

LATTICE EQUABLE PARALLELOGRAMS

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ABSTRACT. This paper studies equable parallelograms whose vertices lie on the integer lattice. Using Rosenberger's Theorem on generalised Markov equations, we show that the g.c.d. of the side lengths of such parallelograms can only be 3, 4 or 5, and in each of these cases the set of parallelograms naturally forms an infinite tree all of whose vertices have degree 4, bar the root. The paper then focuses on what we call Pythagorean equable parallelograms. These are lattice equable parallelograms whose complement in a circumscribing rectangle consists of two Pythagorean triangles. We prove that for these parallelograms the shortest side can only be 3, 4, 5, 6 or 10, and there are five infinite families of such parallelograms, given by solutions to corresponding Pell-like equations.

1. INTRODUCTION

A polygon with integer sides is said to be *equable* if its perimeter equals its area. Equable polygons have fascinated recreational mathematical amateurs at least as far back as Ozanam [7]:

Problème XXIII. *Décrire un triangle rectangle dont l'aire en nombres soit égale au contour.*¹

Problème XXVII. *Décrire un parallélogramme rectangle dont l'aire en nombres soit égale au contour.*²

These exercises can easily be done in high school today, since they rely merely on the ability of *completing a rectangle*, instead of the usual *square*.

(a) For the Pythagorean triangle with side lengths $b < c < \sqrt{b^2 + c^2}$ we have

$$\begin{aligned} \frac{1}{2}bc = b + c + \sqrt{b^2 + c^2} &\implies 0 = (bc - 2b - 2c)^2 - 4(b^2 + c^2) \\ &= bc(bc - 4b - 4c + 8) \\ &\implies (b - 4)(c - 4) = 8, \end{aligned}$$

and hence $(b, c) = (5, 12)$ and $(6, 8)$ are the only two possibilities.

(b) For a rectangle with integer side lengths b, c with $b \leq c$ we have

$$bc = 2(b + c) \iff 0 = bc - 2b - 2c = (b - 2)(c - 2) - 4 \iff (b - 2)(c - 2) = 4,$$

giving $(b, c) = (4, 4)$ and $(3, 6)$.

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¹Determine which right triangles have the same area as perimeter.

²Determine which rectangles have the same area as perimeter.

The striking feature of equable integer sided triangles is that apart from the two Pythagorean triangles given above, there are only 3 other possibilities; see the Appendix. In this paper we study equable parallelograms. Unlike equable triangles, at first sight equable parallelograms are sadly disappointing, as they are just far too common. Indeed, it is easier to itemise the non-equable cases.

Proposition 1. *If b, c are positive integers with $b \leq c$, then there is an equable parallelogram with sides b, c unless one of the following holds:*

- (a) $b = 1$ or 2 , and c arbitrary with $b \leq c$.
- (b) $b = 3$ and $c = 3, 4$ or 5 .

Proof. Repeating the argument we used above for Ozanam’s Problème XXVII, the area is less than the perimeter when

$$bc < 2(b + c) \iff 4 > (b - 2)(c - 2),$$

and this condition is satisfied for precisely for the values of b, c itemised in the possible conditions of the proposition. So in these cases, there is no equable parallelogram with side lengths b, c . Conversely, if $bc \geq 2(b + c)$, start with the rectangle with side lengths b, c , and hence area bc , and gradually push the parallelogram over, while maintaining its side lengths, as in Figure 1. Ultimately, when the parallelogram becomes flat, the area is zero. So by continuity, somewhere in the process the area equals $2(b + c)$ and the parallelogram is equable. \square

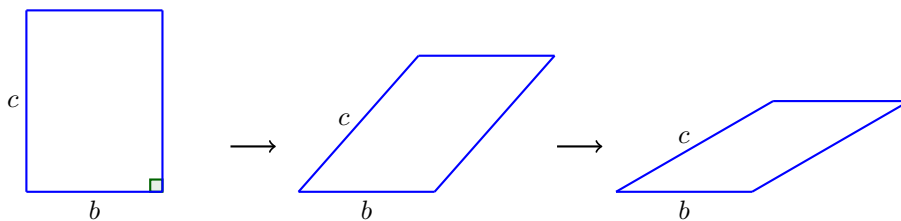


FIGURE 1. Collapsing a parallelogram

Given the above, to make life interesting we restrict ourselves to considering a subclass of equable parallelograms that has a wealth of interesting members, without becoming mundane.

Definition 1. A *lattice equable parallelogram* (or *LEP*, for short) is a parallelogram whose perimeter equals its area and whose vertices lie on the integer lattice \mathbb{Z}^2 .

Notice that in the above definition, we don’t require that the side lengths be integers. Indeed, as we show in Lemma 1 below, this fact can be deduced from the other hypotheses. Throughout this paper we will denote the side lengths by the symbols b and c (b for *base* and c for *côté*). Moreover, for brevity, we will at times drop the word *length* and simply refer to b, c as the *sides*. Note that a LEP is completely determined, up to a Euclidean motion, by its sides b, c . Indeed, if θ denotes one of the angles between the sides, then the area is $bc \sin \theta$ and so by

equability, $\sin \theta = 2(b+c)/bc$ is determined by b and c . So our main aim is this paper in to study the values of b, c for which a LEP exists with sides b, c .

In Section 2 we use a result of Paul Yiu on Heronian triangles to prove the following criteria.

Theorem 1. *Given positive integers b, c , a lattice equable parallelogram with sides b, c exists if and only if $b^2c^2 - 4(b+c)^2$ is a square.*

In Section 3 we use Rosenberger's Theorem on generalised Markov equations to prove the following result.

Theorem 2. *If a lattice equable parallelogram has sides b, c , then $\gcd(b, c)$ is 3, 4 or 5.*

In Section 4 we describe how all LEPs can be derived from three fundamental examples by successive applications of four functions. This gives the forest of LEPs, consisting of three trees corresponding to the three possible values of $\gcd(b, c)$: see Figures 6 and 7. We investigate various aspects and applications of these results in Sections 5 and 6. Then in Section 7 we analyse a large natural family of LEPs that we call *Pythagorean equable parallelograms*. These are LEPs whose complement in a circumscribing rectangle consists of two Pythagorean triangles; see Definition 3. We use Theorem 2 to prove the following:

Theorem 3. *If a Pythagorean equable parallelogram has sides b, c with $b \leq c$ then $b = 3, 4, 5, 6$ or 10 . There are 5 infinite families of such parallelograms, which are given by the solutions of corresponding Pell or Pell-like equations:*

- (F1) $b = 3, c = \frac{3(x^2+y^2)}{2}$, where $y^2 - 5x^2 = 4$.
- (F2) $b = 4, c = 4(x^2 + y^2)$, where $y^2 - 3x^2 = 1$.
- (F3) $b = 5, c = \frac{x^2+y^2}{2}$, where $3y^2 - 7x^2 = 20$.
- (F4) $b = 6, c = 3(x^2 + y^2)$, where $y^2 - 2x^2 = 1$.
- (F5) $b = 10, c = x^2 + y^2$, where $2y^2 - 3x^2 = 5$.

At the end of Section 7 we locate the Pythagorean equable parallelograms on certain branches of the trees of LEPs given in Figures 6 and 7.

The paper concludes with an appendix that revisits the classic theorem on equable triangles that dates to 1904.

2. LEPs: GENERAL PROPERTIES AND SPECIAL CASES

Let us begin with a trivial remark that we will use on several occasions.

Remark 1. If the square root of an integer is rational, then it is an integer. Consequently, if the distance d between two integer lattice points is rational, then d is an integer.

Lemma 1. *LEPs have integer side lengths.*

Proof. Consider a LEP P with vertices $O(0, 0), A(x, y), B(z, w), C(u, v)$, in anti-clockwise order, where $z = x + u, w = y + v$. Let b denote the length of OA and c the length of OC . The area of P is $xv - yu$, which is an integer. By the equability hypothesis, $2(b+c)$ is an integer. So, as b^2, c^2 are integers, $b - c = (b^2 - c^2)/(b+c)$

is rational. Thus $b = \frac{b+c}{2} + \frac{b-c}{2}$ is rational, and hence b is an integer, by Remark 1. So c is also rational, and hence an integer. \square

Proposition 2. *No LEP can be partitioned into two congruent right triangles.*

Proof. Suppose a LEP P can be partitioned into two congruent right triangles. Then P can be cut in half and recombined to form an equable isosceles triangle with integer sides, as in Figure 2. The proposition then follows from the fact that the complete list of 5 equable triangles includes no isosceles triangles; see the Appendix below.

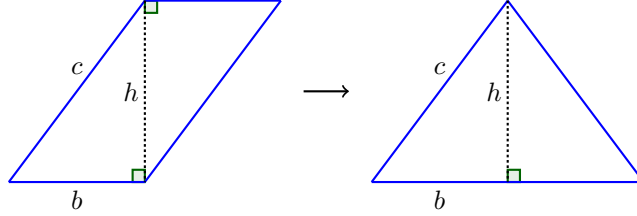


FIGURE 2. A particular non-equable parallelogram

A simple and more direct proof of the proposition is as follows. The equability hypothesis gives

$$(1) \quad \begin{aligned} hb = 2b + 2\sqrt{b^2 + h^2} &\implies b^2(h-2)^2 = 4(b^2 + h^2) \iff 0 = b^2h - 4b^2 - 4h \\ &\iff (b^2 - 4)(h - 4) = 16. \end{aligned}$$

Consequently h is rational and so by Remark 1, h is an integer. Then by (1), $b^2 - 4$ is a factor of 16. But there is obviously no such positive integer b . \square

Lemma 2. *Suppose a LEP P has sides b, c . Then the lengths of the diagonals of P are given by the following formula:*

$$d^2 = (b^2 + c^2) \pm 2\sqrt{b^2c^2 - 4(b+c)^2}.$$

In particular, $b^2c^2 - 4(b+c)^2$ is a square.

Proof. Consider a diagonal of length d . By Heron's formula, the triangle with sides b, c, d has area

$$\frac{1}{4}\sqrt{(b+c+d)(-b+c+d)(b-c+d)(b+c-d)}.$$

Hence the equability hypothesis is $(b+c+d)(-b+c+d)(b-c+d)(b+c-d) = 16(b+c)^2$. Rearranging, this gives $d^4 - 2(b^2+c^2)d^2 + (b^2-c^2)^2 + 16(b+c)^2 = 0$, so $d^2 = (b^2+c^2) \pm \sqrt{(b^2+c^2)^2 - (b^2-c^2)^2 - 16(b+c)^2}$. Simplifying, we have

$$d^2 = (b^2+c^2) \pm \sqrt{4b^2c^2 - 16(b+c)^2},$$

as required. In particular, as b, c, d^2 are integers, $4b^2c^2 - 16(b+c)^2$ is a square. \square

Remark 2. Suppose a LEP P has sides b, c and diagonals d_1, d_2 . Then the above lemma gives $d_1d_2 = (b+c)\sqrt{16 + (b-c)^2}$.

Proof of Theorem 1. The necessity of the condition was shown in Lemma 2. Therefore, assume that b, c are positive integers such that $4b^2c^2 - 16(b+c)^2$ is a square. Consider a triangle T with sides b, c and $d := \sqrt{b^2 + c^2 + 2\sqrt{b^2c^2 - 4(b+c)^2}}$. Notice that such a triangle exists because $b, c < d < b + c$, where the latter inequality holds as $b^2 + c^2 + 2\sqrt{b^2c^2 - 4(b+c)^2} < (b+c)^2$ since $\sqrt{b^2c^2 - 4(b+c)^2} < bc$. Let θ denote the angle between sides b, c and note that θ is obtuse since $d^2 \geq b^2 + c^2$. So

$$d^2 = b^2 + c^2 - 2bc \cos \theta = b^2 + c^2 + 2\sqrt{b^2c^2 - b^2c^2 \sin^2 \theta}.$$

So, from the definition of d , we have $bc \sin \theta = 2(b+c)$. Hence, since $bc \sin \theta$ is twice the area of T , the area of T is $b+c$. Now consider the parallelogram P made from two copies of T . From what we have just seen, P is equable. It remains to show that P can be realised as a lattice parallelogram, or equivalently, that T can be realised as a lattice triangle. But since b, c are integers, and d^2 is an integer, and the area of T is an integer, T is *geodetic*, in the terminology of Paul Yiu. Thus T can be realised as a lattice triangle; see the last paragraph of [12]. \square

Remark 3. Observe that LEPs are not determined up to congruence by their area. Indeed, by Theorem 1, there are LEPs with $(b, c) = (3, 87), (5, 85), (25, 65)$ that each have area 180. A larger example is given by $(b, c) = (85, 1525)$ and $(445, 1165)$.

Corollary 1.

- (a) *The only rhombi that are LEPs are the 4×4 square and the rhombus with side length 5 and area 20.*
- (b) *If a LEP has sides b, c with $c = 2b$, then $b = 3$ or 5.*

Proof. From Theorem 1, a rhombus with side length b is a LEP if and only if $b^4 - 16b^2$ is a square, that is, if $b^2 - 16$ is a square. But obviously this only occurs when b is 4 or 5. The first case is the 4×4 square, while the second case is exhibited in Figure 3. Similarly, if $c = 2b$, then $4b^4 - 36b^2$ is a square and so $b^2 - 9$ is a square. Hence $b = 3$ or 5. \square

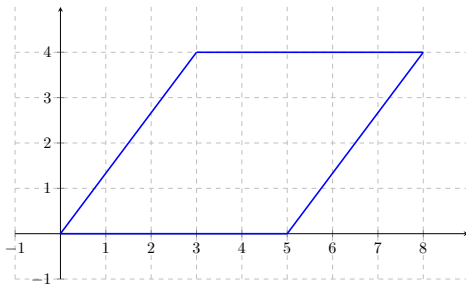


FIGURE 3. Equable Rhombus

We conclude this section with an elementary result whose proof will hopefully give the reader a better feel for the nature of LEPs.

Theorem 4. *There are only three LEPs lying in the first quadrant with one vertex at the origin and a diagonal lying on the line $x = y$. They are the 4×4 square, the LEP with vertices $(0, 0)$, $(3, 0)$, $(12, 12)$, $(9, 12)$, and its reflection in the line $y = x$.*

Proof. Consider a parallelogram P with vertices $O(0, 0)$, $A(x, y)$, $B(z, z)$, $A'(z - x, z - y)$, in clock-wise order, where x, y, z are non-negative integers and $y < x \leq z$, as in diagram on the left of Figure 4. Let us first calculate the area α of the triangle OAB . The diagonal OB has length $\sqrt{2}z$ and the distance of A to the diagonal is $(x - y)/\sqrt{2}$. So $\alpha = \frac{1}{2}z(x - y)$. The sum σ of the lengths OA and AB is

$$(2) \quad \sigma = \sqrt{x^2 + y^2} + \sqrt{(z - x)^2 + (z - y)^2}.$$

Moreover, σ is no greater than the sum of the lengths of the segments $O(z, 0)$ and $(z, 0)B$. That is, $\sigma \leq 2z$. Assume that P is a LEP. The equable hypothesis, $\alpha = \sigma$, gives

$$(3) \quad \frac{1}{2}z(x - y) \leq 2z,$$

and hence $y < x \leq y + 4$. So we have 4 cases to consider:

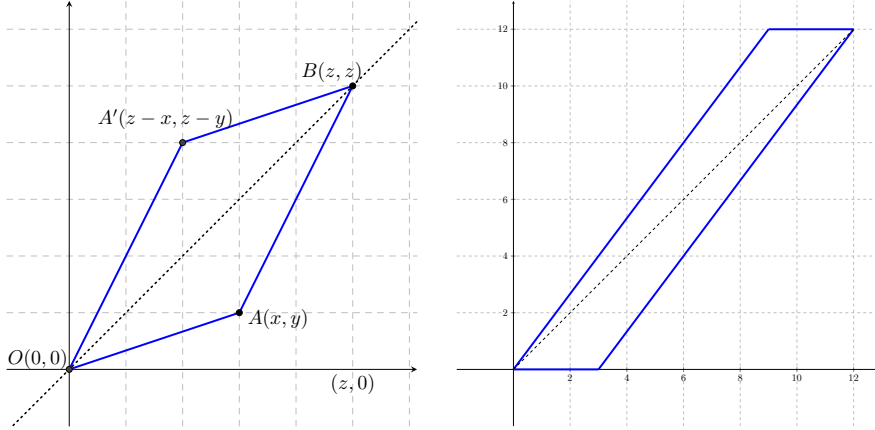


FIGURE 4. Parallelograms with diagonal on line $y = x$

Case $x = y + 4$. Here (15) is an equality, from which we conclude that the given parallelogram is a square, with side length z . So, by equability, $z^2 = 4z$ and hence $z = 4$. So $A = (4, 0)$, $B = (4, 4)$.

Case $x = y + 3$. Substituting this in (2), equability gives

$$\frac{3}{2}z = \sqrt{x^2 + (x - 3)^2} + \sqrt{(z - x)^2 + (z - x + 3)^2}.$$

Writing this as $3z - 2\sqrt{x^2 + (x - 3)^2} = 2\sqrt{(z - x)^2 + (z - x + 3)^2}$ and squaring both sides and simplifying gives $9z^2 - 12z\sqrt{x^2 + (x - 3)^2} = 4(2z^2 - 2z(2x - 3))$. Rearranging and dividing by z gives $z = 4(6 - 4x + 3\sqrt{x^2 + (x - 3)^2})$. In particular,

$x^2 + (x-3)^2$ is a square. Moreover, $x \leq z$ gives $4(6-4x+3\sqrt{x^2+(x-3)^2}) \geq x$ and so $144(x^2+(x-3)^2) \geq (17x-24)^2$ and simplifying,

$$0 \geq x^2 + 48x - 720 = (x+60)(x-12).$$

Thus $x \leq 12$. Calculations show that the only such values for which $x^2 + (x-3)^2$ is a square are $x = 3$ and $x = 12$. These values correspond to the parallelogram $(0,0), (3,0), (12,12), (9,12)$, and its reflection in the line $y = x$; see the diagram on the right of Figure 4.

Cases $x = y + 2$ and $x = y + 1$. The area is respectively z and $z/2$ in these cases, but the sum σ of the sides is greater than the length of the diagonal, so $\sigma > \sqrt{2}z$. Thus equability is impossible in both cases. \square

Remark 4. The above result does not hold without the assumption that the LEPs lies in the first quadrant. For example, consider the following 6 LEPs with vertices $O(0,0), A(x,y), B(z,z), A'(z-x, z-y)$, in clock-wise order:

$$\begin{aligned} A &= (0, -3), B = (60, 60), \\ A &= (0, -4), B = (12, 12), \\ A &= (-12, -16), B = (68, 68), \\ A &= (-9, -12), B = (348, 348), \\ A &= (-60, -63), B = (2028, 2028), \\ A &= (-80, -84), B = (396, 396). \end{aligned}$$

Notice incidentally that in all these cases $x - y \leq 4$, as in the proof of the above theorem, even though in these cases the diagonal on $y = x$ is the short diagonal of the LEP. This is a general fact; see Theorem 5 below.

3. THE LEP RESTRICTION: $\gcd(b, c) = 3, 4, 5$

The main tool we use in this section is Rosenberger's Theorem on generalised Markov equations. Recall that in [9] Rosenberger considered equations of the form

$$(4) \quad ax^2 + by^2 + cz^2 = dxyz,$$

where a, b, c are pairwise relatively prime positive integers with $a \leq b \leq c$ such that a, b, c all divide d . We are only interested in positive integer solutions, that is, $x, y, z \in \mathbb{N}$, so we use the word *solution* to mean positive integer solution. Rosenberger's remarkable result is that only 6 such equations have a solution and when such a solution exists, there are infinitely many solutions. We use the R1–R5 notation of [1].

Rosenberger's Theorem ([9]). *Equation (4) only has a solution in the following 6 cases:*

$$\begin{aligned} \text{M: } & x^2 + y^2 + z^2 = 3xyz \text{ (Markov's equation),} \\ \text{R1: } & x^2 + y^2 + 2z^2 = 4xyz, \\ \text{R2: } & x^2 + 2y^2 + 3z^2 = 6xyz, \\ \text{R3: } & x^2 + y^2 + 5z^2 = 5xyz, \\ \text{R4: } & x^2 + y^2 + z^2 = xyz, \\ \text{R5: } & x^2 + y^2 + 2z^2 = 2xyz, \end{aligned}$$

Remark 5. As mentioned in [9], it is easy to see by considering R4 modulo 3, that if (x, y, z) is a solution to R4, then x, y, z are each divisible by 3, and (x, y, z) is a solution to Markov's equation if and only if $(3x, 3y, 3z)$ is a solution to R4. Similarly, by considering R5 modulo 4, if (x, y, z) is a solution to R5, then x, y, z are each even, and (x, y, z) is a solution to R1 if and only if $(2x, 2y, 2z)$ is a solution to R5.

Returning to LEPs, suppose that b, c are positive integers such that $b^2c^2 - 4(b+c)^2$ is a square. Using the standard characterisation of Pythagorean triples we have relatively prime positive integers m, n and a positive integer k such that either

$$(5) \quad bc = k(m^2 + n^2), \quad 2(b+c) = 2kmn, \quad \sqrt{b^2c^2 - 4(b+c)^2} = k(n^2 - m^2),$$

or

$$(6) \quad bc = k(m^2 + n^2), \quad \sqrt{b^2c^2 - 4(b+c)^2} = 2kmn, \quad 2(b+c) = k(n^2 - m^2).$$

We will show that the second possibility can be reduced to the first. Indeed, suppose that (6) holds. Assume for the moment that k is odd. Since $\sqrt{b^2c^2 - 4(b+c)^2} = 2kmn$, it follows that bc is even. Consider the equation $bc = k(m^2 + n^2)$. As bc is even and k is odd and m, n are relatively prime, m, n must both be odd. Thus, from $bc = k(m^2 + n^2)$, we have $bc \equiv 2 \pmod{4}$. So exactly one of b, c is even. Thus $2(b+c) \equiv 2 \pmod{4}$. But as m, n are all odd, $n^2 - m^2 \equiv 0 \pmod{4}$, contradicting $2(b+c) = k(n^2 - m^2)$. Thus k is even, $k = 2\kappa$ say. Let $y = m + n, x = n - m$, so $n = \frac{y+x}{2}, m = \frac{y-x}{2}$. Then (6) can be written

$$bc = k(m^2 + n^2) = \kappa(x^2 + y^2), \quad 2(b+c) = k(n^2 - m^2) = 2\kappa xy, \quad 2kmn = \kappa(y^2 - x^2),$$

which has the same form as (5). Since m, n are relatively prime, $\gcd(x, y) = 1$ or 2. In the later case we can replace x and y by $x = x/2$ and $y = y/2$ respectively, and replace κ by $\kappa = 4\kappa$. We then have the same form as (5) again but now x, y are relatively prime. We may therefore assume (5) for what follows.

Note that b, c are solutions to the equation

$$(7) \quad x^2 - kmnx + k(m^2 + n^2) = 0.$$

In particular, we have

$$(8) \quad km^2 + kn^2 + b^2 = kbm.$$

Let $k = fs^2$, where f is square-free.

Lemma 3. *k is either 5, 8 or 9.*

Proof. From (8), f divides b^2 and hence f divides b . Let $b = f\beta$. Dividing (8) by f gives

$$(9) \quad s^2m^2 + s^2n^2 + f\beta^2 = fs^2mn\beta.$$

From (9), s^2 divides $f\beta^2$ and hence s^2 divides β^2 , and thus s divides β . Let $\beta = sq$. Dividing (9) by s^2 gives

$$(10) \quad m^2 + n^2 + fq^2 = fsmnq.$$

Thus by Rosenberger's Theorem, using $x = m, y = n, z = q$, there are five possibilities:

- (a) $f = 1, s = 3$ (corresponding to M),
- (b) $f = 2, s = 2$ (corresponding to R1),
- (c) $f = 5, s = 1$ (corresponding to R3),
- (d) $f = 1, s = 1$ (corresponding to R4),
- (e) $f = 2, s = 1$ (corresponding to R5).

Cases (a) to (e) give $k = 9, 8, 5, 1, 2$ respectively. It remains to eliminate the last two cases.

If $k = 1$, then (10) is $m^2 + n^2 + q^2 = mnq$. But then as noted in Remark 5, the values m, n, q are all divisible by 3, which contradicts our assumption that m, n are relatively prime. Similarly, if $k = 2$, then (10) is $m^2 + n^2 + 2q^2 = 2mnq$. But then as noted in Remark 5, the values m, n, q are all even, again contradicting our assumption that m, n are relatively prime. \square

Proof of Theorem 2. Suppose as before that

$$bc = k(m^2 + n^2), \quad 2(b + c) = 2kmn, \quad \sqrt{b^2c^2 - 4(b + c)^2} = k(n^2 - m^2),$$

so from the previous lemma, k is either 5, 8 or 9.

Lemma 4. *The numbers k and $\gcd(b, c)$ have the same prime divisors.*

Proof. From (7), $b^2 - kmnb + k(m^2 + n^2) = 0$ and $c^2 - kmnc + k(m^2 + n^2) = 0$. So if p is any prime divisor of k , then p divides b^2 and c^2 , so p divides b and c . Hence p divides $\gcd(b, c)$.

Conversely, let p be any prime divisor of $\gcd(b, c)$, and suppose that p doesn't divide k . Then p^2 divides $m^2 + n^2$ and p divides mn so p divides $(m + n)^2$ and $(m - n)^2$, so p divides $m + n$ and $m - n$. Hence p divide $2m$ and $2n$. If p were odd we would have that p divides m and n , which is impossible as m, n are relatively prime. So $p = 2$; that is, b, c are both even. Since p doesn't divide k , we have that k is odd and $m^2 + n^2$ is even. So, as m, n are relatively prime, m, n are both odd. But then $b + c = kmn$ implies that k is even, a contradiction. So p divides k . \square

Consider the case where $k = 5$. From Lemma 4, $\gcd(b, c) = 5^i$ for some $i \geq 1$. Suppose that $i > 1$, so b, c are both divisible by 25. From $bc = k(m^2 + n^2)$ we have that $m^2 + n^2$ is divisible by 5^3 . From $b + c = kmn$ we have that mn is divisible by 5, so exactly one of m or n is divisible by 5 since m, n are relatively prime. But this is a contradiction since $m^2 + n^2$ would not be divisible by 5. Hence $\gcd(b, c) = 5$.

Now suppose $k = 9$. From Lemma 4, $\gcd(b, c) = 3^i$ for some $i \geq 1$. Suppose that $i > 1$, so b, c are both divisible by 9. From $bc = k(m^2 + n^2)$ we have that $m^2 + n^2$ is divisible by 9. In \mathbb{Z}_9 the squares are 0, 1, 4, 7. Because m, n are relatively prime, they are not both congruent to 0 modulo 3. So modulo 9,

$$m^2 + n^2 \in \{0 + 1, 0 + 4, 0 + 7, 1 + 1, 1 + 4, 1 + 7, 4 + 4, 4 + 7, 7 + 7\} \equiv \{1, 4, 7, 2, 5, 8\},$$

contradicting the fact that $m^2 + n^2$ is divisible by 9. Hence $\gcd(b, c) = 3$.

Finally, suppose $k = 8$. From Lemma 4, $\gcd(b, c) = 2^i$ for some $i \geq 1$. As 2^6 divides $(b + c)^2 = k^2m^2n^2$ and 2^5 divides $4bc = 4k(m^2 + n^2)$, thus 2^5 divides $(b + c)^2 - 4bc = (b - c)^2$. Consequently, as $(b - c)^2$ is a square, 2^6 divides $(b - c)^2$. Hence, 2^3 divides $b - c$. Thus, as 2^3 divides $b - c$ and $b + c = kmn$, so 2^3 divides both $2b$ and $2c$. Hence 2^2 divides both b and c , and hence $\gcd(b, c) \geq 2^2$. Thus

$i \geq 2$. Now suppose $i > 2$. Thus 2^6 divides $bc = 8(m^2 + n^2)$ and hence 8 divides $m^2 + n^2$. In \mathbb{Z}_8 the squares are 0,1,4. Because m, n are relatively prime, they are not both even. So modulo 8,

$$m^2 + n^2 \in \{0 + 1, 1 + 1, 1 + 4\} \equiv \{1, 4, 2, 5\},$$

contradicting the fact that $m^2 + n^2$ is divisible by 8. Hence $\gcd(b, c) = 2^2$.

This completes the proof of Theorem 2. \square

Remark 6. In the case $\gcd(b, c) = 3$, we have $k = 3^2$ and $b + c = kmn$, so $b + c$ is divisible by 3^2 . Then neither b nor c is divisible by 3^2 since otherwise both b and c would be divisible by 3^2 , contradicting $\gcd(b, c) = 3$. Similarly, if $\gcd(b, c) = 4$, then $b + c$ is divisible by 2^3 but neither b nor c is divisible by 2^3 . But a similar result does not hold in the $\gcd(b, c) = 5$ case. For example, for the LEP with $b = 85, c = 1525$ one has $b = 5 \cdot 17, c = 5^2 \cdot 61$.

Let us summarise the results of this section. We have seen in the proof of Theorem 2 that for LEPs with sides b, c , we have $\gcd(b, c) = 3, 4$ or 5 . The possible values of b, c are determined by the corresponding Markov-Rosenberger equations M,R1 and R3, as follows:

- (a) $\gcd(b, c) = 3$: $m^2 + n^2 + q^2 = 3mnq$, where $q = b/3$,
- (b) $\gcd(b, c) = 4$: $m^2 + n^2 + 2q^2 = 4mnq$, where $q = b/4$,
- (c) $\gcd(b, c) = 5$: $m^2 + n^2 + 5q^2 = 5mnq$, where $q = b/5$,

and $bc = k(m^2 + n^2), b + c = kmn$ and $k = 9, 8, 5$ respectively. If (m, n, q) is a solution to one of the above equations, then by Rosenberger's Theorem, the triple m, n, q is pairwise relatively prime. In particular, m, n are relatively prime.

4. THE FOREST OF LEPs

For a given k ($= 5, 8$ or 9) consider the set S_k of solutions of the corresponding Markov-Rosenberger equation. Let us briefly recall Rosenberger's theory. For the reader's convenience, we recall (10):

$$m^2 + n^2 + fq^2 = fsmnq.$$

Following the presentation given in [1], from a solution $x = (m, n, q)$ to (10), one can generate three new solutions by applying the involutions:

$$\phi_1(x) = (fsnq - m, n, q)$$

$$\phi_2(x) = (m, fsmq - n, q)$$

$$\phi_3(x) = (m, n, smn - q).$$

The group of transformations of S_k generated by the maps ϕ_i is the free product of three copies of \mathbb{Z}_2 , and this group acts transitively on S_k . Moreover, the maps ϕ_i give the set S_k of solutions the structure of an infinite binary tree: each solution is a vertex and two solutions are connected by an edge if one of the maps ϕ_i sends one solution to the other. The *fundamental solutions* have the smallest values of $m + n + q$; in M,R1,R3 they are $(m, n, q) = (1, 1, 1), (1, 1, 1), (1, 2, 1)$ respectively.

Having recalled Rosenberger's theory, we now describe how the solution trees S_k of the Markov-Rosenberger equations determine the induced structure on the set T_{gcd} of LEPs, where $gcd = 3, 4, 5$ for $k = 9, 8, 5$ respectively. We identify T_{gcd}

as the set of ordered pairs (b, c) of possible side lengths with $\gcd(b, c) = gcd$ and $b \leq c$. A solution (m, n, q) to the corresponding Markov-Rosenberger equation corresponds to a LEP if $n \geq m$. Let τ denote the involution of S_k given by $\tau : (m, n, q) \mapsto (n, m, q)$. The map τ commutes with ϕ_3 , and conjugates ϕ_1 to ϕ_2 . We form the quotient tree $\bar{S}_k = S_k/\tau$. The elements of \bar{S}_k can be regarded as triples (m, n, q) with $m \leq n$. The map ϕ_3 induces an involution $\bar{\phi}_3$ on \bar{S}_k . Note that by definition ϕ_3 leaves m and n unchanged and sends q to $smn - q$, or equivalently, b to $kmn - b = b + c - b = c$. Thus (m, n, q) and $\phi_3(m, n, q)$ correspond to the same LEP. Consequently we form T_{gcd} by contracting each of the edges of \bar{S}_k that are given by the map $\bar{\phi}_3$; see Figure 5. Note that contraction of edges of a tree produces another tree. So T_{gcd} is a tree for each $gcd = 3, 4, 5$. Notice also that contraction turns vertices of degree 3 into vertices of degree 4. The trees T_3 and T_5 have the same form; they are shown in Figure 6. The tree T_4 is shown in Figure 7. The *fundamental solutions* are the solutions (b, c) for which $b + c$ is minimal. For $\gcd(b, c) = 3, 4, 5$, the fundamental solutions are $(3, 6), (4, 4), (5, 5)$ respectively. Apart from the fundamental solutions, the vertices of T_{gcd} all have degree 4. The fundamental solutions of T_3 and T_5 have degree 2, while the fundamental solution of T_4 has degree 1.

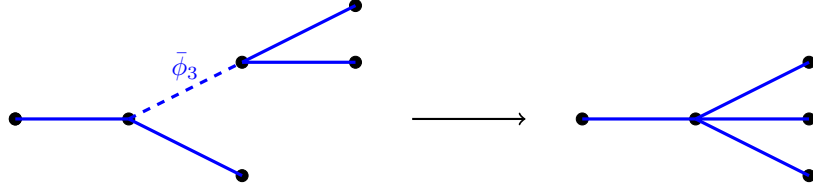


FIGURE 5. Contraction of the $\bar{\phi}_3$ edges

Let us look in more detail at the edges of the LEP solution trees T_{gcd} . Consider a LEP (b, c) . As we saw in the previous section, k is determined by $\gcd(b, c)$; $k = 9, 8, 5$ for $\gcd(b, c) = 3, 4, 5$ respectively. Writing $bc = k(m^2 + n^2)$, $b + c = kmn$ as before, and imposing $n \geq m$, the values m, n are given by the following formulas:

$$(11) \quad kn^2 = \frac{bc + \sqrt{b^2c^2 - 4(b+c)^2}}{2}$$

$$(12) \quad km^2 = \frac{bc - \sqrt{b^2c^2 - 4(b+c)^2}}{2}.$$

For each $i = 1, 2$, let φ_i denote the function induced by ϕ_i on the set T_{gcd} of LEP pairs (b, c) . Note that in the above notation, as $b + c = kmn$, we have $(b, c) = (fsq, kmn - fsq)$. From the definition of ϕ_1 , the map φ_1 leaves b and n unchanged and m is changed to $m' = bn - m$. Then under φ_1 , the value of c is changed to

$$\begin{aligned} c' &= km'n - b = kbn^2 - knm - b = kbn^2 - (b + c) - b \\ &= \frac{b^2c + b\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2b - c. \quad (\text{from (11)}) \end{aligned}$$

Thus

$$\varphi_1 : (b, c) \mapsto \left(b, \frac{b^2c + b\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2b - c \right).$$

Similarly, one finds that

$$\varphi_2 : (b, c) \mapsto \left(b, \frac{b^2c - b\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2b - c \right).$$

A calculation shows that $\varphi_2 \circ \varphi_1 = \text{id}$. However it is not true that $\varphi_1 \circ \varphi_2(b, c) = (b, c)$ for all (b, c) . In fact, the function φ_2 is not injective and the function φ_1 is not surjective; there is no solution (b, c) with $b < c$ for which $\varphi_1(b, c)$ is the fundamental solution.

Similarly, analogous to φ_1, φ_2 , interchanging the roles of b and c , we have two further maps:

$$\begin{aligned} \psi_1 : (b, c) &\mapsto \left(\frac{bc^2 + c\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2c - b, c \right), \\ \psi_2 : (b, c) &\mapsto \left(\frac{bc^2 - c\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2c - b, c \right). \end{aligned}$$

After applying the maps φ_i, ψ_i , one reverses the image (b', c') if necessary so that $b' \leq c'$. The edges in the trees T_{gcd} are all obtained by applications of the four maps $\varphi_1, \varphi_2, \psi_1, \psi_2$.

5. RECURRENCE RELATIONS IN THE BRANCHES OF THE LEP TREES

The trees T_3, T_4, T_5 contain a raft of interesting patterns that connect with phenomena that have been observed in other areas. We will give some simple examples. First we require a technical lemma concerning recurrence relations. Consider the real function

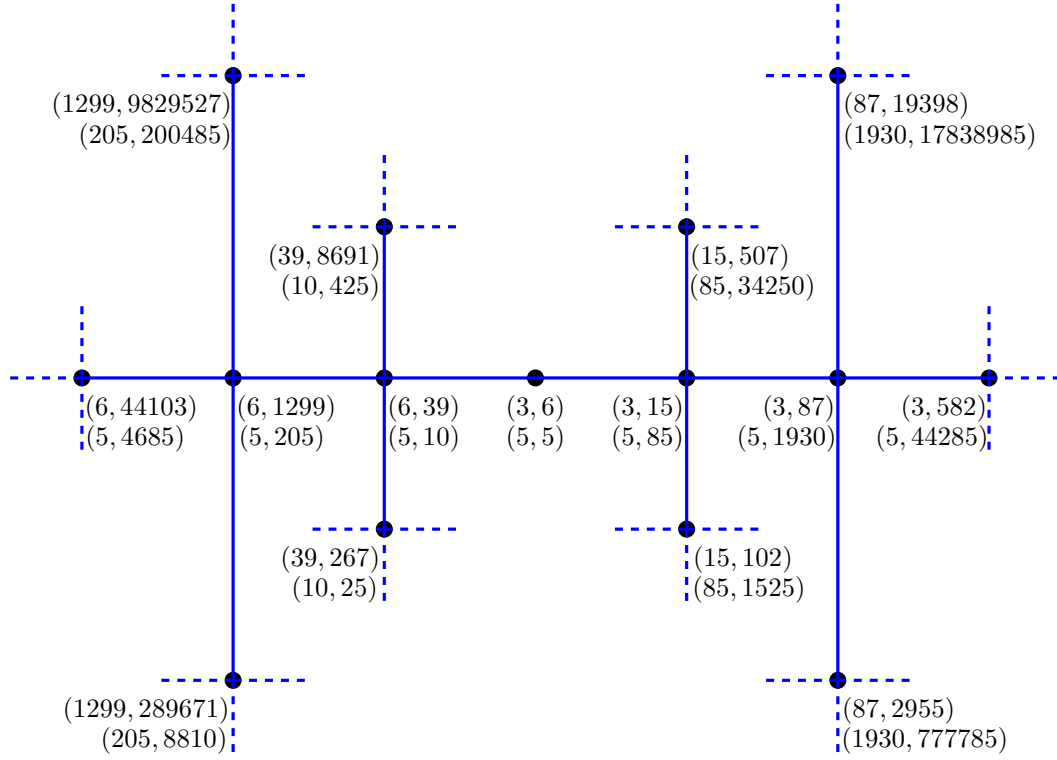
$$f(x) = ux + v + \sqrt{(u^2 - 1)x^2 + 2(u + 1)vx + w},$$

where u, v, w are reals with $u > 1$. Notice that for large x the function $f(x)$ is asymptotic to the linear function $(u + \sqrt{u^2 - 1})x + v$, which is monotonically increasing as $u > 1$. If necessary, we restrict the domain of $f(x)$ to a maximal half-infinite interval $[a, \infty)$ on which $f(x)$ is defined, and has a well defined inverse.

Lemma 5. *Consider the function $f(x) = ux + v + \sqrt{(u^2 - 1)x^2 + 2(u + 1)vx + w}$ with the $u > 1$. Choose a_0 in the domain of $f(x)$ as described above and define a_i recursively by $a_i = f(a_{i-1})$. Then a_i satisfies the 3rd order recurrence relation*

$$a_i = (2u + 1)a_{i-1} - (2u + 1)a_{i-2} + a_{i-3}.$$

Remark 7. It is not difficult to see that if a function $f(x)$ is of the form required by Lemma 5, then for any affine change of variable, $X = rx + s$, the function induced by f on X also has the same form, with the same value of u , while v changes to $rv - us + s$. In particular, as $u > 1$, one can choose $s = rv/(u - 1)$ to make v become 0.

FIGURE 6. The two trees of LEPs with $\gcd(b, c) = 3$ and 5 respectively

Proof of Lemma 5. The required recurrence relation is clearly invariant under affine transformations. So, by the above remark, we may suppose that $v = 0$. Thus

$$f(x) = ux + \sqrt{(u^2 - 1)x^2 + w}.$$

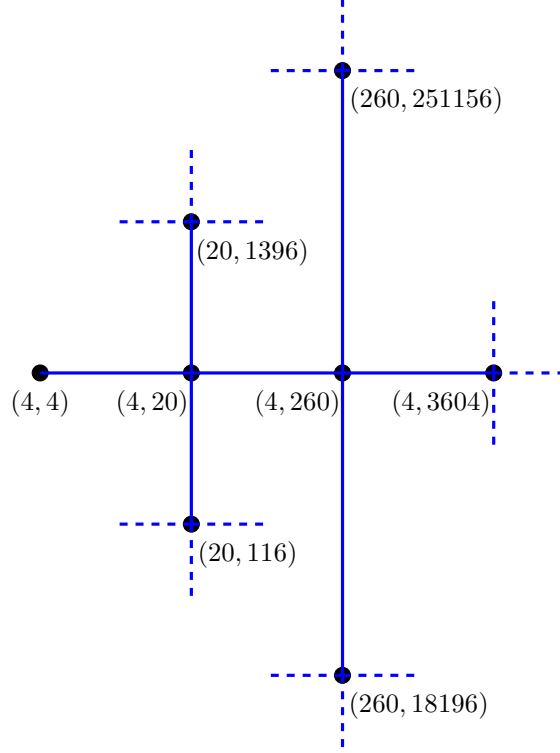
We start by showing that $f^{-1}(x) = ux - \sqrt{(u^2 - 1)x^2 + w}$. Indeed, if $f(y) = x$, then $uy + \sqrt{(u^2 - 1)y^2 + w} = x$. Rearranging and squaring gives $y^2 - 2uxy + x^2 - w = 0$, and solving for y gives $y = ux \pm \sqrt{(u^2 - 1)x^2 + w}$. As $u > 1$, we have $f(x) > x$ for sufficiently large x , so $f^{-1}(x) < x$ and we take the negative root in the formula for y . So $f^{-1}(x)$ is as claimed.

Let $E := f(f(x)) - ((2u + 1)f(x) - (2u + 1)x + f^{-1}(x))$. We claim that $E = 0$. Indeed, let $D = (u^2 - 1)x^2 + w$ and note that

$$f(f(x)) = u^2x + u\sqrt{D} + \sqrt{(u^2 - 1)(ux + \sqrt{D})^2 + w}$$

and

$$(2u + 1)f(x) - (2u + 1)x + f^{-1}(x) = (2u^2 - 1)x + 2u\sqrt{D}.$$

FIGURE 7. The tree of LEPs with $\gcd(b, c) = 4$

So E simplifies to

$$-[(u^2 - 1)x + u\sqrt{D}] + \sqrt{(u^2 - 1)(ux + \sqrt{D})^2 + w}.$$

Note that as $u > 1$, we have $(u^2 - 1)x + u\sqrt{D} > 0$. Hence to show that $E = 0$ we need to see that

$$[(u^2 - 1)x + u\sqrt{D}]^2 - [(u^2 - 1)(ux + \sqrt{D})^2 + w] = 0.$$

Expanding and simplifying the left-hand-side we are left with

$$-(u^2 - 1)x^2 + D - w,$$

which is 0, as claimed, by the definition of D .

As $E = 0$ we have $f(f(x)) = (2u + 1)f(x) - (2u + 1)x + f^{-1}(x)$, for all x in the appropriate domain. Replacing x by $f(x)$ gives

$$f(f(f(x))) = (2u + 1)f(f(x)) - (2u + 1)f(x) + x.$$

Applying this equation to $x = a_0$ gives the required recurrence relation. \square

For the function $\varphi_1(b, c) = (b, \frac{b^2c + b\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2b - c)$ defined in the previous section, let us fix b and look at the resulting function of c :

$$(13) \quad f : c \mapsto \frac{b^2c + b\sqrt{b^2c^2 - 4(b+c)^2}}{2} - 2b - c.$$

One readily verifies that this function f has the form required by Lemma 5, with $u = \frac{b^2-2}{2}, v = -2b, w = -b^4$. Then Lemma 5 gives the recurrence relation

$$(14) \quad c_i = (b^2 - 1)c_{i-1} - (b^2 - 1)c_{i-2} + c_{i-3}.$$

Remark 8. While the function f of (13) is useful in its own right, we now briefly outline a proof of the recurrence relation (14) that doesn't require Lemma 5. As we saw in the previous Section, the map φ_1 leaves b and n unchanged and m is changed to $m' = bn - m$. But this gives $m' > n$ so we have to then interchange m' and n . Thus we can write $n' = bn - m$ and $m' = n$. It follows that under two applications, we have $m'' = n' = bn - m = bm' - m$. Consequently, the value m satisfies the second order recurrence relation $m_{i+1} = bm_i - m_{i-1}$. Similarly, $n_{i+1} = bn_i - n_{i-1}$. According to a well known theorem of E. S. Selmer (see [2]) if one has a second order recurrence relation $a_{i+1} = Aa_i + Ba_{i-1}$ and $x^2 - Ax - B = (x - \alpha)(x - \beta)$ with $\alpha \neq \beta$, then a_i^2 satisfies the third order recurrence relation $a_{i+1}^2 = Ca_i^2 + Da_{i-1}^2 + Ea_{i-2}^2$, where $x^3 - Cx^2 - Dx - E = (x - \alpha^2)(x - \beta^2)(x - \alpha\beta)$. In particular, if $a_{i+1} = ba_i - a_{i-1}$ with $b \geq 3$, then $a_{i+1}^2 = (b^2 - 1)a_i^2 - (b^2 - 1)a_{i-1}^2 + a_{i-2}^2$. So the sequences for m^2 and n^2 both satisfy this third order recurrence relation, and hence so too does $c = k(m^2 + n^2)/b$.

We now examine the horizontal branches of the trees T_3, T_4, T_5 that start at the fundamental solutions and head towards the right. These solutions have constant b and are defined by repeated applications of the map φ_1 , or equivalently, repeated applications of the function f defined in (13).

Example 1. For $b = 3$, with the above notation, (13) gives

$$f : c \mapsto \frac{7}{2}c - 6 + \sqrt{\frac{45}{4}c^2 - 54c - 81}.$$

Setting $c_0 = 6$, corresponding to the fundamental LEP $b = 3, c = 6$, the first few terms of the sequence are 6, 15, 87, 582, 3975. The first 4 of these are visible in Figure 6; they lie on the central horizontal branch to the right of the fundamental solution (3, 6). We employ Lemma 5 with $u = \frac{7}{2}, v = -6, w = -81$. Thus

$$c_i = 8c_{i-1} - 8c_{i-2} + c_{i-3}, \quad \text{for } i \geq 3.$$

By Remark 6, the area of the LEP, $2(b+c)$, is divisible by 9, and it is obviously even. Let $A = 2(b+c)/18 = (3+c)/9$. This is an affine transformation of c , so by Remark 7, the function induced by f on the variable A also has the form required by Lemma 5 with the same value of u . So by Lemma 5,

$$A_i = 8A_{i-1} - 8A_{i-2} + A_{i-3}, \quad \text{for } i \geq 3.$$

Setting $A_0 = 2$, corresponding to the fundamental LEP $b = 3, c = 6$, the first few terms of the sequence are 1, 2, 10, 65, 442. This sequence is related to the sequence A064170, in OEIS [10], which is conjectured to satisfy the same recurrence relation.

Example 2. For $b = 4$, (13) gives

$$f : c \mapsto 7c - 8 + 4\sqrt{3c^2 - 8c - 16}.$$

Setting $c_0 = 4$, corresponding to the fundamental LEP $b = 4, c = 4$, the first few terms of the sequence are 4, 20, 260, 3604, 50180. The first 4 of these are visible in Figure 7; they lie on the central horizontal branch to the right of the fundamental solution $(4, 4)$.

We use Lemma 5 with $u = 7, v = -8, w = -256$. Thus

$$c_i = 15c_{i-1} - 15c_{i-2} + c_{i-3}, \quad \text{for } i \geq 3.$$

Dividing the c -values by 4 we have the sequence 1, 5, 65, 901, 12545, \dots . This is the sequence A103974 in OEIS [10]; it is the sequence of smaller sides a in $(a, a, a+1)$ -integer triangles with integer area.

Further, by Remark 6, the area, $2(b+c)$ is divisible by 16. Let $A = (b+c)/8 = (4+c)/8$. As in the previous example, this is an affine transformation of c , so by Remark 7, the function induced by f on the variable A also has the form required by Lemma 5 with the same value of u . Thus

$$A_i = 15A_{i-1} - 15A_{i-2} + A_{i-3}, \quad \text{for } i \geq 3.$$

Setting $A_0 = 1$, corresponding to the fundamental LEP $b = 4, c = 4$, the first few terms of the sequence are 1, 3, 33, 451, 6273. This is the sequence A011922 in OEIS [10].

Example 3. For $b = 5$, (13) gives

$$f : c \mapsto \frac{23}{2}c - 10 + \sqrt{\frac{21 \cdot 25}{4}c^2 - 250c - 625}.$$

Setting $c_0 = 5$, corresponding to the fundamental LEP $b = 5, c = 5$, the first few terms of the sequence are 5, 85, 1930, 44285, 1016605. The first 4 of these are visible in Figure 6; they lie on the central horizontal branch to the right of the fundamental solution $(5, 5)$.

We use Lemma 5 with $u = \frac{23}{2}, v = -10, w = -625$. Thus

$$c_i = 24c_{i-1} - 24c_{i-2} + c_{i-3}, \quad \text{for } i \geq 3.$$

The area $2(b+c)$ is divisible by 5. Let $A = 2(b+c)/5$. We employ Lemma 5 with the same value of u . Thus

$$A_i = 24A_{i-1} - 24A_{i-2} + A_{i-3}, \quad \text{for } i \geq 3.$$

Setting $A_0 = 4$, corresponding to the fundamental LEP $b = 5, c = 5$, the first few terms of the sequence are 4, 36, 774, 17716, 406644.

6. DIAGONALS, HEIGHTS AND ALTITUDES

Let us first fix some terminology and notation; see Figure 8.

Definition 2. Consider a non-square LEP P . We denote the length of its long (resp. short) diagonal d_l (resp. d_s). The *heights* of P are the distances between opposite sides; we denote the long (resp. short) height h_l (resp. h_s). Each diagonal d partitions P into two congruent triangles T . We will call the distance from d to

the third vertex of T an *altitude* of P . We call the altitude from d_s (resp. d_l) the *long* (resp. *short*) altitude and denote it η_l (resp. η_s).

Remark 9. Our notion of altitude is not universal. Some authors use the term altitude for the concept we have called height.

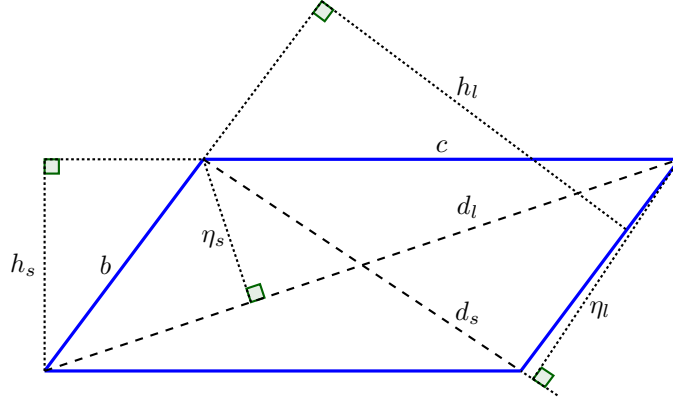


FIGURE 8. Diagonals, heights and altitudes

Lemma 6. *Suppose a LEP P has sides b, c with $b \leq c$. Then*

$$(a) \quad h_l = \frac{2(b+c)}{b}, \quad h_s = \frac{2(b+c)}{c}.$$

$$(b) \quad \eta_l = \frac{2(b+c)}{d_s}, \quad \eta_s = \frac{2(b+c)}{d_l}.$$

Proof. By equability, the area of P is $2(b+c)$, but the area is obviously also bh_l and ch_s . This gives (a). But the area of P is also twice the area of the triangle determined by each diagonal. So the area of P is both $\eta_l d_s$ and $\eta_s d_l$. This gives (b). \square

Remark 10. If h_s is an integer, then $h_s = 2 + \frac{2b}{c}$ by Lemma 6(a), and so either $c = b$ or $c = 2b$. These cases were treated in Corollary 1. Note also that the above lemma also gives $(h_s - 2)(h_l - 2) = 4$. So h_s, h_l are both integers only in the cases $h_s = h_l = 4$ and $h_s = 3, h_l = 6$. The first is the 4×4 square. The second case is the 3×6 rectangle.

Equable parallelograms tend to be very thin in form. The following result gives a sharp statement of this thinness.

Theorem 5. *For every LEP the altitudes satisfy $2 < \eta_l \leq 2\sqrt{5}$ and $2 < \eta_s \leq 2\sqrt{2}$.*

Proof. Suppose a LEP P has sides b, c with $b \leq c$. With the notation used in Section 3, we have $bc = k(m^2 + n^2), b + c = kmn, \sqrt{b^2c^2 - 4(b+c)^2} = k(n^2 - m^2)$, for some positive integer k and relatively prime integers m, n with $m \leq n$. From

Lemma 2, the diagonals are given by

$$\begin{aligned} d_l^2 &= (b^2 + c^2) + \sqrt{4b^2c^2 - 16(b+c)^2} \\ &= k^2m^2n^2 - 2k(m^2 + n^2) + 2k(n^2 - m^2) = m^2(k^2n^2 - 4k), \\ d_s^2 &= (b^2 + c^2) - \sqrt{4b^2c^2 - 16(b+c)^2} \\ &= k^2m^2n^2 - 2k(m^2 + n^2) - 2k(n^2 - m^2) = n^2(k^2m^2 - 4k). \end{aligned}$$

The long altitude is

$$\eta_l = \frac{2(b+c)}{d_s} = \frac{2knm}{n\sqrt{k^2m^2 - 4k}}.$$

Squaring and rearranging gives $km^2 = \frac{4\eta_l^2}{\eta_l^2 - 4}$. As $m \geq 1$ and $k \geq 5$ by Lemma 3, we have $\frac{4\eta_l^2}{\eta_l^2 - 4} \geq 5$ and so $20 \geq \eta_l^2$, that is $\eta_l \leq 2\sqrt{5}$. Furthermore,

$$\eta_l = \frac{2knm}{n\sqrt{k^2m^2 - 4k}} > \frac{2knm}{n\sqrt{k^2m^2}} = 2.$$

Arguing in the same manner one finds $\eta_s > 2$ and $kn^2 = \frac{4\eta_s^2}{\eta_s^2 - 4}$. As we saw in Section 4, for $k = 9$ (resp. 8, resp. 5), the fundamental solution has $(m, n) = (1, 1)$ (resp. $(1, 1)$, resp. $(1, 2)$). So the minimum value of kn^2 is 8. Rearranging $8 \leq \frac{4\eta_s^2}{\eta_s^2 - 4}$ gives $\eta_s^2 \leq 8$, that is $\eta_s \leq 2\sqrt{2}$. \square

Remark 11. The bound $\eta_l = 2\sqrt{5}$ is attained by the rhombus of Figure 3, and the bound $\eta_s = 2\sqrt{2}$ is attained by the 4×4 square.

We complete this section with some results that use either the above theorem, or ideas from its proof.

Proposition 3. *For every LEP, the two diagonals and the two altitudes are irrational.*

Proof. As we saw in the above proof, $d_l^2 = m^2(k^2n^2 - 4k)$. So the longer diagonal has integer length only when $k^2n^2 - 4k$ is a square. But $k = 5, 8$ or 9 , by Lemma 3. For $k = 5$ (resp. 8), the expression $k^2n^2 - 4k$ is divisible by 5 (resp. $4k = 2^5$) but not by 5^2 (resp. 2^6) and is hence never a square. For $k = 9$, one has $k^2n^2 - 4k = 3^2(9n^2 - 4)$, and $9n^2 - 4$ is not a square for any integer n . Thus the long diagonal is never an integer, and hence by Remark 1, the long diagonal is irrational for every LEP. Similarly, the short diagonal is always irrational as $k^2m^2 - 4k$ is never a square. Consequently, by Lemma 6(b), the altitudes η_l, η_s are also irrational for every LEP. \square

The following immediate corollaries of Proposition 3 are each generalisations of Proposition 2.

Corollary 2. *No LEP can be partitioned into the union of two Heronian triangles.*

Corollary 3. *There exists no LEP having a pair of opposite vertices with the same x -coordinate, or the same y -coordinate.*

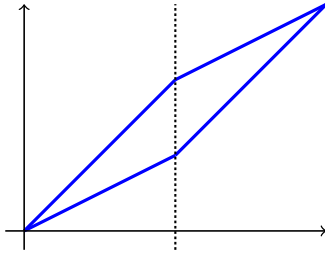


FIGURE 9. A parallelogram that isn't a LEP

Proposition 4. *If a LEP P contains the origin O and vertices O, A, B, C in cyclic order with long diagonal $OB = d_l$ belonging to the 1st quadrant, then all four vertices belong to the first quadrant.*

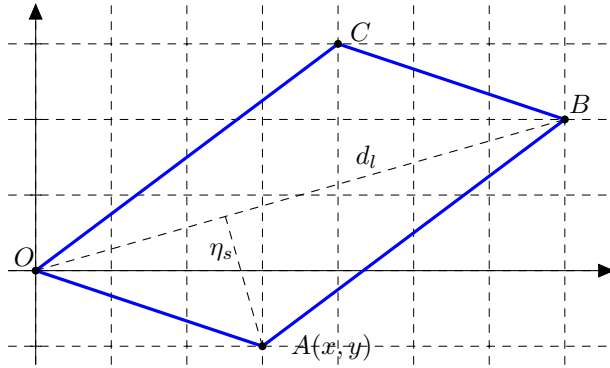


FIGURE 10. A LEP configuration that can't exist

Proof. By applying if necessary a symmetry along the $x = y$ axis we suppose that A lies in the 4th quadrant as in Figure 10. Since the diagonals are irrational the vertex B can't be on the x -axis. Moreover, as P has an acute angle at O , the foot of the altitude from A to d_l also lies in the first quadrant. Hence, letting $A = (x, y)$, the distance from A to d_l is greater than $|y|$. Thus, by Theorem 5, $|y| < \eta_s \leq 2\sqrt{2}$. Hence, as y is an integer, $|y|$ is 1 or 2. Let OA have length b . Then b^2 is $x^2 + 1$ or $x^2 + 2$. But this is impossible for $x > 0$. \square

Corollary 4. *Every LEP is congruent to a LEP in the 1st quadrant with a vertex at the origin.*

7. PYTHAGOREAN EQUABLE PARALLELOGRAMS

In general there are two ways in which parallelograms can be circumscribed by a rectangle so that the rectangle and parallelogram share a common diagonal; the rectangle may have sides extending the long sides of the parallelogram, or the short sides of the parallelogram; see Figure 11.

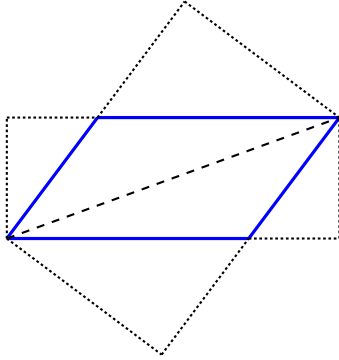


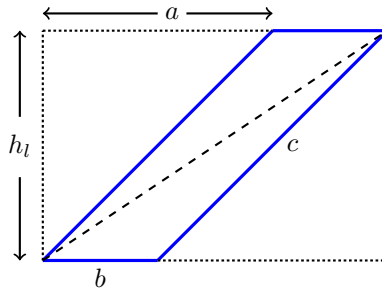
FIGURE 11. Rectangles circumscribing a parallelogram

Definition 3. A LEP is said to be *Pythagorean* if it is circumscribed by a rectangle having integer side lengths so that the rectangle and parallelogram share a common diagonal as in Figure 11.

This terminology is justified by the equivalent condition (b) in the following result.

Proposition 5. Consider a LEP P having sides b, c with $b \leq c$. The following conditions are equivalent:

- (a) P is Pythagorean,
- (b) P is circumscribed by a rectangle R such that the complement of P in R is the union of two Pythagorean triangles,
- (c) b divides $2c$,
- (d) P can be drawn as a LEP with a horizontal pairs of sides.

FIGURE 12. Integer $h_l \implies$ integer a

Proof of Proposition 5. Suppose that P is a LEP with vertices O, A, B, C , in cyclic order, and that P has sides b, c with $b \leq c$.

(a) \implies (d). Suppose that P is Pythagorean and that the distance between sides OA and BC is an integer, h say. Rotate and translate P so that it has the

side OA along the positive x -axis, with the vertex O at the origin. We want to verify that moved in this way, P is still a LEP. Since P has integer side lengths, A has integer coordinates. By assumption, B, C have integer y -coordinate, equal to h . From the Pythagorean hypothesis, B has integer x -coordinate, z say. Then x -coordinate of C is z minus the x -coordinate of A . So P is a LEP.

(d) \implies (c). If a LEP P has a horizontal pairs of sides, then the fact that the coordinates of the vertices are integers implies that the distance h between the horizontal sides is an integer. So either h_l or h_s is an integer. If h_s is an integer, then as $h_s = 2 + \frac{2b}{c}$ by Lemma 6(a), and $b \leq c$, we have either $c = b$ or $c = 2b$. In either case, by Lemma 6(a) again, $h_l = 2 + \frac{2c}{b}$ is also an integer. Thus b divides $2c$.

(c) \implies (b). Since $h_l = 2 + \frac{2c}{b}$, hence h_l is an integer. Consider the rectangle R that circumscribes P , sharing a common diagonal with P and having sides that are extensions of the sides of P of length b , as in Figure 12 (but not necessarily with the sides b horizontal). So R has sides h_l and $b + a$, for some a , and we are required to show that a is an integer. By Lemma 6(a), we have

$$a^2 = c^2 - h_l^2 = c^2 - \frac{4(b+c)^2}{b^2} = \frac{b^2c^2 - 4(b+c)^2}{b^2},$$

so $a = \frac{\sqrt{b^2c^2 - 4(b+c)^2}}{b}$, which is a rational by Theorem 1. Since $h_l = 2 + \frac{2c}{b}$ is an integer, $2c = bi$ for some integer i . Then

$$a^2 = \frac{b^2c^2 - 4(b+c)^2}{b^2} = c^2 - (2+i)^2,$$

which is an integer. So a is rational and the square root of an integer. Hence a is an integer.

(b) \implies (a). This is immediate. \square

Remark 12. The smallest non-Pythagorean LEP has area 180 and sides 25 and 65; it can be constructed in the first quadrant with vertices $(0, 0)$, $(25, 60)$, $(32, 84)$, $(7, 24)$.

Proof of Theorem 3. Consider a Pythagorean LEP P with sides b, c . By Proposition 5, b divides $2c$, and by Theorem 2, $\gcd(b, c) = 3, 4$ or 5 . If b is odd, then b divides c , so $b = \gcd(b, c)$ and b is 3 or 5. If b is even, say $b = 2b'$, then b' divides c so b' divides $\gcd(b, c)$. Thus b' is 2, 3, 4 or 5 and thus b is 4, 6, 8 or 10. But as observed in Remark 6, if $\gcd(b, c) = 4$ neither b nor c is divisible by 8. So b is 3, 4, 5, 6 or 10, as claim in Theorem 3. It remains to exhibit a family of such Pythagorean LEPs in each case.

In the notation of previous sections, $bc = k(m^2 + n^2)$, $b + c = kmn$ for positive integers k, m, n where m, n are relatively prime with $m < n$. For convenience, let us restate (8):

$$km^2 + kn^2 + b^2 = kbm n.$$

Moreover, by Lemma 3, k is either 5, 8 or 9, corresponding to $\gcd(b, c) = 3, 4, 5$ respectively.

Note that for $b = 3$ we have $k = 9$ and (8) gives $m^2 + n^2 + 1 = 3mn$. Setting $x = n - m$, $y = m + n$ we obtain the Pell-like equation $y^2 - 5x^2 = 4$. Conversely, note that if (x, y) is a solution to this equation, then x, y necessarily have the same

parity and we can set $m = \frac{y-x}{2}, n = \frac{y+x}{2}$ to obtain an integer solution (m, n) to $m^2 + n^2 + 1 = 3mn$. Then $c = \frac{k}{b}(m^2 + n^2) = \frac{3(x^2+y^2)}{2}$.

For $b = 4$ we have $k = 8$ and (8) gives $m^2 + n^2 + 2 = 4mn$. Note then that m, n must have same parity. Setting $2x = n - m, 2y = n + m$ we obtain the Pell equation $y^2 - 3x^2 = 1$. Conversely, note that if (x, y) is a solution to this equation, then we can set $m = y - x, n = y + x$ to obtain an integer solution (m, n) to $m^2 + n^2 + 2 = 4mn$. Then $c = \frac{k}{b}(m^2 + n^2) = 4(x^2 + y^2)$.

For $b = 5$ we have $k = 5$ and (8) gives $m^2 + n^2 + 5 = 5mn$. Setting $x = n - m, y = m + n$ we obtain the Pell-like equation $3y^2 - 7x^2 = 20$. Conversely, note that if (x, y) is a solution to this equation, then x, y necessarily have the same parity and we can set $m = \frac{y-x}{2}, n = \frac{y+x}{2}$ to obtain an integer solution (m, n) to $m^2 + n^2 + 5 = 5mn$. Then $c = \frac{k}{b}(m^2 + n^2) = \frac{x^2+y^2}{2}$.

Similarly, for $b = 6$ (resp. $b = 10$), we have $k = 9$ (resp. $k = 5$) and (8) gives $m^2 + n^2 + 4 = 6mn$ (resp. $m^2 + n^2 + 20 = 10mn$). Note that m, n necessarily have the same parity and we can set $x = \frac{n-m}{2}, y = \frac{m+n}{2}$. This gives the Pell (resp. Pell-like) equation $y^2 - 2x^2 = 1$ (resp. $2y^2 - 3x^2 = 5$). Conversely, note that if (x, y) is a solution to this equation, then we can set $m = y - x, n = y + x$ to obtain an integer solution (m, n) to $m^2 + n^2 + 4 = 6mn$ (resp. $m^2 + n^2 + 20 = 10mn$). Using $c = \frac{k}{b}(m^2 + n^2)$, for $b = 6$ we have $c = 3(x^2 + y^2)$ and for $b = 10$ we have $c = x^2 + y^2$. \square

We complete this section by connecting the above solution families F1-F5 to the material in the earlier Sections. In particular, we locate the Pythagorean equable parallelograms in the trees of LEPs of Figure 6 and 7.

F1: $b = 3$. The solutions (x, y) to the Pell-like equation $y^2 - 5x^2 = 4$ are well known to be (L_{2i}, F_{2i}) , starting at $(F_0, L_0) = (0, 2), (F_2, L_2) = (1, 3)$, where F stands for the Fibonacci numbers and L for the Lucas numbers. Both L_{2i} and F_{2i} satisfy the recurrence relation $a_i = 3a_{i-1} - a_{i-2}$, with different initial conditions. So the first few solutions of equation $y^2 - 5x^2 = 4$ are: $(0, 2), (1, 3), (3, 7), (8, 18), (21, 47)$. The resulting c -values, given by $c = \frac{3(x^2+y^2)}{2}$, are: 6, 15, 87, 582, 3975. We saw the corresponding LEPs $(3, c)$ previously in Example 1; in Figure 6 they lie on the central horizontal branch to the right of the fundamental solution $(3, 6)$. As we saw, these c -values are generated recursively by the formula $f(c) = \frac{7}{2}c - 6 + \sqrt{\frac{45}{4}c^2 - 54c - 81}$, or alternatively, using the recurrence relation $c_i = 8c_{i-1} - 8c_{i-2} + c_{i-3}$. Notice that by Theorem 1, these c values are precisely those numbers $c = 3i$ for which $5i^2 - 8i - 4$ is a square.

F2: $b = 4$. The solutions (x, y) to Pell's equation $y^2 - 3x^2 = 1$ are well known to be given by the recurrence relation $a_i = 4a_{i-1} - a_{i-2}$. See OEIS entries A001075 and A001353 [10]. The first few solutions of $y^2 - 3x^2 = 1$ are: $(0, 1), (1, 2), (4, 7), (15, 26), (56, 97)$. The resulting c -values, given by $c = 4(x^2+y^2)$, are: 4, 20, 260, 3604, 50180. We saw the corresponding LEPs $(4, c)$ in Example 2; in Figure 7 they lie on the central horizontal branch to the right of the fundamental solution $(4, 4)$. The c -values can be generated recursively by the formula $f(c) = 7c - 8 + 4\sqrt{3c^2 - 8c - 16}$, or by the recurrence relation $c_i = 15c_{i-1} - 15c_{i-2} + c_{i-3}$.

Notice that by Theorem 1, these c values are precisely those numbers $c = 4i$ for which $3i^2 - 2i - 1$ is a square.

F3: $b = 5$. The Pell-like equation $3y^2 - 7x^2 = 20$ is less common. Its solutions (x, y) are given by the recurrence relation $a_i = 5a_{i-2} - a_{i-4}$. The first few solutions are: $(1, 3), (2, 4), (7, 11), (11, 17), (34, 52), (53, 81), (163, 249)$. The resulting c -values, given by $c = \frac{x^2+y^2}{2}$, are: 5, 10, 85, 205, 1930, 4685, 44285. The position of the corresponding LEPs $(5, c)$ is more complicated than what we saw for $b = 3$ and $b = 4$. Every second solution, starting at $(5, 5)$, appeared in Example 3; in Figure 6 they lie on the central horizontal branch to the right of the fundamental solution $(5, 5)$. These c -values can be generated recursively by the formula

$$f(c) = \frac{23}{2}c - 10 + \sqrt{\frac{21 \cdot 25}{4}c^2 - 250c - 625},$$

or by the recurrence relation $c_i = 24c_{i-1} - 24c_{i-2} + c_{i-3}$. The other solutions are on the central horizontal branch of Figure 6 to the left of the fundamental solution $(5, 5)$. These c -values are generated recursively by the same formula (and recurrence relation) but starting at $c = 10$.

Note that in the proof of Theorem 3 the equation $3y^2 - 7x^2 = 20$ was derived from $m^2 + n^2 + 5 = 5mn$. This latter equation is well known; indeed, it is easy to see that for the solutions m, n , the first component comprise those numbers m for which $21m^2 - 20$ is a square. See entry A237254 in [10]. By Theorem 1, the c values for $b = 5$ are precisely those numbers $c = 5i$ for which $21i^2 - 8i - 4$ is a square.

F4: $b = 6$. The solutions (x, y) to Pell's equation $y^2 - 2x^2 = 1$ are well known to be given by the recurrence relation $a_i = 6a_{i-1} - a_{i-2}$. See OEIS entries A001541, A001542 [10]. The first few solutions are: $(0, 1), (2, 3), (12, 17), (70, 99), (408, 577)$. The resulting c -values, given by $c = 3(x^2 + y^2)$, are: 3, 39, 1299, 44103, 1498179. The corresponding LEPs $(6, c)$ occur on the central horizontal branch of Figure 6, to the left of the fundamental solution $(3, 6)$. To generate the c -values of the solutions on this branch we can use Equation 13 with $b = 6$. These c -values can be generated recursively by the formula

$$f(c) = 17c - 12 + 6\sqrt{9c^2 - (6+c)^2},$$

and by the recurrence relation $c_i = 35c_{i-1} - 35c_{i-2} + c_{i-3}$. By Theorem 1, these c values are precisely those numbers $c = 3i$ for which $2i^2 - i - 1$ is a square.

F5: $b = 10$. Like the $b = 5$ case, the Pell-like equation $2y^2 - 3x^2 = 5$ is not very common. Its solutions (x, y) are given by the recurrence relation $a_i = 10a_{i-2} - a_{i-4}$. The first few solutions are: $(1, 2), (3, 4), (13, 16), (31, 38), (129, 158), (307, 376)$. The resulting c -values, given by $c = x^2 + y^2$, are: 5, 25, 425, 2405, 41605, 235625. The corresponding LEPs $(10, c)$ occur on the first vertical branch to the left of the fundamental solution $(5, 5)$ in Figure 6. Every second solution, starting at $(10, 5)$ appears above the central horizontal axis. To generate these c -values we can use Equation 13 with $b = 10$. This gives the equation

$$f(c) = 49c - 20 + 10\sqrt{25c^2 - (10+c)^2},$$

and by (14), the recurrence relation $c_i = 99c_{i-1} - 99c_{i-2} + c_{i-3}$. The other solutions appears below the central horizontal axis. Their c -values are generated recursively by the same formula (and recurrence relation) but starting at $c = 25$.

Note that in the proof of Theorem 3 the equation $2y^2 - 3x^2 = 5$ was derived from $m^2 + n^2 + 20 = 10mn$. This latter equation is well known; indeed, it is easy to see that for the solutions m, n , the first component comprise those numbers m for which $6m^2 - 5$ is a square. See entry A080806 in [10]. By Theorem 1, the c values for $b = 10$ are precisely those numbers $c = 5i$ for which $6i^2 - i - 1$ is a square.

Remark 13. The Pell-like equations in families F1 – F5 can all be put in the following form:

$$(15) \quad ay^2 - (a+4)x^2 = d,$$

where a, d are integers and d is even if a is odd. Table 1 gives the values of a, d for the five families.

A direct calculation shows that if (x, y) is a solution to (15), then another solution is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a+2 & a \\ a+4 & a+2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \frac{a(x+y)}{2} \\ 2x + y + \frac{a(x+y)}{2} \end{pmatrix}.$$

This was proved by Réalis for $d = \pm 4$ [8] (see also [3, p. 407]), but it holds in the general case. (Note that if a is odd, then from (15), as d is even, the numbers x, y must have the same parity, so x', y' are integers). Furthermore, provided (15) has one solution, (x_0, y_0) say, this process gives a sequence of solutions (x_i, y_i) , and it is easy to verify that it satisfies the following recurrence relation:

$$(x_i, y_i) = (a+2)(x_{i-1}, y_{i-1}) - (x_{i-2}, y_{i-2}).$$

Note that (15) does not have a solution for all a and d . For example, for $a = 3, d = 4$, the equation is $3y^2 - 7x^2 = 4$. Modulo 3 this is $-x^2 \equiv 1$, which has no solution.

Family	F1	F2	F3	F4	F5
a	1	2	3	4	8
d	4	2	20	4	20

TABLE 1.

APPENDIX: THE EQUABLE TRIANGLES THEOREM

The Equable Triangles Theorem says that there are only 5 equable triangles. For the early history of the theorem, Dickson [3, pp. 195, 199] cites Whitworth and Bridle in *Math. Quest. Educational Times* 5, 1904, 54–56, 62–63, but we have been unable to locate this/these works. Equable triangles have been investigated in several works [5, 6, 11, 13]. However the only proof of the Equable Triangles Theorem we have seen in the literature is by Arthur H. Foss [4]. In this appendix we supply a different proof.

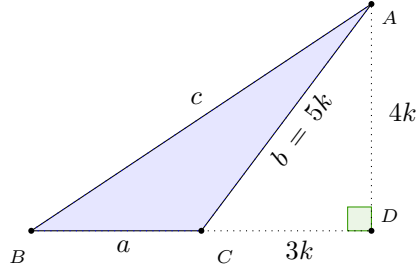


FIGURE 13. Equable triangles

For equable triangles with sides a, b, c , Heron's formula gives

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 16(a + b + c).$$

Let $u = -a + b + c, v = a - b + c, w = a + b - c$, so that $a = \frac{v+w}{2}, b = \frac{u+w}{2}, c = \frac{u+v}{2}$. Then our equation is

$$(16) \quad uvw = 16(u + v + w),$$

and we look for solutions u, v, w , all of the same parity. Further, we may assume $u \leq v \leq w$. Note that, since u, v, w , have the same parity, so from (16), u, v, w are necessarily even. Let $u = 2x, v = 2y, w = 2z$, so $a = y + z, b = x + z, c = x + y$. Then $xyz = 4(x + y + z)$. Thus

$$y \leq z = \frac{4(x + y)}{xy - 4},$$

so $xy^2 - 8y - 4x \leq 0$. Hence

$$x \leq y \leq \frac{4 + 2\sqrt{4 + x^2}}{x}$$

so $x^2 \leq 4 + 2\sqrt{4 + x^2}$. Hence $(x^2 - 4)^2 \leq 4(4 + x^2)$. Thus $x^4 - 12x^2 \leq 0$, which gives $x \leq 3$. Then

$$y \leq \frac{4 + 2\sqrt{4 + x^2}}{x} \leq 4 + 2\sqrt{4 + 1},$$

since the function $\frac{4 + 2\sqrt{4 + x^2}}{x}$ is decreasing for positive x . So, as y is an integer, $y \leq 8$. Then, considering the values $x \leq 3, y \leq 8$ and $z = \frac{4(x+y)}{xy-4}$, we find the following integer values for x, y, z :

$$1, 5, 24 \quad 1, 6, 14 \quad 1, 8, 9 \quad 2, 3, 10 \quad 2, 4, 6,$$

which give the values for a, b, c :

$$6, 25, 29 \quad 7, 15, 20 \quad 9, 10, 17 \quad 5, 12, 13 \quad 6, 8, 10.$$

This completes the proof. Note that the last two are Pythagorean triples, while intriguingly, for the other three cases, the corresponding triangle is the complement of a Pythagorean triangle of the form $3k, 4k, 5k$ in a larger Pythagorean triangle; see Figure 13. The three cases correspond to the values $k = 2, 3, 5$.

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