THE KIRCH SPACE IS TOPOLOGICALLY RIGID

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Dedicated to Professor Jerzy Mioduszewski

ABSTRACT. The Golomb space (resp. the Kirch space) is the set \mathbb{N} of positive integers endowed with the topology generated by the base consisting of arithmetic progressions $a+b\mathbb{N}_0=\{a+bn:n\geq 0\}$ where $a,b\in\mathbb{N}$ and b is a (square-free) number, coprime with a. It is known that the Golomb space (resp. the Kirch space) is connected (and locally connected). By a recent result of Banakh, Spirito and Turek, the Golomb space has trivial homeomorphism group and hence is topologically rigid. In this paper we prove the topological rigidity of the Kirch space.

In the AMS Meeting announcement [4] M. Brown introduced an amusing topology τ_G on the set \mathbb{N} of positive integers turning it into a connected Hausdorff space. The topology is generated by the base consisting of arithmetic progressions $a+b\mathbb{N}_0 := \{a+bn : n \in \mathbb{N}_0\}$ with coprime parameters $a, b \in \mathbb{N}$. Here by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ we denote the set of non-negative integer numbers.

In [14] Brown's topology is called the relatively prime integer topology. This topology was popularized by Solomon Golomb [6], [7] who observed that the classical Dirichlet theorem on primes in arithmetic progressions is equivalent to the density of the set Π of prime numbers in the topological space (\mathbb{N}, τ_G) . In honour of Golomb the topological space $\mathbb{G} := (\mathbb{N}, \tau_G)$ is known in General Topology as the Golomb space, see [15], [16].

The problem of studying the topological structure of the Golomb space was posed to the first author (Banakh) by the third author (Turek) in 2006. In his turn, Turek learned about this problem from Jerzy Mioduszewski who listened to the lecture of Solomon Golomb on the first Toposym in 1961.

The topological structure of the Golomb space was studied by the first and third authors in [2] and [3]. In particular, they proved that the Golomb space admits continuum many continuous self-maps but has only one homeomorphism (the identity). Topological spaces having trivial homeomorphism group are called *topologically rigid*. Therefore, the Golomb space is topologically rigid.

It is known that the Golomb space is connected but not locally connected. In [9] Kirch introduced a topology $\tau_K \subseteq \tau_G$ turning $\mathbb N$ into a connected and locally connected space. The Kirch topology τ_K on $\mathbb N$ is generated by the subbase consisting of the arithmetic progressions $a+p\mathbb N_0$ were p is prime and $a\in\mathbb N$ is not divided by p. The base of the Kirch topology consists of the arithmetic progressions $a+b\mathbb N_0$ were $a,b\in\mathbb N$ are coprime and b is square-free (i.e., b is not divisible by the square of a prime number).

The main result of this note is the following rigidity theorem.

Theorem 1. The Kirch space (\mathbb{N}, τ_K) is topologically rigid.

The proof of Theorem 1 is long and technical. It is divided into 22 lemmas. A crucial role in the proof belongs to the superconnectedness of the Kirch space and the superconnecting poset of the Kirch space, which is defined in Section 2.

1. Four classical number-theoretic results

By Π we denote the set of prime numbers. For a number $x \in \mathbb{N}$ by Π_x we denote the set of all prime divisors of x. Two numbers $x, y \in \mathbb{N}$ are *coprime* iff $\Pi_x \cap \Pi_y = \emptyset$.

In the proof of Theorem 1 we will exploit the following four known results of Number Theory. The first one is the famous Chinese Remainder Theorem (see. e.g. [8, 3.12]).

Theorem 2 (Chinese Remainder Theorem). If $b_1, \ldots, b_n \in \mathbb{N}$ are pairwise coprime numbers, then for any numbers $a_1, \ldots, a_n \in \mathbb{Z}$, the intersection $\bigcap_{i=1}^n (a_i + b_i \mathbb{N})$ is infinite.

The second classical result is not elementary and is due to Dirichlet [5, S.VI], see also [1, Ch.7].

Theorem 3 (Dirichlet). For any coprime numbers a, b the arithmetic progression $a + b\mathbb{N}$ contains a prime number.

The third classical result is a famous theorem of Mihăilescu [10], see also [13].

Theorem 4 (Mihăilescu). If $a, b \in \{m^n : n, m \in \mathbb{N} \setminus \{1\}\}$, then |a - b| = 1 if and only if $\{a, b\} = \{2^3, 3^2\}$.

The fourth classical result we use is due to Karl Zsigmondy [17], see also [12, Theorem 3].

Theorem 5 (Zsigmondy). For integer numbers $a, n \in \mathbb{N} \setminus \{1\}$ the inclusion $\Pi_{a^n-1} \subseteq \bigcup_{0 < k < n} \Pi_{a^k-1}$ holds if and only if one of the following conditions is satisfied:

- (1) n=2 and $a=2^k-1$ for some $k \in \mathbb{N}$; then $a^2-1=(a+1)(a-1)=2^k(a-1)$;
- (2) n = 6 and a = 2; then $a^n 1 = 2^6 1 = 63 = 3^2 \times 7 = (a^2 1)^2 \times (a^3 1)$.

2. Superconnected spaces and their superconnecting posets

In this section we discuss superconnected topological spaces and some order structures related to such spaces. First let us introduce some notation and recall some notions.

For a set A and $n \in \omega$ let $[A]^n = \{E \subseteq A : |A| = n\}$ be the family of n-element subsets of A, and $[A]^{<\omega} = \bigcup_{n \in \omega} [A]^n$ be the family of all finite subsets of A. For a function $f: X \to Y$ and a subset $A \subseteq X$ by f[A] we denote the image $\{f(a) : a \in A\}$ of the set A under the function f.

For a subset A of a topological space (X, τ) by \overline{A} we denote the closure of A in X. For a point $x \in X$ we denote by $\tau_x := \{U \in \tau : x \in U\}$ the family of all open neighborhoods of x in (X, τ) . A poset is an abbreviation for a partially ordered set.

A family \mathcal{F} of subsets of a set X is called a *filter* if

- ∅ ∉ F;
- for any $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
- for any sets $F \subseteq E \subseteq X$ the inclusion $F \in \mathcal{F}$ implies $E \in \mathcal{F}$.

A topological space (X, τ) is called *superconnected* if for any $n \in \mathbb{N}$ and non-empty open sets U_1, \ldots, U_n the intersection $\overline{U_1} \cap \cdots \cap \overline{U_n}$ is non-empty. This allows us to define the filter

$$\mathcal{F}_{\infty} = \{ B \subseteq X : \exists U_1, \dots, U_n \in \tau \setminus \{\emptyset\} \ (\overline{U_1} \cap \dots \cap \overline{U_n} \subseteq B) \},$$

called the superconnecting filter of X.

For every finite subset E of X consider the subfilter

$$\mathcal{F}_E := \{ B \subseteq X : \exists (U_x)_{x \in E} \in \prod_{x \in E} \tau_x \ (\bigcap_{x \in E} \overline{U_x} \subseteq B) \}$$

of \mathcal{F}_{∞} . Here we assume that $\mathcal{F}_{\emptyset} = \{X\}$. It is clear that for any finite sets $E \subseteq F$ in X we have $\mathcal{F}_E \subseteq \mathcal{F}_F$. The family

$$\mathfrak{F} = \{ \mathcal{F}_E : E \in [X]^{<\omega} \} \cup \{ \mathcal{F}_{\infty} \}$$

is endowed with the inclusion partial order and is called the *superconnecting poset* of the superconnected space X. The filters \mathcal{F}_{\emptyset} and \mathcal{F}_{∞} are the smallest and largest elements of the poset \mathfrak{F} , respectively.

The following obvious lemma shows that the superconnecting poset \mathfrak{F} is a topological invariant of the superconnected space.

Proposition 1. For any homeomorphism h of a superconnected topological space X, the map

$$\tilde{h}: \mathfrak{F} \to \mathfrak{F}, \quad \tilde{h}: \mathcal{F} \mapsto \{h[A]: A \in \mathcal{F}\},\$$

is an order isomorphism of the superconnecting poset \mathfrak{F} .

In the following sections we will study the order properties of the poset \mathfrak{F} for the Kirch space (\mathbb{N}, τ_K) and will exploit the obtained information in the proof of the topological rigidity of the Kirch space.

3. Proof of Theorem 1

We divide the proof of Theorem 1 into 22 lemmas. Our first lemma describes the closure of an arithmetic progression in the Kirch topology.

Lemma 1. For any $a, b \in \mathbb{N}$ the closure $\overline{a + b\mathbb{N}_0}$ of the arithmetic progression $a + b\mathbb{N}_0$ in the Kirch space (\mathbb{N}, τ_K) is equal to

$$\mathbb{N} \cap \bigcap_{p \in \Pi_b} (\{0, a\} + p\mathbb{Z}).$$

Proof. First we prove that $\overline{a+b\mathbb{N}_0} \subseteq \{0,a\} + p\mathbb{Z}$ for every $p \in \Pi_b$. Take any point $x \in \overline{a+b\mathbb{N}_0}$ and assume that $x \notin p\mathbb{Z}$. Then $x+p\mathbb{N}_0$ is a neighborhood of x and hence the intersection $(x+p\mathbb{N}_0) \cap (a+b\mathbb{N}_0)$ is not empty. Then there exist $u, v \in \mathbb{N}_0$ such that x+pu=a+bv. Consequently, $x-a=bv-pu \in p\mathbb{Z}$ and $x \in a+p\mathbb{Z}$.

Next, take any point $x \in \mathbb{N} \cap \bigcap_{p \in \Pi_b} (\{0, a\} + p\mathbb{Z})$. Given any neighborhood O_x of x in (\mathbb{N}, τ_K) , we should prove that $O_x \cap (a + b\mathbb{N}_0) \neq \emptyset$. By the definition of the Kirch topology there exists a square-free number $d \in \mathbb{N}$ such that d, x are coprime and $x + d\mathbb{N}_0 \subseteq O_x$.

If $\Pi_b \subseteq \Pi_x$, then b,d are coprime and by Chinese Remainder Theorem $\emptyset \neq (x+d\mathbb{N}_0) \cap (a+b\mathbb{N}_0) \subseteq O_x \cap (a+b\mathbb{N}_0)$. So, we can assume $\Pi_b \setminus \Pi_x \neq \emptyset$. The choice of $x \in \bigcap_{p \in \Pi_b \setminus \Pi_x} (\{0,a\} + p\mathbb{Z})$ guarantees that $x \in \bigcap_{p \in \Pi_b \setminus \Pi_x} (a+p\mathbb{Z}) = a+q\mathbb{Z}$ where $q = \prod_{p \in \Pi_b \setminus \Pi_x} p$. Since the numbers x and d are coprime and d is squarefree, the greatest common divisor of b and d divides the number q. Since $x-a \in q\mathbb{Z}$, the Euclides algorithm yields two numbers $u, v \in \mathbb{N}_0$ such that x-a = bu-dv, which implies that $O_x \cap (a+b\mathbb{N}_0) \supset (x+d\mathbb{N}_0) \cap (a+b\mathbb{N}_0) \neq \emptyset$. \square

Lemma 1 implies that the Kirch space (\mathbb{N}, τ_K) is superconnected and hence possesses the superconnecting filter

$$\mathcal{F}_{\infty} = \left\{ F \subseteq \mathbb{N} : \exists U_1, \dots, U_n \in \tau_K \setminus \{\emptyset\} \quad \left(\bigcap_{i=1}^n \overline{U_i} \subseteq F\right) \right\}$$

and the superconnecting poset

$$\mathfrak{F} = \{ \mathcal{F}_E : E \in [\mathbb{N}]^{<\omega} \} \cup \{ \mathcal{F}_{\infty} \}$$

consisting of the filters

$$\mathcal{F}_E = \big\{ F \subseteq \mathbb{N} : \exists (U_x)_{x \in E} \in \prod_{x \in E} \tau_x \ \big(\bigcap_{x \in E} \overline{U_x} \subseteq F \big) \big\}.$$

Here for a point $x \in \mathbb{N}$ by $\tau_x := \{U \in \tau_K : x \in U\}$ we denote the family of open neighborhoods of x in the Kirch topology τ_K .

For a nonempty finite subset $E \subseteq \mathbb{N}$, let $\Pi_E = \bigcap_{x \in E} \Pi_x$ be the set of common prime divisors of numbers in the set E. Also let

$$A_E = \{ p \in \Pi : \exists k \in \mathbb{N} \ (E \subset \{0, k\} + p\mathbb{Z}) \}.$$

Observe that $\Pi_E \subseteq A_E$ and $A_E \neq \emptyset$ because $2 \in A_E$. If E is a singleton, then $A_E = \Pi$; if $|E| \geq 2$, then $A_E \subseteq \{1, \ldots, \max E\}$.

Indeed, assuming that A_E contains some prime number $p > \max E$, we can find a number $k \in \{1, \dots, p-1\}$ such that $E \subseteq \{0, k\} + p\mathbb{Z}$. Then for any distinct numbers $x, y \in E$ we get $x, y \in k + p\mathbb{Z}$ and hence $x - y \in p\mathbb{Z}$ which is not possible as $p > \max E \ge |x - y|$.

Let $\alpha_E : A_E \to \omega$ be the unique function satisfying the following conditions:

- (i) $0 \le \alpha_E(p) < p$ for all $p \in A_E$;
- (ii) $E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$ for all $p \in A_E$;
- (iii) $\alpha_E(2) = 1$ and $\alpha_E(p) = 0$ for all $p \in \Pi_E \setminus \{2\}$.

Lemma 2. For any two-element set $E = \{x, y\} \subset \mathbb{N}$ we have $A_E = \{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y}$.

Proof. The number p=2 belongs to A_E because $E \subset \{0,1\} + \mathbb{Z}$. Each number $p \in \Pi_x$ (resp. $p \in \Pi_y$) belongs to A_E because $\{x,y\} \subset \{0,y\} + p\mathbb{Z}$ (resp. $\{x,y\} \subset \{0,x\} + p\mathbb{Z}\}$). Each number $p \in \Pi_{x-y}$ belongs to A_E because $\{x,y\} \subset x + p\mathbb{Z} \subset \{0,x\} + p\mathbb{Z}$. This proves that $\{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} \subseteq A_E$.

Now take any prime number $p \in A_E$ and assume that $p \notin \Pi_x \cup \Pi_y$. It follows from $\{x,y\} = E \subset \{0,\alpha_E(p)\} + p\mathbb{Z}$ that $\{x,y\} \subseteq \alpha_E(p) + p\mathbb{Z}$ and hence $x-y \in p\mathbb{Z}$ and $p \in \Pi_{x-y}$.

The following lemma yields an arithmetic description of the filters \mathcal{F}_E .

Lemma 3. Let $A \subset \Pi$ be a finite set such that $2 \in A \neq \{2\}$ and $\alpha : A \to \mathbb{N}_0$ be a function such that $\alpha(2) = 1$ and $\alpha(p) \in \{0, \ldots, p-1\}$ for all $p \in A \setminus \{2\}$. Let x be the product of odd prime numbers in the set A and A be any number in the set A and A be A be A be A and A be A be A and A be A be

Proof. For every prime number $p \in A$ we have $E = \{y, x, 2x\} \subset \{0, y\} + p\mathbb{Z}$, which implies that $p \in A_E$. Assuming that $A_E \setminus A$ contains some prime number p, we conclude that $x \notin p\mathbb{Z}$ and hence the inclusion $\{y, x, 2x\} = E \subset \{0, \alpha_E(p)\} + p\mathbb{Z}$ implies $\{x, 2x\} \subset \alpha_E(p) + p\mathbb{Z}$ and $x = 2x - x \in p\mathbb{Z}$. This contradiction shows that $A_E = A$. To show that $\alpha_E = \alpha$, take any prime number $p \in A = A_E$. If p = 2, then $\alpha(p) = 1 = \alpha_E(p)$. So, we assume that $p \neq 2$. If $\alpha(p) = 0$, then $y \in \alpha(p) + p\mathbb{Z} = p\mathbb{Z}$ and hence $p \in \Pi_E$. In this case $\alpha_E(p) = 0 = \alpha(p)$. If $\alpha(p) \neq 0$, then the number $y \in \alpha(p) + p\mathbb{Z}$ is not divisible by p and then the inclusions $\{y, x, 2x\} \subseteq \{0, \alpha(p)\} + p\mathbb{Z}$ and $\{y, x, 2x\} = E \subset \{0, \alpha_E(p)\} + p\mathbb{Z}$ imply that $\alpha(p) = \alpha_E(p)$.

Lemma 4. For any finite subset $E \subseteq \mathbb{N}$ with $|E| \ge 2$ we have

$$\mathcal{F}_E = \big\{ B \subseteq \mathbb{N} \colon \exists L \in [\Pi \setminus A_E]^{<\omega} \quad \bigcap_{p \in L} p \mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p \mathbb{Z}) \subseteq B \big\}.$$

Here we assume that $\bigcap_{p \in \emptyset} p\mathbb{N} = \mathbb{N}$.

Proof. It suffices to verify two properties:

(1) for any $(U_x)_{x\in E}\in \prod_{x\in E}\tau_x$ there exists a finite set $L\subseteq \Pi\setminus A_E$ such that

$$\bigcap_{p \in L} p \mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p \mathbb{Z}) \subseteq \bigcap_{x \in E} \overline{U_x};$$

(2) for any finite set $L \subseteq \Pi \setminus A_E$ there exists a sequence of neighborhoods $(U_x)_{x \in E} \in \prod_{x \in E} \tau_x$ such that

$$\bigcap_{x \in E} \overline{U_x} \subseteq \bigcap_{p \in L} p\mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z}).$$

1. Given a sequence of neighborhoods $(U_x)_{x\in E} \in \prod_{x\in E} \tau_x$, for every $x\in E$ find a square-free number $q_x>x$ such that $\Pi_{q_x}\cap \Pi_x=\emptyset$ and $x+q_x\mathbb{N}_0\subseteq U_x$. We claim that the finite set $L=\bigcup_{x\in E}\Pi_{q_x}\setminus A_E$ has the required property. Given any number $z\in\bigcap_{p\in L}p\mathbb{N}\cap\bigcap_{p\in A_E\setminus\Pi_E}(\{0,\alpha_E(p)\}+p\mathbb{Z})$, we should prove that $z\in\overline{U_x}$ for every $x\in E$. By Lemma 1,

$$\mathbb{N} \cap \bigcap_{p \in \Pi_{q_x}} (\{0, x\} + p\mathbb{Z}) = \overline{(x + q_x \mathbb{N}_0)} \subseteq \overline{U_x}.$$

So, it suffices to show that $z \in \{0, x\} + p\mathbb{Z}$ for any $p \in \Pi_{q_x}$. Since the numbers x and q_x are coprime, $p \notin \Pi_x$ and hence $p \notin \Pi_E$. If $p \notin A_E$, then $p \in \Pi_{q_x} \setminus A_E \subseteq L$ and hence $z \in p\mathbb{N} \subseteq \{0, x\} + p\mathbb{Z}$. If $p \in A_E$, then $x \in E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$ and $x \in \alpha_E(p) + p\mathbb{Z}$ (as $p \notin \Pi_x$). Then $x + p\mathbb{Z} = \alpha_E(p) + p\mathbb{Z}$ and $z \in \{0, \alpha_E(p)\} + p\mathbb{Z} = \{0, x\} + p\mathbb{Z}$.

2. Fix any finite set $L \subseteq \Pi \setminus A_E$. For every $x \in E$ consider the neighborhood $U_x = \bigcap_{p \in L \cup A_E \setminus \Pi_x} (x + p \mathbb{N}_0)$ of x in the Kirch topology. By Lemma 1,

$$\overline{U_x} = \mathbb{N} \cap \bigcap_{p \in L \cup A_E \backslash \Pi_x} (\{0, x\} + p\mathbb{Z}).$$

Given any number $z \in \bigcap_{x \in E} \overline{U_x}$, we should show that $z \in \bigcap_{p \in L} p\mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p\mathbb{Z})$. This will follow as soon as we check that $z \in p\mathbb{N}$ for all $p \in L$ and $z \in \{0, \alpha_E(p)\} + p\mathbb{Z}$ for all $p \in A_E \setminus \Pi_E$.

Given any $p \in A_E \setminus \Pi_E$, we can find a point $x \in E \setminus p\mathbb{Z}$ and observe that $x \in E \subseteq \{0, \alpha_E(p)\} + p\mathbb{Z}$. Then $z \in \overline{U_x} \subseteq \overline{x + p\mathbb{N}_0} \subseteq \{0, x\} + p\mathbb{Z} = \{0, \alpha_E(p)\} + p\mathbb{Z}$.

Now take any prime number $p \in L$. Since $L \cap A_E = \emptyset$, we conclude that $E \not\subseteq p\mathbb{Z}$. So, we can fix a number $x \in E \setminus p\mathbb{Z}$. Taking into account that $p \notin A_E$, we conclude that $E \not\subseteq \{0, x\} + p\mathbb{Z}$ and hence there exists a number $y \in E$ such that $p\mathbb{Z} \neq y + p\mathbb{Z} \neq x + p\mathbb{Z}$. Then

$$z \in \overline{U_x} \cap \overline{U_y} \subseteq (\{0, x\} + p\mathbb{Z}) \cap (\{0, y\} + p\mathbb{Z}) = p\mathbb{Z}.$$

We shall use Lemma 4 for an arithmetic characterization of the partial order of the superconnecting poset \mathfrak{F} of the Kirch space.

Lemma 5. For two finite subsets $E, F \subseteq \mathbb{N}$ with $\min\{|E|, |F|\} \ge 2$ we have $\mathcal{F}_E \subseteq \mathcal{F}_F$ if and only if

$$A_F \subseteq A_E, \ \Pi_F \setminus \{2\} \subseteq \Pi_E \ and \ \alpha_E \upharpoonright A_F \setminus \Pi_E = \alpha_F \upharpoonright A_F \setminus \Pi_E.$$

Proof. To prove the "only if" part, assume that $\mathcal{F}_E \subseteq \mathcal{F}_F$. By Lemma 4, the set

$$\bigcap_{p \in A_F \setminus A_E} p \mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p \mathbb{Z})$$

belongs to the filter $\mathcal{F}_E \subseteq \mathcal{F}_F$. By Lemma 4, there exists a finite set $L \subseteq \Pi \setminus A_F$ such that

(1)
$$\bigcap_{p \in L} p \mathbb{N} \cap \bigcap_{p \in A_F \setminus \Pi_F} (\{0, \alpha_F(p)\} + p \mathbb{Z}) \subseteq \bigcap_{p \in A_F \setminus A_E} p \mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p \mathbb{Z}).$$

This inclusion combined with the Chinese Remainder Theorem 2 implies

 $A_F \setminus A_E \subseteq L \subset \Pi \setminus A_F$, $A_E \setminus (\Pi_E \cup \{2\}) \subseteq L \cup (A_F \setminus \Pi_F)$ and $\alpha_E(p) = \alpha_F(p)$ for any $p \in (A_F \setminus \Pi_F) \cap (A_E \setminus \Pi_E)$, and

(2)
$$A_F \subseteq A_E, \ \Pi_F \setminus \{2\} \subseteq \Pi_E \ \text{and} \ \alpha_E \upharpoonright A_F \setminus \Pi_E = \alpha_F \upharpoonright A_F \setminus \Pi_E.$$

To prove the "if" part, assume that the condition (2) holds. To prove that $\mathcal{F}_E \subseteq \mathcal{F}_F$, fix any set $\Omega \in \mathcal{F}_E$ and using Lemma 4, find a finite set $L \subseteq \Pi \setminus A_E$ such that

$$\bigcap_{p\in L} p\mathbb{N} \cap \bigcap_{p\in A_E\backslash \Pi_E} (\{0,\alpha_E(p)\} + p\mathbb{Z}) \subseteq \Omega.$$

Consider the finite set $\Lambda = (L \cup A_E) \setminus A_F = L \cup (A_E \setminus A_F) \supseteq L$ and observe that the condition (2) implies the inclusion

(3)
$$\mathcal{F}_F \ni \bigcap_{p \in \Lambda} p \mathbb{N} \cap \bigcap_{p \in A_F \setminus \Pi_F} (\{0, \alpha_F(p)\} + p \mathbb{Z}) \subseteq \bigcap_{p \in L} p \mathbb{N} \cap \bigcap_{p \in A_E \setminus \Pi_E} (\{0, \alpha_E(p)\} + p \mathbb{Z}) \subseteq \Omega,$$

yielding $\Omega \in \mathcal{F}_F$.

Lemma 6. For two nonempty subsets $E, F \subseteq \mathbb{N}$ with $\min\{|E|, |F|\} = 1$ the relation $\mathcal{F}_E \subseteq \mathcal{F}_F$ holds if and only if |E| = 1 and $E \subseteq F$.

Proof. The "if" part is trivial. To prove the "only if" part, assume that $\mathcal{F}_E \subseteq \mathcal{F}_F$. First we prove that |E| = 1. Assuming that |E| > 1 and taking into account that $\min\{|E|, |F|\} = 1$, we conclude that |F| = 1. Choose a prime number $p > \max(E \cup F)$. Since $\bigcap_{y \in E} \overline{y + p\mathbb{N}_0} \in \mathcal{F}_E \subseteq \mathcal{F}_F$, for the unique number x in the set F, there exists a square-free number d such that $\Pi_d \cap \Pi_x = \emptyset$ and $\overline{x + dp\mathbb{N}_0} \subseteq \bigcap_{y \in E} \overline{y + p\mathbb{N}_0}$. By Lemma 1,

$$x+qp\mathbb{N}\subseteq \overline{x+dp\mathbb{N}_0}\subseteq \bigcap_{y\in E}\overline{y+p\mathbb{N}_0}=\bigcap_{y\in E}(\{0,y\}+p\mathbb{N}_0)=p\mathbb{N}_0.$$

The latter equality follows from $p > \max E$ and |E| > 1. Then $x + dp\mathbb{N} \subseteq p\mathbb{N}_0$ implies $x \in p\mathbb{N}_0$, which contradicts the choice of $p > \max(E \cup F) \ge x$. This contradiction shows that |E| = 1. Let z be the unique element of the set E.

It remains to prove that $z \in F$. To derive a contradiction, assume that $z \notin F$. Take any odd prime number $p > \max(E \cup F)$ and consider the set $\{0, z\} + p\mathbb{N}_0 = \overline{z + p\mathbb{N}_0} \in \mathcal{F}_E \subseteq \mathcal{F}_F$. By the definition of the filter \mathcal{F}_F , for every $x \in F$ there exists a square-free number d_x such that $\Pi_{d_x} \cap \Pi_x = \emptyset$ and

$$\bigcap_{x \in F} \overline{x + d_x \mathbb{N}_0} \subseteq \overline{z + p \mathbb{N}_0} = \{0, z\} + p \mathbb{N}_0.$$

Consider the set $P = \bigcup_{x \in F} \Pi_{d_x}$. If $p \in \Pi_{d_x}$ for some $x \in F$, we can use the Chinese Remainder Theorem 2 and find a number

$$c \in (x + p\mathbb{N}_0) \cap \bigcap_{q \in P \setminus \{p\}} q\mathbb{N} \subseteq \bigcap_{y \in F} \overline{y + d_y \mathbb{N}_0} \subseteq \{0, z\} + p\mathbb{N}_0.$$

Taking into account that x is not divisible by p, we conclude that $c \in (x+p\mathbb{Z}) \cap (z+p\mathbb{Z})$ and hence $x-z \in p\mathbb{Z}$, which contradicts the choice of $p > \max(E \cup F)$. This contradiction shows that $p \notin P$. Since $p \geq 3$, we can find a number $z' \notin \{0, z\} + p\mathbb{Z}$ and using the Chinese Remainder Theorem 2, find a number

$$u \in (z' + p\mathbb{N}_0) \cap \bigcap_{q \in P} q\mathbb{N} \subseteq \bigcap_{x \in F} \overline{x + d_x \mathbb{N}_0} \subseteq \{0, z\} + p\mathbb{Z},$$

which is a desired contradiction showing that $E \subseteq F$.

As we know, the largest element of the superconnecting poset \mathfrak{F} is the superconnecting filter \mathcal{F}_{∞} . This filter can be characterized as follows.

Lemma 7. The superconnecting filter \mathcal{F}_{∞} of the Kirch space is generated by the base consisting of the sets $q\mathbb{N}$ for odd square-free numbers $q \in \mathbb{N}$, i.e.

$$\mathcal{F}_{\infty} = \{ B \subseteq \mathbb{N} \colon (\exists q) (q \text{ is an odd square-free}) \land q \mathbb{N} \subseteq B \}.$$

Proof. Lemma 1 implies that each element $F \in \mathcal{F}_{\infty}$ contains the set $q\mathbb{N}$ for some odd square-free number q. Conversely, let q be an odd square-free number. Then $U_1 = 1 + q\mathbb{N}_0$, $U_2 = 2 + q\mathbb{N}_0 \in \tau_K$. By Lemma 1 we have

$$\overline{U_1} \cap \overline{U_2} = \mathbb{N} \cap \bigcap_{p \in \Pi_q} (\{0,1\} + p\mathbb{Z}) \cap (\{0,2\} + p\mathbb{Z}) = \mathbb{N} \cap \bigcap_{p \in \Pi_q} p\mathbb{Z} = q\mathbb{N}.$$

Hence $q\mathbb{N} \in \mathcal{F}_{\infty}$.

Lemma 8. For a nonempty subset $E \subseteq \mathbb{N}$ the following conditions are equivalent:

- (1) $\mathcal{F}_E = \mathcal{F}_{\infty}$;
- (2) $A_E = \{2\}.$

If |E| = 2, then the conditions (1), (2) are equivalent to

(3) $E = \{2^n, 2^{n+1}\}$ for some $n \in \omega$.

Proof. (1) \Rightarrow (2): Assume $\mathcal{F}_E = \mathcal{F}_{\infty}$. Consider $F = \{1, 2\}$. It is clear that $A_F = \{2\}$ and $\Pi_F = \emptyset$. Thus $A_F \subseteq A_E$, $\Pi_F \setminus \{2\} \subseteq \Pi_E$ and $\alpha_F \upharpoonright A_F \setminus \Pi_E = \alpha_E \upharpoonright A_F \setminus \Pi_E$. Lemma 5 implies $\mathcal{F}_E \subseteq \mathcal{F}_F$. Since $\mathcal{F}_E = \mathcal{F}_{\infty}$ is the largest element of \mathfrak{F} we get $\mathcal{F}_E = \mathcal{F}_F$. By using again Lemma 5 we get $A_E \subseteq A_F$ which implies that $A_E = \{2\}$.

 $(2) \Rightarrow (1)$: If $A_E = \{2\}$, then by the Lemma 4, the filter \mathcal{F}_E is generated by the base consisting of the sets $q\mathbb{N}$ for an odd square-free number $q \in \mathbb{N}$. Therefore $\mathcal{F}_E = \mathcal{F}_{\infty}$ by the Lemma 7.

If
$$|E|=2$$
, then the equivalence (2) \Leftrightarrow (3) follows from Lemma 2.

Lemma 9. For every $n \in \omega$, the number 2^n is a fixed point of any homeomorphism h of the Kirch space.

Proof. Consider the graph $\Gamma_2 = (V_2, \mathcal{E})$ with set of vertices $V_2 = \{2^n : n \in \omega\}$ and set of edges $\mathcal{E} = \{\{2^n, 2^{n+1}\} : n \in \omega\}$.

By Lemma 8, for every edge $E \in \mathcal{E}$ of the graph \mathcal{E} we have $\mathcal{F}_E = \mathcal{F}_{\infty}$ and hence $\mathcal{F}_{h[E]} = \tilde{h}(\mathcal{F}_E) = \tilde{h}(\mathcal{F}_{\infty}) = \mathcal{F}_{\infty}$ by the topological invariance of the filter \mathcal{F}_{∞} . Applying Lemma 8 once more, we conclude that $h[E] \in \mathcal{E}$. The same argument applied to the homeomorphism h^{-1} ensures that $\tilde{h}^{-1}[E] \in \mathcal{E}$ for any $E \in \mathcal{E}$. This means that \tilde{h} induces an isomorphism of the graph Γ_2 . Now observe that the number $2^0 = 1$ is a unique vertex of the graph Γ_2 that has order 1. This graph-theoretic property of the vertex 2^0 in Γ_2 ensures that $h(2^0) = 2^0$. Next, observe that 2^1 is a unique vertex of Γ_2 that is connected with 2^0 and hence $h(2^1) = 2^1$. Proceeding by induction, we can show that $h(2^n) = 2^n$ for all $n \in \omega$.

In the following lemmas by \mathfrak{F}' we denote the set of maximal elements of the poset $\mathfrak{F} \setminus \{\mathcal{F}_{\infty}\}$.

Lemma 10. For a finite subset $E \subseteq \mathbb{N}$ the filter \mathcal{F}_E belongs to the family \mathfrak{F}' if and only if there exists an odd prime number $p \notin \Pi_E$ such that $A_E = \{2, p\}$.

Proof. To prove the "if" part, assume that $A_E = \{2, p\}$ and $p \notin \Pi_E$ for some odd prime number p. By Lemma 8, $\mathcal{F}_E \neq \mathcal{F}_{\infty}$. To show that the filter \mathcal{F}_E is maximal in $\mathfrak{F} \setminus \{\mathcal{F}_{\infty}\}$, take any finite set $F \subset \mathbb{N}$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \neq \mathcal{F}_{\infty}$. By Lemmas 5 and 8, $\{2\} \neq A_F \subseteq A_E = \{2, p\}$, $\Pi_F \subseteq \Pi_E \cup \{2\} = \{2\}$, and $\alpha_F \upharpoonright A_F \setminus \Pi_E = \alpha_E \upharpoonright A_F \setminus \Pi_E$. It follows that $A_F = \{2, p\} = A_E$, $\Pi_F \cup \{2\} = \Pi_E \cup \{2\}$ and $\alpha_F = \alpha_E$. Applying Lemma 5, we conclude that $\mathcal{F}_E = \mathcal{F}_F$, which means that the filter \mathcal{F}_E is a maximal element of the poset $\mathcal{F} \setminus \{\mathcal{F}_{\infty}\}$.

To prove the "only if" part, assume that $\mathcal{F}_E \in \mathfrak{F}'$. By Lemma 8, $A_E \neq \{2\}$ and hence there exists an odd prime number $p \in A_E$. We claim that $p \notin \Pi_E$. To derive a contradiction, assume that $p \in \Pi_E$ and consider the sets $F = \{p, 2p\}$ and $G = \{1, p, 2p\}$. By Lemma 2, $A_F = A_G = \{2, p\}$, $\Pi_F = \{p\}$, and $\Pi_G = \emptyset$. Taking into account that $F \subset G$, $A_F = \{2, p\} \subseteq A_E$, $\Pi_F \setminus \{2\} = \{p\} \subseteq \Pi_E$ and $A_F \setminus \Pi_E \subseteq \{2\}$, we can apply Lemmas 5, 8 and conclude that $\mathcal{F}_E \subseteq \mathcal{F}_F \subseteq \mathcal{F}_G \neq \mathcal{F}_\infty$. The maximality of \mathcal{F}_E implies $\mathcal{F}_E = \mathcal{F}_F = \mathcal{F}_G$. By Lemma 5, the equality $\mathcal{F}_G = \mathcal{F}_F$ implies $p \in \Pi_F \setminus \{2\} \subseteq \Pi_G = \emptyset$, which is a contradiction showing that $p \notin \Pi_E$.

Now consider the set $H = \{\alpha_E(p), p, 2p\}$ and observe that $A_H = \{2, p\}$, $\Pi_H = \emptyset$ and $\alpha_H(p) = \alpha_E(p)$. Lemmas 5 and 8 guarantee that $\mathcal{F}_E \subseteq \mathcal{F}_H \neq \mathcal{F}_{\infty}$. By the maximality of \mathcal{F}_E , we have $\mathcal{F}_E = \mathcal{F}_H$. Applying Lemma 5 once more, we conclude that $A_E = A_H = \{2, p\}$.

Lemma 10 implies the following description of the set \mathfrak{F}' .

Lemma 11. $\mathfrak{F}' = \{ \mathcal{F}_{\{a,p,2p\}} : p \in \Pi \setminus \{2\}, \ a \in \{1,\ldots,p-1\} \}.$

Let \mathfrak{F}'' be the set of maximal elements of the poset $\mathfrak{F} \setminus (\mathfrak{F}' \cup \{\mathcal{F}_{\infty}\})$

Lemma 12. For a finite set $E \subset \mathbb{N}$, the filter \mathcal{F}_E belongs to the family \mathfrak{F}'' if and only if one of the following conditions holds:

- (1) there exists an odd prime number p such that $p \in \Pi_E$ and $A_E = \{2, p\}$;
- (2) there are two distinct odd prime numbers p, q such that $A_E = \{2, p, q\}$ and $\Pi_E \subseteq \{2\}$.

Proof. To prove the "only if" part, assume that $\mathcal{F}_E \in \mathfrak{F}''$. By Lemma 8, there is an odd prime number $p \in A_E$. If $A_E = \{2, p\}$, then $p \in \Pi_E$ by Lemma 10, and condition (1) is satisfied. So, we assume that $\{2, p\} \neq A_E$ and find an odd prime number $q \in A_E \setminus \{2, p\}$. By Lemma 3, there is a number $x \in \mathbb{N}$ such that for the set $F = \{x, pq, 2pq\}$ we have $A_F = \{2, p, q\}$, $\Pi_F = \emptyset$, $\alpha_F(p) = a$ and $\alpha_F(q) = b$. Then $\mathcal{F}_E \subseteq \mathcal{F}_F$ by Lemma 5, and $\mathcal{F}_F \in \mathfrak{F} \setminus (\mathfrak{F}' \cup \{\mathcal{F}_\infty\})$ by Lemma 10. Now the maximality of the filter \mathcal{F}_E implies that $\mathcal{F}_E = \mathcal{F}_F$ and hence $A_E = A_F = \{2, p, q\}$ and $\Pi_E \subset \Pi_F \cup \{2\} = \{2\}$, see Lemma 5.

To prove the "if" part, we consider two cases. First we assume that $A_E = \{2, p\}$ for some $p \in \Pi_E$. By Lemmas 8 and 10, $\mathcal{F}_E \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. To prove that \mathcal{F}_E is a maximal element of $\mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$, take any finite set $F \subseteq \mathbb{N}$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. Lemma 6 implies that $\min\{|E|, |F|\} \geq 2$ and then by Lemmas 5 and 10, we have $A_F = \{2, p\}$, $\Pi_F \setminus \{2\} \subseteq \{p\}$ and $\alpha_E \upharpoonright A_F \setminus \{p\} = \alpha_F \upharpoonright A_F \setminus \{p\}$. Now notice that $p \in \Pi_F$ since otherwise $\mathcal{F}_F \in \mathfrak{F}'$ by Lemma 10. By using again Lemma 5 we get $\mathcal{F}_F = \mathcal{F}_E$ which means that $\mathcal{F}_E \in \mathfrak{F}''$.

Now assume that there are two distinct odd prime numbers p,q such that $A_E = \{2,p,q\}$ and $\Pi_E \subseteq \{2\}$. By Lemmas 8 and 10, $\mathcal{F}_E \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. To prove that \mathcal{F}_E is a maximal element of $\mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$, take any finite set $F \subseteq \mathbb{N}$ such that $\mathcal{F}_E \subseteq \mathcal{F}_F \in \mathfrak{F} \setminus (\{\mathcal{F}_\infty\} \cup \mathfrak{F}')$. Lemma 5 implies that $A_F \subseteq \{2,p,q\}$, $\Pi_F \subseteq \{2\}$ and $\alpha_E \upharpoonright A_F \setminus \Pi_E = \alpha_F \upharpoonright A_F \setminus \Pi_E$. Taking into account that $\mathcal{F}_F \notin \mathfrak{F}' \cup \{\mathcal{F}_\infty\}$ and $\Pi_F \subseteq \{2\}$, we can apply Lemmas 10, 8 and conclude that $A_F = \{2,p,q\}$. We therefore know that $A_F = A_E$, $\Pi_E \cup \{2\} = \Pi_F \cup \{2\}$ and $\alpha_F \upharpoonright A_E \setminus \Pi_F = \alpha_E \upharpoonright A_E \setminus \Pi_F$. By Lemma 5, $\mathcal{F}_E = \mathcal{F}_F$ and hence $\mathcal{F}_E \in \mathfrak{F}''$.

Lemma 13. For any homeomorphism h of the Kirch space and any odd prime number p we have

$$\tilde{h}(\mathcal{F}_{\{p,2p\}}) = \mathcal{F}_{\{p,2p\}}.$$

Proof. By Proposition 1, the homeomorphism h induces an order isomorphism \tilde{h} of the superconnecting poset \mathfrak{F} on the Kirch space. Then $\tilde{h}[\mathfrak{F}'] = \mathfrak{F}'$ and $\tilde{h}[\mathfrak{F}''] = \mathfrak{F}''$.

By Lemmas 12 and 3, $\mathfrak{F}'' = \mathfrak{F}_2'' \cup \mathfrak{F}_3''$ where

$$\mathfrak{F}_2'' = \left\{ \mathcal{F}_{\{p,2p\}} : p \in \Pi \setminus \{2\} \right\} \quad \text{and}$$

$$\mathfrak{F}_3'' = \left\{ \mathcal{F}_{\{x,pq,2pq\}} : p,q \in \Pi \setminus \{3\}, \ p \neq q, \ x \in \{0,\dots,pq-1\} \setminus (p\mathbb{Z} \cup q\mathbb{Z}) \right\}.$$

By Lemmas 5 and 10, for every filter $\mathcal{F}_{\{p,2p\}} \in \mathfrak{F}_2''$ the set $\uparrow \mathcal{F}_{\{p,2p\}} = \{\mathcal{F} \in \mathfrak{F}' : \mathcal{F}_{\{p,2p\}} \subset \mathcal{F}_E\}$ coincides with the set $\{\mathcal{F}_{\{a,p,2p\}} : a \in \{1,\ldots,p-1\}\}$ and hence has cardinality p-1.

On the other hand, for any filter $\mathcal{F}_{\{x,pq,2pq\}} \in \mathfrak{F}_3''$, the set $\uparrow \mathcal{F}_{\{x,pq,2pq\}} = \{\mathcal{F} \in \mathfrak{F}' : \mathcal{F}_{\{x,pq,2pq\}} \subset \mathcal{F}\}$ coincides with the doubleton $\{\mathcal{F}_{\{x,p,2p\}}, \mathcal{F}_{\{x,q,2q\}}\}$.

These order properties uniquely determine the filters $\mathcal{F}_{\{p,2p\}}$ for $p \in \Pi \setminus \{3\}$ and ensure that $\tilde{h}(\mathcal{F}_{\{p,2p\}}) = \mathcal{F}_{\{p,2p\}}$ for every $p \in \Pi \setminus \{3\}$.

Next, observe that $\mathcal{F}_{\{3,6\}}$ is a unique element \mathcal{F} of \mathfrak{F}'' such that $\uparrow \mathcal{F} \cap \bigcup_{p \in \Pi \setminus \{3\}} \uparrow \mathcal{F}_{\{p,2p\}} = \emptyset$. This uniqueness order property of $\mathcal{F}_{\{3,6\}}$ ensures that $\tilde{h}(\mathcal{F}_{\{3,6\}}) = \mathcal{F}_{\{3,6\}}$.

Lemmas 9 and 13 imply

Lemma 14. For any homeomorphism h of the Kirch space and any odd prime number p we have

$$\tilde{h}(\mathcal{F}_{\{1,p,2p\}}) = \mathcal{F}_{\{1,p,2p\}} \quad and \quad \tilde{h}(\mathcal{F}_{\{2,p,2p\}}) = \mathcal{F}_{\{2,p,2p\}}.$$

Lemma 15. For an integer number $x \geq 3$ and an odd prime p, the following conditions are equivalent:

- (1) $p \in \Pi_x$;
- (2) $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$ and $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$.

Proof. If $p \in \Pi_x$, then $A_{\{1,p,2p\}} = \{2,p\} \subseteq A_{\{1,x\}}$, $\Pi_{\{1,x\}} = \emptyset = \Pi_{\{1,p,2p\}}$ and $\alpha_{\{1,x\}}(p) = 1 = \alpha_{\{1,p,2p\}}(p)$. By Lemma 5, $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$. By analogy we can prove that $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$.

Conversely, assume $\mathcal{F}_{\{1,x\}} \subseteq \mathcal{F}_{\{1,p,2p\}}$ and $\mathcal{F}_{\{2,x\}} \subseteq \mathcal{F}_{\{2,p,2p\}}$. By Lemmas 5 and 2, we have

$$\{2,p\} = A_{\{1,p,2p\}} \subseteq A_{\{1,x\}} = \Pi_x \cup \Pi_{x-1} \text{ and } \{2,p\} = A_{\{2,p,2p\}} \subseteq A_{\{2,x\}} = \{2\} \cup \Pi_x \cup \Pi_{x-2}$$
 and hence $p \in (\Pi_x \cup \Pi_{x-1}) \cap (\Pi_x \cap \Pi_{x-2}) \setminus \{2\} \subseteq \Pi_x$.

Proposition 1 and Lemmas 9, 14, 15 imply

Lemma 16. For every homeomorphism h of the Kirch space and any number $x \in \mathbb{N}$ we have

$$\Pi_x \cup \{2\} = \Pi_{h(x)} \cup \{2\}.$$

For every prime number p consider the set

$$V_p = \{2^{n-1}p^m : n, m \in \mathbb{N}\}\$$

of numbers $x \in \mathbb{N}$ such that $p \in \Pi_x \subseteq \{2, p\}$. Lemmas 9 and 16 imply that $h[V_p] = V_p$ for every homeomorphism h of the Kirch space.

Consider the graph $\Gamma_p = (V_p, \mathcal{E}_p)$ on the set V_p with the set of edges

$$\mathcal{E}_p := \{ E \in [V_p]^2 : A_E = \{2, p\} \}.$$

Lemma 17. For every prime number p and every homeomorphism h of the Kirch space, the restriction of h to V_p is an isomorphism of the graph Γ_p .

Proof. Let $E \in \mathcal{E}_p$. Since $p \in \Pi_E$, we can apply Lemma 12 and conclude that $\mathcal{F}_E \in \mathfrak{F}''$. Using fact that \tilde{h} is an order isomorphism of \mathfrak{F} we get $\mathcal{F}_{h[E]} = \tilde{h}(\mathcal{F}_E) \in \mathfrak{F}''$. Since $h[E] \subseteq h[V_p] = V_p$, we obtain $p \in \Pi_{h[E]}$. Using Lemma 12 once more, we obtain that $A_{h[E]} = \{2, p\}$, which means that $h[E] \in \mathcal{E}_p$. By analogical reasoning we can prove that $h^{-1}[E] \in \mathcal{E}_p$ for every $E \in \mathcal{E}_p$. This means that $h \upharpoonright V_p$ is isomorphism of the graph Γ_p .

The structure of the graph Γ_p depends on properties of the prime number p.

A prime number p is called

- Fermat prime if $p = 2^n + 1$ for some $n \in \mathbb{N}$;
- Mersenne prime if $p = 2^n 1$ for some $n \in \mathbb{N}$;
- Fermat–Mersenne if p is Fermat prime or Mersenne prime.

It is known (and easy to see) that for any Fermat prime number $p=2^n+1$ the exponent n is a power of 2, and for any Mersenne prime number $p=2^n-1$ the power n is a prime number. It is not known whether there are infinitely many Fermat-Mersenne prime numbers. All known Fermat prime numbers are the numbers $2^{2^n}+1$ for $0 \le n \le 4$ (see oeis.org/A019434 in [11]). At the moment only 51 Mersenne prime numbers are known, see the sequence oeis.org/A000043 in [11].

Lemma 18. Let p be an odd prime number.

- (1) If p=3, then the set \mathcal{E}_p of edges of the graph Γ_p coincides with the set of doubletons $\{2^{a-1}3^b, 2^{a-1}3^{b+1}\}$, $\{2^{a-1}3^b, 2^{a-1}3^{b+2}\}$, $\{2^{a-1}3^b, 2^a3^b\}$, $\{2^{a-1}3^b, 2^a3^{b+1}\}$, $\{2^{a+3}3^b, 2^a3^{b+2}\}$ for some $a, b \in \mathbb{N}$.
- (2) If $p = 2^m + 1 > 3$ is Fermat prime, then $\mathcal{E}_p = \{\{2^{a-1}p^b, 2^{a-1}p^{b+1}\}, \{2^{a-1}p^b, 2^ap^b\}, \{2^{m+a-1}p^b, 2^{a-1}p^{b+1}\} : a, b \in \mathbb{N}\}.$
- (3) If $p = 2^m 1 > 3$ is Mersenne prime, then $\mathcal{E}_p = \{\{2^{a-1}p^b, 2^ap^b\}, \{2^{a-1}p^b, 2^{m+a-1}p^b\}, \{2^{a-1}p^{b+1}, 2^{m+a-1}p^b\} : a, b \in \mathbb{N}\}.$
- (4) If p is not Fermat-Mersenne, then $\mathcal{E}_p = \{\{2^a p^b, 2^{a-1} p^b\} : a, b \in \mathbb{N}\}.$

Proof. Proof of Lemma 18 in each of cases (1)–(4) will be similar. Edges of the graph Γ_p are 2-element subsets of the set V_p such that $A_E = \{2, p\}$. Since vertices of the graph Γ_p are numbers of the form $2^{n-1}p^m$, where $n, m \in \mathbb{N}$, we can apply Lemma 2 and conclude that a doubleton $\{x, y\} \subset V_p$ belongs to \mathcal{E}_p if and only if $\{2, p\} = \{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y}$. In subsequent proofs, we will intensively use the Mihăilescu Theorem 4 saying that $2^3, 3^2$ is a unique pair of consecutive powers.

1. First we consider the case of p=3. It is easy to see that the doubletons $\{x,y\}$ written in the statement (1) have $\Pi_x \cup \Pi_y \cup \Pi_{x-y} \subseteq \{2,3\}$, which implies that $\{x,y\} \in \mathcal{E}_3$. It remains to show that every doubleton $\{x,y\} \in \mathcal{E}_3$ is of the form indicated in the statement (1). Write $\{x,y\}$ as $\{2^{a-1}3^b, 2^{c-1}3^d\}$ for some $a,b,c,d \in \mathbb{N}$ such that $2^{a-1}3^b < 2^{c-1}3^d$.

If a = c, then b < d and the inclusion $\Pi_{x-y} \subseteq \{2,3\}$ implies that $\Pi_{3^{d-b}-1} \subseteq \{2,3\}$ and hence $3^{d-b}-1$ is a power of 2. By the Mihăilescu Theorem $4, d-b \in \{1,2\}$, which means that $\{x,y\}$ is equal to $\{2^{a-1}3^b, 2^{a-1}3^{b+1}\}$ or $\{2^{a-1}3^b, 2^{a-1}3^{b+2}\}$.

If b=d, then a < c and the inclusion $\Pi_{x-y} \subseteq \{2,3\}$ implies that $\Pi_{2^{c-a}-1} \subseteq \{2,3\}$ and hence $2^{c-a}-1$ is a power of 3. By the Mihăilescu Theorem 4, $c-a \in \{1,2\}$, which means that $\{x,y\}$ is equal to $\{2^{a-1}3^b, 2^a3^b\}$ or $\{2^{a-1}3^b, 2^{a+1}3^b\}$.

So, we assume that $a \neq c$ and $b \neq d$. In this case we should consider four subcases.

If a < c and b < d, then $\Pi_{x-y} \subseteq \{2,3\}$ implies that each prime divisor of $2^{c-a}3^{d-b} - 1$ is equal to 2 or 3, which is not possible.

If a < c and b > d, then $\Pi_{x-y} \subseteq \{2,3\}$ and $2^{a-1}3^b < 2^{c-1}3^d$ imply that $2^{c-a} - 3^{b-d} = 1$ and hence c - a = 2 and b - d = 1 by the Mihăilescu Theorem 4. In this case $\{x,y\} = \{2^{a-1}3^{d+1}, 2^{a+1}3^d\}$.

If a > c and b < d, then $\Pi_{x-y} \subseteq \{2,3\}$ and $2^{a-1}3^b < 2^{c-1}3^d$ imply that $3^{d-b} - 2^{a-c} = 1$ and hence $(d-b,a-c) \in \{(1,1),(2,3)\}$ by the Mihăilescu Theorem 4. In this case $\{x,y\}$ is equal to $\{2^{c+1}3^b,2^c3^{b+1}\}$ or $\{2^{c+3}3^b,2^c3^{b+2}\}$.

The subcase a > c and b > d is forbidden by the inequality $2^{a-1}3^b < 2^{c-1}3^d$.

2. Assume that $p = 2^m + 1 > 3$ is a Fermat prime. In this case m > 1. It is easy to check that every doubleton $\{x,y\} \in \{\{2^{a-1}p^b, 2^{a-1}p^{b+1}\}, \{2^{a-1}p^b, 2^ap^b\}, \{2^{m+a-1}p^b, 2^{a-1}p^{b+1}\} : a,b \in \mathbb{N}\}$ has $A_{\{x,y\}} = \{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} = \{2,p\}$ and hence $\{x,y\} \in \mathcal{E}_p$.

Now assume that $\{x,y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p . Then $\{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2,p\}$ and $\{x,y\}$ can be written as $\{2^{a-1}p^b, 2^{c-1}p^d\}$ for some $a,b,c,d \in \mathbb{N}$ with $2^{a-1}p^b < 2^{c-1}p^d$.

If a = c, then b < d and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{p^{d-b}-1} \subseteq \{2, p\}$ and hence $p^{d-b}-1$ is a power of 2. By the Mihăilescu Theorem 4, d-b=1, which means that $\{x,y\}$ is equal to $\{2^{a-1}p^b, 2^{a-1}p^{b+1}\}$.

If b=d, then a< c and the inclusion $\Pi_{x-y}\subseteq\{2,p\}$ implies that $\Pi_{2^{c-a}-1}\subseteq\{2,p\}$ and hence $2^{c-a}-1$ is a power of p. By the Mihăilescu Theorem 4, $2^{c-a}-1\in\{1,p\}=\{1,2^m+1\}$ and hence c-a=1, which means that $\{x,y\}$ is equal to $\{2^{a-1}p^b,2^ap^b\}$.

So, we assume that $a \neq c$ and $b \neq d$. By analogy with the case of p = 3, we can show that the subcases (a < c and b < d) and (a > c and b > d) are impossible.

If a < c and b > d, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $2^{c-a} - p^{b-d} = 1$. In this case the Mihăilescu Theorem 4 ensures that b - d = 1 and hence $2^{c-a} = p + 1 = 2^m + 2$ which is not possible (as m > 1).

If a > c and b < d, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $p^{d-a} - 2^{a-c} = 1$. In this case the Mihăilescu Theorem 4 implies that d-b=1 and hence $2^{a-c}=p-1=2^m$ and a-c=m. In this case $\{x,y\}=\{2^{c+m}2^b,2^cp^{b+1}\}$.

3. Assume that $p = 2^m - 1 > 3$ is a Mersenne prime. In this case m > 2. It is easy to check that every doubleton $\{x,y\} \in \{\{2^ap^b,2^{a-1}p^b\},\{2^{a-1}p^b,2^{m+a-1}p^b\},\{2^{a-1}p^{b+1},2^{m+a-1}p^b\}:a,b\in\mathbb{N}\}$ has $A_{\{x,y\}} = \{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} = \{2,p\}$ and hence $\{x,y\} \in \mathcal{E}_p$.

Now assume that $\{x,y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p . Then $\{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2,p\}$ and $\{x,y\}$ can be written as $\{2^{a-1}p^b, 2^{c-1}p^d\}$ for some $a,b,c,d \in \mathbb{N}$ with $2^{a-1}p^b < 2^{c-1}p^d$.

If a = c, then b < d and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{p^{d-b}-1} \subseteq \{2, p\}$ and hence $p^{d-b}-1$ is a power of 2. By the Mihăilescu Theorem 4, d-b=1 and hence $2^m-2=p-1$ is a power of 2, which is not true as m > 2.

If b = d, then a < c and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{2^{c-a}-1} \subseteq \{2, p\}$ and hence $2^{c-a}-1$ is a power of p. By the Mihăilescu Theorem $4, 2^{c-a}-1 \in \{1, p\} = \{1, 2^m-1\}$ and hence $c-a \in \{1, m\}$, which means that $\{x, y\}$ is equal to $\{2^{a-1}p^b, 2^ap^b\}$ or $\{2^{a-1}p^b, 2^{m+a-1}p^b\}$.

So, we assume that $a \neq c$ and $b \neq d$. By analogy with the case of p = 3, we can show that the subcases (a < c and b < d) and (a > c and b > d) are impossible.

If a < c and b > d, then $\Pi_{x-y} \subseteq \{2,p\}$ implies that $2^{c-a} - p^{b-d} = 1$. In this case the Mihăilescu Theorem 4 ensures that b-d=1 and hence $2^{c-a}=p+1=2^m$ and c-a=m. In this case $\{x,y\}=\{2^{a-1}p^{d+1},2^{m+a-1}p^d\}$.

If a > c and b < d, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $p^{d-a} - 2^{a-c} = 1$. In this case the Mihăilescu Theorem 4 implies that d - b = 1 and hence $2^{a-c} = p - 1 = 2^m - 2$, which is not possible as m > 2.

4. Assume that p is not Fermat-Mersennne. It is easy to check that every doubleton $\{x,y\} \in \{\{2^{a-1}p^b,2^{a-1}p^{b+1}\}: a,b\in\mathbb{N}\}$ has $A_{\{x,y\}}=\{2\}\cup\Pi_x\cup\Pi_y\cup\Pi_{x-y}=\{2,p\}$ and hence $\{x,y\}\in\mathcal{E}_p$.

Now assume that $\{x,y\} \in \mathcal{E}_p$ is an edge of the graph Γ_p . Then $\{2\} \cup \Pi_x \cup \Pi_y \cup \Pi_{x-y} = A_{\{x,y\}} = \{2,p\}$ and $\{x,y\}$ can be written as $\{2^{a-1}p^b, 2^{c-1}p^d\}$ for some $a,b,c,d \in \mathbb{N}$ with $2^{a-1}p^b < 2^{c-1}p^d$.

If a = c, then b < d and the inclusion $\Pi_{x-y} \subseteq \{2, p\}$ implies that $\Pi_{p^{d-b}-1} \subseteq \{2, p\}$ and hence $p^{d-b}-1$ is a power of 2. By the Mihăilescu Theorem 4, d-b=1 and hence p is a Fermat prime, which is not true.

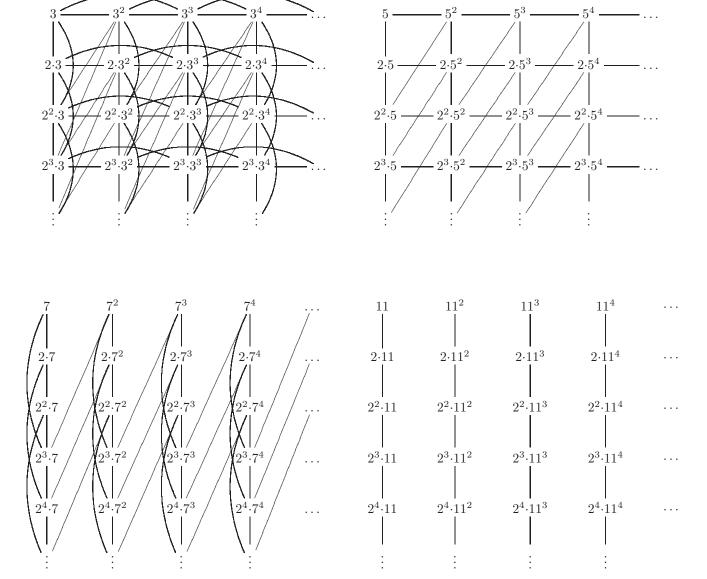
If b=d, then a< c and the inclusion $\Pi_{x-y}\subseteq\{2,p\}$ implies that $\Pi_{2^{c-a}-1}\subseteq\{2,p\}$ and hence $2^{c-a}-1$ is a power of p. By the Mihăilescu Theorem 4, $2^{c-a}-1\in\{1,p\}$. Taking into account that p is not Mersenne prime, we conclude that $2^{c-a}-1=1$ and hence c-a=1. Then $\{x,y\}=\{2^{a-1}p^b,2^ap^b\}$.

So, we assume that $a \neq c$ and $b \neq d$. By analogy with the case of p = 3, we can show that the subcases (a < c and b < d) and (a > c and b > d) are impossible.

If a < c and b > d, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $2^{c-a} - p^{b-d} = 1$. In this case the Mihăilescu Theorem 4 ensures that b - d = 1 and hence $p = 2^{c-a} - 1$ is a Mersenne prime, which is not true.

If a > c and b < d, then $\Pi_{x-y} \subseteq \{2, p\}$ implies that $p^{d-a} - 2^{a-c} = 1$. In this case the Mihăilescu Theorem 4 implies that d-b=1 and hence $p=1+2^{a-c}$ is a Fermat prime, which is not true.

In the following diagrams we draw the graphs Γ_p for p equal to 3, 5, 7, 11. Observe that 3 is both Fermat and Mersenne prime, 5 is Fermat prime, 7 is Mersenne prime and 11 is not Fermat–Mersenne.



Lemma 19. Let p be an odd prime number and h be a homeomorphism of the Kirch space.

- (1) If p is Fermat-Mersenne, then h(p) = p;
- (2) If p is not Fermai-Mersenne, then $h[p^{\mathbb{N}}] = p^{\mathbb{N}}$.

Proof. Given an odd prime number p, consider the graph $\Gamma_p = (V_p, \mathcal{E}_p)$.

First we consider the case p=3. In this case Lemma 18(1) ensures that the degree of the vertex 3 in the graph Γ_3 is equal to 4 but the other vertices have degree at least 5. Hence h(3) = 3.

Next, we assume that p > 3 is Fermat–Mersenne prime. In this case Lemma 18(2,3) implies that the degree of the vertex p in the graph Γ_p is 2 but the other vertices have degree at least 3. Hence h(p) = p.

Finally, assume that p is not Fermat-Mersenne. Then Lemma 18(4) ensures that the set $p^{\mathbb{N}}$ coincides with the set of vertices of order 1 in the graph Γ_p . Taking into account that $h \upharpoonright V_p$ is an isomorphism of the graph Γ_p , we conclude that $h[p^{\mathbb{N}}] = p^{\mathbb{N}}$.

To prove that h(p) = p for any prime number p, we will need the following lemma.

Lemma 20. For any integer number $x \in \mathbb{N} \setminus \{1\}$ the filter $\mathcal{F}_{\{1,x\}}$ is the greatest element of the subset

$$\mathfrak{F}_x = \{\mathcal{F}_{\{1,x^n\}} : n \in \mathbb{N}\}$$

of the poset \mathfrak{F} . If $x \notin \{2m : m \in \mathbb{N}\} \cup \{2^m - 1 : m \in \mathbb{N}\}$, then $\{n \in \mathbb{N} : \mathcal{F}_{\{1,x^n\}} = \mathcal{F}_{\{1,x\}}\} = \{1\}$.

Proof. Observe that for every $n \in \mathbb{N}$ the number x-1 divides x^n-1 , which implies

$$A_{\{1,x\}} = \{2\} \cup \Pi_x \cup \Pi_{x-1} \subseteq \{2\} \cup \Pi_{x^n} \cup \Pi_{x^n-1} = A_{\{1,x^n\}}.$$

Observe also that $\Pi_{\{1,x\}} = \emptyset = \Pi_{\{1,x^n\}}$ and $\alpha_{\{1,x\}}(p) = 1 = \alpha_{\{1,x^n\}}(p)$ for every $p \in A_{\{1,x\}}$. By Lemma 5, $\mathcal{F}_{\{1,x^n\}} \subseteq \mathcal{F}_{\{1,x\}}$, which means that $\mathcal{F}_{\{1,x\}}$ is the largest element of the poset \mathfrak{F}_x . Now assume that $x \notin \{2m : m \in \mathbb{N}\} \cup \{2^m - 1 : n \in \mathbb{N}\}$ and $\mathcal{F}_{\{1,x\}} = \mathcal{F}_{\{1,x^n\}}$ for some number n. We should

prove that n = 1. To derive a contradiction, assume that $n \ge 2$. By Lemmas 5 and 2,

$$\{2\} \cup \Pi_x \cup \Pi_{x^n - 1} = A_{\{1, x^n\}} = A_{\{1, x\}} = \{2\} \cup \Pi_x \cup \Pi_{x - 1}$$

and hence $\Pi_{x^n-1} \subseteq \{2\} \cup \Pi_{x-1} = \Pi_{x-1} \subseteq \bigcup_{0 < k < n} \Pi_{x^k-1}$. By Zsigmondy Theorem 5, $x \in \{2\} \cup \{2^m-1\}_{m \in \mathbb{N}}$, which contradicts our assumption.

Lemma 21. For any homeomorphism h of the Kirch space and any prime number p we have h(p) = p.

Proof. If p=2, then h(p)=p by Lemma 9. If p is Fernat-Mersenne, then h(p)=p by Lemma 19. So, we assume p is not Fermat-Mersenne. By Lemma 19, $h[p^{\mathbb{N}}] = p^{\mathbb{N}}$, which implies $\tilde{h}[\mathfrak{F}_p] = \mathfrak{F}_p$ where

$$\mathfrak{F}_p = \{ \mathcal{F}_{\{1,p^n\}} : n \in \mathbb{N} \}.$$

By Proposition 1, h induces an order isomorphism of the poset \mathfrak{F}_p (endowed with the inclusion order, inherited from the poset \mathfrak{F}).

By Lemma 20, n=1 is a unique number such that $\mathcal{F}_{\{1,p^n\}}$ coincides with the greatest element $\mathcal{F}_{\{1,p\}}$ of the poset \mathfrak{F}_x . This order characterization of the filter $\mathcal{F}_{\{1,p\}}$ implies that h(p)=p.

Our final lemma completes the proof of Theorem 1.

Lemma 22. The homeomorphism group of the Kirch space is trivial.

Proof. To derive a contradiction, assume that the Kirch space admits a homeomorphism h such that $h(x) \neq x$ for some number x. By the Hausdorff property of the Kirch space and the continuity of h, there exists a neighborhood O_x of x in the Kirch topology such that $h[O_x] \cap O_x = \emptyset$. By the Dirichlet Theorem 3, the open set O_x contains some prime number p. Then $h[O_x] \cap O_x = \emptyset$ implies $h(p) \neq p$, which contradicts Lemma 21. \square

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