

# Counting Standard Young Tableaux With Restricted Runs

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*Dedicated to super-enumerators Ian Goulden and David Jackson<sup>1</sup>*

**Abstract.** The number of standard Young tableaux whose shape is a  $k$  by  $n$  rectangle is famously  $\frac{(nk)!0!1!\dots(k-1)!}{(n+k-1)!(n+k-2)!\dots n!}$ , implying that for each specific  $k$ , that sequence satisfies a linear recurrence equation with polynomial coefficients of the **first** order. But what about counting standard Young tableaux where certain “run lengths” are forbidden? Then things seem to get much more complicated. In this tribute to the legendary enumerative pair *Goulden & Jackson* we investigate these intriguing sequences, and conjecture that if the number of rows is larger than two, then these sequences are generally **not**  $P$ -recursive. On the positive side, we conjecture that these sequences have ‘nice’ asymptotic behavior. We pledge donations to the OEIS in honor of the first solvers of these conjectures.

## Preface

Some combinatorial families are easy to count, for example the number of subsets of an  $n$ -element set, that can be computed in *logarithmic* time (in base 2). Also easy is counting the number of permutations, that can be computed in linear-time. Then you have the really hard ones, for example the number of  $n \times n$  Latin squares and the number of self-avoiding walks of length  $n$ , for which we will probably never know the exact value of the 1000-th term.

Both the number of subsets of an  $n$ -element set,  $2^n$ , and the number of permutations,  $n!$ , as well as the famous Catalan numbers  $(2n)!/(n!(n+1)!)$  (OEIS sequence A108) are said to have a **closed-form** formula. They satisfy a *first-order* linear recurrence equation with polynomial coefficients

$$a(n+1) - 2a(n) = 0 \quad , \quad a(n+1) - na(n) = 0 \quad , \quad (n+2)a(n+1) - 2(2n+1)a(n) = 0 \quad .$$

(Note that the first equation is even better, it is *constant coefficients*).

Many natural families satisfy the next-best thing to being closed-form, they satisfy a **linear-recurrence equation** with **polynomial** coefficients, but not necessarily of first order. Such sequences, called  $P$ -recursive, or *holonomic* (see [KP]), satisfy an equation of the form

$$\sum_{i=0}^L p_i(n)a(n+i) = 0 \quad ,$$

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for some positive integer  $L$  and some **polynomials** in  $n, p_0(n), \dots, p_L(n)$ .

The most famous such sequences that are **not** closed-forms are the Fibonacci numbers,  $F_n$  (OEIS sequence A45), the number of involutions of an  $n$ -element set (permutations that are equal to their inverse)  $w_n$  (OEIS sequence A85), and the Motzkin numbers,  $M_n$ , the number of words of length  $n$  in the alphabet  $\{0, -1, 1\}$  that add-up to 0 and all whose partial sums are non-negative (OEIS sequence A1006). They satisfy, respectively, the recurrences

$$a(n+2) - a(n+1) - a(n) = 0 \quad , \quad a(n+2) - a(n+1) - (n+1)a(n) = 0 \quad ,$$

$$(n+4)a(n+2) - (2n+5)a(n+1) - (3n+3)a(n) = 0 \quad .$$

Given a natural combinatorial family, parameterized by  $n$ , it is very interesting to know whether or not the sequence of integers that enumerates it happens to be  $P$ -recursive. This is interesting both conceptually and computationally, since a linear recurrence makes it easy to compute many terms, as well as deriving the asymptotics. Sometimes proving that a given sequence, or family of sequences, is (are)  $P$ -recursive is highly non-trivial, see for example [GJR], [GJ], and [Ge2].

In this modest tribute to Ian Goulden and David Jackson we will raise the question whether a certain very *natural* family of combinatorial sequences is  $P$ -recursive, and give ample computational evidence that generally they are, *probably*, **not**.

## Maple packages

This article is accompanied by the Maple packages `YoungT.txt` and `Tableaux3R.txt`, available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cyt.html> .

## Counting Standard Young Tableaux with Restricted Runs

Recall that a partition of a positive integer  $m$  is a weakly-decreasing list of positive integers  $\lambda = [\lambda_1, \dots, \lambda_k]$ , that add-up to  $m$ , also called a *shape*, and a *standard Young tableau* of shape  $\lambda$  is a left-justified array of  $k$  rows with  $\lambda_i$  boxes in the  $i$ -th row, where the integers  $\{1, \dots, m\}$  are filled in the boxes so that both rows and columns are increasing. For example, here is a standard Young tableau of shape  $[3, 3, 2]$

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 6 & 8 & \end{array} ,$$

and here is one of shape  $[5, 5, 4]$

$$\begin{array}{cccccc} 1 & 3 & 4 & 6 & 7 & \\ 2 & 5 & 8 & 9 & 11 & . \\ 10 & 12 & 13 & 14 & & \end{array}$$

The number of standard Young tableaux of shape  $[\lambda_1, \dots, \lambda_k]$  is famously given by the **Young-Frobenius** formula (equivalent to the *hook-length formula*)

$$\frac{(\lambda_1 + \dots + \lambda_k)!}{(\lambda_1 + k - 1)! \cdots \lambda_k!} \cdot \prod_{1 < i < j \leq k} (\lambda_i - \lambda_j + j - i) \quad .$$

In particular, setting  $\lambda_1 = \dots = \lambda_k = n$ , we get that the number of standard Young tableaux whose shape is a  $k$  by  $n$  rectangle is given by

$$\frac{(nk)! 0! 1! \cdots (k-1)!}{(n+k-1)!(n+k-2)! \cdots n!} \quad .$$

It follows that for each **fixed**  $k$ , this sequence, in  $n$ , is  $P$ -recursive, in fact it even satisfies a **first-order** linear recurrence.

Let's define a **run** in a standard Young tableau to be a **maximal** string of **consecutive integers**. For example, in the following tableau of shape  $[5, 5, 5]$

$$\begin{array}{ccccc} 1 & 3 & 4 & 6 & 7 \\ 2 & 5 & 8 & 9 & 11 \\ 10 & 12 & 13 & 14 & 15 \end{array} \quad ,$$

we have

- First row: one run of length 1, (1), and two runs of length 2 (34 and 67) ;
- Second row: three runs of length 1, (2, 5 and 11) and one run of length 2 (89) ;
- Third row: one run of length 1, (10) and one run of length 4 (12, 13, 14, 15) .

We are interested in the following general question. Fix  $k > 1$ . Given *arbitrary* finite sets of positive integers  $A_1, \dots, A_k$ , (or infinite arithmetical progressions), compute the following integer sequence, let's call it

$$G_{A_1, \dots, A_k}(n) \quad ,$$

defined as the number of standard Young tableaux of shape  $(n, \dots, n)$  such that in each row  $i$ ,  $1 \leq i \leq k$ , none of the runs belongs to  $A_i$ . Of course if all the  $A_i$ 's are the empty set, we are back to counting unrestricted standard Young tableaux, for which there is a nice closed-form formula, and of course it is  $P$ -recursive.

The case of **two** rows can be shown [EZ] to always give  $P$ -recursive sequences (in fact even something stronger is true: the generating functions are algebraic formal power series). This is the case since  $2 \times n$  standard Young tableaux are in easy bijection with *Dyck paths* of semi-length  $n$ .

More generally, as is well-known, and immediate to see, standard Young tableaux are in easy bijection with **lattice paths**. A  $k$ -rowed standard Young tableau of shape  $[\lambda_1, \dots, \lambda_k]$  corresponds

to a lattice path in the  $k$ -dimensional hyper-cubical lattice, from the origin  $[0, \dots, 0]$  to the point  $[\lambda_1, \dots, \lambda_k]$ , with **unit positive steps**  $\mathbf{e}_i := [0^{i-1}, 1, 0^{k-i}]$ , that **always** stay in the region  $x_1 \geq x_2 \geq \dots \geq x_k$ . Given a standard Young tableau, the corresponding path is obtained by executing the step  $\mathbf{e}_i$  at the  $m$ -th step, if  $m$  is located at the  $i$ -th row. So  $G_{A_1, \dots, A_k}(n)$  is also the number of  $k$ -dimensional lattice paths from the origin to  $[n, \dots, n]$ , always staying in  $x_1 \geq \dots \geq x_k \geq 0$ , such that the walker never has a run-length parallel to the  $i$ -th axis that belongs to the set  $A_i$ .

The same question makes sense for general walks, not necessarily those confined to  $x_1 \geq x_2 \geq \dots \geq x_k$ . It turns out that for this analogous question the sequences are **always**  $P$ -recursive, as we will now show.

### Counting Lattice Walks with Restricted Runs

In order to motivate the general case, let's first give yet another proof, a bit more complicated than the usual one, of the very easy fact that the generating function for the number of **all** walks, without restrictions, is given by the generating function

$$\frac{1}{1 - x_1 - \dots - x_k} .$$

Every walk corresponds to a **word** in the alphabet  $\{1, \dots, k\}$ , indicating which  $\mathbf{e}_i$  it went through. For example, the walk

$$[0, 0, 0] \rightarrow [1, 0, 0] \rightarrow [1, 0, 1] \rightarrow [1, 1, 1] \rightarrow [1, 2, 1] \rightarrow [1, 2, 2] \rightarrow [1, 3, 2]$$

corresponds to the word

$$132232 .$$

Given a word in  $\{1, \dots, k\}$ , we can write it in **frequency notation**  $b_1^{r_1} \dots b_k^{r_k}$ , where  $b_{j+1} \neq b_j$ , and  $r_j \geq 1$ . for example, the above word 132232 is abbreviated  $1^1 3^1 2^2 3^1 2^1$ , and the word 1113322211 is written  $1^3 3^2 2^3 1^2$ .

Let  $F_i = F_i(x_1, \dots, x_k)$  be the **weight-enumerator** of all words that end with the letter  $i$ . Then obviously, for  $i = 1, \dots, k$

$$F_i = \frac{x_i}{1 - x_i} \left( 1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} F_j \right) .$$

This is a system of  $k$  linear equations with  $k$  unknowns  $F_1, \dots, F_k$ , whose solution is easily seen to be given explicitly by

$$F_i = \frac{x_i}{1 - x_1 - \dots - x_k} .$$

Finally the full generating function,  $F$ , is gotten by adding the weight of the **empty** word, 1, to the sum of the  $F_i$ 's, getting

$$F = 1 + \sum_{i=1}^k F_i ,$$

that implies the deep theorem

$$F = \frac{1}{1 - x_1 - \dots - x_k} \quad .$$

Note, in particular that the  $F_i$ , (and  $F$ ) are **rational functions** of the variables  $x_1, \dots, x_k$ .

To handle the restricted case, to find the weight-enumerator of all words in  $1^{\lambda_1} \dots k^{\lambda_k}$  such that when written in frequency notation  $b_1^{r_1} \dots b_m^{r_m}$  we have that if  $b_\alpha = i$  then  $r_\alpha \notin A_i$  (i.e. runs in the  $\mathbf{e}_i$  direction can't be of a length that belongs to  $A_i$ ), we have the modified system:

$$F_i = \left( \frac{x_i}{1 - x_i} - \sum_{\beta \in A_i} x_i^\beta \right) \left( 1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} F_j \right) \quad .$$

This is a system of  $k$  linear equations in the  $k$  unknowns  $F_1, \dots, F_k$ , with coefficients that are **rational functions** in  $x_1, \dots, x_k$ . Hence, by Cramer's rule, the  $F_i$  are all rational functions of  $x_1, \dots, x_k$ , and hence so is  $F = 1 + \sum_{i=1}^k F_i$ .

Our sequence of interest is the sequence of coefficients of the **diagonal** of this rational function. Since the diagonal of any formal power series that is a rational function is  $D$ -finite (see [Gel][Z1][L][Z2]), it follows that the sequence itself is  $P$ -recursive.

## Back to Tableaux

We strongly doubt that the multi-variable generating functions for restricted Young tableaux (alias restricted walks confined to  $x_1 \geq \dots \geq x_k$ ) are rational. In order to explore these sequences, we need to generate as many terms as possible. Here is how to do it. Let us fix  $R_1, \dots, R_k$  and denote by  $g(\lambda_1, \dots, \lambda_k)$  the number of standard Young tableaux of shape  $[\lambda_1, \dots, \lambda_k]$  with no runs in the  $i$ -th row that belong to  $R_i$ , or equivalently the number of walks in  $x_1 \geq \dots \geq x_k \geq 0$ , from the origin to the point  $[\lambda_1, \dots, \lambda_k]$  with no run-length in the  $x_i$ -direction that belongs to  $R_i$  (for  $i = 1, \dots, k$ ).

In order to compute  $g(\lambda_1, \dots, \lambda_k)$ , we need the more refined quantities ( $1 \leq i \leq k$ )  $g^{(i)}(\lambda_1, \dots, \lambda_k)$ , that enumerate those walks that end with a step in the  $x_i$ -direction.

We have the **dynamic programming** recurrences ( $1 \leq i \leq k$ )

$$g^{(i)}(\lambda_1, \dots, \lambda_k) = \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \sum_{\substack{1 \leq r \leq \lambda_i \\ r \notin A_i}} g^{(j)}(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - r, \lambda_{i+1}, \dots, \lambda_k) \quad ,$$

with the obvious initial conditions, and the **boundary conditions**

$$g^{(i)}(\lambda_1, \dots, \lambda_k) = 0 \quad ,$$

whenever  $\lambda_1 < \lambda_2$  or  $\lambda_2 < \lambda_3, \dots$ , or  $\lambda_{k-1} < \lambda_k$ , or  $\lambda_k < 0$ . Finally

$$g(\lambda_1, \dots, \lambda_k) = \sum_{i=1}^k g^{(i)}(\lambda_1, \dots, \lambda_k) \quad .$$

This is all implemented in the Maple package `YoungT.txt` mentioned above.

## Two Case Studies

In spite of the fact that we were unable to think of a good reason why these sequences should be  $P$ -recursive, we still hoped that they would be for a non-obvious reason. We focused on two special cases to generate as many terms as we could.

- $G(n)$ , the number of standard Young tableaux of shape  $[n, n, n]$  where each run, in each of the three rows, must have length at least 2. This is the case  $A_1 = A_2 = A_3 = \{1\}$  in the above notation.
- $H(n)$ , the number of standard Young tableaux of shape  $[n, n, n]$  where all the run-lengths, in each row are always odd. This is the case  $A_1 = A_2 = A_3 = \{2r + 2 : r \geq 0\}$  in the above notation.

Regarding  $G(n)$ , using the Maple package

<http://www.math.rutgers.edu/~zeilberg/tokhniot/Tableaux3R.txt>

we got that the sequence starts with (starting at  $n = 1$ )

0, 1, 1, 5, 15, 69, 304, 1518, 7807, 42314, 236621, 1364570, 8062975, 48680547, 299388670, 1871463427, ... .

The first 200 terms may be viewed here:

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTableaux3R1.txt> .

The first 996 terms are available here:

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/CYT/GseqList.txt> .

Regarding  $H(n)$ , we got that the sequence starts with (starting at  $n = 1$ )

1, 2, 9, 46, 306, 2252, 18308, 158872, 1454570, 13888112, 137277741, 1396638636, 14561307281, 155040525128, ... .

The first 200 terms can be viewed here:

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/oTableaux3R2.txt> .

The first 965 terms are available here:

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/CYT/HseqList.txt> .

Even that many terms were not enough to guess a linear recurrence with polynomial coefficients, so if such a recurrence exists, it would be extremely complicated. But we can do better! The existence of a non-zero linear recurrence of a given order and degree boils down to the existence of a non-zero solution to a certain system of linear equations with integer coefficients. If a non-trivial solution exists, then doing everything modulo any prime would also have a solution. Conversely, if there is no solution modulo that prime, there is no solution at all. Now we can generate many more terms, and using the prime  $p = 45007$  we (or rather our computer) generated 5000 terms, and even these

did not suffice. In other words if there exists such a recurrence of order  $\leq K$  and degree, in  $n$ , of the coefficients of degree  $\leq K$ , then  $(K + 1)^2 + 5 \geq 5000$ , i.e.  $K \geq 70$ .

This leads us to make the following conjectures. One of us (DZ) is pledging a donation of 200 US dollars to the On-Line Encyclopedia of Integer Sequences (OEIS) in honor of the first prover, for each of the following four conjectures.

**Conjecture 1a:** The sequence  $G(n)$  is **not**  $P$ -recursive.

**Conjecture 1b:** The sequence  $H(n)$  is **not**  $P$ -recursive.

Surprisingly, the asymptotics seems to be very nice. Using the nearly 1000 terms in these sequences we are safe in making the following conjectures.

**Conjecture 2a:** There exists a constant  $C_1$  (if possible, find it!) such that

$$G(n) \asymp C_1 \frac{8^n}{n^4} .$$

We estimate  $C_1$  to be close to 0.521286 .

**Conjecture 2b:** There exists a constant  $C_2$  (if possible, find it!) such that

$$H(n) \asymp C_2 \frac{(7 + 5\sqrt{2})^n}{n^4} .$$

We estimate  $C_2$  to be close to 0.63892.

This raises the more general question about these sequences. Is the asymptotics always of the form  $C\mu^n n^\theta$  with  $\mu$  an algebraic number, and  $\theta$  a rational number?

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