THE REFLECTION REPRESENTATION IN THE HOMOLOGY OF SUBWORD ORDER

SHEILA SUNDARAM

Abstract. We investigate the homology representation of the symmetric group on rank-selected subposets of subword order. We show that the homology module for words of bounded length decomposes into a sum of tensor powers of the S_n-irreducible indexed by the partition $(n-1,1)$, recovering, as a special case, a theorem of Björner and Stanley for words of length at most k. For arbitrary ranks we show that the homology is an integer combination of positive tensor powers of the reflection representation, and conjecture that this combination is nonnegative. We exhibit a curious duality in homology in the case when one rank is deleted. Our most definitive result describes the Frobenius characteristic of the homology for an arbitrary set of ranks, plus or minus one copy of the reflection representation $S_{(n-1,1)}$, as an integer combination of the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. We conjecture that this combination is nonnegative, and establish this for particular cases.

Keywords: Subword order, reflection representation, h-positivity, Whitney homology, Kronecker product, internal product, Stirling numbers.

1. Introduction

Let A^* denote the free monoid of words of finite length in an alphabet A. Subword order is defined on A^* by setting $u \leq v$ if u is a subword of v, that is, the word u is obtained by deleting letters of the word v. This makes $(A^*, \leq)$ into a graded poset with rank function given by the length $|w|$ of a word w, the number of letters in w. The topology of this poset was first studied by Farmer (1979) and then by Björner, who showed in [8, Theorem 3] that any interval of this poset admits a dual CL-shelling. The intervals are thus homotopy Cohen-Macaulay, as well as all rank-selected subposets obtained by considering only words whose rank belongs to a finite set S. Suppose now that the alphabet A is finite, of cardinality n. The symmetric group S_n acts on A, and thus on A^*. To avoid trivialities we will assume $n \geq 2$.

In this paper we describe the homology representation of intervals $[r, k]$ of consecutive ranks in A^*, as well as some other rank-selected subposets, using the Whitney homology technique and other methods developed in [18]. We refer the reader to [15] for general facts about rank-selection. We show that the unique nonvanishing homology of the rank-selected subposet $A^*_{[r,k]}$ decomposes as a direct sum of copies of r consecutive tensor powers of the reflection representation of S_n, that is, the irreducible representation $S_{(n-1,1)}$ indexed by the partition $(n-1,1)$. Theorem 13 on consecutive ranks generalises a theorem in [8] (conjectured by Björner and proved by Stanley) on the homology representation of the poset of all words of length at most k. We establish similar results for the Whitney and dual Whitney homology modules. The Whitney homology turns out to be a permutation module in each degree, with pleasing orbit stabilisers. Theorem 16

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establishes the nonnegativity property with respect to tensor powers of $S_{(n-1,1)}$ for the case when one rank is deleted from the interval $[1,k]$, and leads to a curious homology isomorphism (Proposition 17) suggesting an equivariant homotopy equivalence between the simplicial complexes associated to the rank sets $[1,k]\backslash\{r\}$ and $[1,k]\backslash\{k-r\}$, for fixed $r, 1 \leq r \leq k-1$.

More generally, we show in Theorem 19 that for any nonempty subset S of ranks $[1,k]$, the homology representation of S_n may be written as an integer combination of positive tensor powers of the reflection representation. Based on our determination of this and other cases of rank-selection, we propose the following conjecture:

Conjecture 1. *Let A be an alphabet of size $n \geq 2$. Then the S_n-action on the homology of any finite nonempty rank-selected subposet of subword order on A^* is a nonnegative integer combination of positive tensor powers of the irreducible indexed by the partition $(n-1,1)$.*

These considerations lead us to examine the tensor powers of the reflection representation (see Section 6), and the question of how many tensor powers are linearly independent characters. In answering these questions, we are led to a decomposition showing that the kth tensor power of $S_{(n-1,1)}$ plus or minus one copy of $S_{(n-1,1)}$, has Frobenius characteristic equal to a nonnegative integer combination of the homogeneous symmetric functions $\{h_{n-r,1^r)} : r \geq 2\}$. It is "almost" an h-positive permutation module. Inspired by this phenomenon, we prove, in Theorem 31, that in fact for all rank subsets T, the homology module $\tilde{H}(T)$ has the property that $\tilde{H}(T) + (-1)^{|T|}S_{(n-1,1)}$ has Frobenius characteristic equal to an *integer* combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geq 2\}$. Theorem 33 establishes the truth of the following conjecture for all the rank-selected homology computed in this paper.

Conjecture 2. *Let A be an alphabet of size $n \geq 2$. Then the homology of any finite nonempty rank-selected subposet of subword order on A^*, plus or minus one copy of the reflection representation of S_n, is a permutation module. In fact its Frobenius characteristic is h-positive and supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$.*

We give a simple criterion for when Conjecture 1 will imply Conjecture 2 in Lemma 39.
The following theorem summarises the main results of this paper.

Theorem 1. *Let $T \subseteq [1,k]$ be one of the following rank sets:*
(1) $[r,k], k \geq r \geq 1$; (2) $[1,k]\backslash\{r\}, k \geq r \geq 1$; (3) $\{1 \leq s_1 < s_2 \leq k\}$.
Then

- **(A):** *The rank-selected homology $\tilde{H}_{k-2}(A^*_{n,k}(T))$ is a nonnegative integer combination of positive tensor powers of the reflection representation $S_{(n-1,1)}$, with simple formulas for the coefficients;*
- **(B):** *The module $V_T = \tilde{H}_{k-2}(A^*_{n,k}(T)) + (-1)^{|T|}S_{(n-1,1)}$ is a nonnegative integer combination of transitive permutation modules with orbit stabilisers of the form $S_1^d \times S_{n-d}, d \geq 2$. Equivalently, the Frobenius characteristic of V_T is supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$ with nonnegative integer coefficients.*

2. Subword order

The subword order poset A^* has a unique least element at rank 0, namely the empty word $\emptyset$ of length zero. In this section we collect the main facts on subword order from

[8] that we will need. For general facts about posets, Möbius functions, etc. we refer the reader to [16].

Definition 2. [9] A word α in A^* is *normal* if no two consecutive letters of α are equal.

For example, *aabbccaabbcc* is not normal, while *abcabc* is normal. Normal words are also called *Smirnov* words in the recent literature. Observe that the number of normal words of length i is $n(n-1)^i$.

Theorem 3. *(Farmer [9])*

(1) Let α be any word in A^*. Then the Möbius function of subword order satisfies
$$\mu(\hat{0},\alpha)=\begin{cases}(-1)^{|\alpha|}, & \text{if } \alpha \text{ is a normal word}\\ 0, & \text{otherwise.}\end{cases}$$
(2) (See also [20].) Let $|A|=n$ and let $A^*_{n,k}$ denote the subposet of A^* consisting of the first k nonzero ranks and the empty word, i.e. of words of length at most k, with an artificially appended top element $\hat{1}$. Then
$$\mu(A^*_{n,k})=\mu(\hat{0},\hat{1})=(-1)^{k-1}(n-1)^k. \tag{2.1}$$
(3) [9, Theorem 5 and preceding Remark] *$A^*_{n,k}$ has the homology of a wedge of $(n-1)^k$ spheres of dimension $(k-1)$.*

Björner generalised Part (1) above to give a simple formula for the Möbius function of an arbitrary interval (β,α), as follows.

Definition 4. [8] Given a word $\alpha=a_1a_2\ldots a_n$ in A^*, its repetition set is $R(\alpha)=\{i : a_{i-1}=a_i\}$. An embedding of β in α is a sequence $1\le i_1<i_2<\ldots<i_k\le n$ such that $\beta=a_{i_1}a_{i_2}\ldots a_{i_k}$. It is called a *normal embedding* if in addition $R(\alpha)\subseteq\{i_1,i_2,\ldots,i_k\}$.

Denote by $\binom{\alpha}{\beta}$ the number of embeddings of β in α, and by $\binom{\alpha}{\beta}_n$ the number of normal embeddings of β in α.

Theorem 5. [8, Theorem 1] *For all $\alpha,\beta\in A^*$,*
$$\mu(\beta,\alpha)=(-1)^{|\alpha|-|\beta|}\binom{\alpha}{\beta}_n.$$

Observing that the word α is normal if and only if its repetition set $R(\alpha)$ is empty, one sees that this generalises Farmer's formula for $\mu(\hat{0},\alpha)$.

Recall that the zeta function [16] of a poset is defined by $\zeta(\beta,\alpha)=1$ if $\beta\le\alpha$, and equals zero otherwise.

Theorem 6. [8] *Let A be an alphabet of size n, and β a word in A^* of length k. The following generating functions hold:*

(1) [8, Theorem 2 (i)] *For the Möbius function of subword order:*
$$\sum_{\alpha\in A^*}\mu(\beta,\alpha)t^{|\alpha|}=\frac{t^k(1-t)}{(1+(n-1)t)^{k+1}}.$$
(2) [8, 3. Remark.] *The number of words of length p in the interval $[\beta,\infty]$ depends only on the length k of β, and equals*
$$\sum_{i=0}^{p-k}\binom{p}{i}(n-1)^i.$$

(3) [8, 3. Remark.(i)] *For the zeta function of subword order:*

$$\sum_{\alpha \in A^*} \zeta(\beta, \alpha) t^{|\alpha|} = \frac{t^k}{(1-nt)(1-(n-1)t)^k}.$$

Farmer's result on the homology of $A^*_{n,k}$ was strengthened by Björner, who showed the following (see [4], [7], and also [21] for a survey of lexicographic shellability):

Theorem 7. *(Björner* [8, Theorem 3, Corollary 2]*) Every interval* (β, α) *in the subword order poset* A^* *is dual CL-shellable, and hence homotopy Cohen-Macaulay. In particular, for a finite alphabet* A*, the poset* $A^*_{n,k}$ *of nonempty words of length at most* k*, which may be viewed as the result of rank-selection from an appropriate interval of* A^**, is also dual CL-shellable and hence also homotopy Cohen-Macaulay.*

We point out three details about Farmer's original paper:

a: The order used in the present paper is what Farmer calls the *embedding* order (see [9, p.609]). Farmer's "subword order" differs from ours and [8].

b: All homology in Farmer's paper is ordinary homology, as opposed to reduced homology in the present paper and [15]. In keeping with his definition of a graded poset, for the rank function d of subword order, Farmer defines $d(w) = j - 1$ if w is a word of length j; in this paper we use the length of the word as its rank.

c: In particular, his definition of the k-skeleton X^k of a poset of words X corresponds to our A^*_{k+1}, i.e. to taking the words of length at most $(k+1)$.

3. Rank-selection in A^*

In this section we will assume the alphabet A is finite of size n.

We follow the standard convention as in [15], [16]: By the homology of a poset P with greatest element $\hat{1}$ and least element $\hat{0}$, we mean the reduced homology $\tilde{H}(P)$ of the simplicial complex whose faces are the chains of $P \backslash \{\hat{0}, \hat{1}\}$. In order to determine the homology of rank-selected subposets of $A^*_{n,k}$, we will use the techniques developed in [18]. For an elementary treatment of these and more general methods, see [19].

The Whitney homology of a poset was originally defined by Baclawski [1]. Björner showed [5] that the ith Whitney homology of a graded poset P with least element $\hat{0}$ is given by the isomorphism

$$WH_i(P) \simeq \bigoplus_{x:\mathrm{rank}(x)=i} \tilde{H}_{i-2}(\hat{0}, x). \tag{3.1}$$

Note that if P has a top element $\hat{1}$, then the top Whitney homology coincides with the top homology of P.

If G is a group acting on the poset P, then $WH_i(P)$ is also a (possibly virtual) G-module. The present author observed that the isomorphism (3.1) is in fact group-equivariant, and also established the equivariant acyclicity of Whitney homology (see [18]). Thus (3.1) becomes an effective tool for computing both the G-module structure of Whitney homology as well as the homology of the full poset, as an equivariant analogue of the inherently recursive structure in the Möbius function. This technique was then exploited in [18] and [19] to determine group actions on the homology of posets.

In the special case when P and all its intervals $(\hat{0}, x)$ have unique nonvanishing homology (e.g. if P is Cohen-Macaulay), then each $WH_i(P)$ is also a true G-module.

It is also computationally useful to consider the *dual* Whitney homology of P when P has a top element $\hat{1}$, that is, the Whitney homology of the dual poset P^*, which we denote by $WH^*(P)$. Note that we now have an equivariant isomorphism

$$WH_i^*(P) \simeq \bigoplus_{x:\mathrm{rank}(x)=r-i} \tilde{H}_{i-2}(x, \hat{1}), 0 \leq i \leq r. \tag{3.2}$$

Here r is the length of the longest chain from $\hat{0}$ to $\hat{1}$.

See [18] and [19] for a more general version of the following theorem (for arbitrary posets).

Theorem 8. [18, Lemma 1.1, Theorem 1.2, Proposition 1.9] *Let P be a graded Cohen-Macaulay poset of rank r carrying an action of a group G. Then the unique nonvanishing top homology of P coincides with the top Whitney homology module $WH_r(P)$, and as a G-module, can be computed as an alternating sum of Whitney homology modules:*

$$\tilde{H}_{r-2}(P) \simeq \bigoplus_{i=0}^{r-1} (-1)^i WH_{r-1-i}(P). \tag{3.3}$$

In particular, if $P(\underline{k})$ denotes the subposet consisting of the first k nonzero ranks, with a bottom and top element attached, then one has the G-module decomposition

$$\tilde{H}_{k-2}(P(\underline{k-1})) \oplus \tilde{H}_{k-1}(P(\underline{k})) \simeq WH_k(P), r \geq k \geq 1. \tag{3.4}$$

Note that $WH_0(P)$ is the trivial G-module, while $WH_r(P)$ gives the reduced top homology of the poset P.

In the following proposition, we formalise Stanley's insight into subword order, as used in the proof of [8, Theorem 4]. For more background on the Hopf trace formula and its use in poset homology, see [19]. Recall that the *Lefschetz module* of a poset P is the alternating sum (by degree) of the homology modules of (the order complex) of P.

Denote by S_λ the irreducible representation of the symmetric group S_n indexed by the partition λ of n, and write $S_\lambda^{\otimes i}$ for the ith tensor power of the module S_λ.

Proposition 9. *Let $\{P_n\}$ be any sequence of finite posets each carrying an action of the symmetric group S_n, such that*

(1) For any $g \in S_n$, the fixed-point subposet P_n^g is isomorphic to the poset $P_{\mathrm{fix}(g)}$, where $\mathrm{fix}(g)$ is the number of fixed points of g, and
(2) the Möbius number $\mu(P_n)$ is a polynomial in $(n-1)$, say $\sum_{i\geq 0} b_i(n-1)^i$.

Then the Lefschetz module of P_n decomposes as a sum of ith tensor powers of the irreducible indexed by the partition $(n-1,1)$, with coefficient equal to $b_i, i \geq 0$. (Note that the 0th tensor power corresponds to the trivial S_n-module $S_{(n)}$.) In particular, the S_n-module structure of the Lefschetz module of P_n is completely determined by its Möbius number.

More generally, if for all $k \geq 0$, the Betti number of $WH_k(P_n)$ is a polynomial in $(n-1)$, then this polynomial determines the trace of $g \in S_n$ on the kth Whitney homology of P_n. The action of S_n on $WH_k(P_n)$ is therefore a linear combination of tensor powers of the irreducible $S_{(n-1,1)}$.

Proof. This is clear since

(1) ([15], [16], [19]) the Lefschetz module of P_n has (virtual) degree $\mu(P_n)$, the Euler characteristic of the order complex of P_n;
(2) ([16], [19]) by the Hopf trace formula, the trace of an element $g \in S_n$ on this Lefschetz module is the Möbius number $\mu(P_n^g)$ of the fixed-point poset P_n^g, since it is the Euler characteristic of the order complex of P_n^g;
(3) by hypothesis, $\mu(P_n^g) = \mu(P_{\mathrm{fix}(g)}) = \sum_i b_i(\mathrm{fix}(g)-1)^i$, and finally
(4) the trace of g on the irreducible S_n-module indexed by $(n-1,1)$ is $\mathrm{fix}(g)-1$.

Similar conclusions hold for Whitney homology. The key observation here is that from Björner's formulation eqn. (3.1), it follows that the Whitney homology of the fixed-point subposet P_n^g coincides with the Whitney homology of $P_{\mathrm{fix}(g)}$. □

Our motivating example for the poset P_n satisfying the conditions of Proposition 9 is clearly subword order A^* when $|A| = n$. More generally, fix an integer $k \geq 1$, and let S be any subset of the ranks $[1,k]$. Then the rank-selected subposet $A^*_{n,k}(S)$ of A^* consisting of elements with ranks belonging to S also satisfies the conditions of Proposition 9. Recall that when $S = [1,k]$ we denote this rank-selected subposet $A^*_{[1,k]}$ simply by $A^*_{n,k}$.

Using the generating function for the Möbius function of A^* given in Theorem 6, the proposition below computes all but the top Whitney homology S_n-modules for subword order. The proof requires a key formula, which we derive from the generating function for the Möbius function of A^* given in Theorem 6. We isolate this computation in the following lemma.

Lemma 10. *Let β be any element of $A^*_{n,k}\backslash\{\hat{1}\}$, where the alphabet A has cardinality n. then*

$$\mu(\beta,\hat{1})_{A^*_{n,k}}(-1)^{k+1-|\beta|} = \binom{k}{|\beta|}(n-1)^{k-|\beta|}.$$

In particular this Möbius number depends only on the rank (length) of the word β.

Proof. For convenience let $|\beta| = i$. We have, using the defining recurrence for the Möbius function,

$$\mu(\beta,\hat{1})_{A^*_{n,k}}(-1)^{k+1-|\beta|} = (-1)^{k+1-i}(-1)\sum_{\substack{\alpha\in A^*_{n,k}\\ \alpha<\hat{1}}}\mu(\beta,\alpha) = (-1)^{k-i}\sum_{j=i}^{k}\sum_{\substack{\alpha\in A^*_{n,k}\\ |\alpha|=j}}\mu(\beta,\alpha)$$

$$= (-1)^{k-i}\sum_{j=i}^{k}[t^j](1-t)t^i(1+t(n-1))^{-(i+1)} = (-1)^{k-i}\sum_{j=i}^{k}[t^{j-i}](1-t)(1+t(n-1))^{-(i+1)}$$

(using the generating function in (1) of Theorem 6).

Setting $u = j - i$, this in turn equals

$$(-1)^{k-i} \sum_{u=0}^{k-i} [t^u](1-t)(1+t(n-1))^{-(i+1)}, \text{ (setting } u = j-i)$$
$$= (-1)^{k-i}[t^{k-i}](1+t(n-1))^{-(i+1)},$$
$$\text{(since for any power series } f(t), \text{ one has } \sum_{j=0}^{m} [t^j](1-t)f(t) = [t^m]f(t)),$$
$$= (-1)^{k-i}\binom{-(i+1)}{k-i}(n-1)^{k-i} = \binom{i+1+k-i-1}{k-i}(n-1)^{k-i}.$$

The last line follows by using the fact that $\binom{-m}{j} = (-1)^j\binom{m+j-1}{j}$, thereby completing the proof. $\square$

Theorem 11. *Consider the subword order poset $A^*_{n,k}$, with $|A| = n$. As S_n-modules, the Whitney homology $WH(A^*_{n,k})$ and the dual Whitney homology $WH^*(A^*_{n,k})$, for $1 \le i \le k$, are as follows. Note that $WH_0(A^*_{n,k}) = S_{(n)} = WH^*_{k+1}(A^*_{n,k})$ (the trivial S_n-module).*

$$WH_i(A^*_{n,k}) = S^{\otimes i}_{(n-1,1)} \oplus S^{\otimes (i-1)}_{(n-1,1)}; \tag{3.5}$$

$$WH^*_{k+1-i}(A^*_{n,k}) = \binom{k}{i} S^{\otimes (k-i)}_{(n-1,1)} \otimes (S_{(n-1,1)} \oplus S_{(n)})^{\otimes i} \tag{3.6}$$

$$= \bigoplus_{j=0}^{i} \binom{k}{i}\binom{i}{j} S^{\otimes j+(k-i)}_{(n-1,1)}. \tag{3.7}$$

Proof. For fixed k, we will show that the Betti number of the kth Whitney homology is a polynomial in $(n-1)$ with nonnegative coefficients. By Proposition 9, to compute the action of S_n, it is enough to carry out the appropriate Möbius number (in effect, Betti number) computations.

For the Whitney homology, for $0 \le i \le k$ we have $WH_i(A^*_{n,k}) = \bigoplus_{x:|x|=i} \tilde{H}(\hat{0}, x)_{A^*_{n,k}}$. Hence we compute $\sum_{x:|x|=i}(-1)^{i-2}\mu(\hat{0}, x)_{A^*_{n,k}}$. By Theorem 3, we need only sum over the normal words of length i, and these are clearly $n(n-1)^{i-1}$ in number. Continuing the computation, we have

$$\sum_{\substack{x:|x|=i \\ x \text{ is a } normal \text{ word}}} (-1)^{i-2} \cdot (-1)^i = n(n-1)^{i-1} = (n-1)^i + (n-1)^{i-1}, 0 \le i \le k. \tag{3.8}$$

Invoking Proposition 9 now gives the first isomorphism.

For the dual Whitney homology, for $0 \le i \le k$ we have

$$WH^*_{k+1-i}(A^*_{n,k}) = \bigoplus_{x:|x|=i} \tilde{H}(x, \hat{1})_{A^*_{n,k}}.$$

Computing Betti numbers, and using Lemma 10, we have that the dimension of the dual Whitney homology module equals

$$\sum_{\substack{x \text{ any word} \\ |x|=i}} (-1)^{k+1-i}\mu(x, \hat{1})_{A^*_{n,k}} = \sum_{\substack{x \text{ any word} \\ |x|=i}} \binom{k}{i}(n-1)^{k-i} = n^i\binom{k}{i}(n-1)^{k-i}$$

This expression translates into the one in the statement of the proposition, since the trace of g on $S_{(n-1,1)} \oplus S_{(n)}$ is the number of fixed points of g. The second expression is obtained from the binomial expansion of n^i into powers of $(n-1)$. □

Now apply acyclicity of Whitney homology, Theorem 8, to deduce the top homology:

Corollary 12. *The top homology of $A^*_{n,k}$ as an S_n-module is given by*

(1) *(Björner-Stanley* [8, Theorem 4]*)*

$$S_{(n-1,1)}^{\otimes k}.$$

(2) *It is also equal to the alternating sums*

$$\sum_{i=0}^{k}(-1)^{k-i}\left(S_{(n-1,1)} \oplus S_{(n)}^{\otimes i} \otimes \binom{k}{i} S_{(n-1,1)}^{\otimes(k-i)}\right)$$

$$= \sum_{i=0}^{k}(-1)^{k-i} \bigoplus_{j=0}^{i} \binom{k}{i}\binom{i}{j} S_{(n-1,1)}^{\otimes j+(k-i)}.$$

Proof. The first result is an immediate consequence of the fact that the alternating sum of Whitney homology modules in (3.5) telescopes to a single term, namely $(n-1)^k$.

The second and third expressions are simply the alternating sums of dual Whitney homology modules in (3.6). □

We can now prove the main result of this section, which generalises the preceding corollary to the rank-set $[r, k]$ consisting of the interval of consecutive ranks $r, r+1, \ldots, k$. To do this, we must rewrite the partial alternating sums of terms appearing in the dual Whitney homology (3.6) as a nonnegative linear combination rather than a signed sum. The poset of words in an alphabet of size n, with lengths bounded above by k and below by r, has homology as follows.

Theorem 13. *Fix $k \geq 1$ and let S be the interval of consecutive ranks $[r, k]$ for $1 \leq r \leq k$. Then the rank-selected subposet $A^*_{n,k}(S)$ has unique nonvanishing homology in degree $k - r$, and the S_n-homology representation on $\tilde{H}_{k-r}(A^*_{n,k}(S))$ is given by the decomposition*

$$\bigoplus_{i=1+k-r}^{k} b_i\, S_{(n-1,1)}^{\otimes i}, \quad \text{where } b_i = \binom{k}{i}\binom{i-1}{k-r}, i = 1+k-r, \ldots, k. \tag{3.9}$$

Proof. For brevity we will simply write $\tilde{H}([i, k])$ for the homology of the subposet $A^*_{n,k}(S)$ when $S = [i, k]$.

Recall again from Proposition 9 that it suffices to work with the Betti numbers, for which (3.4) in Theorem 8, in conjunction with Theorem 11, gives the following recurrence for $1 \leq i \leq k-1$:

$$\dim \tilde{H}([i, k]) \oplus \dim \tilde{H}([i+1, k]) = \dim WH^*_{k+1-i}(A^*_{n,k}) = n^i \binom{k}{i} (n-1)^{k-i}. \tag{3.10}$$

We will prove the Betti number version of (3.9) by induction on i. Note that the result is true for $i = 1$, since in that case the formula in (3.9) gives simply $S_{(n-1,1)}^{\otimes k}$, with Betti number $(n-1)^k$, in agreement with Theorem 3.

When $r=k$, the formula (3.9) reduces to $\sum_{i=1}^{k}\binom{k}{i}\binom{i-1}{0}(n-1)^i$, which equals n^k-1. This is easily seen to be the correct Möbius number (up to sign) since we then have a single rank consisting of the n^k words of length k. Also observe that when $i=k-1$, the recurrence (3.10) gives

$$\dim \tilde{H}([k-1,k]) = n^{k-1}k(n-1)-n^k+1=(k-1)n^k-kn^{k-1}+1.$$

Let $i=1$. The recurrence (3.10) gives

$$\begin{aligned}\dim \tilde{H}([2,k]) &= n\binom{k}{1}(n-1)^{k-1}-\dim \tilde{H}([1,k])\\ &= kn(n-1)^{k-1}-(n-1)^k=(k-1)(n-1)^k+k(n-1)^{k-1},\end{aligned}$$

and this coincides with (3.9) for $r=2$.

Assume that (3.9) holds for the rank-set $S=[r,k]$. We will show that it must hold for $S=[r+1,k]$. By hypothesis we have $\dim \tilde{H}([r,k])=\sum_{j=1+(k-r)}^{k}\binom{k}{j}\binom{j-1}{k-r}(n-1)^j$, and hence the recurrence (3.10) gives, for $\dim\tilde{H}([r+1,k])$, the expression

$$\binom{k}{r}(n-1)^{k-r}n^r-\sum_{j=1+(k-r)}^{k}\binom{k}{j}\binom{j-1}{k-r}(n-1)^j.$$

Expanding n^r in powers of $(n-1)$, we obtain

$$\binom{k}{r}(n-1)^{k-r}\sum_{i=0}^{r}\binom{r}{i}(n-1)^i-\sum_{j=1+(k-r)}^{k}\binom{k}{j}\binom{j-1}{k-r}(n-1)^j.$$

The coefficient of $(n-1)^{k-r}$ is clearly $\binom{k}{r}=\binom{k}{k-r}\binom{r}{0}$, in agreement with (3.9). For $j=1+(k-r),\dots r+(k-r)$, the term $(n-1)^j$ appears with coefficient c_j where

$$\begin{aligned}c_j &= \binom{k}{r}\binom{r}{j-k+r}-\binom{k}{j}\binom{j-1}{k-r}\\ &= \binom{k}{j}\left(\frac{j!(k-j)!}{r!(k-r)!}\frac{r!}{(j-k+r)!(k-j)!}-\binom{j-1}{k-r}\right)\\ &= \binom{k}{j}\left(\binom{j}{k-r}-\binom{j-1}{k-r}\right)=\binom{k}{j}\binom{j-1}{k-r-1},\end{aligned}$$

which is precisely as predicted by (3.9) for $S=[r+1,k]$. This finishes the inductive step, and hence the proof. □

This proof establishes the following combinatorial identity, which will be instrumental in the proof of Theorem 33 later in the paper.

Corollary 14.

$$\begin{aligned}&\sum_{i=0}^{k+1-r}(-1)^i\dim WH^*_{k+1-(r+i)}(A^*_{n,k})\\ &=\sum_{i=0}^{k-r}(-1)^i\binom{k}{r+i}n^{r+i}(n-1)^{k-(r+i)}+(-1)^{k+1-r}=\sum_{i=1+k-r}^{k}\binom{k}{i}\binom{i-1}{k-r}(n-1)^i.\end{aligned}$$

4. Deleting one rank from $A^*_{n,k}$: a curious isomorphism of homology

In this section we will determine the homology representation of the rank-selected subposet $A^*_{n,k}(S)$ of $A^*_{n,k}$ when S is obtained by deleting one rank from the interval $[1,k]$. In this special case the computation will reveal a curious duality in homology.

Again we use a method developed in [18] which is particularly useful for Lefschetz homology computations when the deleted set is an antichain. The version below is the special case when homology is concentrated in a single degree.

Theorem 15. *([18, Theorem 1.10], [19]) Let P be a Cohen-Macaulay poset of rank r, G a group of automorphisms of P and let Q be a subposet obtained by deleting a G-invariant antichain T in P. Then Q is also G-invariant. Assume Q is graded and has homology concentrated in the highest degree $rank(Q)-2$. Then one has the G-equivariant decomposition*

$$(4.1)\quad (-1)^{r-rank(Q)}\tilde{H}(Q)-\tilde{H}_{r-2}(P)=\bigoplus_{x\in T/G}(-1)\cdot(\tilde{H}(\hat{0},x)_P\otimes\tilde{H}(x,\hat{1})_P)\uparrow^G_{stab(x)},$$

where the sum runs over one element $x\in T$ in each orbit of G.

Here $stab(x)$ denotes the stabiliser subgroup of G which fixes the element x.

We apply this theorem to the poset $A^*_{n,k}$ and the rank-set $S=[1,k]\backslash\{r\}$, removing all words of length r, for a fixed r in $[1,k]$.

Theorem 16. *As an S_n-module, we have*

$$\tilde{H}_{k-2}(A^*_{n,k}(S))\simeq\left[\binom{k}{r}-1\right]S^{\otimes k}_{(n-1,1)}\oplus\binom{k}{r}S^{\otimes k-1}_{(n-1,1)}.$$

Proof. We invoke Proposition 9 by fixing a rank-set S and considering the family of posets $P_n=A^*_{n,S}=A^*_{n,k}(S)$, where $n=|A|$. Once again we need only compute Möbius numbers in Theorem 15. Writing simply $\mu(P)$ for the Möbius number of the poset P, the Betti number identity given by the theorem is

$$-(-1)^{k-2}\mu(A^*_{n,k}(S))-(-1)^{k-1}\mu(A^*_{n,k})=(-1)\cdot\sum_{x:|x|=r}(-1)^r\mu(\hat{0},x)_{A^*_{n,k}}\cdot(-1)^{k+1-r}\mu(x,\hat{1})_{A^*_{n,k}},$$

or equivalently, clearing signs,

$$\mu(A^*_{n,k}(S))-\mu(A^*_{n,k})=(-1)\cdot\sum_{x:|x|=r}\mu(\hat{0},x)_{A^*_{n,k}}\cdot\mu(x,\hat{1})_{A^*_{n,k}}.$$

The summand corresponding to a word x of length r in the right-hand side of this equation is nonzero only if x is a normal word, by Theorem 3. We therefore obtain, using Lemma 10

$$\begin{aligned}\mu(A^*_{n,k}(S))-\mu(A^*_{n,k})&=(-1)\cdot(-1)^r n(n-1)^{r-1}\mu(x_0,\hat{1})_{A^*_{n,k}}\\&=(-1)\cdot(-1)^r n(n-1)^{r-1}(-1)^{k-r+1}(n-1)^{k-r}\binom{k}{r}=(-1)^k n(n-1)^{k-1}\binom{k}{r},\end{aligned}$$

for any fixed normal word x_0 of length r.

Hence

$$(-1)^k\mu(A^*_{n,k}(S)) = (-1)^k\mu(A^*_{n,k}) + n(n-1)^{k-1}\binom{k}{r} = -(n-1)^k + \binom{k}{r}n(n-1)^{k-1}$$
$$= \left[\binom{k}{r} - 1\right](n-1)^k + \binom{k}{r}(n-1)^{k-1}$$

Since $A^*_{n,k}(S)$ has rank k, this is precisely the Betti number version of the statement of the theorem, thereby completing the proof. □

An immediate and intriguing corollary is the following.

Proposition 17. *Let $|A| = n$. Fix a rank $r \in [1, k-1]$. Then the homology modules of the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$ are S_n-isomorphic.*

It would be interesting to explain this isomorphism topologically. More precisely:

Question 3. Is there an S_n-homotopy equivalence between the simplicial complexes associated to the subposets $A^*_{n,k}([1,k]\backslash\{r\})$ and $A^*_{n,k}([1,k]\backslash\{k-r\})$?

5. The action on chains, and arbitrary rank-selected homology

Assume $|A| = n$. For a subset $S \subseteq [1, k]$, denote by $\alpha_n(S)$ the permutation module of S_n afforded by the maximal chains of the rank-selected subposet $A^*_{n,k}(S)$. In this section we derive a recurrence for the action, and hence an explicit formula. We begin with an analogue of Proposition 9 for the chains.

Proposition 18. *Let $\{P_n\}$ be any sequence of finite posets each carrying an action of the symmetric group S_n, such that for any $g \in S_n$, the fixed-point subposet P_n^g is isomorphic to the poset $P_{\mathrm{fix}(g)}$, where $\mathrm{fix}(g)$ is the number of fixed points of g. Suppose that the number of chains of P_n is a polynomial in $(n-1)$, say $\sum_{i\geq 0} a_i(n-1)^i$. Then the permutation action of S_n on the chains of P_n decomposes as a sum of ith tensor powers of the irreducible indexed by the partition $(n-1,1), i \geq 0$, with coefficient equal to a_i. In particular, the S_n-module structure of the chains of P_n is completely determined by its dimension.*

Proof. Since S_n acts by permuting the chains, the trace of $g \in S_n$ on the chains of P_n is equal to the number of chains fixed by g. As in the proof of Proposition 9, the key point is that this in turn is the number of chains in the fixed-point poset P_n^g, and the latter coincides with $P_{\mathrm{fix}(g)}$. □

Theorem 19. *For any subset $S \subseteq [1, k]$, the action of S_n on the maximal chains of the rank-selected subposet $A^*_{n,k}(S)$ is a nonnegative integer combination of tensor powers of the irreducible indexed by $(n-1,1)$. Hence the S_n-action on the homology of the rank-selected subposet $A^*_{n,k}(S)$ is an integer combination of positive tensor powers of the irreducible indexed by $(n-1,1)$. The highest tensor power that can occur is the mth, where $m = \max(S)$.*

Proof. Let $S = \{1 \leq s_1 < s_2 < \ldots < s_p \leq k\}$. By Part (2) of Theorem 6, the number of words in $[\beta, \infty]$ depends only on $|\beta|$. This immediately gives the recurrence

$$\alpha_n(S) = \alpha_n(S\backslash\{s_p\}) \cdot \sum_{i=0}^{s_p - s_{p-1}} \binom{s_p}{i}(n-1)^i. \tag{5.1}$$

Since $\alpha_n(\{s_1\}) = n^{s_1} = \sum_{i=0}^{s_1} \binom{s_1}{i}(n-1)^i$, by Proposition 18 and an induction argument, it is clear that $\alpha_n(S)$ is a nonnegative integer combination of $S_{(n-1,1)}^{\otimes j}, 0 \le j \le m = \max(S)$. It is also clear that the 0th tensor power, that is, the trivial module $S_{(n)}$, occurs exactly once in each $\alpha_n(S)$.

Note that when $S = \emptyset$, the homology is simply the trivial module. The claim about the decomposition of the homology into tensor powers of $S_{(n-1,1)}$ now follows from Stanley's theory of rank-selected homology representations [15]. We have

$$\alpha_n(T) = \sum_{S \subseteq T} \beta_n(S),$$

and thus

$$\beta_n(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} \alpha_n(S), \tag{5.2}$$

where $\beta_n(S)$ is the representation of S_n on the homology of the rank-selected subposet $A^*_{n,k}(S)$ of $A^*_{n,k}$. When T is nonempty, it is clear from the previous paragraph that the occurrences of the 0th tensor power, which equals $S_{(n)}$, all cancel in (5.2); the trivial module occurs with coefficient $\sum_{S \subseteq T}(-1)^{|T|-|S|}$, which is zero. Hence only positive tensor powers will appear. $\square$

Thus Theorem 19 supports Conjecture 1. Note that it is easy to concoct signed integer combinations of tensor powers that are not true S_n-modules. For instance, the integer combination $S_{(n-1,1)}^{\otimes 2} - 2\, S_{(n-1,1)}$ decomposes into $S_{(n)} + S_{(n-2,2)} S_{(n-2,1,1)} - S_{(n-1,1)}$, while $S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)} = S_{(n)} + S_{(n-2,2)} + S_{(n-2,1,1)}$ is a true S_n-module. See Theorem 25 below.

It is worth pointing out the special case for the full poset $A^*_{n,k}$.

Theorem 20. *The action of S_n on the maximal chains of $A^*_{n,k}$ decomposes into the sum*

$$S_{(n)} \oplus \bigoplus_{j=1}^{k} c(k+1,j) S_{(n-1,1)}^{k+1-j},$$

where $c(k+1,j)$ is the number of permutations in S_{k+1} with exactly j cycles in its disjoint cycle decomposition.

Proof. Specialising (5.1) to the case $S = [1,k]$ gives the recurrence $\alpha_n([1,k]) = \alpha_n([1,k-1])(1+k(n-1))$, and clearly $\alpha_n([1,1]) = n$. It follows that

$$\alpha_n([1,k]) = \prod_{i=1}^{k}(1+i(n-1)),$$

a formula due to Viennot [20, Lemma 4.1, Proposition 4.2].

Using the generating function (see [16]) $\sum_{j=1}^{m} c(m,j)t^j = t(t+1)(t+2)\dots(t+(m-1))$, we find that

$$\alpha_n([1,k]) = 1 + \sum_{j=1}^{k} c(k+1,j)(n-1)^{k+1-j}.$$

Invoking Proposition 18, the result follows, noting that the constant term in the above expression corresponds to the occurrence of the trivial representation. $\square$

By expanding the expression for $\alpha_n(S)$ in (5.1), we have the following observation. Although $S_{(n-1,1)}$ is a quotient of two permutation modules, it is not clear how to deduce this corollary directly. Later in the paper we will examine these tensor powers more carefully; see Proposition 39.

Corollary 21. *The S_n-module $S_{(n)} \oplus \bigoplus_{j=1}^{k} c(k+1,j) S_{(n-1,1)}^{k+1-j}$ is in fact a permutation module. More generally, for any subset $S = \{1 \leq s_1 < \ldots < s_p \leq k\}$ of $[1,k]$, the S_n-module*

$$\bigotimes_{r=1}^{p} \left(\bigoplus_{i=0}^{s_r - s_{r-1}} \binom{s_r}{i} S_{(n-1,1)}^{\otimes i} \right), s_0 = 1,$$

is a permutation module.

Proof. The expression gives the S_n-action on the chains of the rank-selected subposet $A^*_{n,k}(S)$ and is therefore a permutation module. □

We have the following two descriptions of the action on chains between two ranks.

Proposition 22. *For $S = \{1 \leq s_1 < s_2 \leq k\}$, $\alpha_n(\{s_1 < s_2\})$ is given by*

(1)

$$\bigoplus_{j=0}^{s_1} \binom{s_1}{j} S_{(n-1,1)}^{\otimes j} \otimes \left(\bigoplus_{i=0}^{s_2 - s_1} \binom{s_2}{i} S_{(n-1,1)}^{\otimes i} \right);$$

(2)

$$(S_{(n-1,1)} \oplus S_{(n)})^{\otimes s_1} \otimes \bigoplus_{i,j \geq 0, i+j = s_2 - s_1} \binom{s_1 + j - 1}{j} S_{(n-1,1)}^{\otimes j} \otimes (S_{(n-1,1)} \oplus S_{(n)})^{\otimes i}$$

Proof. From Eqn. (5.1), the dimension of $\alpha_n(\{s_1 < s_2\})$ is given by

$$n^{s_1} \sum_{i=0}^{s_2 - s_1} \binom{s_2}{i} (n-1)^i,$$

while Part (3) of Theorem 6) gives (by extracting the coefficient of t^{s_2} in the right-hand side of Part (3)):

$$n^{s_1} \sum_{i,j \geq 0, i+j = s_2 - s_1} \binom{s_1 + j - 1}{j} (n-1)^j n^i.$$

It is easily seen that these coincide with the S_n-module decompositions in the statement. □

By expanding in powers of $(n-1)$, the equivalence of the two expressions for the dimension of $\alpha_n(\{s_1 < s_2\})$ is equivalent to the following binomial coefficient identity:

$$\binom{a+r}{i} = \sum_{j=0}^{i} \binom{r-j}{i-j} \binom{a+j-1}{j} \text{ for } 0 \leq i \leq r, \tag{5.3}$$

or equivalently, putting $k = r - i$,

$$\binom{a+r}{a+k} = \sum_{j=0}^{r-k} \binom{r-j}{k} \binom{a+j-1}{a-1} \text{ for } 0 \leq k \leq r.$$

The latter identity can be established by a bijection mapping an $(a+k)$-subset $T = \{t_1 < \ldots < t_{a+k}\}$ of $\{1, 2, \ldots, a+r\}$ to a pair of subsets (U, V) as follows: Note that t_a must equal $a+j$ for some $0 \le j \le r$. Then $V = \{t_1 < \ldots < t_{a-1}\}$ is a subset of $\{1, \ldots, a+j-1\}$ which can be chosen in $\binom{a+j-1}{a-1}$ ways, and $U = \{t_{a+1} < \ldots < t_{a+k}\}$ is a subset of $\{a+j+1, \ldots a+r\}$ (hence necessarily contained in $\{a+1, \ldots a+r\}$) which can be chosen in $\binom{r-j}{k}$ ways.

We can now show that Conjecture 1 is true for rank sets of size 2.

Theorem 23. *Let $S = \{1 \le s_1 < s_2 \le k\}$ be a rank-set of size 2 in $A^*_{n,k}$. The homology representation of $A^*_{n,k}(S)$ is given by*

$$\sum_{v=1}^{s_1} S_{(n-1,1)}^{\otimes v}\Big(c_v - \binom{s_1}{v}\Big) \oplus \sum_{v=1+s_1}^{s_2} S_{(n-1,1)}^{\otimes v} c_v,$$

where c_v is the following positive integer:

$$c_v = \sum_{j=1}^{\min(v, s_2-s_1)} \binom{s_2-j}{v-j}\binom{s_1+j-1}{j}.$$

Moreover $c_v \ge \binom{s_1}{v}$ when $v \le s_1$, and hence the homology is a nonnegative integer combination of tensor powers of $S_{(n-1,1)}$.

Proof. In order to establish the positivity, it is (curiously) easier to work with the second formulation of Proposition 22. We have

$$\beta_n(\{s_1 < s_2\}) = \alpha_n(\{s_1 < s_2\}) - \alpha_n(\{s_1\}) - \alpha_n(\{s_2\}) + \alpha_n(\emptyset)$$

which equals (in terms of Betti numbers)

$$\begin{aligned} & n^{s_1} \sum_{j=0}^{s_2-s_1} n^{s_2-s_1-j}(n-1)^j \binom{s_1+j-1}{j} - n^{s_2} - n^{s_1} + 1 \\ &= \sum_{j=1}^{s_2-s_1} n^{s_2-j}(n-1)^j \binom{s_1+j-1}{j} - (n^{s_1}-1). \end{aligned}$$

Expanding n^{s_1} and n^{s_2-j} in nonnegative powers of $(n-1)$ gives $\beta_n(\{s_1 < s_2\}) =$

$$\sum_{j=1}^{s_2-s_1} \sum_{u=0}^{s_2-j} \binom{s_2-j}{u}(n-1)^{u+j}\binom{s_1+j-1}{j} - \sum_{j=1}^{s_1}\binom{s_1}{j}(n-1)^j \tag{5.4}$$

$$= \sum_{v=1}^{s_2}(n-1)^v c_v - \sum_{j=1}^{s_1}\binom{s_1}{j}(n-1)^j, \tag{5.5}$$

where

$$c_v = \sum_{\substack{(u,j): u+j=v \\ 1 \le j \le s_2-s_1,\, 0 \le u \le s_2-j}} \binom{s_2-j}{v-j}\binom{s_1+j-1}{j}.$$

The latter sum runs over all j such $1 \le j \le s_2 - s_1$ and $0 \le v - j \le s_2 - j$, i.e. over all $j = 1, \ldots, \min(v, s_2 - s_1)$, as stated.

Now c_v is a sum of nonnegative integers for each $v = 1, \ldots, s_2$. When $v \le s_1$, the $j = 1$ summand of c_v can be seen to be $\binom{s_2-1}{v-1}s_1$, and so

$$c_v - \binom{s_1}{v} \ge \binom{s_2-1}{v-1}s_1 - \binom{s_1}{v} = \frac{s_1!}{v!(s_2-v)!}\left(v\frac{(s_2-1)!}{(s_1-1)!} - \frac{(s_2-v)!}{(s_1-v)!}\right)$$
$$= \frac{s_1!(s_2-s_1)!}{v!(s_2-v)!}\left(v\binom{s_2-1}{s_2-s_1} - \binom{s_2-v}{s_2-s_1}\right)$$

and this is clearly nonnegative, since $\binom{s_2-1}{s_2-s_1} \ge \binom{s_2-v}{s_2-s_1}$ for $v \ge 1$. We have shown that the Betti number of $\beta_n(\{s_1 < s_2\})$ is a nonnegative integer combination of positive powers of $(n-1)$, as claimed. $\square$

Remark 24. Let $s_1 = 1$, and consider the two ranks $\{1 < s_2\}$. The homology of the rank-selected subposet is then

$$\bigoplus_{v=2}^{s_2-1}\binom{s_2}{v-1}S_{(n-1,1)}^{\otimes v}\bigoplus(s_2-1)S_{(n-1,1)}^{\otimes s_2}.$$

6. Tensor powers of the reflection representation

In this section we explore the tensor powers $S_{(n-1,1)}$. The paper [10] gives a combinatorial model for determining the multiplicity of an irreducible in the kth tensor power, and an explicit formula in the case when n is sufficiently larger than k. We give general formulas that apply to the case of arbitrary tensor powers.

Burnside proved that given a faithful representation V of a finite group G, every G-irreducible occurs in some tensor power of V. A simple and beautiful proof of a generalisation of this was given by Brauer in [3]. In the present context, it states that since $S_{(n-1,1)}$ is a faithful representation whose character takes on n distinct values (viz. $-1, 0, 1, \ldots, n-1$, but not $n-2$) every irreducible S_λ occurs in at least one of the n tensor powers $S_{(n-1,1)}^{\otimes j}, 0 \le j \le n-1$.

In view of these results, the next fact is interesting. We were unable to find it in the literature. Our proof mimics Brauer's elegant argument.

Theorem 25. *Let G be any finite group and X any character of G. Suppose X takes on k distinct nonzero values b_i. Then the first k tensor powers of X are linearly independent functions on G, and form a basis for the subspace of class functions spanned by all the positive tensor powers. If $X^{k+1} = \sum_{i=1}^k c_i X^i$, then the polynomial $P(t) = t^{k+1} - \sum_{i=1}^k c_i t^i$ has the factorisation $t\prod_{i=1}^k (t - b_i)$.*

Let $X^0 = 1_G$ denote the trivial character of G. Then $X^k = \sum_{i=1}^k c_i X^{i-1}$ if and only if the character X never takes the value zero.

Proof. Let U be the vector space spanned by the characters X^j of the positive tensor powers of X. Suppose X takes the distinct values $\{b_i \ne 0 : 1 \le i \le k\}$. For each $i = 1, \ldots, k$, choose an arbitrary element in the preimage of b_i, that is, $a_i \in X_n^{-1}(b_i)$. Let $A = \{a_i : 1 \le i \le k\}$. We may thus view U as a subspace of the space of functions defined on the set A of size k; this space has dimension exactly k, and hence $\dim(U) \le k$.We will show that the characters $X^i, 1 \le i \le k$, are linearly independent.

We now claim that the k functions

$$\{X^j\downarrow_A, 1 \le j \le k\}$$

are linearly independent. Suppose c_j are scalars such that $\sum_{j=1}^k c_j X^j$ is the zero function. This implies $\sum_{j=1}^k c_j X^j(a_j) = 0$. But $X^j(a_i) = b_i^j$, so the coefficient matrix $(X^j(a_i))$ is a k by k Vandermonde with determinant $(b_1 \dots b_k) \prod_{1\le i<j\le k}(b_j - b_i)$, which is nonzero by hypothesis. Hence $c_j = 0$ for $1 \le j \le k$. This establishes the first statement.

Now let $X^{k+1} = \sum_{i=1}^k c_i X^i$ for some scalars c_i. If X never takes on the value zero we can clearly simplify the dependence relation to $X^k = \sum_{i=1}^k c_i X^{i-1}$. If however 0 is a value of X, the set $\{1_G, X^i : 1 \le i \le k\}$ must be linearly independent, since the trivial character equals 1 everywhere. This finishes the proof. $\square$

Remark 26. For $k \ge 2$, the representation $S_{(n-1,1)}^{\otimes(k-1)}$ is contained in $S_{(n-1,1)}^{\otimes k}$. For this it suffices to note that

(1) this is true for $k = 2$, since $S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)} = S_{(n-2,2)} \oplus S_{(n-1,1^2)} \oplus S_{(n)}$, and thus
(2) $S_{(n-1,1)}^{\otimes k} - S_{(n-1,1)}^{\otimes(k-1)} = S_{(n-1,1)}^{\otimes(k-2)} \otimes (S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)})$ is a true module.

We use symmetric functions to describe some of the results that follow. See [12] and [17, Chapter 7]. The homogeneous symmetric function h_n is the Frobenius characteristic, denoted ch, of the trivial representation of S_n. Also let $*$ denote the internal product on the ring of symmetric functions, so that the Frobenius characteristic of the inner tensor product,or Kronecker product, of two S_n-modules is the internal product of the two characteristics. Recall that the natural representation of S_n is the permutation action on a set of n objects. The stabiliser of any one object is the Young subgroup $S_1 \times S_{n-1}$, and hence the natural representation is given by the induced module $1 \uparrow_{S_1\times S_{n-1}}^{S_n}$, with Frobenius characteristic $h_1 h_{n-1}$. In particular we have the decomposition

$$1 \uparrow_{S_1\times S_{n-1}}^{S_n} = S_{(n)} \oplus S_{(n-1,1)} = S_{(n,1)} \downarrow_{S_n}^{S_{n+1}} .$$

The following lemma is an easy exercise in permutation actions. We sketch a proof for completeness.

Lemma 27. *Let $V_{j,n}$ denote the permutation module obtained from the S_n-action on the cosets of the Young subgroup $S_1^j \times S_{n-j}$. Then the kth tensor power of the natural representation $V_{1,n}$ of S_n decomposes into a sum of $S(k,j)$ copies of $V_{j,n}$, where $S(k,j)$ is the Stirling number of the second kind:*

$$V_{1,n}^{\otimes k} = \sum_{j=1}^{\min(n,k)} S(k,j)\, V_{j,n}, \qquad \text{and thus} \quad (h_1 h_{n-1})^{*k} = \sum_{j=1}^{\min(n,k)} S(k,j)\, h_1^j h_{n-j}. \tag{6.1}$$

Proof. If the module $V_{1,n}$ is realised as $\mathbb{C}^n$ with basis $\{v_1 \dots, v_n\}$, say, then $V_{1,n}^{\otimes k}$ is realised by the kth tensor power of $\mathbb{C}^n$, with n^k basis elements $v_{i_1} \otimes \dots \otimes v_{i_k}, \quad 1 \le i_1, \dots, i_k \le n$. The S_n-action now permutes these n^k basis elements. To determine the orbits, note that there is a surjection from this basis of tensors to the set partitions of a k-element set into nonempty blocks. Each such partition with j blocks indexes an orbit of the S_n-action, with stabiliser (conjugate to) $S_1^j \times S_{n-j}$.

An example will make this clear. With $n = 5$ and $k = 7$, the tensor $v_5 \otimes v_2 \otimes v_2 \otimes v_4 \otimes v_2 \otimes v_4 \otimes v_5$ maps to the partition $17 - 235 - 46$ of a set of size $k = 7$ into $j = 3$ blocks, corresponding to the three distinct basis elements v_2, v_4, v_5 of $\mathbb{C}^n$. Its orbit under

S_5 consists of all basis tensors $v_{i_1} \otimes \ldots \otimes v_{i_7}$ such that $v_{i_1} = v_{i_7}, v_{i_2} = v_{i_3} = v_{i_5}$, and $v_{i_4} = v_{i_6}$. Writing S_A for the permutations of the elements of A, for any subset A of positive integers, the stabiliser is $S_{\{5\}} \times S_{\{2\}} \times S_{\{4\}} \times S_{\{1,3\}}$, conjugate to the Young subgroup indexed by the integer partition $(2,1,1,1)$ of 5.

The last statement is now immediate. $\square$

Remark 28. This lemma can also be proved by iterating a standard representation theory result, namely that for finite groups G and H with H a subgroup of G, and G-module W, H-module V, $W \otimes (V \uparrow_H^G) = (W \downarrow_H \otimes V) \uparrow_H^G$.

Theorem 29. *The top homology of $A^*_{n,k}$ has Frobenius characteristic*

$$\sum_{i=0}^{\min(n,k)} h_1^i h_{n-i} \left(\sum_{r=0}^{k-i} (-1)^r \binom{k}{r} S(k-r,i) \right).$$

Proof. Observe that the Frobenius characteristic of $(S_{(n-1,1)})^{\otimes k}$ is the k-fold internal product of $(h_1 h_{n-1} - h_n)$. Standard properties of the tensor product make $*$ a commutative and associative product in the ring of symmetric functions, so we have

$$\begin{aligned}
&(h_1 h_{n-1} - h_n)^{*k} \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (h_1 h_{n-1})^{*j} * (h_n)^{*(k-j)} = (-1)^k h_n + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} (h_1 h_{n-1})^{*j} \\
&= (-1)^k h_n + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \sum_{i=1}^{\min(n,j)} S(j,i) h_1^i h_{n-i} \quad \text{from Lemma 27} \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{i=0}^{\min(n,j)} S(j,i) h_1^i h_{n-i} = \sum_{i=0}^{\min(n,k)} h_1^i h_{n-i} \left(\sum_{j=i}^{k} (-1)^{k-j} \binom{k}{j} S(j,i) \right).
\end{aligned}$$

Note that $S(0,0) = 1$ and $S(j,0) = 0$ for $j \geq 1$. Putting $r = k - j$ in the last step gives the result. $\square$

Theorem 37 gives a different description of this module, from which it will be evident that the coefficients of $h_1^i h_{n-i}$ are positive for $i \geq 2$.

We can now determine the multiplicity of the trivial representation in the top homology of $A^*_{n,k}$:

Corollary 30. *Let $n \geq 2$. The following are equal:*

(1) the multiplicity of the trivial representation in $S_{(n-1,1)}^{\otimes k}$;
(2) the multiplicity of the irreducible $S_{(n-1,1)}$ in $S_{(n-1,1)}^{\otimes k-1}$;
(3) the number

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{i=0}^{\min(n,k)} S(k-r,i).$$

When $n \geq k$, this multiplicity equals the number of set partitions of $\{1,\ldots,k\}$ with no singleton blocks.

Proof. The first two multiplicities are equal by standard properties of the tensor product, since

$$\langle V\otimes W, S_{(n)}\rangle = \langle V, S_{(n)}\otimes W\rangle = \langle V, W\rangle$$

The equivalence with the third formula follows from Theorem 29, since $\langle h_1^i h_{n-i}, h_n\rangle = 1$ for all i (alternatively, $1\uparrow_{S_1^i\times S_{n-i}}^{S_n}$ is a transitive permutation module). Let $B_n^{\geq 2}$ denote the number of set partitions of $[n]=\{1,\ldots,n\}$ with no blocks of size 1, and let B_n denote the nth Bell number, that is, the total number of set partitions of $[n]$. Inclusion-exclusion shows that

$$B_n^{\geq 2} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}B_{n-r}, \tag{6.2}$$

since the number of partitions containing a fixed set of r singleton blocks is B_{k-r}. When $n\geq k$, the formula in Part (3) simplifies to

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}\sum_{i=0}^{k-r}S(k-r,i) = \sum_{r=0}^{k}(-1)^r\binom{k}{r}B_{k-r}.$$

That this number is $B_k^{\geq 2}$, the number of partitions of $[k]$ with no singleton blocks, now follows from Eqn. (6.2). (This is sequence A000296 in OEIS.) □

Corollary 38 in the next section will give a different expression for the multiplicity of the trivial representation, for arbitrary n,k, as a sum of positive integers.

7. "Almost" an h-positive permutation module

The goal of this section is to prove the following theorem.

Theorem 31. *Let $T\subseteq[1,k]$ be any nonempty subset of ranks in $A^*_{n,k}$. The following statements hold for the Frobenius characteristic $F_n(T)$ of the homology representation $\tilde{H}(A^*_{n,k}(T))$:*

1. *its expansion in the basis of homogeneous symmetric functions is an integer combination supported on the set $T_1(n)=\{h_\lambda:\lambda=(n-r,1^r), r\geq 1\}$.*
2. *$F_n(T)+(-1)^{|T|}s_{(n-1,1)}$ is supported on the set $T_2(n)=\{h_\lambda:\lambda=(n-r,1^r), r\geq 2\}$.*

When $F_n(T)+(-1)^{|T|}s_{(n-1,1)}$ is in fact a nonnegative integer combination of $T_2(n)=\{h_\lambda:\lambda=(n-r,1^r), r\geq 2\}$, we may view $F_n(T)$ as being almost a permutation module, hence the title of this section. First we prove a stronger result for the action on the chains.

Proposition 32. *Let $S\subseteq[1,k]$. If $|S|\geq 1$, the action of S_n on the space of chains $\alpha_n(S)$ has h-positive Frobenius characteristic supported on the set $T_1(n)=\{h_\lambda:\lambda=(n-r,1^r), r\geq 1\}$. Furthermore, h_1h_{n-1} always appears with coefficient 1 in the h-expansion of $\alpha_n(S)$.*

Proof. Recall that $*$ denotes the inner tensor product. Note that the case of a single rank has already been established in Lemma 27: if $S=\{s_1\}$, then $\operatorname{ch}\alpha_n(S)=(h_1h_{n-1})^{*s_1}$ since the dimension of the module is n^{s_1}, and the coefficient of h_1h_{n-1} in the h-expansion is the Stirling number $S(s_1,1)=1$.

We proceed by induction, using the decomposition (5.1) of Theorem 19. We have, with $|S| \geq 2$,

$$\alpha_n(S) = \alpha_n(S \backslash \{s_p\}) \otimes \sum_{i=0}^{s_p - s_{p-1}} \binom{s_p}{i} S_{(n-1,1)}^{\otimes i} = \sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} S_{(n-1,1)}^{\otimes i} \oplus S_{(n)}.$$

Translating Remark 28 into Frobenius characteristics gives the well-known symmetric function formula $(h_1 h_{n-1}) * f = h_1 \frac{\partial}{\partial p_1} f$ for any symmetric function f of homogeneous degree n [17, Exercise 7.81, p. 477], [12, Example 3 (c), p. 75]. It is easy to check that for $a \geq 1$,

$$(h_1^a h_b) * s_{(n-1,1)} = \begin{cases} (a-1) h_1^a h_b + h_1^{a+1} h_{b-1}, \ a \geq 2; \\ h_1^{a+1} h_{b-1}, \ a = 1. \end{cases} \tag{7.1}$$

Note the factor h_1^2 when $a \neq 0$. Let $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. Iterating (7.1) gives the fact that when $\mu \in T_1(n-1)$, $(h_1 h_\mu) * s_{(n-1,1)}^{*k}$ is a nonnegative integer combination of terms in $T_2(n)$.

Assume $S = \{1 \leq s_1 < \ldots < s_p \leq k\}$, and $|S| \geq 2$. In terms of symmetric functions, the decomposition (5.1) of Theorem 19 becomes

$$\begin{aligned} \operatorname{ch} \alpha_n(S) &= \operatorname{ch} \alpha_n(S \backslash \{s_p\}) * (h_n + \sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} s_{(n-1,1)}^{*i}) \\ &= \operatorname{ch} \alpha_n(S \backslash \{s_p\}) + \operatorname{ch} \alpha_n(S \backslash \{s_p\}) * \left(\sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} s_{(n-1,1)}^{*i} \right). \end{aligned} \tag{7.2}$$

Suppose now that the first term above, $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$, is a nonnegative integer combination of terms in $T_1(n)$, in which $h_1 h_{n-1}$ appears with coefficient 1. By (7.1), the h-expansion of the second term contains only terms in $T_2(n)$; the crucial point here is that, since $p \geq 2$, the h-expansion of $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$ does not contain the function h_n. It follows that $\operatorname{ch} \alpha_n(S)$ must be a nonnegative integer combination of terms in $T_1(n)$, and the coefficient of $h_1 h_{n-1}$ is inherited from $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$. It is therefore equal to 1. This completes the induction. □

Proof of Theorem 31:

Proof. Using Stanley's equation for rank-selected homology, Equation (5.2) in Theorem 19, we have

$$F_{n,k}(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} \operatorname{ch} \alpha_n(S).$$

From Proposition 32, this has an expansion in the h-basis in which h_n appears only in $\alpha_n(\emptyset)$ with coefficient 1, and $h_1 h_{n-1}$ appears in $\alpha_n(S)$ with coefficient 1 for all nonempty S. Hence the coefficient of h_n in the right-hand side above is $(-1)^{|T|}$, while the coefficient of $h_1 h_{n-1}$ is

$$\sum_{S \subseteq T, S \neq \emptyset} (-1)^{|T| - |S|} = (-1)^{|T|} [\sum_{i=0}^{|T|} \binom{|T|}{i} (-1) - 1] = (-1)^{|T| - 1}.$$

But $(-1)^{|T|} s_{(n-1,1)} = (-1)^{|T|} h_1 h_{n-1} - (-1)^{|T|} h_n$, and the conclusion follows. □

The preceding theorem motivates Conjecture 2 in the Introduction. We will show that Conjecture 2 is true in the following cases of rank-selection:

Theorem 33. *For any nonempty rank set $T \subseteq [1,k]$, consider the module $V_T = \tilde{H}_{k-2}(A^*_{n,k}(T)) + (-1)^{|T|}S_{(n-1,1)}$. In each of the following cases, V_T is a nonnegative combination of transitive permutation modules with orbit stabilisers of the form $S_1^d \times S_{n-d}, d \geq 2$. Equivalently, the symmetric function $F_{n,k}(T) + (-1)^{|T|}s_{(n-1,1)}$ is supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$ with nonnegative integer coefficients in each of the following cases:*

(1) $T = [r,k], k \geq r \geq 1$.
(2) $T = [1,k]\backslash\{r\}, k \geq r \geq 1$.
(3) $T = \{1 \leq s_1 < s_2 \leq k\}$.

The proof relies on the homology computations of the preceding sections, but we also need to develop additional tools. In particular Theorem 37 will be crucial to the proof. We begin by deriving a different expression for the Whitney homology modules of $A^*_{n,k}$, thereby obtaining a new expression for the top homology module as well.

Proposition 34. *Let α be a nonempty word in $A^*_{n,k}$. Then*

(1) If α is not a normal word, the (order complex of the) interval $(\hat{0}, \alpha)$ is contractible and hence its homology vanishes in all degrees;
(2) If α is a normal word, the (order complex of the) interval $(\hat{0}, \alpha)$ is homotopy equivalent to a single sphere in the top dimension, and the stabiliser subgroup of α acts trivially on the homology.

Proof. The topological conclusions in both parts are immediate from the formula for the Möbius number in Theorem 3 and Björner's dual CL-shellability result of Theorem 7.

If α is not normal, then it consists of some $m \leq n$ distinct letters $\{x_1, \ldots, x_m\}$, and consecutive letters are distinct. The stabiliser is the subgroup which fixes each x_i pointwise, and permutes the remaining $n-m$ letters arbitrarily. It is thus a product of m copies of the trivial group S_1 and the group S_{n-m}. Clearly this subgroup fixes every element in the interval $(\hat{0}, \alpha)$ pointwise, and hence the action on the unique nonvanishing homology is trivial. $\square$

Let $S^*(j,d)$ denote the number of set partitions of $[j]$ into d blocks, with the property that no block contains consecutive integers (a reduced Stirling number).

Lemma 35. *There is a surjection ψ from the set of words of length j in an alphabet of size n to the set partitions into d blocks of $[j]$, where d is the number of distinct letters in α. This surjection maps normal words onto set partitions with the property that no two consecutive integers are in the same block. In particular, the number of normal words of length $j \geq 2$ on an alphabet of size n is*

$$\sum_{d=1}^{\min(n,j)} \frac{n!}{(n-d)!} S^*(j,d) = n(n-1)^{j-1}.$$

Proof. It suffices to illustrate with an example. Let $\alpha = abbcbca$, of length 7 with 3 distinct letters. Then the set partition of [7] associated to α is $\psi(\alpha) = 17 - 235 - 46$. Positions corresponding to integers in the same block have equal letters in α.

Now suppose α is normal. Then no two consecutive positions have equal letters, which is precisely the condition that no block of the set partition $\psi(\alpha)$ contains consecutive integers.

The last statement is verified by observing that a normal word in the pre-image of every set partition of $[j]$ with d blocks contains d distinct letters chosen out of n, which can be permuted amongst themselves in d ways. $\square$

Recall that the ordinary Stirling numbers of the second kind satisfy the recurrence $S(n+1,d) = S(n,d-1) + dS(n,d)$ with initial conditions $S(0,0) = 1$ and $S(n,0) = 0 = S(0,d)$ if $n,d > 0$. It is easy to verify similarly that the reduced Stirling numbers $S^*(j,d)$ satisfy the recurrence $S^*(n+1,d) = S^*(n,d-1) + (d-1)S^*(n,d)$, by examining the possibilities for inserting $(n+1)$ into a partition of $[n]$ into d blocks. A comparison of the recurrences immediately shows that in fact

$$S^*(n+1,d) = S(n,d-1) \quad \text{for all } n \geq 0, d \geq 1.$$

See [13] for generalisations of these numbers. Recall that in Theorem 11, the jth Whitney homology of $A^*_{n,k}$ $j \geq 2$, was determined as a sum of two consecutive tensor powers of $S_{(n-1,1)}$. From the preceding Lemma and Proposition 34 we now have the following surprising result.

Proposition 36. *Each Whitney homology module of subword order, and hence the sum of two consecutive tensor powers of the reflection representation, has h-positive Frobenius characteristic, and in particular it is a permutation module. We have* $\mathrm{ch}\, Wh_0 = h_n, \mathrm{ch}\, Wh_1 = h_1 h_{n-1}$, *and for* $k \geq j \geq 2$, *the* j*th Whitney homology of* $A^*_{n,k}$ *has Frobenius characteristic*

$$\sum_{d=2}^{j} S(j-1,d-1)\, h_1^d h_{n-d} = \sum_{d=2}^{j} S^*_{j,d}\, h_1^d h_{n-d}, \tag{7.3}$$

a permutation module with orbits whose stabilisers are Young subgroups indexed by partitions of the form $(n-d, 1^d), d \geq 0$.

Proof. From Theorem 11, for $j \geq 2$, we have $WH_j(A^*_{n,k}) = S_{(n-1,1)}^{\otimes j} \oplus S_{(n-1,1)}^{\otimes j-1}$ (see Eqn. (3.5)). Now by definition we also have

$$WH_j(A^*_{n,k}) = \sum_{x \in A^*_{n,k}, |x|=j} \tilde{H}(\hat{0}, x).$$

Proposition 34 says that the sum runs over only normal words x, and each homology module is trivial for the stabiliser of x. Collecting the summands into orbits and using the preceding lemma gives Eqn. (7.3). $\square$

Recall [12] that the homogeneous symmetric functions h_λ form a basis for the ring of symmetric functions.

Theorem 37. *Fix* $k \geq 1$. *The* k*th tensor power of the reflection representation* $S_{(n-1,1)}^{\otimes k}$, *i.e. the homology module* $\tilde{H}_{k-1}(A^*_{n,k})$, *has the following property:* $S_{(n-1,1)}^{\otimes k} \oplus (-1)^k S_{(n-1,1)}$ *is a permutation module* $U_{n,k}$ *whose Frobenius characteristic is h-positive, and is supported on the set* $\{h_\lambda : \lambda = (n-r, 1^r), r \geq 2\}$. *If* $k = 1$, *then* $U_{n,1} = 0$.

More precisely, the k-fold internal product $s_{(n-1,1)}^{*k}$ has the following expansion in the basis of homogeneous symmetric functions h_λ :

$$\sum_{d=0}^{n} g_n(k,d) h_1^d h_{n-d}, \tag{7.4}$$

where $g_n(k,0) = (-1)^k, g_n(k,1) = (-1)^{k-1}$, and for $2 \le d \le n$,

$$g_n(k,d) = \sum_{i=d}^{k} (-1)^{k-i} S(i-1, d-1), \text{ for } 2 \le d \le n.$$

The integers $g_n(k,d)$ are independent of n for $k \le n$, nonnegative for $2 \le d \le k$, and $g_n(k,d) = 0$ if $d > k$. Also:

(1) $g_n(k,2) = \frac{1+(-1)^k}{2}$.
(2) $g_n(k,k-1) = \binom{k-1}{2} - 1, k \le n$.
(3) $g_n(k,k) = 1, k < n$.

In particular the coefficient of h_1^n in the expansion (7.4) of $\mathrm{ch}\ S_{(n-1,1)}^{\otimes k}$ is

$$\begin{cases} g_n(k,n) + g_n(k,n-1) & \text{if } k > n, \\ \binom{n-1}{2} & \text{if } k = n, \\ 1 & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $k = 1$, the terms for $d \ge 2$ in the summation in (7.4) vanish and thus the right-hand side equals the characteristic of the top homology.

The statement follows from Proposition 36 and Theorem 8. Fix m and d such that $m \ge d \ge 2$. Let $\bar{g}(m,d)$ be the alternating sum of Stirling numbers $\bar{g}(m,d) = \sum_{i=d}^{m} (-1)^{m-i} S(i-1, d-1)$. Note that $\bar{g}(m,d)$ equals

$$[S(m-1,d-1) - S(m-2,d-1)] + [S(m-3,d-1) - S(m-4,d-1)] + \dots$$
$$\dots + \begin{cases} [S(d+1,d-1) - S(d,d-1)] + S(d-1,d-1), & m-d \text{ even,} \\ [S(d,d-1)) - S(d-1,d-1)], & m-d \text{ odd.} \end{cases}$$

Since $S(n,d)$ is an increasing function of $n \ge d$ for fixed d, the coefficient $\bar{g}(m,d)$ is always nonnegative. It is also clear that $\bar{g}(m,d) = S(m-1,d-1) - \bar{g}(m-1,d)$ for all $m \ge d \ge 2$.

The remaining parts follow from the facts that $S(k,k-1) = \binom{k}{2}, S(k,k) = 1$, and the observation that for $k \ge n$, the coefficient of h_1^n is $g_k(k,n) + g_k(k,n-1)$. This equals $S(n-1,n-2) - 1$ when $k = n$. ☐

Corollary 38. *Let $k \ge 2$.*

(1) For $\min(n,k) \ge d \ge 2$, the coefficient of $h_1^d h_{n-d}$ is the nonnegative integer $g(k,d)$ given by the two equal expressions:

$$\sum_{j=d}^{k} (-1)^{k-j} S(j-1,d-1) = \sum_{r=0}^{k-d} (-1)^r \binom{k}{k-r} S(k-r,d). \tag{7.5}$$

In particular, when $n \ge k$, this multiplicity is independent of n.

(2) The positive integer $\beta_n(k) = \sum_{d=2}^{\min(n,k)} g_n(k,d)$ is the multiplicity of the trivial representation in $S_{(n-1,1)}^{\otimes k}$. When $n \ge k$, it equals the number of set partitions

$B_k^{\geq 2}$ of the set $\{1,\ldots,k\}$ with no singleton blocks. We have $\beta_n(n+1) = B_{n+1}^{\geq 2} - 1$ and $\beta_n(n+2) = B_{n+2}^{\geq 2} - \binom{n+1}{2}$.

Proof. This follows by comparing with Theorem 29 and Corollary 30.

We have $\beta_n(n) = \sum_{d=2}^{n} g(n,d) = B_n^{\geq 2} = \beta_n(k)$ for $n \geq k$, and from (7.4),
$\beta_n(n+1) = \sum_{d=2}^{n} g_n(n+1,d) = \sum_{d=2}^{n+1} g_{n+1}(n+1,d) - g_{n+1}(n+1,n+1) = B_{n+1}^{\geq 2} - 1$,
$\beta_n(n+2)$
$= \sum_{d=2}^{n+2} g_{n+2}(n+2,d) - g_{n+2}(n+2,n+2) - g_{n+2}(n+2,n+1)$
$= B_{n+2}^{\geq 2} - 1 - [\binom{n+1}{2} - 1] = B_{n+2}^{\geq 2} - \binom{n+1}{2}$. □

We need one final observation in order to prove Theorem 33.

Lemma 39. *Suppose V is an S_n-module which can be written as an integer combination $V = \oplus_{k=1}^{m} c_k S_{(n-1,1)}^{\otimes k}$ of positive tensor powers of $S_{(n-1,1)}$, $c_k \geq 0$. If $\sum_{k=1}^{m}(-1)^{k-1}c_k = 0$, then the Frobenius characteristic of V is h-positive and supported on the set $\{h_\lambda : \lambda = (n-r,1^r), r \geq 2\}$. If in addition $c_k \geq 0$ for all k, it is hpositive and hence V is a permutation module.*

Proof. Immediate from Theorem 37, since we have

$$V = (\sum_{k=1}^{m}(-1)^{k-1}c_k)S_{(n-1,1)} \oplus \sum_{k=2}^{m} c_k U_{n,k} = \sum_{k=2}^{m} c_k U_{n,k},$$

where $U_{n,k}$ is h-positive with support $\{h_\lambda : \lambda = (n-r,1^r), r \geq 2\}$.. Note that $U_{n,1} = 0$. □

In particular from Theorem 20, this gives a direct proof that the action of S_n on the chains in $A_{n,k}^*$ is also a nonnegative linear combination of $\{h_\lambda : \lambda = (n-r,1^r), r \geq 2\}$.

Proof of Theorem 33:

Proof. Note that in all cases, the module V_T has been shown to be a nonnegative sum of tensor powers of $S_{(n-1,1)}$. Hence, by Lemma 39, it remains only to verify that the alternating sum of coefficients of the tensor powers vanishes for V_T in each case.

Consider Part (1), the case $T = [r,k]$. From Theorem 13, we must show that $(-1)^{k-r+1}$ added to the signed sum of the $(-1)^{i-1}b_i$, for the coefficients $b_i = \binom{k}{i}\binom{i-1}{k-r}$, is zero, i.e.

$$\sum_{i=1+k-r}^{k} b_i(-1)^{i-1} = (-1)^{k-r}. \tag{7.6}$$

It is easiest to use the combinatorial identity of Corollary 14. Consider the two polynomials of degree $k \geq 2$ in x defined by

$$F(x) = \sum_{i=0}^{k-r}(-1)^i\binom{k}{r+i}n^{r+i}(n-1)^{k-(r+i)} + (-1)^{k+1-r},$$

$$G(x) = \sum_{i=1+k-r}^{k}\binom{k}{i}\binom{i-1}{k-r}(n-1)^i.$$

Corollary 14 says $F(x)$ and $G(x)$ agree for all $n \geq 2$, and hence $F(x) = G(x)$ identically.

In particular $F(0) = G(0)$. But $F(0) = (-1)^{k+r-1}$ and clearly $(-1)G(0)$ is precisely the expression in (7.6). The claim follows.

For Part (2), T is the rank-set $[1, k]\backslash\{r\}$, and from Theorem 16 the alternating sum of coefficients in V_T is clearly

$$(-1)^{k-1} + \left[\binom{k}{r} - 1\right](-1)^{k-1} + \binom{k}{r}(-1)^{k-2} = 0.$$

For Part (3), the rank set is $T = \{1 \le s_1 < s_2 \le k\}$. From the homology formula in Theorem 23, we need to show that the following sum, the alternating sum of coefficients in V_T, vanishes:

$$1 + \sum_{v=1}^{s_1}(-1)^{v-1}\left[c_v - \binom{s_1}{v}\right] \oplus \sum_{v=1+s_1}^{s_2}(-1)^{v-1}c_v,$$

where

$$c_v = \sum_{j=1}^{\min(v, s_2-s_1)} \binom{s_2-j}{v-j}\binom{s_1+j-1}{j}.$$

But $1 + \sum_{v=1}^{s_1}(-1)^{v-1}(-\binom{s_1}{v}) = 0$, so this reduces to showing that $\sum_{v=1}^{s_2}(-1)^v c_v = 0$. Split the summation over v at $s_2 - s_1$. This gives that $\sum_{v=1}^{s_2}(-1)^v c_v$ equals

$$\underbrace{\sum_{v=1}^{s_2-s_1}(-1)^v \sum_{j=1}^{v}\binom{s_2-j}{v-j}\binom{s_1+j-1}{j}}_{(A)} + \underbrace{\sum_{v>s_2-s_1}^{s_2}(-1)^v \sum_{j=1}^{s_2-s_1}\binom{s_2-j}{v-j}\binom{s_1+j-1}{j}}_{(B)}.$$

Switching the order of summation, (A) is equal to

$$\sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{v=j}^{s_2-s_1}\binom{s_2-j}{v-j}(-1)^v,$$

while (B) is

$$\sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{v>s_2-s_1}^{s_2}(-1)^v\binom{s_2-j}{v-j}(-1)^v.$$

Hence $\sum_{v=1}^{s_2}(-1)^v c_v$ equals

$$\sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{v=j}^{s_2}\binom{s_2-j}{v-j}(-1)^v = \sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{w=0}^{s_2-j}\binom{s_2-j}{w}(-1)^{w-s_2},$$

where we have put $w = s_2 - v$. But $1 \le j \le s_2 - s_1 < s_2$, so the inner sum vanishes. $\square$

We conclude this section by pointing out a representation-theoretic consequence, and some enumerative implications, of the expansion (7.4). Fix $n \ge 3$ and consider the n by $n-1$ matrix D_n whose kth column consists of the coefficients $g_n(n-k, n-d), d = 1, \ldots, n-1$. Thus the kth column contains the coefficients in the expansion of $S_{(n-1,1)}^{\otimes n-k}$ in the h-basis: we have $\mathrm{ch}\, S_{(n-1,1)}^{\otimes k} = \sum_{d=1}^{n} g_n(k, n-d)h_1^{n-d}h_d, 1 \le k \le n-1$. From Theorem 37 it is easy to see that the matrix D_n has rank $(n-1)$; the last two rows, consisting of alternating ± 1s, differ by a factor of (-1), and the matrix is lower triangular with 1's on the diagonal, hence it has rank $(n-1)$. Similarly the $(n+1)$ by

$(n-1)$ matrix obtained by appending to D_n a first column consisting of the h-expansion of the nth tensor power of $S_{(n-1,1)}^{\otimes n}$ also has rank $(n-1)$. We therefore have a second proof of Theorem 25, for the modules $S_{(n-1,1)}$. In this special case we can now be more precise about the linear combination of tensor powers:

Theorem 40. *The first $n-1$ tensor powers of $S_{(n-1,1)}$ are an integral basis for the vector space spanned by the positive tensor powers. The nth tensor power of $S_{(n-1,1)}$ is an integer linear combination of the first $(n-1)$ tensor powers:*

$$S_{(n-1,1)}^{\otimes n} = \bigoplus_{k=1}^{n-1} a_k(n) S_{(n-1,1)}^{\otimes k},$$

with $a_{n-1}(n) = \binom{n-1}{2}$. The coefficients $a_k(n)$ are determined by the polynomial $P(t) = t^n - \sum_{k=1}^{n-1} a_k(n) t^k$, defined by

$$P(t) = \frac{t+1}{t-(n-2)} \sum_{j=1}^{n} c(n,j) t^j (-1)^{n-j} \tag{7.7}$$

where $c(n,j)$ is the number of permutations in S_n with exactly j disjoint cycles.

Proof. We invoke Theorem 25. The linear combination of tensor powers in the statement translates into a polynomial equation for the character values, whose zeros are the n distinct values $-1, 0, 1, \ldots, n-3, n-1$, taken by the character of $S_{(n-1,1)}$. Hence we have

$$P_n(t) = t^n - \sum_{k=1}^{n-1} a_k(n) t^k = (t+1)t \prod_{i=1, i\neq n-2}^{n-1} (t-i) = \frac{t+1}{(t-(n-2))} \prod_{i=0}^{n-1} (t-i).$$

But $\prod_{j=0}^{n-1}(t-j)$ is the generating function for the Stirling numbers of the first kind [16], so the result follows. □

The preceding result gives a recurrence for the coefficients $a_k(n)$; we have

$$\begin{aligned}
a_{n-1}(n) &= \binom{n-1}{2};\\
(n-2)a_j(n) - a_{j-1}(n) &= (-1)^{n-j}[c(n,j) - c(n,j-1)], 2 \le j \le n-1;\\
(n-2)a_1(n) &= c(n,1)(-1)^{n-1}\\
\Longrightarrow a_1(n) &= \frac{(n-1)!}{n-2}(-1)^{n-1} = (-1)^{n-1}[(n-2)! + (n-3)!]
\end{aligned}$$

Question 4. The identity (7.5) holds for all $d = 2, \ldots, k$. Is there a combinatorial explanation?

Question 5. For fixed k and n, what do the positive integers $g_n(k,d)$? Is there a combinatorial interpretation for $\beta_n(k) = \sum_{j=d}^{\min(n,k)} g_n(k,d)$, the multiplicity of the trivial representation in the top homology of $A_{n,k}^*$, in the nonstable case $k > n$. Recall that for $k \le n$ this is the number $B_k^{\ge 2}$ of set partitions of $[k]$ with no singleton blocks, and is sequence OEIS A000296.

Question 6. Recall that $a_{n-1}(n) = \binom{n-1}{2}$. Is there a combinatorial interpretation for the signed integers $a_i(n)$? There are many interpretations for $(-1)^{n-1}a_1(n) = (n-2)! + (n-3)!$, see OEIS A001048. For $n \geq 4$ it is the size of the largest conjugacy class in S_{n-1}. We were unable to find the other sequences $\{a_i(n)\}_{n\geq 3}$ in OEIS.

Example 41. Write X_n^k for $S_{(n-1,1)}^{\otimes k}$. Maple computations with Stembridge's SF package show that

(1) $X_3^3 = X_3^2 + 2X_3$.
(2) $X_4^4 = 3X_4^3 + X_4^2 - 3X_4$.
(3) $X_5^5 = 6X_5^4 - 7X_5^3 - 6X_5^2 + 8X_5$.
(4) $X_6^6 = 10X_6^5 - 30X_6^4 + 20X_6^3 + 31X_6^2 - 30X_6$
(5) $X_7^7 = 15X_7^6 - 79X_7^5 + 165X_7^4 - 64X_7^3 - 180X_7^2 + 144X_7$
(6) $X_8^8 = 21X_8^7 - 168X_8^6 + 630X_8^5 - 1029X_8^4 + 189X_8^3 + 1198X_8^2 - 840X_8$.

8. The subposet of normal words

Let $N_{n,k}$ denote the poset of normal words of length at most k in $A_{n,k}^*$, again with an artificial top element $\hat{1}$ appended. Farmer showed that

Theorem 42. *(Farmer [9]) $\mu(N_{n,k}) = (-1)^{k-1}(n-1)^k = \mu(A_{n,k}^*)$, and $A_{n,k}, N_{n,k}$ both have the homology of a wedge of $(n-1)^k$ $(k-1)$-dimensional spheres.*

Björner and Wachs [7] showed that $N_{n,k}$ is dual CL-shellable and hence homotopy Cohen-Macaulay; it is therefore homotopy-equivalent to a wedge of $(n-1)^k$ $(k-1)$-spheres. The order complexes of the posets $A_{n,k}^*$ and $N_{n,k}$ are thus homotopy-equivalent.

Using Quillen's fibre theorem ([14]) we can establish a slightly stronger result:

Lemma 43. *Let $\alpha \in N_{n,k}^*$. Then the intervals $(\hat{0}, \alpha)_{N_{n,k}}$ and $(\hat{0}, \alpha)_{A_{n,k}^*}$ are $G_{stab(\alpha)}$-homotopy equivalent, for the stabiliser subgroup $G_{stab(\alpha)}$ of α. In particular the homology groups are all $G_{stab(\alpha)}$-isomorphic.*

If $\alpha \in A_{n,k}^*$, but $\alpha \notin N_{n,k}^*$, then we know that the interval $(\hat{0}, \alpha)_{A_{n,k}^*}$ is contractible.

Proof. Let J_m be the set of words of length m that are *not* normal. Let

$$B_j = (\hat{0}, \alpha)_{A_{n,k}^*} \backslash (\cup_{m=j}^k J_m)$$

be the subposet obtained by removing all normal words at rank j and higher. Thus $B_1 = (\hat{0}, \alpha)_{N_{n,k}}$. Set $B_{k+1} = (\hat{0}, \alpha)_{A_{n,k}^*}$. We claim that the inclusion maps

$$(\hat{0}, \alpha)_{N_{n,k}} = B_1 \subset B_2 \subset \ldots B_j \subset B_{j+1} \subset B_{k+1} = (\hat{0}, \alpha)_{A_{n,k}^*} \tag{8.1}$$

are group equivariant homotopy equivalences. Note that

$$B_j = B_{j+1} \backslash \{\text{non-normal words of length } j+1\},$$

and B_j coincides with B_{j+1} for the first j ranks. The fibres to be checked are $F_{\leq w} = \{\beta \in B_j : \beta \leq \alpha\}$, for $w \in B_{j+1}$. If w is a normal word in B_{j+1}, then $w \in B_j$ and the fibre is the half-closed interval $(\hat{0}, w]$ in B_j; it is therefore contractible. If $w \in B_{j+1}$ is not a normal word, then $w \notin B_j$ and the interval $(\hat{0}, w)_{B_j}$ coincides with the same interval in $A_{n,k}^*$, so by Part (1) of Proposition 34, it is contractible. Hence by Quillen's fibre theorem, the inclusion induces a homotopy equivalence. $\square$

Proposition 44. *The Whitney homology modules of $A^*_{n,k}$ and $N_{n,k}$ are S_n-isomorphic, and are given by Theorem 11 and Proposition 36.*

However, this statement does not hold for the dual Whitney homology. For instance, $\mu(ab, abab)_{A^*_{n,k}} = +3$, but $\mu(ab, abab)_{N_{n,k}} = +1$. The first interval is a rank-one poset consisting of the four elements bab, aab, abb, aba, whereas the second is a rank-one poset consisting of two elements aba, bab.

It is also easy to find examples showing that the rank-selected homology is not the same for each poset.

Example 45. Let $n = 2$ and consider the rank-set $\{1, 3\}$ for the poset $A^*_{2,k}$ and for its subposet of normal words $N_{2,k}$.

The words of length 3, all of which cover the two rank 1 elements a and b, are

$$aaa, aab, aba, abb, baa, bab, bba, bbb.$$

It is clear that the Möbius function values are

$$\mu(\hat{0}, aaa) = 0 = \mu(\hat{0}, bbb), \mu(\hat{0}, w) = -1 \text{ for all } w \notin \{aaa, bbb\}.$$

Hence the Möbius number of the rank-selected subposet of all words is -5, and from Theorem 16 the S_2-representation on homology is $3\, S_{(2)} \oplus 2\, S_{(1,1)}$. The order complex is a wedge of 5 one-dimensional spheres.

Now consider the corresponding rank-selected subposet of normal words: there are only two normal words of length 3, namely aba, bab and hence the Möbius number of the rank-selected subposet of normal words is -1, with trivial homology representation. The order complex is a one-dimensional sphere.

Example 46. More generally, let S be the rank-set $[2, k]$, and consider the posets $A_{2,k}(S)$ and $N_{2,k}(S)$ obtained by deleting the atoms. Then by Theorem 16 the homology of $A_{2,k}(S)$ is

$$(k-1)S_{(1^2)}^{\otimes k} + kS_{(2)},$$

while the homology of the normal word subposet $N_{2,k}(S)$ is seen to be $S_{(1^2)}^{\otimes k}$, which is either the trivial or the sign module, depending on the parity of k.

Remark 47. In fact it is easy to see that $N_{2,k}$ is the ordinal sum [16] of k copies of an antichain of size 2, with a bottom and top element attached. Hence for any subset T of $[1, k]$, there is an S_2-equivariant poset isomorphism between $N_{2,k}(T)$ and $N_{2,|T|}$. Since the S_n-homology of $N_{2,k}$ is easily seen to be the k-fold tensor power of the sign representation, this determines $\tilde{H}(N_{2,k}(T))$ for all rank-sets T.

Recall from [16] that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is Eulerian if its Möbius function μ_P satisfies $\mu(P) = (-1)^{\text{rank}(y)-\text{rank}(x)}$ for all intervals $(x, y) \subseteq (\hat{0}, \hat{1})$. It is known that all intervals $(x, y), y \neq \hat{1}$, in $N_{n,k}$ are Eulerian (see e.g. [16, Exercise 188]). In fact Björner and Wachs observed in [7] that for a finite alphabet $A = \{a_i : 1 \leq i \leq n\}$, the poset of normal words without the top element $N_{n,k} \backslash \{\hat{1}\}$ is simply Bruhat order on the Coxeter group with n generators a_i and relations $a_i^2 = 1$. Thus by lexicographic shellability [6], all intervals $(x, y), y \neq \hat{1}$ are homotopy equivalent to a sphere.

An EL-labelling of the dual poset of normal (or Smirnov) words appears in [11].

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Pierrepont School, One Sylvan Road North, Westport, CT 06880
E-mail address: shsund@comcast.net