any edge $(n,d) \to (n+1,d')$ in $\Gamma_{\mathcal{B}}$ we have $(n+1,d')-(n,d) \in \{(1,1),(1,0),(1,-1)\}$ corresponding to up-steps, level-steps and down-steps. $\square$

As was the case for the ballot path interpretation of the branching graph of $\mathrm{TL}(\infty,\delta)$, Proposition 5.7 yields a convenient reinterpretation of central measures on the space $(\Omega_{\Gamma_{\mathcal{B}^+}}, \mathcal{F}_{\Gamma_{\mathcal{B}^+}})$ of infinite paths on $\Gamma_{\mathcal{B}^+}$. A central measure μ is a *random (infinite) Motzkin paths* $(w_i)_{i\geq 0}$ starting at $(0,0)$ such that the conditional distribution given $w_n = (n,d)$ is uniform on $\mathrm{Mo}((0,0),(n,d))$.

The formula for the number of Motzkin paths in Theorem 2.4 thus yields the following corollary.

Corollary 5.8. *The dimension of the simple module $V_{(n,d)}$ of $A_{(\mathcal{B},\delta)}(n)$ is*

$$\dim V_{(n,d)} = \sum_{k=0}^{n} \binom{n}{k} \left(\binom{n-k}{(n-k+d)/2} - \binom{n-k}{(n-k+d+2)/2} \right) =: m_{n,d}.$$

In particular the dimension of $V_{(n,0)}$ is the n-th Motzkin number m_n.

Remark 5.9. (1) More generally, our description of the branching graph through Motzkin paths together with Theorem 2.4 provides an exact formula for the multiplicity of $V_{(n,d)}$ in the induced representation $\mathrm{Ind}_{A_{(\mathcal{B},\delta)}(m)}^{A_{(\mathcal{B},\delta)}(n)} V_{(m,c)}$ for arbitrary choices of (m,c), (n,d), as this multiplicity is equal to $|\,\mathrm{Mo}((m,c),(n,d))|$.

(2) Alternatively, one can deduce the formula for $\dim V_{(n,d)}$ from known results on easy quantum groups: the bistochastic group B_N^+ with fundamental representation u_B is isomorphic as a matrix quantum group to the free orthogonal group O_{N-1}^+ with representation $u_O \oplus 1$, where u_O (u_B) denotes the fundamental representation of O_{N-1}^+ (B_N^+). Since the dimension of $V_{n,d}$ is equal to the multiplicity of the irreducible representation u_d in $(u_O \oplus 1)^{\otimes k} = \sum_{k=0}^{n} \binom{n}{k} u_O^{\otimes k}$, $\dim V_{(n,d)}$ can be calculated by using the well-known formulas for the multiplicies in O_{N-1}^+. These exactly correspond to the dimensions of the simple modules of the Temperley-Lieb algebras.

5.2.2 Classification of random Motzkin paths

We will now classify ergodic central random Motzkin paths (and thereby extremal traces on $A_{(\mathcal{B},\delta)}(\infty)$) in analogy to Theorem 5.1. Given an ergodic central probability measure ν let us denote the probability of arriving at vertex (n,d) by

$$v_{n,d} \;:=\; \nu(\{\omega \in \Omega_{\mathcal{B}+} \;;\; \omega_n = (n,d)\}).$$

Then, centrality implies that ν is uniquely determined by the values $v_{n,0},\ n \geq 0$ and the recursive rules

$$\frac{m_{n-1,0}}{m_{n,1}}\; v_{n,1} = v_{n-1,0}\left(1 - \frac{m_{n-1,0}}{m_{n,0}} v_{n,0}\right)$$

and

$$\frac{m_{n-1,d}}{m_{n,d+1}}\; v_{n,d+1} = v_{n-1,d}\left(1 - m_{n-1,d}\left(\frac{v_{n,d-1}}{m_{n,d-1}} + \frac{v_{n,d}}{m_{n,d}}\right)\right)$$

for $1 < d < n$. To see that these recursions are in fact correct, one simply notes that when dividing by $v_{n-1,d}$ one gets an expression for the probability under ν of the transition $(n-1,d) \to (n,d+1)$.

Now, let $\lambda_1, \lambda_2 \in [0,1]$. By $M^{(\lambda_1,\lambda_2)} = (M_n^{(\lambda_1,\lambda_2)})_{n\geq 0}$, we will denote the unique random Motzkin path that is Markov and whose transition probabilities are determined by

$$v_{n,0}^{(\lambda_1,\lambda_2)} = \mathbb{P}(M_n^{(\lambda_1,\lambda_2)} = (n,0)) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2l)!(l+1)!l!} \lambda_1^l \lambda_2^l (1-\lambda_1-\lambda_2)^{n-2l}$$

and the recursion above. Denote the law of $M^{(\lambda_1,\lambda_2)}$ by $\nu_{(\lambda_1,\lambda_2)}$.

Let

$$U := \{(\lambda_1,\lambda_2) \in [0,1]\times[0,1] \;;\; \lambda_1 \geq \lambda_2,\ 0 \leq \lambda_1+\lambda_2 \leq 1\}$$

and note that for two distinct elements of U, the associated random Motzkin paths are distinct.

Theorem 5.10. *A (rooted) random Motzkin path is central and ergodic if and only if it is equal to $M^{(\lambda_1,\lambda_2)}$ for some $(\lambda_1,\lambda_2) \in U$.*

In particular, every central random Motzkin path M with law ν is a mixing of the Markov chains $M^{(\lambda_1,\lambda_2)}$, $(\lambda_1,\lambda_2) \in U$, that is to say there is a probability measure μ on U such that

$$\nu = \int_U \nu_{(\lambda_1,\lambda_2)}\; d\mu(\lambda_1,\lambda_2).$$

Proof. We will adapt the methods of [Was81] to our situation to classify the traces on $A_{(B,\delta)}(\infty)$. Let us first extend the standard representation $\pi : SU(2) \to M_2(\mathbb{C})$ of $SU(2)$ to a representation

$$\tilde{\pi} : SU(2) \to M_3(\mathbb{C}), \qquad \tilde{\pi}(g) = \begin{pmatrix} \pi(g) & 0 \\ 0 & 1. \end{pmatrix}$$

Note that the compact matrix quantum group (B_N^+, u_B) (with standard fundamental representation u_B) is isomorphic to $(O_{N-1}^+, u_O \oplus 1_O)$ (with standard representation u_O and trivial representation 1_O) and that the parameter $\delta = N$ is generic for all $N \geq 3$. Hence we can just work with the choice $N = 3$. Since $O_2^+ \cong (SU(2), \pi)$, it follows that $A_{(\mathcal{B},3)}(\infty)$ is the fixed point algebra of the infinite tensor product action

$$\bigotimes_{n=0}^{\infty} \mathrm{Ad}(\tilde{\pi})(g) : \bigotimes_{n=0}^{\infty} M_3(\mathbb{C}) \to \bigotimes_{n=0}^{\infty} M_3(\mathbb{C}), \qquad g \in SU(2),$$

where Ad denotes the adjoint action. By [Was81, Theorem 4], a trace on $A_{(B,3)}(\infty)$ is extremal if and only if it is the restriction of a product state

$$\varphi_{(\lambda_1,\lambda_2)} := \bigotimes_{n=0}^{\infty} \mathrm{Tr}_3(\,\cdot\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)),$$

with $\lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 \leq 1,\ \lambda_3 = 1 - \lambda_1 - \lambda_2$. We will denote the extremal trace on $A_{(B,3)}(\infty)$ arising as the restriction of $\varphi_{(\lambda_1,\lambda_2)}$ by $\tau_{(\lambda_1,\lambda_2)}$.

Since the central projection $p_{(n,0)}$ in the irreducible module $V_{(n,0)} \subset A_{(\mathcal{B},3)}(n)$ is the projection onto the space of invariant vectors of the representation $\tilde{\pi}^{\otimes n}$ of $SU(2)$, $p_{(n,0)}$ is given by the formula

$$p_{(n,0)} = \int_{SU(2)} \tilde{\pi}^{\otimes n}(g)^{\otimes n}\, dg,$$

where integration is understood w.r.t. the Haar state on $SU(2)$. Using the standard parametrisation $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, $|\alpha|^2 + |\beta|^2 = 1$, for elements of $SU(2)$, we arrive at

$$\begin{aligned}
\tau_{(\lambda_1,\lambda_2)}(p_{(n,0)}) &= \int_{SU(2)} \langle \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \tilde{\pi}(g) \rangle^n\, dg \\
&= \int_{SU(2)} \mathrm{Tr}\left(\left(\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 - \lambda_1 - \lambda_2 \end{pmatrix}, \begin{pmatrix} \alpha & \beta & 0 \\ -\overline{\beta} & \overline{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right)^n dg(\alpha, \beta) \\
&= \int_{SU(2)} (\lambda_1 \alpha + \lambda_2 \overline{\alpha} + 1 - \lambda_1 - \lambda_2)^n\, dg(\alpha, \beta).
\end{aligned}$$

To this expression, we now apply the formula

$$\int_{SU(2)} f(\alpha, \beta)\, dg(\alpha, \beta) = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^{\pi} f(\alpha, \beta) \sin(\theta)\, d\theta\, d\phi\, d\psi$$

for the Haar state on $SU(2)$, with the reparametrisation $\alpha = \cos(\theta/2)\exp(i(\phi + \psi)/2),\ \beta = i\sin(\theta/2)\exp(i(\phi - \psi)/2)$, see e.g. [Vi68, p.159]. After a routine computation, this yields

$$\tau_{(\lambda_1,\lambda_2)}(p_{(n,0)}) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2l)!(l+1)!l!} \lambda_1^l \lambda_2^l (1 - \lambda_1 - \lambda_2)^{n-2l}.$$

Since the process $(M_n^{(\lambda_1,\lambda_2)})_n$ with law $\nu_{(\lambda_1,\lambda_2)}$ satisfies

$$\mathbb{P}(M_n^{(\lambda_1,\lambda_2)} = (n, 0)) = \tau_{(\lambda_1,\lambda_2)}(p_{(n,0)}),$$

the theorem now follows by decomposing central measures into ergodic components. $\square$

5.3 Determining the branching graph of the algebra $A_{(\mathcal{B}^{\#},\delta)}(\infty)$

We will now turn to the branching graph of the algebras $A_{(\mathcal{B}^{\#},\delta)}(n)$ that come with the category of partitions $\mathcal{B}^{\#} := \mathcal{C}_{\mathcal{B}^{\#+}}$. Recall that $A_{(\mathcal{B}^{\#},\delta)}(n)$ is spanned by partitions with n upper and n lower boundary points whose blocks are of size one or two with a counterclockwise labelling of the boundary points alternating between two symbols a and b. The labelling starts with a in the upper right corner and two blocks always have to connect two points which are labelled differently. We call set partition of this form *ab-admissible*.

Our goal is to describe the branching graph of the sequence of finite-dimensional algebras $A_{(\mathcal{B}^{\#},\delta)}(n)$, $n \geq 0$ and we will do so in Corollary 5.17. Unsurprisingly, the computations resemble those for $A_{(\mathcal{B}^{+},\delta)}(n)$, $n \geq 0$ in Subsection 5.2.1 with the additional caveat of having to deal with the labelling. We will again describe the irreducible modules of $A_{(\mathcal{B}^{\#},\delta)}(n)$ through an appropriate notion of link states.

Definition 5.11. A *n-link state* of $\mathcal{B}^{\#}$ is an ab-admissible set partition of n points into pairs and singletons of two types called *proper singletons* and *defects*. A defect is not allowed to be braced by a pair, i.e. if $i < j < k$ and i and k are paired, then j must be a proper singleton. Moreover, a defect is called a-defect, respectively b-defect, if it is labelled a, resp. b.

A (n, k, l)-link state of $\mathcal{B}^{\#}$ is a n-link state with k a-defects. and l b-defects.

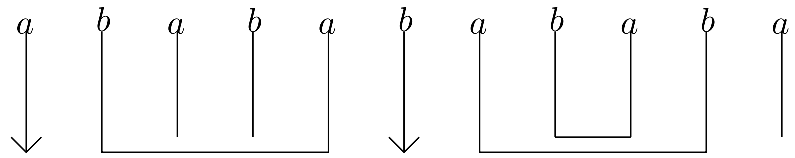

Figure 5: A $(11, 1, 1)$-link state of $\mathcal{B}^{\#}$.

As in the previous cases, a partition p in $A_{(\mathcal{B}^{\#},\delta)}(n)$ acts on an n-link state v by drawing p above v and connecting lines (without any additional restrictions by the labelling). Note that a point of v labelled a always meets a lower boundary point of p labelled b and the other way round. The fact that this operation returns an n-link state is easily verified in the exact same way as in the proof that composition of partitions in $\mathcal{B}^{\#}$ is well-defined, see [We13, Proof of Prop. 2.7].

Definition 5.12. We denote the vector space freely spanned by all n-link states by M_n and the subspace of M_n spanned by all n-link states with at most d defects by $M_{(n,d)}$ where $d = 0, \ldots n$. Note that $M_{(n,n)} = M_n$.

Let $\alpha(n)$ be the word of length n with alternating letters a, b that ends in a. If $n = 0$, we set $\alpha(0) = \emptyset$. We denote by $W_{\alpha(n)}$ the set of subwords of $\alpha(n)$, that is the set of words that are obtained from $\alpha(n)$ by deleting letters. The inverted word $\bar{w}$ of w is the word arising from w by applying the map $a \mapsto b$, $b \mapsto a$ to every letter of w. In particular, if $w \in W_{\alpha(n)}$, then $\bar{w} \in W_{\overline{\alpha(n)}}$.

Definition 5.13. Let v be an n-link state. The *boundary word* $w(v)$ is the subword of $\alpha(n)$ that labels the defects of v from left to right.

Lemma 5.14. *Let v_1 and v_2 be n-link states with the same number of defects d. There exists a partition p in $\mathcal{B}^{\#}(n, n)$ such that $p \cdot v_1 = v_2$ if and only if $w(v_1) = w(v_2)$.*

Proof. Suppose first that a partition as in the lemma exists. If m is a defect of v_2 then there must be a through-string of p connecting of p whose lower boundary point is a defect for v_1. Since the number of defects is the same for v_1 and v_2, all defects of v_1 are met. Moreover, since p is non-crossing, the k-th defect of v_2 has to connect to the k-th defect of v_1. Since the word labelling lower boundary points of p that meet defects of v_1 is $\overline{w(v_1)}$ and since through strings always connect opposite labels, it follows that $w(v_2) = \overline{\overline{w(v_1)}} = w(v_1)$.

Conversely, if v_1 and v_2 are such that $w(v_1) = w(v_2)$ we can define the partition p by placing v_2 above v_1, flipping v_1 horizontally and inverting the boundary labelling of v_1. We then connect the k-th defect of v_1 to the k-th defect of v_2 to not produce any crossing. Since $w(v_1) = w(v_2)$ and since through-strings connect boundary points of opposite labels (we inverted $w(v_1)$), p is well-defined. $\square$

In analogy to Subsection 5.2.1 we make M_n into a left $A_{(\mathcal{B}\#,\delta)}(n)$-module by extending the action of the base partitions of $A_{(\mathcal{B}\#,\delta)}(n)$ on n-link states linearly to all of $A_{(\mathcal{B}\#,\delta)}(n)$ and M_n. Since the action of base partitions of $A_{(\mathcal{B}\#,\delta)}(n)$ can never add a defect to a link state, the vector spaces $M_(n,d),\ d = 0, \dots n$ also inherit a left $A_{(\mathcal{B}\#,\delta)}(n)$-module structure.

We write $W^{(k,l)}_{\alpha(n)} \subset W_{\alpha(n)}$ for the set of subwords of $\alpha(n)$ consisting of k letters a and l letters b where $k = 0, \dots \lceil \frac{n}{2} \rceil,\ l = 0, \dots \lfloor \frac{n}{2} \rfloor$.

Theorem 5.15. *(1) The module $V_{(n,0,0)} := M_{(n,0)}$ is irreducible.*

(2) The quotient module $V_{(n,d)} := {M_{(n,d)}}/{M_{(n,d-1)}},\ d = 1, \dots n$ decomposes as a direct sum

$$V_{(n,d)} = \bigoplus_{k,l:\ k+l=d} V_{(n,k,l)}$$

of submodules $V_{(n,k,l)},\ k = 0, \dots \lceil \frac{n}{2} \rceil,\ l = 0, \dots \lfloor \frac{n}{2} \rfloor,\ k + l = d$. The submodule $V_{(n,k,l)}$ is spanned by the equivalence classes of (n, k, l)-link states in $V_{(n,d)}$.

(3) The submodule $V_{(n,k,l)}$ decomposes further into inequivalent irreducible submodules

$$V_{(n,k,l)} = \bigoplus_{w \in W^{(k,l)}_{\alpha(n)}} V_w,$$

where V_w is spanned by the equivalence classes of (n, k, l)-link states v with $w(v) = w$.

(4) The modules $V_w,\ w \in W_{\alpha(n)}$ form a full family of inequivalent irreducible modules for $A_{(\mathcal{B}\#,\delta)}(n)$.

Proof. We note first that, for $d > 0$,the equivalence classes $v + M_{(n,d-1)},\ v \in \mathcal{L}_{n,d}$, where $\mathcal{L}_{n,d}$ is the set of n-link states with d defects, form a basis of $V_{(n,d)}$. For $d = 0$, the same statement is true with the equivalence classes may be dropped since we are not in a quotient. We show first that $V_{(n,k,l)}$ is in fact invariant under the action of $A_{(\mathcal{B}\#,\delta)}(n)$ on $V_{(n,d)}$ and thus a submodule for $k + l = d$. Let $p \in \mathcal{B}^{\#}(n, n)$ a partition and v a (n, k, l)-link state. Since p has $2n$ boundary points, the rightmost lower point is labelled b. Thus, when p acts on v, an a-labelled boundary point of v meets a b-labelled lower point of p. There are now two possibilities: either a defect of v is erased by the action of p in which case $p \cdot v$ is annihilated by the quotient or all defects are kept, which means that they must connect to an upper point of p. By the observation we just made on the matching of boundary points, the upper point to which a defect connects, carries the same label as the defect. Therefore, $p \cdot v$ is also a (n, k, l)-link state and $V_{(n,k,l)}$ is invariant under the action of $A_{(\mathcal{B}\#,\delta)}(n)$. By a dimension count, it follows that $V_{(n,d)}$ decomposes in the stated manner which proves (2). The refined decomposition of $V_{(n,k,l)}$ stated in (3), then follows from Lemma 5.14, as the orbits of (n,k,l)-link states under the action of $A_{(\mathcal{B}\#,\delta)}(n)$ are fully classified by the words in $W^{(k,l)}_{\alpha(n)}$. This also shows that the module V_w is irreducible for arbitrary $w \in W_{\alpha(n)}$. In particular, this proves (1) since $V_{n,0,0} = V_\emptyset$.

To finish the proof of (3) and (4), we still have to show that the modules V_w, $w \in W_{\alpha(n)}$ are pairwise inequivalent. To see this, we invoke [FrWe16, Theorem 5.5] which tells us that every minimal projection in $A_{(\mathcal{B}\#,\delta)}(n)$ is a projective partition. The minimal projections that act nontrivially on the module V_w are the projections onto $\langle v\rangle$ for n-link states v with $w(v) = w$. By Lemma 5.14 two minimal projective partitions mapping onto $\langle v_1\rangle, \langle v_2\rangle$ respectively, cannot be equivalent if $w(v_1) \neq w(v_2)$.

□

As always, we define the inclusion $A_{(\mathcal{B}\#,\delta)}(n-1) \hookrightarrow A_{(\mathcal{B}\#,\delta)}(n)$ by a adding a through string on the right of every base partition (and extending this mapping linearly). Note that under this embedding the labelling of a boundary point is inverted: a boundary point that was labbeled a is now labelled b and the other way round.

For a word $w \in W_{\alpha(n)}$, we write $|w|$ for its length, $ll(w)$ for its last letter and $w\backslash ll(w)$ for the word obtained from w by deleting its last letter.

Proposition 5.16. *As an $A_{(\mathcal{B}\#,\delta)}(n-1)$-module, V_w, $w \in W_{\alpha(n)}$ decomposes as the direct sum of irreducible modules*

$$V_w^{\downarrow} = \delta_{ll(w)=a} V_{\overline{w\backslash ll(w)}} \oplus \delta_{|w|<n-1} V_{\overline{w}b} \oplus \delta_{|w|<n} V_{\bar{w}},$$

where $\delta_{ll(w)=a}$ (resp. $\delta_{|w|<m}$) is the characteristic functions of the set $\{w \in W_{\alpha(n)}\ ;\ ll(w) = a\}$ (resp. $\{w \in W_{\alpha(n)}\ ;\ |w| < m\}$).

Proof. The isomorphism

$$\delta_{ll(w)=a} V_{\overline{w\backslash ll(w)}} \oplus \delta_{|w|<n-1} V_{\overline{w}b} \oplus \delta_{|w|<n} V_{\bar{w}} \to V_w^{\downarrow}$$

is defined in the same way as in Proposition 5.6. We leave the details of the proof to the reader. □

We are now ready to describe the branching graph $\Gamma_{\mathcal{B}\#}$ of $A_{(\mathcal{B}\#,\delta)}(\infty)$. Since the inversion of words that appears in Proposition 5.16 is somewhat inconvenient, we will do the following. Define the word

$$\beta(n) = \begin{cases} \alpha(n) & n \text{ odd} \\ \overline{\alpha(n)} & n \text{ even} \end{cases}.$$

In other words, $\beta(n)$ is simply the alternating word $abab\ldots$ with n letters starting with a. We also identify $W_{\alpha(n)}$ with $W_{\beta(n)}$ through the bijection $w \mapsto \overline{w}$ for even n. Under this identification, $u \in W_{\beta(2n-1)}$ is connected to $w \in W_{\beta(2n)}$ for $n \geq 1$ if and only if

$$w = u, \qquad \text{or} \qquad w = ub \qquad \text{or} \qquad u = wa.$$

$u \in W_{\beta(2n)}$ is connected to $w \in W_{\beta(2n+1)}$ for $n \geq 0$ if and only if

$$w = u, \qquad \text{or} \qquad w = ua \qquad \text{or} \qquad u = wb.$$

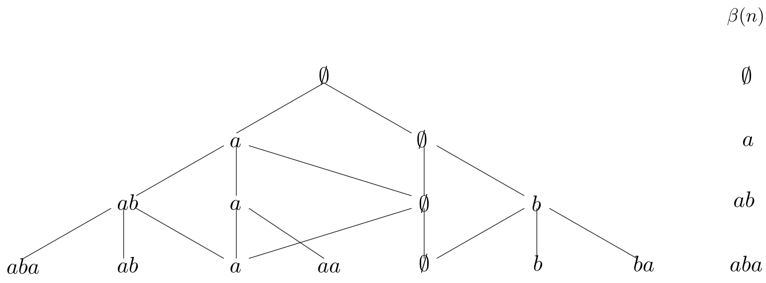

Figure 6: The first four levels of $\Gamma_{\mathcal{B}\#}$ and the words $\beta(n)$.

The first few levels of $\Gamma_{\mathcal{B}\#}$ are depicted below.

Again, the graph $\Gamma_{\mathcal{B}\#}$ can be thought of the pascalization of a smaller graph which is the *derooted Fibonacci tree* $\mathbb{FT}^*$. Let us therefore remind the reader that the (non-derooted) Fibonacci tree $\mathbb{FT}$ is given as follows. On the 0-th level, we start with the root $\emptyset$ ('a newborn pair of rabbits'). The rabbits now age one month creating a vertex on level one on level 1 and are now able to procreate so that they have two descendants on level two corresponding to themselves and a new pair. This process continues inductively with each new pair only being able to procreate after having lived for one month. More formally, let us describe the vertices of $\mathbb{FT}$ through words in letters a and b. The n-th level set $\mathbb{FT}_n,\ n \geq 1$ contains words of length n starting with a such that two copies of the letter b never appear next to each other. A word $w_1 \in \mathbb{FT}_n$ is connected to a word in $w_2 \in \mathbb{FT}_{n+1}$ if $w_2 = w_1 b$ or $w_2 = w_1 a$. The derooted Fibonacci tree $\mathbb{FT}^*$ is just $\mathbb{FT}$ with the root deleted, i.e. $\mathbb{FT}^*_n = \mathbb{FT}_{n+1},\ n \geq 0$. Note that any word in $\mathbb{FT}^*$ begins with the letter a and in particular, the root of $\mathbb{FT}^*$ is the single letter word a. The first few levels of $\mathbb{FT}$ and $\mathbb{FT}^*$ are depicted in Figure 7.

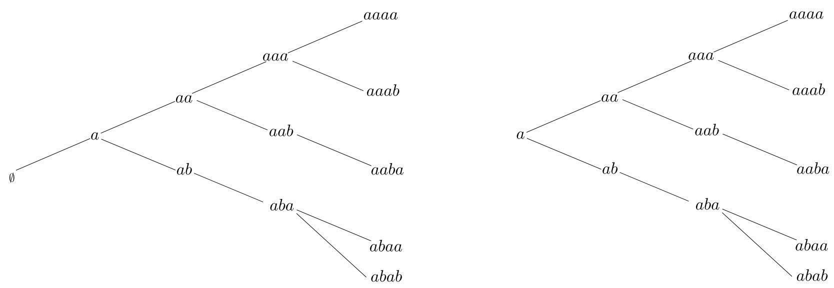

Figure 7: The first levels of the Fibonacci and the derooted Fibonacci tree.

Corollary 5.17. *The branching graph $\Gamma_{\mathcal{B}\#}$ is the pascalization of the derooted Fibonacci tree $\mathbb{FT}^*$.*

Proof. Since every word w in $\mathbb{FT}^*$ begins with the letter a, we can identify the word w with the word $\tilde{w}$ obtained from w by deleting the first letter a. We denote the graph with the new labelling by $\widetilde{\mathbb{FT}}^*$. Let us define a map $\phi_n : \widetilde{\mathbb{FT}}^*_n \to W_{\beta(n)},\ \tilde{w} \mapsto \phi_n(\tilde{w})$ in the following way. First replace every letter a of $\tilde{w}$ that is in an even position by the auxilliary letter c and delete all letters b. Then replace all instances of c again by b. The map ϕ_n is injective for all n and that its image is the set

$$\phi_n(\widetilde{\mathbb{FT}}^*_n) = W_{\beta(n)} \Bigg\backslash \left(\bigcup_{\substack{k<n,\\ k\equiv n \mod 2}} W_{\beta(k)} \right).$$

Therefore, we can define a new branching graph $\phi(\tilde{\mathbb{FT}}^*)$ with $\phi(\widetilde{\mathbb{FT}}^*)_n = \phi_n(\widetilde{\mathbb{FT}}^*_n)$ and the connection rule $\phi_n(u) \to \phi_{n+1}(w)$ if and only if $\tilde{u} \to \tilde{w}$ in $\widetilde{\mathbb{FT}}^*$ for $\tilde{u} \in \widetilde{\mathbb{FT}}^*_n, \tilde{w} \in \widetilde{\mathbb{FT}}^*_{n+1}$. If $\tilde{w} = \tilde{u}b$, then $\phi_{n+1}(\tilde{w}) = \phi_n(\tilde{u})$. If n is even and $\tilde{w} = \tilde{u}a$, then $\phi_{n+1}(\tilde{w}) = \phi_n(\tilde{u})a$. Lastly, if n is odd and $\tilde{w} = \tilde{u}a$, then $\phi_{n+1}(\tilde{w}) = \phi_n(\tilde{u})b$. Now, combining the definition of pascalization with the connection rules found for $\Gamma_{\mathcal{B}\#}$, it follows that $\Gamma_{\mathcal{B}\#} = \mathcal{P}(\phi(\tilde{\mathbb{FT}}^*)) \cong \mathcal{P}(\mathbb{FT}^*)$. □

5.4 The branching graph of the Fuss-Catalan algebras

The hyperoctahedral quantum group H_n^+ is isomorphic to the free wreath product $\mathbb{Z}_2 \wr_* S_n^+$ and the endomorphism spaces $\mathrm{End}(w^{\otimes k})$ of its fundamental representation w are known to be isomorphic to the *s-Fuss-Catalan algebras* of Bisch and Jones for $s = 2$ and a suitable choice of loop parameters. The s-Fuss-Catalan algebras $\mathrm{FC}_s(k, \delta_1, \ldots, \delta_s), s \geq 2$ were introduced and in [BiJo95] and further analyzed in [La01] in the context of intermediate subfactors. For higher values of s, these algebras still admit an interpretation as commutants of the fundamental representation of a compact quantum group, namely the free wreath product quantum group $\mathbb{Z}_s \wr_* S_n^+$.
As in the Temperley-Lieb case, there is a nice description of these algebras in terms of generators and relations, see [La01], but we will stick to their description as diagram algebras, as this makes it easier to identify $\mathrm{FC}_2(k, \delta)$ with $A_{(\mathcal{C}_{H^+},\delta)}(k)$. Many fundamental aspects of the representation theory of the Fuss-Catalan algebras including their branching graph (for fixed s and increasing k) have been computed in [BiJo95] and [La01]. We will thus start this section with a summary of the results needed for a discussion of the minimal boundary of the branching graphs.

Definition 5.18. Let $s \geq 2,\ k \geq 1$ and let p be a non-crossing pair partition diagram with $s \cdot k$ upper and equally many lower points. Fix a set of labels $a_1, \ldots a_s$ and label the boundary points of p clockwise starting at the upper left corner by

$$a_1, a_2, \ldots, a_s, a_s, \ldots a_2, a_1, a_1, a_2 \ldots .$$

We call p a *(s, k)-Fuss-Catalan diagram* if for any two boundary points of p that are in the same block, their labels coincide. We will denote the set of all (s, k)-Fuss-Catalan-diagrams by $\mathcal{FC}(s, k)$.

As diagram algebras, the Fuss-Catalan algebras can be introduced as follows.

Definition 5.19. Let $s \geq 2,\ k \geq 1$ and fix $\delta_1, \ldots, \delta_s \in \mathbb{C}\backslash\{0\}$. The *s-Fuss-Catalan-algebra* $\mathrm{FC}_s(k, \delta_1, \ldots, \delta_s)$ is the $\mathbb{C}$-vector space with basis $e_p,\ p \in \mathcal{FC}(s, k)$ with multiplication $e_p \cdot e_q$ given by

$$e_p \cdot e_q = \delta_1^{l_1} \ldots \delta_s^{l_s} e_{p \cdot q},$$

where $p \cdot q$ is the usual multiplication of partitions by stacking p on top of q connecting lines and erasing loops and where $l_i,\ i = 1, \ldots, s$ is the number of loops of type a_i.

Once again, for fixed s the set of generic parameter vectors $(\delta_1, \ldots, \delta_s)$ is dense in $\mathbb{C}^s$ and contains the set $[2, \infty)^s$, see [BiJo95, Corollary 2.2.5]. From the perspective of easy quantum groups, a good choice of parameters is $s = 2$ and $\delta_1 = \delta_2 := \delta = \sqrt{n}, n \geq 4$ as in this case, the Fuss-Catalan algebra $\mathrm{FC}_2(k, \delta) := \mathrm{FC}_2(k, \sqrt{n}, \sqrt{n})$ is isomorphic to the k-diagram algebra of the category $\mathcal{C}_{H^+}$, see e.g. [BBC07].

Again at a generic parameter and for arbitrary $s \geq 2$, we get an embedding of semisimple algebras $\mathrm{FC}_s(k, \delta_1, \ldots, \delta_s) \subset \mathrm{FC}_s(k + 1, \delta_1, \ldots, \delta_s)$ by adding s through strings of type

$a_1, \dots, a_s$ on the right of every diagram. As before, we denote the inductive limit algebra by $\mathrm{FC}_s(\infty, \delta_1, \dots, \delta_s)$. As the results that will follow will not depend on the parameters (as long as they are generic), we will write $\mathrm{FC}_s(k),\ k = 0, 1, \dots, \infty$ for short. The branching graph $\Gamma^s := \Gamma^{FC_s}$ of the inductive sequence $\mathrm{FC}_s(1, \delta_1, \dots, \delta_s) \subset \mathrm{FC}_s(2, \delta_1, \dots, \delta_s) \subset \dots$ at a generic parameter was computed in [BiJo95] for $s = 2$ and [La01] for $s \geq 3$. It is the pascalization of the following tree.

Definition 5.20. The *s-Fuss-Catalan tree* $\mathbb{T}^s$ is the tree constructed according to the following inductive procedure. Let $\emptyset$ be the root of the tree and label it with a 1. If a descendent v at distance $k, k \geq 0$ from $\emptyset$ with label i is given, create i children of v and label them $s, s-1, \dots, s-i+1$.

Note that the level of a vertex v of $\mathbb{T}^s$ is simply its number of descendants. For $s = 2$, the tree $\mathbb{T}^s$ is exactly the Fibonacci tree $\mathbb{FT}$ discussed in the previous section. If we define the map $\psi^2 : \{\emptyset, a, b\} \to \{1, 2\},\ \emptyset,\ b \mapsto 1,\ a \mapsto 2$, the label of a word $w \in \mathbb{FT}_n$ is simply $\psi(l(w))$, where $l(w)$ is its last letter. More generally, note that for any s, we can uniquely identify any vertex v of $\mathbb{T}^s$ with a word in s letters $a_1, \dots, a_s$ by identifying a_k with the label $s-k+1$ through a map $\psi^s : a_k \mapsto s-k-1,\ (\emptyset \mapsto 1)$ and recording the labels on the unique path from the first level $\mathbb{T}^s_1$ to v. We will call any word appearing in this manner *s-admissible*, so that we can identify $\mathbb{T}^s_n$ with the set of s-admissible words of length n. Since the branching graph $\Gamma^s = \mathcal{P}(\mathbb{T}^s)$ is the pascalization of $\mathbb{T}^s$, any vertex $v \in \Gamma^s_n$ is described by a pair (n, w) with an s-admissible word $w \in \mathbb{T}^s_k$, of length $k \leq n,\ k \equiv n \mod 2$.

5.5 Walk counting on (derooted) Fuss-Catalan trees and dimension formulas for $\mathrm{FC}_2(\infty, \delta)$ and $A_{(\mathcal{B}\#, \delta)}(\infty)$

The dimension $\dim_{\Gamma^s}(n, w)$ of the irreducible representation of $\mathrm{FC}_s(k)$ indexed by a vertex (n, w) was computed in [La01, Section 9] using the underlying combinatorics of the tree $\mathbb{T}^s$ and Raney's recurrence relation for Fuss-Catalan numbers laid out in [GKP94, p. 360ff]. In this section, we will recall these results and extend them to the derooted Fuss-Catalan trees.

Before we state Landau's formula, we will introduce the following notations which we will use throughout this section.

- The derooted s-Fuss-Catalan tree $\widetilde{\mathbb{T}}^s$ is the s-Fuss-Catalan tree whose root (and root-adjacent edge) have been deleted. The root of the derooted s-Fuss-Catalan tree thus corresponds to the unique first level vertex of $\mathbb{T}^s$.
- As before we will to refer to the length of an s-admissible word w by $|w|$. For two vertices v, w on a tree, we will write $d(v, w)$ for their distance, so that on the Fuss-Catalan tree $d(w, \emptyset) = |w|$.
- For two s-admissible words v, w we denote by $s(v, w)$ the longest word such that there exist words v', w' with $v = s(v, w)v'$ and $w = s(v, w)w'$. Note that $s(v, w)$ is the vertex on the s-Fuss-Catalan tree where the paths from v and w to the root meet for the first time.
- We define the label sum of a word $w = a_{i_1} a_{i_2} \dots a_{i_n}$ as $r(w) := 1 + \sum_{k=1}^{n} \psi^s(a_{i_k})$. Equivalently, $r(w)$ is simply the sum $\sum_{k=0}^{n} l(v_k)$ of labels $l(v_k)$ of the vertices on the unique path $v_0 = \emptyset, v_1, \dots, v_n = w$ from $\emptyset$ to w. Recall that $l(v)$ is nothing but the number of sons of v.

- Note that for a vertex v on the tree $\mathbb{T}^s$, the number of loops of length $2n$ starting and ending at v that never get closer to the root than v only depends on $l(v)$. We will denote this number by $C_n^{s,i}$ for $1 \leq l(v) = i \leq s$.
- The number of walks of length m on the tree $\mathbb{T}$ from v to w will be denoted by $\mathrm{Wk}_{\mathbb{T}}(m, v, w)$.
- For an s-admissible word w and $n \geq |w|/2$, define the quantity

$$\begin{bmatrix} n \\ w \end{bmatrix} = \frac{r(w)}{(s+1)(n - \lceil \frac{|w|}{2} \rceil) + r(w)} \binom{(s+1)(n - \lceil \frac{|w|}{2} \rceil) + r(w)}{n - \lceil \frac{|w|}{2} \rceil}.$$

Recall that by Lemma 3.7, $\dim_{\Gamma^s}(m, w) = \mathrm{Wk}_{\mathbb{T}^s}(m, \emptyset, w)$ for every s-admissible word w and $m \geq 1$.

Theorem 5.21 ([La01])**.** *We have*

$$\begin{bmatrix} n \\ w \end{bmatrix} = \begin{cases} \dim_{\Gamma^s}(2n, w) & \text{if } |w| \text{ is even,} \\ \dim_{\Gamma^s}(2n-1, w) & \text{if } |w| \text{ is odd.} \end{cases}$$

If $|w| \neq m \mod 2$, *then* $\dim_{\Gamma^s}(m, w) = 0$.

Remark 5.22. (1) The number of rooted loops of length $2n$ on the s-Fuss-Catalan tree $\mathbb{T}^s$ is the Fuss-Catalan number

$$\dim_{\Gamma^s}(2n, \emptyset) = C_n^s := \frac{1}{(s+1)n+1} \binom{(s+1)n+1}{n}.$$

(2) The ratio test shows that the generating function

$$G_s(z) = \sum_{n=0}^{\infty} C_n^s z^n$$

has radius of convergence $\frac{s^s}{(s+1)^{s+1}}$. Using Stirling's formula, it follows that also for the critical value $\frac{s^s}{(s+1)^{s+1}}$, the series $\sum_{n=0}^{\infty} C_n^s \left(\frac{s^s}{(s+1)^{s+1}}\right)^n < \infty$ converges. This fact will become important in the proof of Lemma 6.7.

(3) By [GKP94, p. 360ff], the function G_s satisfies the recursion $G_s(z) = zG(z)^{s+1} + 1$. Moreover, it is shown there that the n-th coefficient in the series expansion of G_s^l is given by

$$[z^n]G_s(z)^l = \frac{1}{(s+1)n+l} \binom{(s+1)n+l}{n}.$$

The proof of Theorem 5.21 can be adapted to the derooted s-Fuss-Catalan tree, yielding in particular an explicit formula for the dimensions of the irreducible representations of the algebras $A_{(\mathcal{B}\#,\delta)}(n)$, $n \geq 0$ at the generic parameter. Note that the distance to the root in $\widetilde{\mathbb{T}}^s$ of a word w is $|w| - 1$. Let

$$\begin{bmatrix} n \\ w \end{bmatrix}_{\#} = \frac{r(w) - 1}{(s+1)(n - \lceil \frac{|w|-1}{2} \rceil) + r(w) - 1} \binom{(s+1)(n - \lceil \frac{|w|-1}{2} \rceil) + r(w) - 1}{n - \lceil \frac{|w|-1}{2} \rceil}.$$

for $n \geq \frac{|w|-1}{2}$.

Proposition 5.23. *In the derooted s-Fuss-Catalan tree $\widetilde{\mathbb{T}}^s$, we have*

$$\begin{bmatrix} n \\ w \end{bmatrix}_{\#} = \begin{cases} \mathrm{Wk}_{\widetilde{\mathbb{T}}^s}(2n, \emptyset_{\widetilde{\mathbb{T}}^s}, w) & \text{if } |w| \text{ is even,} \\ \mathrm{Wk}_{\widetilde{\mathbb{T}}^s}(2n-1, \emptyset_{\widetilde{\mathbb{T}}^s}, w) & \text{if } |w| \text{ is odd.} \end{cases}$$

In particular, the dimension of the irreducible representations $A_{(\mathcal{B}\#,\delta)}(2n)$ indexed by a word $(2n, w)$, $|w|$ odd, is

$$\frac{r(w)-1}{3(n-\lceil\frac{|w|-1}{2}\rceil)+r(w)-1}\binom{3(n-\lceil\frac{|w|-1}{2}\rceil)+r(w)-1}{n-\lceil\frac{|w|-1}{2}\rceil}.$$

The same formula holds for the dimension of $(2n-1, w)$ with $|w|$ even.

Proof. By the same argument as in the proof of [La01, Theorem 11], $\mathrm{Wk}_{\widetilde{\mathbb{T}}^s}(2n, \emptyset, w)$ is the coefficient of $z^{n-\lceil\frac{|w|-1}{2}\rceil}$ in the $r(w)-1$ power of the generation function G_s of Remark 5.22. The formula then follows from the third part of said remark. □

Remark 5.24. In the special case, when $w = a$ is the root of the derooted Fibonacci tree, another formula for the dimension of $(2n, a)$ is given in [We13, Proposition 4.3] and one easily checks that both expressions are the same.

6 The minimal boundary for random walks on the Fibonacci tree

The observations of Section 3.2 and Theorem 3.5 turn the problem of classifying extremal traces on $\mathrm{FC}_s(\infty)$ into a classification problem for random walks on Fuss-Catalan trees. For homogeneous trees, central ergodic random walks have been classified in [VM15]. However, the symmetric nature of the homogeneous trees renders the combinatorics of that problem more simple than for the Fuss-Catalan trees and we will need to make regular use of the walk counting formulas of Subsection 5.5. An interesting result that we prove along the way is the law of large numbers mentioned in the introduction (Theorem B). Note that laws of large numbers for central measures on branching graphs have also been proven in other cases such as for instance for the Young graph (see e.g. [Me17]), although these examples behave rather differently from a combinatorial point of view.

Let us start this section by introducing a few notations that we will use down the road. Let $\mathbb{T}$ be a locally finite tree in which every node has at least one successor. An infinite path $(\emptyset = t_1, t_2, \dots)$ on a tree $\mathbb{T}$ is typically called an *end* of $\mathbb{T}$. We will denote the set of ends of $\mathbb{T}$ by $\partial\mathbb{T}$. Note that is no violation of our previous notation $\partial\Gamma$ for the minimal boundary of a branching graph Γ. In fact, if $\Gamma = \mathbb{T}$, the set of ends of $\mathbb{T}$ is in bijection with the minimal boundary of $\mathbb{T}$ as we can associate to every end t the Dirac measure δ_t and all ergodic central measures on the space of ends are of this form. With this identification in mind, we interpret $\partial\mathbb{T}$ as a topological space (with the weak topology).

Definition 6.1.
- We say that a sequence $(x_n)_{n\geq 0}$, $x_n \in \mathbb{T}$ converges to an end $t \in \partial\mathbb{T}^s$ if for every vertex $v \in \mathbb{T}$, the length of the common part $[v, x_n] \cap [v, t\rangle$ of the geodesic path $[v, x_n]$ from v to x_n and the geodesic ray $[v, t\rangle$ from v to t tends to infinity as $n \to \infty$.
- Similarly, we say that a path $(n, x_n)_{n\geq 0}$ in the pascalized graph $\mathcal{P}(\mathbb{T})$ converges to t if $(x_n)_{n\geq 0}$ converges to t in the sense above.

Definition 6.2. Let $\mathbb{T}$ be a locally finite tree, let $t \in \partial\mathbb{T}$ be an end and let v, w be neighboring vertices of $\mathbb{T}$.

- We will call the (ordered) pair (v, w) *t-directed* if w lies on the unique geodesic from v to t.
- Similarly, we will call an edge $((k, v), (k+1, w))$ in the pascalized graph $\mathcal{P}(\mathbb{T})$ t-directed if (v, w) is t-directed in $\mathbb{T}$.

Note that for every pair of neighboring vertices v, w of $\mathbb{T}$, either (v, w) or (w, v) is t-directed.

Proposition 6.3. *Let S be a random walk on a locally finite tree $\mathbb{T}$ starting at the root. Then S is central if and only if there exists a constant $\eta \geq 0$ such that the transition probabilities of S satisfy*

$$p(v, w)p(w, v) = \eta \tag{6.1}$$

for every pair of neighboring vertices v, w.

Proof. Assume first that Equation 6.1 holds for v, w. Let x be another vertex of $\mathbb{T}$ and let $n \equiv |x| \mod 2$. We need to show that S takes all n-step walks from $\emptyset$ to x with the same probability. Let $\emptyset = t_0, t_1, \ldots, t_{|x|} = x$ be the unique path on $\mathbb{T}$ from $\emptyset$ to x. We can then decompose any n-step walks from $\emptyset$ to x as

$$(\text{a loop on } t_0)\ (t_0 \to t_1)\ (\text{a loop on } t_1)\ (t_1 \to t_2) \cdots (t_{|x|-1} \to x)\ (\text{a loop on } x).$$

Since $\mathbb{T}$ is a tree, every edge on one of the loops has to be traversed equally many times in both directions. Since by assumption $p(v, w)p(w, v) = \eta$ is constant for all edges (v, w), it follows that every n-step walk from $\emptyset$ to x is taken with probability

$$\eta^{n-|x|} \prod_{i=0}^{|x|-1} p(t_i, t_{i+1}).$$

Conversely, it suffices to note the following for a given vertex x with $|x| = n$: by centrality the loop of length $2n$ obtained by following the geodesic from $\emptyset$ to x and back has the same probability as the loop obtained by only following this geodesic up to the father of x, returning and taking 2 steps between the root and level 1. The claim then follows by induction over the distance to the root. □

Given an end $t \in \partial\mathbb{T}^s$ and a parameter $\eta \in [0, \frac{s^s}{(s+1)^{s+1}}]$, we will now define a rooted random walk $S_{(t,\eta)}$ on the s-Fuss-Catalan tree as follows. If a is the unique vertex connected to the root, we define the transition probabilities $p_{(t,\eta)}(\emptyset, a) = 1$, $p_{(t,\eta)}(a, \emptyset) = \eta$. Next consider an edge (v, w) that does not lie on the end t. For such an edge, we define the transition probabilities to be

$$p_{(t,\eta)}(v, w) = \begin{cases} G_s(\eta)^{-l(v)} & \text{if } (v, w) \text{ is } t\text{-directed}, \\ \eta \cdot G_s(\eta)^{l(w)} & \text{else}, \end{cases} \tag{6.2}$$

where $G_s(z) = \sum_{n=0}^{\infty} C_n^s z^n$ is the generating function of the s-Fuss-Catalan numbers discussed in Remark 5.22. Since $\eta \leq \frac{s^s}{(s+1)^{s+1}}$, the number $G_s(\eta)$ is well-defined. In addition, $G_s(\eta) \geq 1$, so that the first value for $p_{(t,\eta)}(v, w)$ is bounded by 1. The fact that also $\eta \cdot G_s(\eta)^l \leq 1$ for all $l = 1, \ldots, s$ follows directly from the recursion formula $G_s(z) = zG_s(z)^{s+1} + 1$, see Remark 5.22.

So far, we have defined all transition probabilities except for those between neighboring vertices $t_i, t_{i+1}, i \geq 1$ on our prescribed end $t = (t_i)_{i \geq 0}$.

Lemma 6.4. *Let $t = (t_i)_{i\geq 0} \in \partial\mathbb{T}^s$ and $\eta \in [0, \frac{s^s}{(s+1)^{s+1}}]$. Then there exists a unique central random walk $S_{(t,\eta)}$ on $\mathbb{T}^s$ such that*

- $p_{(t,\eta)}(\emptyset, a) = 1,\ p_{(t,\eta)}(a, \emptyset) = \eta$ *for the unique level* 1 *vertex* a*;*
- *For any edge (v, w) that does not lie on t, the transition probablities $p_{(t,\eta)}(v, w), p_{(t,\eta)}(w, v)$ are given by Equation 6.2;*
- *For any edge (v, w), we have $p_{(t,\eta)}(v, w)p_{(t,\eta)}(w, v) = \eta$.*

Proof. Let x be a vertex that does not lie on t. We have already shown above that $0 \leq p_{(t,\eta)}(x, w) \leq 1$ for every neighbour w of x. Let v be the unique neighbor of x such that (x, v) is t-directed. Since $x \neq t_i$ for all $i \geq 0$, we must have $|v| = |x| - 1$. Therefore

$$\sum_{w \text{ nb. of } x} p_{(t,\eta)}(x, w) = 1$$

is equivalent to

$$G_s(\eta)^{l(x)} = 1 + \sum_{w \text{ descend. of } x} \eta G_s(\eta)^{l(x)+l(w)}$$

and this equality follows from the computation

$$\begin{aligned}
1 + \sum_{w \text{ descend. of } x} \eta G_s(\eta)^{l(x)+l(w)} &= 1 + \sum_{i=1}^{l(x)} \eta G_s(\eta)^{l(x)+s-i+1} \\
&= 1 + \sum_{i=0}^{l(x)-1} \eta G_s(\eta)^{s+1+i} \\
&= 1 + \sum_{i=0}^{l(x)-1} G_s(\eta)^i (G_s(\eta) - 1) \\
&= G_s(\eta)^{l(x)},
\end{aligned}$$

where we use the relation $G_s(z) = zG_s(z)^{s+1}+1$. The fact that the missing transition probabilities $p_{(t,\eta)}(t_i, t_{i+1}), p_{(t,\eta)}(t_{i+1}, t_i)$ are uniquely defined now follows by a straightforward induction on i. The fact that $S_{(t,\eta)}$ is central follows from Proposition 6.3. □

Note that when $\eta = 0$, the random walk $S_{(t,0)}$ converges deterministicly to the end t.

Now for $t \in \partial\mathbb{T}^s$ and $\eta \in [0, \frac{s^s}{(s+1)^{s+1}}]$ define the Markov measure $\nu_{(t,\eta)}$ on the space of infinite paths on $\mathcal{P}(\mathbb{T}^s)$ by

$$p_{\nu_{(t,\eta)}}((n, v), (n + 1, w)) := p_{(t,\eta)}(v, w),$$

so that $\pi(\nu_{(t,\eta)})$ is the distribution of the random walk $S_{(t,\eta)}$. In particular the measure $\nu_{(t,\eta)}$ is time-homogeneous.

We are now ready to state a more precise version of Theorem A. The rest of this section will be devoted to its proof. Most of the partial results leading up to the proof, will be formulated for general Fuss-Catalan trees.

Theorem 6.5. *The set of all ergodic central measures on the space $(\Omega_{\mathcal{P}(\mathbb{FT})}, \mathcal{F}_{\mathcal{P}(\mathbb{FT})})$ of infinite paths on $\mathcal{P}(\mathbb{FT})$ conincides with the family of Markov measures*

$$\Sigma := \{\nu_{(t,\eta)}\ ;\ t \in \partial\mathbb{FT},\ \eta \in [0, 4/27]\}.$$

Combining Theorem 6.5 with Theorem 3.5, we obtain the following. Denote by $\tilde{S}_{(t,\eta)}$ the random walk starting at the root of $\widetilde{\mathbb{FT}}$ with the same transition probabilities as $S_{(t,\eta)}$ outside of the fixed end $t \in \partial\widetilde{\mathbb{FT}} \cong \partial\mathbb{FT}$ and denote by $\tilde{\nu}_{(t,\eta)}$ its pullback to $\mathcal{P}(\widetilde{\mathbb{FT}})$.

Corollary 6.6. *(a) A full list of extremal traces on the infinite Fuss-Catalan algebra* $\mathrm{FC}_2(\infty,\delta)$ *is given by*

$$\{\tau_{(t,\eta)} \ ; \ t \in \partial\mathbb{FT}, \ \eta \in [0, 4/27]\}$$

where

$$\tau_{(t,\eta)}(x) = \sum_{(n,v_i)\in\Gamma^2_n} \nu_{(t,\eta)}\left(X_n = (n, v_i)\right) \frac{\tau_{(n,v_i)}(x)}{\dim_{\Gamma^2}(n, v_i)} \qquad (x \in \mathrm{FC}_2(n, \delta)).$$

Here $\tau_{(n,v_i)}$ *denotes the (unnormalized) trace on the simple direct summand indexed by* (n, v_i) *in the decomposition of* $\mathrm{FC}_2(n, \delta)$*.*

(b) The same statement holds for the extremal traces on the algebra $A_{(\mathcal{B}\#,\delta)}(\infty)$ *if* $\nu_{(t,\eta)}$ *is replaced by* $\tilde{\nu}_{(t,\eta)}$*.*

Recall that a Markov chain on a tree [VM15] is *transient* if its number of returns to $\emptyset$ is finite with probability one.

Lemma 6.7. *Let* ν *be an ergodic central measure on the space* $(\Omega_{\Gamma^s}, \mathcal{F}_{\Gamma^s})$ *of infinite paths on the branching graph* $\Gamma^s = \mathcal{P}(\mathbb{T}^s)$*. Then the Markov chain* $\pi(\nu)$ *on* $\mathbb{T}^s$ *is transient.*

Proof. Consider a ergodic central measure ν on $(\Omega_{\Gamma^s}, \mathcal{F}_{\Gamma^s})$ and suppose that $\pi(\nu)$ is recurrent, i.e. the Markov chain returns to the root $\emptyset$ infinitely many times with positive probability. Translating this statement into a statement for the branching graph $\Gamma^s = \mathcal{P}(\mathbb{T}^s)$, we see that the set

$$W = \{(n_i, w_i)_{i\geq 1} \in \Omega_{\Gamma^s} \ ; \ w_i = \emptyset \text{ for infinitely many i}\}$$

has positive probability $\nu(W) > 0$. By the ergodic method (Theorem 3.6), we know that for almost every path $(n_i, w_i)_{i\geq 1} \in W$, for every neighbouring pair of vertices $v \in \mathbb{T}^s_k, x \in \mathbb{T}^s_l$, $|k - l| = 1$ and any $m \geq k, m = k \mod 2$, the transition probability from v to x on step $m + 1$ is

$$p_\nu((m, v),(m + 1, x)) = \lim_{i\to\infty} \frac{\dim((m + 1, x),(n_i, w_i))}{\dim((m, v),(n_i, w_i))}.$$

By definition of the sequence $(n_i, w_i)_{i\geq 1}$, there exists a subsequence (n_{i_k}, w_{i_k}) such that $w_{i_k} = \emptyset$ for all k, whence

$$p_\nu((m, v),(m + 1, x)) = \lim_{k\to\infty} \frac{\dim((m + 1, x),(n_{i_k}, \emptyset))}{\dim((m, v),(n_{i_k}, \emptyset))}.$$

By the path-walk identification Lemma 3.7, for $n > m$, $\dim((m, v),(n, \emptyset))$ is exactly the number of $n - m$-step walks on $\mathbb{T}_{\mathrm{FC}_s}$ starting at v and ending at $\emptyset$. Reversing the direction of these walks, and setting $l(n) = \lceil n - m/2\rceil$, it follows from Theorem 5.21 that

$$\dim((m, v),(n, \emptyset)) = \dim((0, \emptyset),(n - m, v)) = \begin{bmatrix} l(n) \\ v \end{bmatrix}$$

so that

$$p_\nu((m,v),(m+1,x)) = \lim_{k\to\infty} \frac{\left[{l(n_{i_k}-1) \atop x}\right]}{\left[{l(n_{i_k}) \atop v}\right]}.$$

From this, it is not hard to derive explicit formula for the transition probabilities which will be of the form

$$p_\nu((m,v),(m+1,x)) = K \cdot \frac{r(x)}{r(v)}$$

with a factor $K \geq 0$ that only depends on $s, r(v) - r(w) \in \{-s, \dots, s\}$ and $|v| - |x| \in \{-1, 1\}$. In particular, the transition probabilities $p_\nu((m,v),(m+1,x)) = p_\nu(v,x)$ are independent of m. Now, choosing $x = \emptyset$ and v to be the unique level 1 vertex (for the proof for $\widetilde{\mathbb{T}}^s$ choose any one of the two level 1 vertices), the transition probability $p_\nu((m,v),(m+1,\emptyset))$ takes the value

$$p_\nu(v,\emptyset) = \frac{s^s}{(s+1)^{s+1}}.$$

Consider the $2n$ step rooted loop that jumps from the root to v and back n times. Our computation then implies that the probability that the Markov chain $\pi(\nu)$ follows this walk in the first $2n$ steps is $\left(\frac{s^s}{s+1^{s+1}}\right)^n$. By centrality any other $2n$ step rooted loop must have the same probability. As there are exactly C_n^s of these loops, the probability to return to the root after $2n$ steps is $p_n = C_n^s \left(\frac{s^s}{s+1^{s+1}}\right)^n$. By Remark 5.22, it thus follows that

$$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} C_n^s \left(\frac{s^s}{(s+1)^{s+1}}\right)^n < \infty,$$

which implies that $\pi(\nu)$ is not recurrent, contradictory to our assumption. $\square$

Remark 6.8. Note that the argument of the proof of the previous lemma combined with Remark 5.22 also shows that for any time-homogeneous ergodic central measure ν, the transition probability from the first level vertex v to $\emptyset$ must be bounded above by $p_\nu(v,\emptyset) \leq \frac{s^s}{(s+1)^{s+1}}$.

Lemma 6.9. *Let ν be an ergodic central measure on $(\Omega_{\Gamma^s}, \mathcal{F}_{\Gamma^s})$. Then there exists an end $t \in \partial\mathbb{T}^s$ to which ν-almost every path on $\mathcal{P}(\mathbb{T}^s)$ converges. Moreover, for every ν-typical path $(n, w_n)_{n\geq 0}$, the limit*

$$\lim_{n\to\infty} \frac{n - |w_n|}{2r(w_n)} \in [0, +\infty]$$

exists and does not depend on the choice of $(n, w_n)_{n\geq 0}$.

Proof. Since $\pi(\nu)$ is transient by Lemma 6.7, ν-almost every path $x = (n, w_n)$ of $\mathcal{P}(\mathbb{T}^s)$ converges to some end $t_x \in \partial\mathbb{T}^s$. As for every $t \in \partial\mathbb{T}^s$, the set of paths that converge to t is invariant under the tail relation on $\mathcal{P}(\mathbb{T}^s)$ (i.e. changes of starting paths of finite length), ergodicity implies that $\pi(\nu)$-almost all walks converge to the same end.

Let $(n, w_n)_{n\geq 0}$ be a ν-typical path and $k \geq 0$ even. Arguing as in Lemma 6.7, by the path-walk identification and the ergodic method, the limit

$$\begin{aligned}
p_\nu((k+1,a),(k+2,\emptyset)) &= \lim_{n\to\infty} \frac{\dim((k+2,\emptyset),(n,w_n))}{\dim((k+1,a),(n,w_n))} \\
&= \lim_{n\to\infty} \frac{\mathrm{Wk}(n-k-2,\emptyset,w_n)}{\mathrm{Wk}(n-k-1,a,w_n)} \\
&= \lim_{n\to\infty} \frac{\mathrm{Wk}(n-k-2,\emptyset,w_n)}{\mathrm{Wk}(n-k,\emptyset,w_n)}
\end{aligned}$$

exists. Let$l = k/2$. If $n = 2m$ is even, then

$$\frac{\mathrm{Wk}(n-k-2,\emptyset,w_n)}{\mathrm{Wk}(n-k,\emptyset,w_n)} = \frac{\begin{bmatrix} m-l-1 \\ w_{2m} \end{bmatrix}}{\begin{bmatrix} m-l \\ w_{2m} \end{bmatrix}}$$

and if $n = 2m-1$ is odd then

$$\frac{\mathrm{Wk}(n-k-2,\emptyset,w_n)}{\mathrm{Wk}(n-k,\emptyset,w_n)} = \frac{\begin{bmatrix} m-l-1 \\ w_{2m-1} \end{bmatrix}}{\begin{bmatrix} m-l \\ w_{2m-1} \end{bmatrix}}.$$

Hence, in any case

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathrm{Wk}(n-k-2,\emptyset,w_n)}{\mathrm{Wk}(n-k,\emptyset,w_n)} &= \lim_{n\to\infty} \frac{\begin{bmatrix} \lceil (n-k-1)/2 \rceil \\ w_n \end{bmatrix}}{\begin{bmatrix} \lceil (n-k)/2 \rceil \\ w_n \end{bmatrix}} \\ &= \lim_{n\to\infty} \left(\frac{n-|w_n|}{2r(w_n)}\right) \frac{(s(\frac{n-|w_n|}{2r(w_n)})+1)^s}{((s+1)(\frac{n-|w_n|}{2r(w_n)})+1)^{(s+1)}}, \end{aligned}$$

where we used $\lim_{n\to\infty} r(w_n) = \infty$ which holds since $\pi(\nu)$ is transient. Note that this limit depends no longer on k, i.e. $p_\nu((k+1,a),(k+2,\emptyset)) =: p_\nu(a,\emptyset))$. Set $c_n = \frac{n-|w_n|}{2r(w_n)}$ and $f : [0,+\infty) \to [0,+\infty)$, $f(x) = \frac{x(sx+1)^s}{((s+1)x+1)^{s+1}}$, so that $\eta = \lim_{n\to\infty} f(c_n)$. One checks that f is a strictly increasing continuous function with $\lim_{x\to\infty} f(x) = \frac{s^s}{(s+1)^{s+1}}$ so that as a function onto $[0, \frac{s^s}{(s+1)^{s+1}})$, f is invertible with continuous inverse $f^{-1} : [0, \frac{s^s}{(s+1)^{s+1}}) \to [0,\infty)$. It follows that $\lim_{n\to\infty} c_n = f^{-1}(\eta)$ exists in $[0,+\infty]$, where we set $f^{-1}(\frac{s^s}{(s+1)^{s+1}}) := +\infty$. Note that our argument is independent of the choice of typical path (w_n) (alternatively, apply ergodicity of ν to the limiting random variable). $\square$

Since we have shown in the proof of the previous lemma that the product

$$p_\nu((k+1,a),(k+2),\emptyset) = \eta$$

is independent of k, we can repeat the proof of Proposition 6.3 to get the following.

Lemma 6.10. *If ν is a central ergodic measure on $\mathcal{P}(\mathbb{T}^s)$, then there is a constant $\eta \in [0, \frac{s^s}{(s+1)^{s+1}}]$ such that*

$$p_\nu((k,v),(k+1,w)\cdot p_\nu((k+1,w),(k+2,v)) = \eta$$

for every edge $((k,v),(k+1,w))$ with $|w| = |v|+1$ on $\mathcal{P}(\mathbb{T}^s)$.

We call the constant $\eta = \eta_\nu$ from Lemma 6.10 the *structure constant* of the ergodic central measure ν.

Proposition 6.11. *Let ν be an ergodic central measure on $(\Omega_{\Gamma^s}, \mathcal{F}_{\Gamma^s})$, let $t \in \partial\mathbb{T}^s$ be the end to which ν-almost every path on $\mathcal{P}(\mathbb{T}^s)$ converges and let $\eta \in [0, \frac{s^s}{(s+1)^{s+1}}]$ be the constant from the previous lemma. If $\eta < \frac{s^s}{(s+1)^{s+1}}$, then $\nu = \nu_{(t,\eta)}$.*
Moreover, given an end t, there exists at most one ergodic central measure $\nu^*_{(t,\frac{s^s}{(s+1)^{s+1}})}$ with structure constant $\eta = \frac{s^s}{(s+1)^{s+1}}$ that converges to t. This measure is the pull back of (the law of) a random walk $S_{(t,\frac{s^s}{(s+1)^{s+1}})}$ and thus time-homogeneous.

Proof. If $\eta = 0$, it follows inductively that in every step $\pi(\nu)$ jumps to a descendent of its current location with probability one and thus converges deterministically to t. Hence in this case $\nu = \nu_{(t,0)}$.

Therefore assume now $\eta > 0$. Let $(n, w_n)_{n\geq 0}$ be a ν-typical path, and let $((k, v), (k+1, w))$ be an edge on $\mathcal{P}(\mathbb{T}^s)$ with $|w| = |v| + 1$. Arguing as in Lemma 6.9, by the ergodic method

$$\begin{aligned}
\frac{\mathrm{Wk}(n-k-2, v, w_n)}{\mathrm{Wk}(n-k, v, w_n)} &= \frac{\mathrm{Wk}(n-k-2, v, w_n)}{\mathrm{Wk}(n-k-1, w, w_n)} \frac{\mathrm{Wk}(n-k-1, w, w_n)}{\mathrm{Wk}(n-k, v, w_n)} \\
&= \frac{\dim_{\mathcal{P}(\mathbb{T}^s)}((k+2, v), (n, w_n))}{\dim_{\mathcal{P}(\mathbb{T}^s)}((k+1, w), (n, w_n))} \cdot \frac{\dim_{\mathcal{P}(\mathbb{T}^s)}((k+1, w), (n, w_n))}{\dim_{\mathcal{P}(\mathbb{T}^s)}((k, v), (n, w_n))} \\
&\to \quad p_\nu((k+1, w), (k+2, v)) \cdot p_\nu((k, v), (k+1, w)) = \eta
\end{aligned}$$

as $n \to \infty$. More generally, for fixed $k, l > 0$

$$\frac{\mathrm{Wk}(n-k-2l, v, w_n)}{\mathrm{Wk}(n-k, v, w_n)} = \prod_{i=1}^{l} \frac{\mathrm{Wk}(n-k-2i, w, w_n)}{\mathrm{Wk}(n-k-2(i-1), w, w_n)} \quad \to \quad \eta^l.$$

Assume now that (w, v) is t-directed and does not lie on t. Note that this automatically implies $|w| = |v| + 1$. Since (w_n) converges to t, every walk from w to w_n must pass through v at some point for large enough n. Hence any walk from from w to w_n (for large n) of length m must split into a downward loop of length $2l$ rooted at w and a walk of length $m - 2l - 1$ from v to w_n. Since the number of downward loops of length $2l$ rooted at w is $C_l^{l(w)}$, it follows that

$$\begin{aligned}
\frac{\dim_{\mathcal{P}(\mathbb{T}^s)}((k+1, w), (n, w_n))}{\dim_{\mathcal{P}(\mathbb{T}^s)}((k, v), (n, w_n))} &= \frac{\mathrm{Wk}(n-k-1, w, w_n)}{\mathrm{Wk}(n-k, v, w_n)} \\
&= \sum_{l=0}^{\frac{n-k-d(w,w_n)}{2}} C_l^{l(w)} \frac{\mathrm{Wk}(n-k-2(l+1), v, w_n)}{\mathrm{Wk}(n-k, v, w_n)}.
\end{aligned}$$

On the one hand by the ergodic method this expression converges to $p_\nu((k, v), (k+1, w))$, on the other hand by Lemma 6.9 $\frac{n-k-d(w,w_n)}{2} \to \infty$ as n goes to infinity. If $\eta < \frac{s^s}{(s+1)^{s+1}}$, then we can make use of the uniform convergence of the power series $\sum_{l=0}^{L} C_l^{l(w)} z^l$ to $G_s(z)^{l(w)}$ on $[0, \frac{s^s}{(s+1)^{s+1}}]$, the right hand side therefore converges to $\eta G_s(\eta)^{l(w)}$, so that

$$p_\nu((k, v), (k+1, w)) = \eta G_s(\eta)^{l(w)}$$

for all $k \geq |v|$. By Lemma 6.10, also

$$p_\nu((k, w), (k+1, v)) = G_s(\eta)^{-l(w)}$$

for $k \geq |w|$. Note that since either (v, w) or (v, w) is t-directed, we have determined all transition probabilities outside of the end $t = (t_j)_{j\geq 0}$. Arguing as in Lemma 6.4, it follows by induction on j that the transition probabilities $p_\nu((k, t_j), (k+1, t_{j\pm 1}))$ are uniquely determined once the ones outside of t are given and that they do not depend on k. Hence all transition probabilities coincide with those of the random walk $S_{(t,\eta)}$ whence $\nu = \nu_{(t,\eta)}$. Note that if $\eta = \frac{s^s}{(s+1)^{s+1}}$, our argument also shows that

$$p_\nu((k, v), (k+1, w)) = p_\nu(v, w) = \lim_{n\to\infty} \sum_{l=0}^{\frac{n-k-d(w,w_n)}{2}} C_l^{l(w)} c_n^l > 0,$$

where $c_n = \frac{\mathrm{Wk}(n-2,v,w_n)}{\mathrm{Wk}(n,v,w_n)}$ converges to η and (w,v) is t-directed. Let $S^*_{(t,\frac{s^s}{(s+1)^{s+1}})}$ be the random walk with transition probabilities $p_\nu(v,w)$. Repeating the last part of the argument, it follows that $\nu = \nu^*_{(t,\frac{s^s}{(s+1)^{s+1}})}$ is the pullback of the law of $S^*_{(t,\frac{s^s}{(s+1)^{s+1}})}$ and is thus time-homogeneous with structure constant $\frac{s^s}{(s+1)^{s+1}}$. $\square$

6.1 The random walks $S_{(t,\eta)}$

In this section, we will analyse the random walks $S_{(t,\eta)}$ on the Fibonacci tree $\mathbb{FT} = \mathbb{T}^2$. In order to do so, for $\eta \in (0, 4/27]$, let us introduce an auxillary random walk S^η on the derooted Fibonacci tree $\widetilde{\mathbb{FT}}$. If $x_i, i = 1, 2$ is the unique first level vertex of $\widetilde{\mathbb{FT}}$ with $l(x_i) = i$, we define the transition probabilities from $\emptyset$ to x_i by $p_\eta(\emptyset, x_1) = \frac{\eta G(\eta)}{A(\eta)} = \frac{G(\eta)}{G(\eta)+G(\eta)^2}$ and $p_\eta(\emptyset, x_2) = \frac{\eta G(\eta)^2}{A(\eta)} = \frac{G(\eta)^2}{G(\eta)+G(\eta)^2}$ where $A(\eta) = \eta(G(\eta) + G(\eta)^2)$. The remaining transition probabilities are defined by

$$p_\eta(v,w) = \begin{cases} G(\eta)^{-l(v)} & \text{if } |v| = |w| + 1 \\ \eta \cdot G(\eta)^{l(w)} & \text{if } |v| = |w| - 1 \end{cases}, \tag{6.3}$$

where $G(z) = G_2(z)$ is the generating function for the 2-Fuss-Catalan numbers.

Lemma 6.12. *The random walk S^η on the derooted Fibonacci tree is recurrent.*

Proof. We will prove that $\sum_{n=1}^\infty \mathbb{P}(S^\eta_{2n} = \emptyset) = \infty$. First let

$D_{n,k} :=$ |{loops of length 2n crossing the edges betw. level 0 and 1 exactly k times}|.

By definition of S_η, we have

$$\mathbb{P}(S^\eta_{2n} = \emptyset) = \sum_{k=1}^{n} \eta^n A(\eta)^{-k} D_{n,k}.$$

Fix a vector $(i_1, \ldots, i_k) \in \{1,2\}^k$. and write $e_1 = (\emptyset, x_1), e_1 = (\emptyset, x_1)$. Then the number of loops of length $2n$ crossing the edges e_1 and e_2 in the order $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ is

$$\sum_{n_1+\cdots+n_k=n-k} C^{i_1}_{n_1} \ldots C^{i_k}_{n_k}$$

which is the coefficient of z^{n-k} in $G(z)^{\sum_{j=1}^k i_j}$. Therefore

$$D_{n,k} = \sum_{(i_1,\ldots,i_k)\in\{1,2\}^k} [z^{n-k}]G(z)^{\sum_{j=1}^k i_j} = \sum_{l=0}^{k} \binom{k}{l} [z^{n-k}]G(z)^{2k-l}.$$

Since $\sum_{n=0}^\infty \eta^n < \infty$, the sum $\sum_{n=1}^\infty \mathbb{P}(S^\eta_{2n} = \emptyset)$ is infinite if and only if

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k} \eta^n A(\eta)^{-k} \binom{k}{l} [z^{n-k}]G(z)^{2k-l}$$

is infinite. Reparametrising this sum yields

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{k=l}^{\infty} A(\eta)^{-k}\binom{k}{l}\eta^k \sum_{n=k}^{\infty}\eta^{n-k}[z^{n-k}]G(z)^{2k-l} &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\left(\frac{\eta}{A(\eta)}\right)^k\binom{k}{l}G(\eta)^{2k-l}\\ &= \sum_{k=0}^{\infty}\left(\frac{\eta}{A(\eta)}\right)^k G(\eta)^{2k}\sum_{l=0}^{k}\binom{k}{l}G(\eta)^{-l}\\ &= \sum_{k=0}^{\infty}\left(\frac{\eta}{A(\eta)}(G(\eta)^2+G(\eta))\right)^k\end{aligned}$$

Since $\frac{\eta}{A(\eta)}(G(\eta)^2+G(\eta)) = 1$, the result follows. $\square$

Corollary 6.13. *The random walk $S_{(t,\eta)}$, $t \in \partial\mathbb{FT}$, $\eta \in [0, 4/27]$ converges to t almost surely.*

Proof. Because $\eta \leq 4/27$, $S_{(t,\eta)}$ is transient. If $\eta = 0$, $S_{(t,0)}$ converges deterministically to t. Therefore let $\eta > 0$. Assume that the event

$$A = \{S_{(t,\eta)} \text{ does not converge to } t\}$$

has positive probability and denote by $d(t_k)$ the unique son of t_k that does not lie on t for $k \geq 0$, $l(t_k) = 2$. Then, for every trajectory $\omega = (\omega_n)_{n\geq 0} \in A$, there exists a time $N(\omega)$ and a vertex $v(\omega)$, $l(v(\omega)) = 2$ such that subtree rooted at $v(\omega)$ does not contain any vertex of t and such that ω_n lies on this subtree for all $n \geq N(\omega)$. But then, the transition probabilities of the random walk $S_{(t,\eta)}$ conditioned on $n \geq N$ coincide with those of S. Since S is recurrent by Lemma 6.12 this contradicts the transience of $S_{(t,\eta)}$. $\square$

6.2 A law of large numbers for exit times

In order to prove that the converse of Proposition 6.11 when $s = 2$, i.e. the fact that the measures $\nu_{(t,\eta)}$ are in fact ergodic, in this section, we prove a law for large numbers for normalized *exit times* for the Fibonacci tree $\mathbb{FT} = \mathbb{T}^2$.

Definition 6.14. Let $S = (S_n)_{n\geq 0}$ be a transient random walk on $\mathbb{FT}$ converging to an end $t = (t_k)_{k\geq 0}$. The *exit time* at t_k is the random variable N_k defined as the last moment of passage at t_k, that is to say the unique nonnegative integer such that $S_{N_k} = t_k$ and $|S_m| > k$ for all $m \geq N_k$. Since S converges to t, $N_k < \infty$ is almost surely well-defined.

Let us recall the statement of Theorem B, now that we have clarified our notation.

Theorem 6.15 (Theorem B)**.** *Let $t = (t_k)_{k\geq 0} \in \partial\mathbb{FT}$ an end and let $\eta \in [0, 4/27]$. Denote by N_k the exit time at t_k for the random walk $S_{(t,\eta)}$. Then the limit*

$$\lim_{k\to\infty}\frac{N_k - k}{r(t_k)} \in [0, \infty]$$

exists (and is almost surely constant). It is finite if and only $\eta < 4/27$.

Proof. Let us first define discrete random variables $Y_\eta^{(1)}, Y_\eta^{(2)}$ on $2\mathbb{N}$ that are distributed according to

$$\mathbb{P}[Y_\eta^{(j)} = 2n] = \frac{C_n^j \cdot \eta^n}{G^j(\eta)}.$$

Note that $\mathbb{E}[Y_\eta^j] = \frac{2\eta (G^j)'(\eta)}{G^j(\eta)}$ when $\eta \in [0, 4/27)$ and $\mathbb{E}[Y_{4/27}^j] = \infty$. We observe that with probability one, between $N_{k-1}+1$ and N_k, the random walk $S_{(t,\eta)}$ performs a loop starting and ending at t_k that always stays strictly below t_{k-1}. Since the number of such loops of length $2n$ is $C_n^{l(t_k)}$ and since every loop of length $2n$ is taken by $S_{(t,\eta)}$ with probability η^n, it follows that

$$N_k - N_{k-1} - 1 \sim Y_\eta^{l(t_k)}.$$

Moreover, thanks to the Markov property of the random walk $S_{(t,\eta)}$, the random variables $N_k - N_{k-1} - 1,\ k \geq 1$ are independent. Write

$$F_j(t_k) = |\{0 \leq i \leq k \ ;\ l(t_i) = j\}| \qquad \text{for } j = 1, 2.$$

Then $r(t_k) = F_1(t_k) + 2F_2(t_k)$ and $F_1(t_k) + F_2(t_k) = k+1$. Note that for all $k \geq 0$, $F_2(t_k) \geq k/2$ and $k+1 \leq r(t_k) \leq 2k+1$. Let first $\eta = 4/27$. Then

$$\frac{N_k - k}{r(t_k)} \geq \sum_{\substack{l=1 \\ l(t_l)=2}}^{k} (N_l - N_{l-1} - 1) \sim \frac{F_2(t_k)}{r(t_k)} \frac{\sum_{l=1}^{F_2(t_k)} Y_l^{(2)}}{F_2(t_k)} \geq \frac{1}{3} \frac{\sum_{l=1}^{F_2(t_k)} Y_l^{(2)}}{F_2(t_k)}$$

and by the law of large numbers the righthand side converges to ∞. If $\eta < 4/27$, we write

$$\begin{aligned} N_k - k = \sum_{l=1}^{k} (N_l - N_{l-1} - 1) &= \sum_{\substack{l=1 \\ l(t_l)=1}}^{k} (N_l - N_{l-1} - 1) + \sum_{\substack{l=1 \\ l(t_l)=2}}^{k} (N_l - N_{l-1} - 1) \\ &\sim \sum_{l=1}^{F_1(t_k)} Y_l^{(1)} + \sum_{l=1}^{F_1(t_k)} Y_l^{(2)}. \end{aligned}$$

Therefore

$$\frac{N_k - k}{r(t_k)} \sim \frac{F_1(t_k)}{r(t_k)} \frac{\sum_{l=1}^{F_1(t_k)} Y_l^{(1)}}{F_1(t_k)} + \frac{F_2(t_k)}{r(t_k)} \frac{\sum_{l=1}^{F_2(t_k)} Y_l^{(2)}}{F_2(t_k)}.$$

If $\sup_k F_1(t_k) < \infty$, by the LLN $\frac{N_k - k}{r(t_k)}$ converges a.s. to $\frac{\mathbb{E}[Y_\eta^{(2)}]}{2} < \infty$. If $\sup_k F_1(t_k) = \infty$, we rewrite

$$\frac{N_k - k}{r(t_k)} \sim \frac{\sum_{l=1}^{F_1(t_k)} Y_l^{(1)}}{F_1(t_k)} + \frac{F_2(t_k)}{r(t_k)} \left(\frac{\sum_{l=1}^{F_2(t_k)} Y_l^{(2)}}{F_2(t_k)} - 2 \frac{\sum_{l=1}^{F_1(t_k)} Y_l^{(1)}}{F_1(t_k)} \right)$$

and we observe that

$$\mathbb{E}[Y_\eta^{(2)}] - 2\mathbb{E}[Y_\eta^{(1)}] = \frac{2\eta (G^2)'(\eta)}{G^2(\eta)} - 2 \frac{2\eta G'(\eta)}{G(\eta)} = 0.$$

Hence, by the LLN $\frac{N_k - k}{r(t_k)}$ converges a.s. to $\mathbb{E}[Y_\eta^{(1)}] = \frac{\mathbb{E}[Y_\eta^{(2)}]}{2}$ as well. ☐

Remark 6.16. Note that the previous theorem also holds for the random walk $S^*_{(t,4/27)}$, since we have only used the structure constant but not explicit transition probabilities in the proof.

We are now ready to finish the proof of Theorem A/Theorem 6.5.

Proof of Theorem 6.5. Let ν be an ergodic central measure. By Lemmas 6.9 and 6.10, ν converges to an end $t \in \partial\mathbb{FT}$ and has structure constant $\eta \in [0, 4/27]$. If $\eta < 4/27$, then by Proposition 6.11, $\nu = \nu_{(t,\eta)}$, if $\eta = 4/27$, then $\nu = \nu^*_{(t,4/27)}$. We need to prove that these measures are in fact ergodic and that $\nu^*_{(t,4/27)} = \nu_{(t,4/27)}$. If for given t, we show that $\nu_{(t,\eta)}$ is ergodic for $\eta < 4/27$, then $\nu_{(t,4/27)}$ is ergodic as well since the simplex of ergodic central measures is weakly closed and $\nu_{(t,\eta)} \to \nu_{(t,4/27)}$. As there is at most one ergodic measures with structure constant $4/27$ converging to t, it follows that $\nu^*_{(t,4/27)} = \nu_{(t,4/27)}$.

Thus let us assume that $\eta < 4/27$. To show ergodicity of $\nu_{(t,\eta)}$, we argue as in [VM15, Proof of Proposition 5.1]. We already know that the set of ergodic measures is a subset of

$$\tilde{\Sigma} := \{\nu_{(t,\eta)}\ ;\ t \in \partial\mathbb{FT},\ \eta \in [0, 4/27)\} \cup \{\nu^*_{(t,4/27)},\ t \in \partial\mathbb{FT}\}.$$

Therefore, by ergodic decomposition, we know that $\nu_{(t,\eta)}$ can be decomposed as

$$\nu_{(t,\eta)} = \int_{\tilde{\Sigma}} \xi \ d\rho(\xi)$$

for some probability measure ρ on $\tilde{\Sigma}$. Let $c = \lim_{k\to\infty} \frac{N_k - k}{r(t_k)}$ the limit of Theorem 6.15 for $\nu_{(t,\eta)}$. Again, using Corollary 6.13 and Theorem 6.15, it follows that, if $\rho \neq \delta_{\nu_{(t,\eta)}}$ is not the Dirac measure at $\nu_{(t,\eta)}$, then the set

$$A = \{\omega \in \Omega_{\mathcal{P}(\mathbb{FT})},\ \lim_{n\to\infty} w_n = t \text{ and } \lim_{k\to\infty} \tfrac{N_k(\omega) - k}{r(t_k)} = c\}$$

has probability $\nu_{(t,\eta)}(A) < 1$, a contradiction. Thus, $\nu_{(t,\eta)}$ is ergodic. □

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