

Seven-Point Conformal Blocks in the Extended Snowflake Channel and Beyond

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Seven-point functions have two inequivalent topologies or channels. The comb channel has been computed previously and here we compute scalar conformal blocks in the extended snowflake channel in d dimensions. Our computation relies on the known action of the differential operator that sets up the operator product expansion in embedding space. The scalar conformal blocks in the extended snowflake channel are obtained as a power series expansion in the conformal cross-ratios whose coefficients are a triple sum of the hypergeometric type. This triple sum factorizes into a single sum and a double sum. The single sum can be seen as originating from the comb channel and is given in terms of a ${}_3F_2$ -hypergeometric function, while the double sum originates from the snowflake channel which corresponds to a Kampé de Fériet function. We verify that our results satisfy the symmetry properties of the extended snowflake topology. Moreover, we check that the behavior of the extended snowflake conformal blocks under several limits is consistent with known results. Finally, we conjecture rules leading to a partial construction of scalar M -point conformal blocks in arbitrary topologies.

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1. Introduction

With their extended spacetime symmetry group, conformal field theories (CFTs) are possibly amenable to exact non-perturbative solutions. Indeed, from the operator product expansion (OPE), which expresses the product of two quasi-primary operators in terms of an infinite sum of quasi-primary operators and their descendants, conformal correlation functions can be expanded in terms of the CFT data (the OPE coefficients and operator dimensions, which encode in particular the spectrum of quasi-primary operators of the CFT and their couplings) and conformal blocks. The blocks are functions of the conformal cross-ratios (scalar quantities built from the spacetime coordinates that are invariant under the conformal group) and are completely determined by conformal covariance, although it is notoriously difficult to compute them in all generality. Once the four-point conformal blocks are known, the CFT data is in principle constrained from associativity of the four-point correlation functions, the so-called conformal bootstrap introduced in [1]. For this reason, a lot of effort has been put towards computing four-point conformal blocks, see for example the Casimir equations [2], the shadow formalism [3], the weight-shifting formalism [4], integrability [5], AdS/CFT [6], and the OPE [7–12].

The study of higher-point conformal blocks is a relatively recent field of research in CFT. For example, the scalar M -point blocks in the comb channel in one and two spacetime dimensions as well as the scalar five-point blocks in any spacetime dimensions were first computed in [13, 14]. In any number of dimensions, the scalar M -point conformal blocks in the comb channel were introduced in [15, 16], and the scalar six-point conformal blocks in the snowflake channel were obtained in [17].¹

The method used in [16, 17] to compute the higher-point conformal blocks is based on the embedding space OPE formalism introduced in [9, 10]. Since the embedding space OPE formalism is general, it can in principle be used for any correlation functions in any topology, including internal and external quasi-primary operators in arbitrary irreducible representations of the Lorentz group. In practice, we start with known $(M - 1)$ -point correlation functions, express them in terms of the natural conformal cross-ratios for the OPE differential operator, trivially apply the latter on the known $(M - 1)$ -point correlation functions,² express the resulting M -point correlation functions in terms of the appropriate conformal cross-ratios, and finally evaluate the extra sums using simple hypergeometric identities. All the steps are straightforward, although the final re-summations can be somewhat tedious.

One interesting property of higher-point conformal blocks is the existence of several inequivalent topologies, as exemplified for example by the two topologies appearing at six points, namely the comb channel and the snowflake channel. In [17], we argued that the symmetry group of

¹See also [18] for specific snowflake conformal blocks in two-dimensional CFTs.

²The action of the OPE differential operator has been worked out explicitly in [10].

a given topology, denoted by $H_{M|\text{channel}}$, plays an interesting role in determining the number of equivalent ways of writing the conformal blocks, or in other words the number of identities the conformal blocks must satisfy. This observation also led to an understanding of the number of M -point conformal bootstrap channels one should expect. Moreover, we conjectured an identity between the number of topologies and the symmetry groups, given by

$$\sum_{1 \leq i \leq T_0(M)} \frac{1}{|H_{M|\text{channel } i}|} = \frac{T(M)}{|S_M|} = \frac{(2M-5)!!}{M!}. \quad (1.1)$$

In (1.1), $T_0(M)$ is the number of inequivalent M -point topologies, which corresponds also to the number of unrooted binary trees with M unlabeled leaves;³ $T(M)$ is the number of different ways of expressing the same full M -point correlation functions, which is given by the number of unrooted binary trees with M labeled leaves; and S_M is the symmetry group of the full M -point correlation functions, which is simply the symmetry group of M elements.

Another interesting feature of higher-point conformal blocks is the appearance of extra sums, denoted by the F -function [see (2.5)], in the conformal cross-ratio power series. We argued in [16,17] that the minimal number of extra sums for M -point conformal blocks in an arbitrary channel is $M-4$ and that it is always possible to express F_M as $M-4$ sums (extra sums can be evaluated). This was proven by direct computation in the comb channel and for six-point conformal blocks in the snowflake channel. We also noticed that the F -function factorizes in the comb channel, but not in the six-point snowflake channel.

In this paper, we continue the analysis of higher-point correlation functions by computing scalar seven-point conformal blocks in all topologies, *i.e.* in the comb channel and the so-called extended snowflake channel. From the topologies, it is clear that the symmetry groups are

$$H_{7|\text{comb}} = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2 \quad \text{and} \quad H_{7|\text{snowflake}}^{\text{extended}} = \mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2),$$

in agreement with (1.1). Focusing on the coset cardinalities of the two topologies $|S_7/H_{7|\text{comb}}| = 630$ and $|S_7/H_{7|\text{snowflake}}^{\text{extended}}| = 315$, we conclude that there are 945 different ways of writing the full seven-point correlation functions.

Moreover, we prove that $F_{7|\text{snowflake}}^{\text{extended}}$ has interesting factorization properties. Although this function does not factorize into three independent sums as in the comb channel, we show that it factorizes into one sum and an object with two intertwined sums. In fact, these two factors correspond to $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$, respectively. From this observation, we conjecture rules to partially write down scalar M -point conformal blocks in any topology, up to the knowledge of the conformal cross-ratios.

³Unfortunately, there does not exist an analytic expression for $T_0(M)$. Starting at $M=2$, the first few numbers in the sequence are (1, 1, 1, 1, 2, 2, 4, 6, 11, ...). See *The On-line Encyclopedia of Integer Sequences* at <https://oeis.org/A000672> and <https://oeis.org/A129860> for more details.

This paper is organized as follows: Section 2 summarizes the embedding space OPE formalism. We first review the OPE differential operator in the scalar case and then we give its action on products of conformal cross-ratios. After, we describe our notation for contributions to scalar M -point correlation functions from exchanged quasi-primary operators in trivial irreducible representations, and we discuss the recurrence relation transforming scalar $(M - 1)$ -point correlation functions to scalar M -point correlation functions. We finally review the scalar M -point conformal blocks in the comb channel as well as the scalar six-point conformal blocks in the snowflake channel before introducing the scalar seven-point conformal blocks in the extended snowflake channel. In Section 3, we perform several consistency checks for the scalar seven-point conformal blocks in the extended snowflake channel. We start by verifying that the extended snowflake conformal blocks transform appropriately under the three generators of its topology symmetry group $H_{7|_{\text{snowflake}}}^{\text{extended}}$. We then study the OPE limit, where two embedding space coordinates are taken to coincide, and the limit of unit operator, where one external quasi-primary operator is set to the identity operator. Finally, we conclude in Section 4 with a lengthy discussion of the implications of our results by conjecturing rules to construct scalar conformal blocks in arbitrary topologies from partially fixed building blocks. Appendix A presents the computations necessary to obtain the scalar seven-point conformal blocks in the extended snowflake channel from the scalar six-point conformal blocks in the snowflake channel. Appendix B gives the proofs for the identities of the extended snowflake conformal blocks under the symmetry generators of $H_{7|_{\text{snowflake}}}^{\text{extended}}$. Lastly, Appendix C contains the remaining proofs related to the OPE limit and the limit of unit operator.

2. Higher-Point Conformal Blocks

This section presents the scalar seven-point conformal blocks in the extended snowflake channel. Following [10,16,17], we first give a quick review of the OPE formalism and the scalar conformal blocks in all channels for four-, five-, and six-point correlation functions, as well as for the seven-point correlation functions in the comb channel only. We then give results for the remaining scalar seven-point conformal blocks in the extended snowflake channel, exhausting all the seven-point topologies. The relevant proofs are left for the appendixes.

2.1. M -Point Correlation Functions from the OPE

The form of the OPE as introduced in [9,10] is applicable to operators with any spins in any dimension d . Using the explicit action obtained in [8,10], it is straightforward to compute M -point correlation functions from $(M - 1)$ -point correlation functions. By recurrence, we can thus determine any conformal blocks by simply applying the OPE recursively in the proper order. For example, the OPE has been used to compute two-, three-, and four-point correlation functions in [11] and general rules for four-point conformal blocks in arbitrary irreducible representations,

relevant for the bootstrap, were presented in [12]. Recently, higher-point scalar conformal blocks in the comb channel were obtained in [16] and scalar six-point conformal blocks in the snowflake channel were evaluated in [17] using this method.

The OPE in embedding space simplifies considerably for scalar operators because there are no free Lorentz indices

$$\begin{aligned}\mathcal{O}_i(\eta_1)\mathcal{O}_j(\eta_2) &= c_{ij}^k \frac{1}{(\eta_1 \cdot \eta_2)^{p_{ijk}}} (\mathcal{D}_{12}^2)^{h_{ijk}} \mathcal{O}_k(\eta_2) + \dots, \\ p_{ijk} &= \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k), \quad h_{ijk} = -\frac{1}{2}(\Delta_i - \Delta_j + \Delta_k),\end{aligned}\tag{2.1}$$

where Δ 's are the dimensions of operators $\mathcal{O}(\eta)$, and $\eta_{1,2}$ are the embedding space coordinates. We omitted an infinite sum over quasi-primary operators and kept only the scalar primary in the OPE. The differential operator $(\mathcal{D}_{12}^2)^{h_{ijk}}$ generates the appropriate infinite towers of descendants of $\mathcal{O}_k(\eta_2)$, while c_{ij}^k is the OPE coefficient. All quantities defined in (2.1) are introduced and explained in detail in [10] as is the more complicated case of operators with spin.

The scalar differential operator \mathcal{D}_{12}^2 in (2.1) can be defined for any pairs of coordinates as

$$\mathcal{D}_{ij}^2 = (\eta_i \cdot \eta_j) \partial_j^2 - (d + 2\eta_j \cdot \partial_j) \eta_i \cdot \partial_j,$$

where $\partial_j = \frac{\partial}{\partial \eta_j}$ and d is the dimension of spacetime. \mathcal{D}_{ij}^2 is homogeneous of degree 1 with respect to η_i and of degree -1 with respect to η_j . The scalar OPE differential operator can be made homogeneous of degree 0 with respect to all coordinates, where for brevity $\eta_{ij} \equiv \eta_i \cdot \eta_j$,

$$\bar{\mathcal{D}}_{ij;kl;m}^2 = \frac{\eta_{ij}\eta_{kl}\eta_{im}}{\eta_{ik}\eta_{il}} \mathcal{D}_{ij}^2,$$

for some k, l , and m . While η_m does not appear above, it is singled out in the choice of the cross-ratios below. The action of $\bar{\mathcal{D}}_{ij;kl;m}^2$ on the conformal cross-ratios

$$\begin{aligned}x_m &= \frac{\eta_{ij}\eta_{kl}\eta_{im}}{\eta_{ik}\eta_{il}\eta_{jm}}, \\ y_a &= 1 - \frac{\eta_{im}\eta_{ja}}{\eta_{ia}\eta_{jm}}, \quad 1 \leq a \leq M, \quad a \neq i, j, m, \\ z_{ab} &= \frac{\eta_{ik}\eta_{il}\eta_{ab}}{\eta_{kl}\eta_{ia}\eta_{ib}}, \quad 1 \leq a < b \leq M, \quad a, b \neq i, j,\end{aligned}\tag{2.2}$$

can be written as an infinite series of the hypergeometric type as

$$\begin{aligned}\bar{\mathcal{D}}_{ij;kl;m}^{2h} x_m^{\bar{q}} &\prod_{\substack{1 \leq a \leq M \\ a \neq i, j, m}} (1 - y_a)^{-q_a} \\ &= x_m^{\bar{q}+h} \sum_{\{n_a, n_{am}, n_{ab}\} \geq 0} \frac{(-h)_{\bar{n}_m + \bar{n}} (q_m)_{\bar{n}_m} (\bar{q} + h)_{\bar{n} - \bar{n}}}{(\bar{q})_{\bar{n} + \bar{n}_m} (\bar{q} + 1 - d/2)_{\bar{n}_m + \bar{n}}} \\ &\quad \times \prod_{\substack{1 \leq a \leq M \\ a \neq i, j, m}} \frac{(q_a)_{n_a}}{n_{am}! (n_a - n_{am} - \bar{n}_a)!} y_a^{n_a} \left(\frac{x_m z_{am}}{y_a} \right)^{n_{am}} \prod_{\substack{1 \leq a < b \leq M \\ a, b \neq i, j, m}} \frac{1}{n_{ab}!} \left(\frac{x_m z_{ab}}{y_a y_b} \right)^{n_{ab}},\end{aligned}\tag{2.3}$$

for any real h . We note here that the OPE differential operator (2.3) differs by a numerical scaling factor from the one defined in [10]. Moreover, in (2.3) we introduced $\bar{q} = \sum_{\substack{1 \leq a \leq M \\ a \neq i, j}} q_a$ as well as

$$\begin{aligned} \bar{n} &= \sum_{\substack{1 \leq a \leq M \\ a \neq i, j, m}} n_a, & \bar{n}_m &= \sum_{\substack{1 \leq a \leq M \\ a \neq i, j, m}} n_{am}, \\ \bar{n}_a &= \sum_{\substack{1 \leq b \leq M \\ b \neq i, j, m, a}} n_{ab}, & \bar{\bar{n}} &= \sum_{\substack{1 \leq a < b \leq M \\ a, b \neq i, j, m}} n_{ab}. \end{aligned}$$

On the one hand, the action of the OPE is most elementary with the choice of conformal cross-ratios shown above in (2.2). On the other hand, the expressions for conformal blocks may be simpler when written in terms of *a priori* unknown set of cross-ratios. Once a convenient set of cross-ratios for the conformal blocks has been found one needs to change variables between the different choices for the cross-ratios. First, it is necessary to transform an $(M-1)$ -point correlation function to be given in terms of the cross-ratios that work well with the OPE differential operator. Then, one acts with the OPE. Finally, the result is re-expressed using the cross-ratios that simplify the M -point conformal blocks. This last step implies that we explicitly evaluate series of the hypergeometric type to reach the simplest possible form for the conformal blocks.

2.2. Scalar M -Point Correlation Functions

We now focus on the contributions from exchanged scalar quasi-primary operators with conformal dimensions Δ_{k_a} to M -point correlation functions of external scalar quasi-primary operators with conformal dimensions Δ_{i_a} , denoted by

$$I_{M(\Delta_{k_1}, \dots, \Delta_{k_{M-3}})}^{(\Delta_{i_2}, \dots, \Delta_{i_M}, \Delta_{i_1})} \Big|_{\text{channel}} = L_{M|\text{channel}}^{(\Delta_{i_2}, \dots, \Delta_{i_M}, \Delta_{i_1})} \left[\prod_{1 \leq a \leq M-3} (u_a^M)^{\frac{\Delta_{k_a}}{2}} \right] G_{M|\text{channel}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{u}^M, \mathbf{v}^M). \quad (2.4)$$

Obviously, the full scalar correlation functions are sums of I_M for scalar exchanges as in (2.4) (the focus of this paper) and I_M for exchanges of quasi-primary operators in non-trivial irreducible representations of the Lorentz group (not discussed here).

In (2.4), the quantity L_M corresponds to the external operators, or the “legs,” and is a product of M factors of embedding space coordinates responsible for the proper scaling behavior of the correlation functions. Meanwhile, G_M are the conformal blocks in the channel (or topology) of interest, given by

$$\begin{aligned} G_{M|\text{channel}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{u}^M, \mathbf{v}^M) &= \sum_{\{m_a, m_{ab}\} \geq 0} C_{M|\text{channel}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) F_{M|\text{channel}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) \\ &\times \prod_{1 \leq a \leq M-3} \frac{(u_a^M)^{m_a}}{m_a!} \prod_{1 \leq a < b \leq M-3} \frac{(1 - v_{ab}^M)^{m_{ab}}}{m_{ab}!}. \end{aligned} \quad (2.5)$$

$$I_M^{(\Delta_{i_2}, \dots, \Delta_{i_M}, \Delta_{i_1})} \Big|_{\text{comb}}^{(\Delta_{k_1}, \dots, \Delta_{k_{M-3}})} = \mathcal{O}_{i_2} \text{ --- } \begin{array}{c} \mathcal{O}_{i_3} \quad \mathcal{O}_{i_4} \quad \dots \quad \mathcal{O}_{i_M} \\ | \quad | \quad \quad \quad | \\ \mathcal{O}_{k_1} \end{array} \text{ --- } \mathcal{O}_{i_1}$$

Fig. 1: Topology of scalar M -point conformal blocks in the comb channel.

Equations (2.4) and (2.5) are functions of a set of conformal cross-ratios denoted by the vector \mathbf{u}^M of u_a^M and the matrix \mathbf{v}^M of v_{ab}^M . We stress here that the best sets of conformal cross-ratios are not known *a priori* (there are usually more than one) and that they are dependent on the topology. The scalar conformal blocks (2.5) are thus expressed as series of the hypergeometric type in powers of the conformal cross-ratios, with extra sums encoded in the functions F_M . Although there is freedom in the division between C_M and F_M , we conjectured in [17] that the latter is a function of the vector \mathbf{m} of powers of the cross-ratios \mathbf{u}^M only, implying it would not depend on the matrix \mathbf{m} of powers of the cross-ratios $1 - v^M$. With that in mind, we also construct F_M such that it satisfies interesting symmetry properties related to the symmetries of the associated topology.

Finally, we mention that the vectors of parameters \mathbf{h} and \mathbf{p} originate from the action of the OPE (2.1). They are thus simple linear combinations of the conformal dimensions for the external and internal quasi-primary operators. This statement can be seen directly by recursion of the OPE, from which it is straightforward to obtain the following identity for some choice of k , l and m ,

$$I_M^{(\Delta_{i_2}, \dots, \Delta_{i_M}, \Delta_{i_1})} \Big|_{\text{channel}}^{(\Delta_{k_1}, \dots, \Delta_{k_{M-3}})} \Big|_{M \text{ points}} = \frac{1}{\eta_{1M}^{p_M}} \left(\frac{\eta_{kM} \eta_{lM}}{\eta_{1M} \eta_{kl}} \right)^{h_{M-1}} \bar{\mathcal{D}}_{M1;kl;m}^{2h_{M-1}} I_{M-1}^{(\Delta_{i_2}, \dots, \Delta_{i_{M-1}}, \Delta_{k_{M-3}})} \Big|_{\text{channel}} \Big|_{M-1 \text{ points}}. \quad (2.6)$$

It is important to point out that to reach the desired channel on the LHS of (2.6), it is necessary to start from some appropriate channels on the RHS of (2.6).

Before proceeding with the scalar seven-point conformal blocks in the extended snowflake channel, we now state the results for the scalar M -point correlation functions in the comb channel as well as the scalar six-point correlation functions in the snowflake channel that were obtained previously by using (2.6). While these channels were computed elsewhere, the results will be important for the factorization properties of the extended snowflake channel and for certain general observations made in Section 4.

2.3. Scalar M -Point Correlation Functions in the Comb Channel

The comb channel has the topology depicted in Figure 1. Starting from the scalar four-point correlation functions and using (2.6), we found in [16] that the M -point correlation functions in

the comb channel can be expressed as

$$L_{M|\text{comb}}^{(\Delta_{i_2}, \dots, \Delta_{i_M}, \Delta_{i_1})} = \left(\frac{\eta_{34}}{\eta_{23}\eta_{24}} \right)^{\frac{\Delta_{i_2}}{2}} \left[\prod_{1 \leq a \leq M-2} \left(\frac{\eta_{a+1, a+3}}{\eta_{a+1, a+2}\eta_{a+2, a+3}} \right)^{\frac{\Delta_{i_{a+2}}}{2}} \right] \left(\frac{\eta_{M-1, M}}{\eta_{1, M-1}\eta_{1M}} \right)^{\frac{\Delta_{i_1}}{2}}, \quad (2.7)$$

$$u_a^M = \frac{\eta_{1+a, 2+a}\eta_{3+a, 4+a}}{\eta_{1+a, 3+a}\eta_{2+a, 4+a}}, \quad v_{ab}^M = \frac{\eta_{2-a+b, 4+b}}{\eta_{2+b, 4+b}} \prod_{1 \leq c \leq a} \frac{\eta_{3+b-c, 4+b-c}}{\eta_{2+b-c, 4+b-c}},$$

with

$$C_{M|\text{comb}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) = \frac{(p_3)_{m_1 + \text{tr}_0 \mathbf{m}} (p_2 + h_2)_{m_1 + \text{tr}_1 \mathbf{m}}}{(p_3)_{m_1 + \text{tr}_1 \mathbf{m}}} \left[\prod_{1 \leq a \leq M-3} \frac{(\bar{p}_{a+2} + \bar{h}_{a+2})_{m_a + m_{a+1} + \bar{m}_a + \bar{\bar{m}}_a}}{(\bar{p}_{a+2} + \bar{h}_{a+1})_{2m_a + \bar{m}_{a-1} + \bar{m}_a + \bar{\bar{m}}_a}} \right. \\ \left. \times (p_{a+2} - m_{a-1})_{m_a + \text{tr}_a \mathbf{m}} \frac{(-h_{a+2})_{m_a} (-h_{a+2} + m_a - m_{a+1})_{\bar{m}_{a-1}}}{(\bar{p}_{a+2} + \bar{h}_{a+1} + 1 - d/2)_{m_a}} \right], \quad (2.8)$$

$$\text{tr}_a \mathbf{m} = \sum_b m_{b, a+b}, \quad \bar{m}_a = \sum_{b \leq a} m_{ba}, \quad \bar{\bar{m}}_a = \sum_{b > a} (\bar{m}_b - \text{tr}_b \mathbf{m}),$$

and

$$F_{M|\text{comb}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}) = \prod_{1 \leq a \leq M-4} {}_3F_2 \left[\begin{matrix} -m_a, -m_{a+1}, -\bar{p}_{a+2} - \bar{h}_{a+1} + d/2 - m_a \\ p_{a+3} - m_a, h_{a+2} + 1 - m_a \end{matrix} ; 1 \right]. \quad (2.9)$$

We define $\bar{p}_a = \sum_{b=2}^a p_b$ and $\bar{h}_a = \sum_{b=2}^a h_b$, while the explicit expressions for the vectors \mathbf{h} and \mathbf{p} in the comb channel are given by

$$2h_2 = \Delta_{k_1} - \Delta_{i_2} - \Delta_{i_3}, \quad 2h_a = \Delta_{k_{a-1}} - \Delta_{k_{a-2}} - \Delta_{i_{a+1}}, \quad (2.10)$$

$$p_2 = \Delta_{i_3}, \quad 2p_3 = \Delta_{i_2} + \Delta_{k_1} - \Delta_{i_3}, \quad 2p_a = \Delta_{i_a} + \Delta_{k_{a-2}} - \Delta_{k_{a-3}},$$

where $k_{M-2} \equiv i_1$.

As mentioned earlier, the set of conformal cross-ratios is not unique, and (2.7) was chosen based on the OPE limit while other choice are possible. Moreover, we point out that F_M in the comb channel is a product of $M - 4$ hypergeometric functions. The factorization property of $F_{M|\text{comb}}$ will play an important role later.

2.4. Scalar Six-Point Correlation Functions in the Snowflake Channel

Starting at six points, there are more topologies than the comb channel. For six-point correlation functions, the remaining topology is the snowflake channel, depicted in Figure 2 and studied in [17].

Using the known five-point correlation functions in the comb channel and applying the OPE as in (2.6) on the appropriate quasi-primary operator, we generate the scalar six-point correlation

$$I_{6|\text{snowflake}}^{(\Delta_{i_2}, \dots, \Delta_{i_6}, \Delta_{i_1})} \Big|_{\text{snowflake}} = \begin{array}{c} \mathcal{O}_{i_4} \quad \mathcal{O}_{i_5} \\ \diagdown \quad \diagup \\ \mathcal{O}_{k_2} \\ \diagup \quad \diagdown \\ \mathcal{O}_{i_3} \quad \mathcal{O}_{i_6} \\ \diagdown \quad \diagup \\ \mathcal{O}_{k_1} \quad \mathcal{O}_{k_3} \\ \diagdown \quad \diagup \\ \mathcal{O}_{i_2} \quad \mathcal{O}_{i_1} \end{array}$$

Fig. 2: Topology of the six-point conformal blocks in the snowflake channel.

functions in the snowflake channel for which their decomposition as in (2.4) and (2.5) is

$$L_{6|\text{snowflake}}^{(\Delta_{i_2}, \dots, \Delta_{i_6}, \Delta_{i_1})} = \left(\frac{\eta_{13}}{\eta_{12}\eta_{23}} \right)^{\frac{\Delta_{i_2}}{2}} \left(\frac{\eta_{12}}{\eta_{13}\eta_{23}} \right)^{\frac{\Delta_{i_3}}{2}} \left(\frac{\eta_{35}}{\eta_{34}\eta_{45}} \right)^{\frac{\Delta_{i_4}}{2}} \\ \times \left(\frac{\eta_{34}}{\eta_{35}\eta_{45}} \right)^{\frac{\Delta_{i_5}}{2}} \left(\frac{\eta_{15}}{\eta_{16}\eta_{56}} \right)^{\frac{\Delta_{i_6}}{2}} \left(\frac{\eta_{56}}{\eta_{15}\eta_{16}} \right)^{\frac{\Delta_{i_1}}{2}}, \quad (2.11)$$

$$u_1^6 = \frac{\eta_{15}\eta_{23}}{\eta_{12}\eta_{35}}, \quad u_2^6 = \frac{\eta_{13}\eta_{45}}{\eta_{15}\eta_{34}}, \quad u_3^6 = \frac{\eta_{16}\eta_{35}}{\eta_{13}\eta_{56}}, \\ v_{11}^6 = \frac{\eta_{13}\eta_{25}}{\eta_{12}\eta_{35}}, \quad v_{12}^6 = \frac{\eta_{14}\eta_{35}}{\eta_{15}\eta_{34}}, \quad v_{22}^6 = \frac{\eta_{13}\eta_{24}}{\eta_{12}\eta_{34}}, \\ v_{13}^6 = \frac{\eta_{15}\eta_{26}}{\eta_{12}\eta_{56}}, \quad v_{23}^6 = \frac{\eta_{15}\eta_{36}}{\eta_{13}\eta_{56}}, \quad v_{33}^6 = \frac{\eta_{35}\eta_{46}}{\eta_{34}\eta_{56}}$$

with

$$C_{6|\text{snowflake}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) = \frac{(p_2 + h_3)_{m_1+m_{23}} (p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}} (-h_3)_{m_1+m_{11}+m_{22}+m_{13}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}} (p_2 + 1 - d/2)_{m_1}} \\ \times \frac{(p_3 - h_2 + h_4)_{m_2+m_{11}} (p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}} (-h_4)_{m_2+m_{12}+m_{22}+m_{33}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{33}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\ \times \frac{(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}} (-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}} (-h_5)_{m_3+m_{13}+m_{23}+m_{33}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}}, \quad (2.12)$$

and

$$F_{6|\text{snowflake}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}) = \frac{1}{(p_3)_{-m_1} (-p_3 + h_2 + d/2)_{-m_2} (-h_2)_{-m_3}} \\ \times F_{2,1,0}^{1,3,2} \left[\begin{array}{c} \bar{p}_3 - d/2; -m_2, -h_2, p_3; -m_1, -m_3 \\ -h_2 - m_3, p_3 - m_1; p_3 - h_2 + 1 - d/2; - \end{array} \middle| 1, 1 \right] \\ = \frac{1}{(p_3)_{-m_1} (-p_3 + h_2 + d/2)_{-m_2} (-\bar{p}_3 - h_2 + d/2)_{-m_3}} \\ \times F_{1,1,1}^{2,1,1} \left[\begin{array}{c} \bar{p}_3 - d/2, p_3; -m_2; -m_3 \\ p_3 - m_1; p_3 - h_2 + 1 - d/2; \bar{p}_3 + h_2 + 1 - d/2 \end{array} \middle| 1, 1 \right]. \quad (2.13)$$

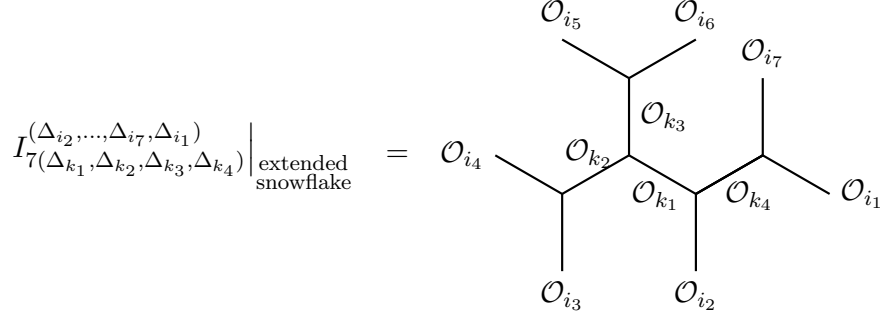


Fig. 3: Topology of the seven-point conformal blocks in the extended snowflake channel.

where the vectors \mathbf{h} and \mathbf{p} are given by

$$\begin{aligned}
2h_2 &= \Delta_{k_3} - \Delta_{k_2} - \Delta_{k_1}, & 2h_3 &= \Delta_{i_3} - \Delta_{i_2} - \Delta_{k_1}, \\
2h_4 &= \Delta_{i_5} - \Delta_{i_4} - \Delta_{k_2}, & 2h_5 &= \Delta_{i_1} - \Delta_{i_6} - \Delta_{k_3}, \\
p_2 &= \Delta_{k_1}, & 2p_3 &= \Delta_{k_2} + \Delta_{k_3} - \Delta_{k_1}, & 2p_4 &= \Delta_{i_2} + \Delta_{i_3} - \Delta_{k_1}, \\
2p_5 &= \Delta_{i_4} + \Delta_{i_5} - \Delta_{k_2}, & 2p_6 &= \Delta_{i_6} + \Delta_{i_1} - \Delta_{k_3}.
\end{aligned} \tag{2.14}$$

Contrary to the comb channel where the extra sums in $F_{M|\text{comb}}$ factorize, the two extra sums appearing in $F_{6|\text{snowflake}}$ do not factorize. In fact, (2.13) is written in terms of Kampé de Fériet functions [19], which are defined as

$$F_{q,s,v}^{p,r,u} \left[\begin{array}{c} \mathbf{a}; \mathbf{c}; \mathbf{f} \\ \mathbf{b}; \mathbf{d}; \mathbf{g} \end{array} \middle| x, y \right] = \sum_{m,n \geq 0} \frac{(\mathbf{a})_{m+n} (\mathbf{c})_m (\mathbf{f})_n x^m y^n}{(\mathbf{b})_{m+n} (\mathbf{d})_m (\mathbf{g})_n m! n!}, \tag{2.15}$$

for

$$\begin{aligned}
(\mathbf{a})_{m+n} &= (a_1)_{m+n} \cdots (a_p)_{m+n}, & (\mathbf{b})_{m+n} &= (b_1)_{m+n} \cdots (b_q)_{m+n}, \\
(\mathbf{c})_m &= (c_1)_m \cdots (c_r)_m, & (\mathbf{d})_m &= (d_1)_m \cdots (d_s)_m, \\
(\mathbf{f})_n &= (f_1)_n \cdots (f_u)_n, & (\mathbf{g})_n &= (g_1)_n \cdots (g_v)_n.
\end{aligned}$$

We now turn to the scalar seven-point correlation functions in the extended snowflake channel.

2.5. Scalar Seven-Point Correlation Functions in the Extended Snowflake Channel

As for six-point correlation functions, there are two different topologies for seven-point correlation functions: the comb channel discussed before and the extended snowflake channel illustrated in Figure 3. The scalar seven-point correlation functions in the extended snowflake channel can be computed from the scalar six-point correlation functions in both channels.

Starting from the comb channel, it is necessary to act with the OPE on one of the two quasi-primary operators inside the comb, *i.e.* on \mathcal{O}_{i_4} or \mathcal{O}_{i_5} in Figure 1 (acting on any one of the other quasi-primary operators leads to the seven-point comb channel). On the contrary, due

to the symmetry properties of the snowflake channel seen in Figure 2, the OPE can be applied anywhere on the snowflake to reach the extended snowflake. In Appendix A we follow the second route to compute the scalar seven-point conformal blocks in the extended snowflake channel.

To summarize, we start from the scalar six-point correlation functions in the snowflake channel (2.12), redefine the quasi-primary operators such that $\mathcal{O}_{i_a}(\eta_a) \rightarrow \mathcal{O}_{i_{a-1}}(\eta_{a-1})$ with the knowledge that $\mathcal{O}_{i_0}(\eta_0) \equiv \mathcal{O}_{i_6}(\eta_6)$, and finally apply (2.6) with $k = 5$, $l = 6$ and $m = 6$, and we obtain

$$\begin{aligned}
L_{7|\text{extended snowflake}}^{(\Delta_{i_2}, \dots, \Delta_{i_7}, \Delta_{i_1})} &= \left(\frac{\eta_{16}}{\eta_{12}\eta_{26}} \right)^{\frac{\Delta_{i_2}}{2}} \left(\frac{\eta_{24}}{\eta_{23}\eta_{34}} \right)^{\frac{\Delta_{i_3}}{2}} \left(\frac{\eta_{23}}{\eta_{24}\eta_{34}} \right)^{\frac{\Delta_{i_4}}{2}} \\
&\quad \times \left(\frac{\eta_{46}}{\eta_{45}\eta_{56}} \right)^{\frac{\Delta_{i_5}}{2}} \left(\frac{\eta_{45}}{\eta_{46}\eta_{56}} \right)^{\frac{\Delta_{i_6}}{2}} \left(\frac{\eta_{16}}{\eta_{17}\eta_{67}} \right)^{\frac{\Delta_{i_7}}{2}} \left(\frac{\eta_{67}}{\eta_{16}\eta_{17}} \right)^{\frac{\Delta_{i_1}}{2}}, \\
u_1^7 &= \frac{\eta_{12}\eta_{46}}{\eta_{16}\eta_{24}}, & u_2^7 &= \frac{\eta_{26}\eta_{34}}{\eta_{23}\eta_{46}}, & u_3^7 &= \frac{\eta_{24}\eta_{56}}{\eta_{26}\eta_{45}}, & u_4^7 &= \frac{\eta_{17}\eta_{26}}{\eta_{12}\eta_{67}}, \\
v_{11}^7 &= \frac{\eta_{14}\eta_{26}}{\eta_{16}\eta_{24}}, & v_{12}^7 &= \frac{\eta_{24}\eta_{36}}{\eta_{23}\eta_{46}}, & v_{13}^7 &= \frac{\eta_{15}\eta_{46}}{\eta_{16}\eta_{45}}, & v_{14}^7 &= \frac{\eta_{16}\eta_{27}}{\eta_{12}\eta_{67}}, \\
v_{22}^7 &= \frac{\eta_{13}\eta_{26}}{\eta_{16}\eta_{23}}, & v_{23}^7 &= \frac{\eta_{25}\eta_{46}}{\eta_{26}\eta_{45}}, & v_{24}^7 &= \frac{\eta_{26}\eta_{37}}{\eta_{23}\eta_{67}}, & v_{33}^7 &= \frac{\eta_{24}\eta_{35}}{\eta_{23}\eta_{45}}, \\
v_{34}^7 &= \frac{\eta_{26}\eta_{47}}{\eta_{24}\eta_{67}}, & v_{44}^7 &= \frac{\eta_{46}\eta_{57}}{\eta_{45}\eta_{67}},
\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}
C_{7|\text{extended snowflake}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) &= \frac{(p_2 + h_3)_{m_1 - m_4 + m_{23}} (p_3)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} (-h_3)_{m_1 + m_4 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}}{(p_2)_{2m_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m_1}} \\
&\times \frac{(p_3 - h_2 + h_4)_{m_2 + m_{11} + m_{34}} (p_2 + h_2)_{m_1 - m_2 + m_3 + m_{13} + m_{23} + m_{44}} (-h_4)_{m_2 + m_{12} + m_{22} + m_{24} + m_{33}}}{(p_3 - h_2)_{2m_2 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\
&\times \frac{(\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}} (-h_2)_{m_1 + m_2 - m_3 + m_{11} + m_{22} + m_{24} + m_{34}} (-h_5)_{m_3 + m_{13} + m_{23} + m_{33} + m_{44}}}{(\bar{p}_3 + h_2)_{2m_3 + m_{12} + m_{13} + m_{23} + m_{33} + m_{44}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
&\times \frac{(p_4 - m_1)_{m_4 + m_{14}} (-h_6)_{m_4 + m_{14} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m_4 + m_{11} + m_{13} + m_{22}}}{(p_2 + h_3 + m_1)_{-m_4} (p_4 - h_3)_{2m_4 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m_4}},
\end{aligned} \tag{2.17}$$

as well as

$$\begin{aligned}
F_{7|\text{extended snowflake}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}) &= \frac{1}{(p_3)_{-m_1} (-p_3 + h_2 + d/2)_{-m_2} (-\bar{p}_3 - h_2 + d/2)_{-m_3}} \\
&\times F_{1,1,1}^{2,1,1} \left[\begin{matrix} \bar{p}_3 - d/2, p_3; -m_2; -m_3 \\ p_3 - m_1; p_3 - h_2 + 1 - d/2; \bar{p}_3 + h_2 + 1 - d/2 \end{matrix} \middle| 1, 1 \right] \\
&\times {}_3F_2 \left[\begin{matrix} -m_1, -m_4, -p_2 + d/2 - m_1 \\ p_4 - m_1, 1 - p_2 - h_3 - m_1 \end{matrix} ; 1 \right],
\end{aligned} \tag{2.18}$$

and finally

$$\begin{aligned}
2h_2 &= \Delta_{k_3} - \Delta_{k_2} - \Delta_{k_1}, & 2h_3 &= \Delta_{i_2} - \Delta_{k_4} - \Delta_{k_1}, & 2h_4 &= \Delta_{i_4} - \Delta_{i_3} - \Delta_{k_2}, \\
2h_5 &= \Delta_{i_6} - \Delta_{i_5} - \Delta_{k_3}, & 2h_6 &= \Delta_{i_1} - \Delta_{i_7} - \Delta_{k_4}, \\
p_2 &= \Delta_{k_1}, & 2p_3 &= \Delta_{k_2} + \Delta_{k_3} - \Delta_{k_1}, & 2p_4 &= \Delta_{k_4} + \Delta_{i_2} - \Delta_{k_1}, \\
2p_5 &= \Delta_{i_3} + \Delta_{i_4} - \Delta_{k_2}, & 2p_6 &= \Delta_{i_5} + \Delta_{i_6} - \Delta_{k_3}, & 2p_7 &= \Delta_{i_7} + \Delta_{i_1} - \Delta_{k_4}.
\end{aligned} \tag{2.19}$$

As conjectured and argued for based on consistency under several limits in [16,17], the number of extra sums for scalar seven-point conformal blocks [see (2.9) with $M = 7$ and (2.18)] is three. Moreover, a comparison between (2.18) on one side and (2.9) and (2.13) on the other side shows that $F_{7|_{\text{snowflake}}}^{\text{extended}} = F_{6|_{\text{snowflake}}} F_{5|_{\text{comb}}}$, a factorization which is reminiscent of the factorization seen in the comb channel. We will come back to this interesting observation in Section 4.

The following sections analyse the extended snowflake results (2.16), (2.17), (2.18), and (2.19). We first verify that the symmetry properties of the extended snowflake topology extend to the scalar conformal blocks G_7 (2.17) and (2.18). We then check that the scalar conformal blocks in the extended snowflake channel exhibit the proper behavior under the OPE limit and the limit of unit operator. All proofs are left for Appendixes B and C.

3. Sanity Checks

Conformal blocks must satisfy several interesting properties. In this section, we verify that the symmetries of the scalar seven-point conformal blocks in the extended snowflake channel coincide with the symmetry group $H_{7|_{\text{snowflake}}}^{\text{extended}} = \mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2)$ of the extended snowflake topology. We also check that the scalar seven-point correlation functions reduce to the corresponding scalar six-point correlation functions under the OPE limit and the limit of unit operator.

3.1. Symmetry Properties

For a given topology, there exists an associated symmetry group which we denote by $H_{M|\text{channel}}$. Each element of the symmetry group is a symmetry transformation of the topology, *i.e.* acting with a symmetry element transforms the topology back to itself. As a consequence, the scalar conformal blocks must verify several identities related to the symmetry group $H_{M|\text{channel}}$ where the conformal cross-ratios and the vectors \mathbf{h} and \mathbf{p} are transformed according to the symmetries. All the identities can be generated from a smaller subset of identities corresponding to the subset of generators of the symmetry group of the associated topology.

For seven-point correlation functions in the extended snowflake channel, the symmetry group is $H_{7|_{\text{snowflake}}}^{\text{extended}} = \mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2)$ since the extended snowflake topology is invariant under the three generators shown in Figure 4. The first \mathbb{Z}_2 subgroup corresponds to dendrite (or OPE)

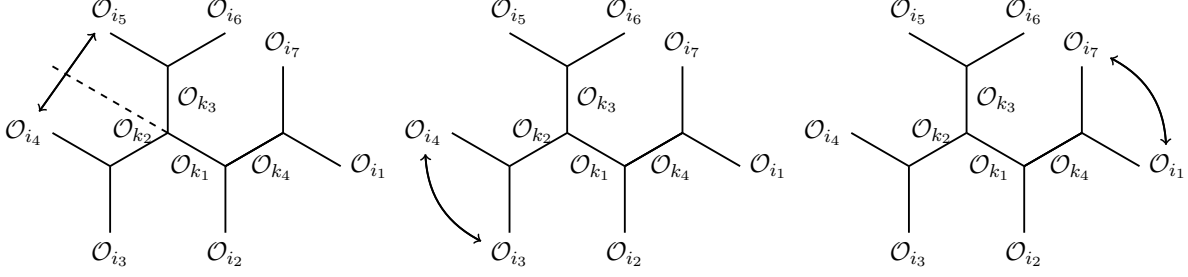


Fig. 4: Symmetries of the scalar seven-point conformal blocks in the extended snowflake channel. The figure shows the three generators with reflection (left), dendrite permutation of the first kind (middle), and dendrite permutation of the second kind (right).

permutations of the second kind, *i.e.* permutations $\mathcal{O}_{i_1} \leftrightarrow \mathcal{O}_{i_7}$. The second and third cyclic groups, denoted by $(\mathbb{Z}_2)^2$ in the symmetry group $H_{7|_{\text{snowflake}}}^{\text{extended}}$, correspond to dendrite permutations of the first kind, *i.e.* permutations $\mathcal{O}_{i_3} \leftrightarrow \mathcal{O}_{i_4}$ as well as $\mathcal{O}_{i_5} \leftrightarrow \mathcal{O}_{i_6}$. Finally, the last \mathbb{Z}_2 subgroup corresponds to simultaneous reflections $\mathcal{O}_{i_3} \leftrightarrow \mathcal{O}_{i_5}$ and $\mathcal{O}_{i_4} \leftrightarrow \mathcal{O}_{i_6}$. The latter can be seen as an internal dendrite permutation, and its effect is to switch the two external dendrites associated to the $(\mathbb{Z}_2)^2$ subgroup. This observation implies the semi-direct nature of the product $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ and distinguishes between the two different kinds (first and second) of external dendrite permutations.

Since the order of the symmetry group for the extended snowflake topology is $|H_{7|_{\text{snowflake}}}^{\text{extended}}| = 16$, there are sixteen equivalent ways of writing the contributions to the scalar seven-point correlation functions I_7 . Hence, there are fifteen identities that the scalar seven-point conformal blocks in the extended snowflake channel G_7 must verify. We now present the three identities associated to the three generators of Figure 4 from which all remaining identities can be obtained. The proofs are left to Appendix B.

For the reflection generator, where we choose

$$\mathcal{O}_{i_3}(\eta_3) \leftrightarrow \mathcal{O}_{i_5}(\eta_5), \quad \mathcal{O}_{i_4}(\eta_4) \leftrightarrow \mathcal{O}_{i_6}(\eta_6), \quad \Delta_{k_2} \leftrightarrow \Delta_{k_3},$$

the legs and conformal cross-ratios (2.16) transform as

$$L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}} \rightarrow (v_{11}^7)^{h_3-h_6} (v_{12}^7)^{h_4} (v_{23}^7)^{h_5} (v_{34}^7)^{h_6} L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}},$$

$$u_1^7 \rightarrow \frac{u_1^7}{v_{11}^7}, \quad u_2^7 \rightarrow \frac{u_3^7}{v_{23}^7}, \quad u_3^7 \rightarrow \frac{u_2^7}{v_{12}^7}, \quad u_4^7 \rightarrow \frac{u_4^7}{v_{34}^7},$$

$$v_{11}^7 \rightarrow \frac{1}{v_{11}^7}, \quad v_{12}^7 \rightarrow \frac{1}{v_{23}^7}, \quad v_{13}^7 \rightarrow \frac{v_{22}^7}{v_{11}^7 v_{12}^7}, \quad v_{14}^7 \rightarrow \frac{v_{11}^7 v_{14}^7}{v_{34}^7},$$

$$v_{22}^7 \rightarrow \frac{v_{13}^7}{v_{11}^7 v_{23}^7}, \quad v_{23}^7 \rightarrow \frac{1}{v_{12}^7}, \quad v_{24}^7 \rightarrow \frac{v_{44}^7}{v_{23}^7 v_{34}^7}, \quad v_{33}^7 \rightarrow \frac{v_{33}^7}{v_{12}^7 v_{23}^7},$$

$$v_{34}^7 \rightarrow \frac{1}{v_{34}^7}, \quad v_{44}^7 \rightarrow \frac{v_{24}^7}{v_{12}^7 v_{34}^7},$$

which leads to the following identity,

$$\begin{aligned}
& G_{7|\text{snowflake}}^{\text{extended}}(d, h_2, h_3, h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7) (u_1^7, u_2^7, u_3^7, u_4^7; v_{11}^7, v_{12}^7, v_{13}^7, v_{14}^7, v_{22}^7, v_{23}^7, v_{24}^7, v_{33}^7, v_{34}^7, v_{44}^7) \\
&= (v_{11}^7)^{h_3-h_6} (v_{12}^7)^{h_4} (v_{23}^7)^{h_5} (v_{34}^7)^{h_6} \\
&\quad \times G_{7|\text{snowflake}}^{\text{extended}}(d, -p_2-h_2, h_3, h_5, h_4, h_6; p_2, p_3, p_4, p_6, p_5, p_7) \left(\frac{u_1^7}{v_{11}^7}, \frac{u_3^7}{v_{23}^7}, \frac{u_2^7}{v_{12}^7}, \frac{u_4^7}{v_{34}^7}, \right. \\
&\quad \left. \frac{1}{v_{11}^7}, \frac{1}{v_{23}^7}, \frac{v_{22}^7}{v_{11}^7 v_{12}^7}, \frac{v_{11}^7 v_{14}^7}{v_{34}^7}, \frac{v_{13}^7}{v_{11}^7 v_{23}^7}, \frac{1}{v_{12}^7}, \frac{v_{44}^7}{v_{23}^7 v_{34}^7}, \frac{v_{33}^7}{v_{12}^7 v_{23}^7}, \frac{1}{v_{34}^7}, \frac{v_{24}^7}{v_{12}^7 v_{34}^7} \right). \tag{3.1}
\end{aligned}$$

With the help of (2.5) and using (2.17) and (2.18), (3.1) implies that F_7 does not change under the reflection generator. As such, there is an associated identity for C_7 which we prove in the appendix.

The generator of dendrite permutations of the first kind that we choose acts as $\mathcal{O}_{i_3}(\eta_3) \leftrightarrow \mathcal{O}_{i_4}(\eta_4)$. Hence, the legs and conformal cross-ratios (2.16) change under this generator as

$$\begin{aligned}
& L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{ka}}{2}} \rightarrow (v_{12}^7)^{-p_3-h_5} (v_{33}^7)^{h_5} L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{ka}}{2}}, \\
& u_1^7 \rightarrow u_1^7 v_{12}^7, \quad u_2^7 \rightarrow \frac{u_2^7}{v_{12}^7}, \quad u_3^7 \rightarrow \frac{u_3^7}{v_{33}^7}, \quad u_4^7 \rightarrow u_4^7, \\
& v_{11}^7 \rightarrow v_{22}^7, \quad v_{12}^7 \rightarrow \frac{1}{v_{12}^7}, \quad v_{13}^7 \rightarrow \frac{v_{12}^7 v_{13}^7}{v_{33}^7}, \quad v_{14}^7 \rightarrow v_{14}^7, \\
& v_{22}^7 \rightarrow v_{11}^7, \quad v_{23}^7 \rightarrow \frac{v_{12}^7 v_{23}^7}{v_{33}^7}, \quad v_{24}^7 \rightarrow v_{34}^7, \quad v_{33}^7 \rightarrow \frac{1}{v_{33}^7}, \\
& v_{34}^7 \rightarrow v_{24}^7, \quad v_{44}^7 \rightarrow \frac{v_{12}^7 v_{44}^7}{v_{33}^7},
\end{aligned}$$

and that translates into the identity

$$\begin{aligned}
& G_{7|\text{snowflake}}^{\text{extended}}(d, h_2, h_3, h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7) (u_1^7, u_2^7, u_3^7, u_4^7; v_{11}^7, v_{12}^7, v_{13}^7, v_{14}^7, v_{22}^7, v_{23}^7, v_{24}^7, v_{33}^7, v_{34}^7, v_{44}^7) \\
&= (v_{12}^7)^{-p_3-h_5} (v_{33}^7)^{h_5} \\
&\quad \times G_{7|\text{snowflake}}^{\text{extended}}(d, h_2, h_3, -p_3+h_2-h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7) \left(u_1^7 v_{12}^7, \frac{u_2^7}{v_{12}^7}, \frac{u_3^7}{v_{33}^7}, u_4^7; \right. \\
&\quad \left. v_{22}^7, \frac{1}{v_{12}^7}, \frac{v_{12}^7 v_{13}^7}{v_{33}^7}, v_{14}^7, v_{11}^7, \frac{v_{12}^7 v_{23}^7}{v_{33}^7}, v_{34}^7, \frac{1}{v_{33}^7}, v_{24}^7, \frac{v_{12}^7 v_{44}^7}{v_{33}^7} \right). \tag{3.2}
\end{aligned}$$

Again, (2.5) with (2.12) and (2.13) imply that F_7 is invariant under dendrite permutations of the first kind. Therefore, there exist a second identity for C_7 .

The last generator $\mathcal{O}_{i_1}(\eta_1) \leftrightarrow \mathcal{O}_{i_7}(\eta_7)$ is the generator for dendrite permutations of the second

kind and it leads to

$$\begin{aligned}
L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}} &\rightarrow (v_{14}^7)^{-p_4} L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}}, \\
u_1^7 &\rightarrow u_1^7 v_{14}^7, & u_2^7 &\rightarrow u_2^7, & u_3^7 &\rightarrow u_3^7, & u_4^7 &\rightarrow \frac{u_4^7}{v_{14}^7}, \\
v_{11}^7 &\rightarrow v_{34}^7, & v_{12}^7 &\rightarrow v_{12}^7, & v_{13}^7 &\rightarrow v_{44}^7, & v_{14}^7 &\rightarrow \frac{1}{v_{14}^7}, \\
v_{22}^7 &\rightarrow v_{24}^7, & v_{23}^7 &\rightarrow v_{23}^7, & v_{24}^7 &\rightarrow v_{22}^7, & v_{33}^7 &\rightarrow v_{33}^7, \\
&& v_{34}^7 &\rightarrow v_{11}^7, & v_{44}^7 &\rightarrow v_{13}^7,
\end{aligned}$$

for the legs and conformal cross-ratios (2.16). The corresponding identity for the conformal blocks is therefore

$$\begin{aligned}
&G_{7|_{\text{snowflake}}}^{\text{extended}}(d, h_2, h_3, h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7)(u_1^7, u_2^7, u_3^7, u_4^7; v_{11}^7, v_{12}^7, v_{13}^7, v_{14}^7, v_{22}^7, v_{23}^7, v_{24}^7, v_{33}^7, v_{34}^7, v_{44}^7) \\
&= (v_{14}^7)^{-p_4} \\
&\times G_{7|_{\text{snowflake}}}^{\text{extended}}(d, h_2, h_3, h_4, h_5, -p_4 + h_3 - h_6; p_2, p_3, p_4, p_5, p_6, p_7) \left(u_1^7 v_{14}^7, u_2^7, u_3^7, \frac{u_4^7}{v_{14}^7}; \right. \\
&\quad \left. v_{34}^7, v_{12}^7, v_{44}^7, \frac{1}{v_{14}^7}, v_{24}^7, v_{23}^7, v_{22}^7, v_{33}^7, v_{11}^7, v_{13}^7 \right). \tag{3.3}
\end{aligned}$$

The decomposition (2.5) with (2.17) and (2.18) shows once again that F_7 given by (2.18) does not transform under this last generator, resulting in a third identity for C_7 .

To summarize, the symmetry group of the extended snowflake topology is $H_{7|_{\text{snowflake}}}^{\text{extended}} = \mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2)$, its order is sixteen and it is generated by the three elements shown in Figure 4. These three generators lead to identities for the scalar seven-point conformal blocks (2.5) in the extended snowflake channel with (2.17) and (2.18). The three identities are (3.1), (3.2) and (3.3) which represent reflections, dendrite permutations of the first kind, and dendrite permutations of the second kind, respectively. It is straightforward to produce the remaining twelve equivalent representations for I_7 in the extended snowflake channel by composing the three identities above.

In the following, we investigate two limits: the OPE limit and the limit of unit operator. Due to the symmetry properties of the extended snowflake topology mentioned above, there are only two (three) limits to verify for the OPE (unit operator) case, the remaining limits being related by the action of the symmetry group.

3.2. OPE Limit

For two external operators that emerge from the same OPE vertex in a given topology, we can define the OPE limit. In this limit, the embedding space coordinates of these two operators coincide. Therefore, the OPE limit reduces the original M -point correlation function to the

appropriate $(M-1)$ -point correlation function, up to a pre-factor originating from the OPE (2.1). In the scalar case, the pre-factor is obtained directly from (2.6).

For scalar seven-point correlation functions in the extended snowflake channel, as depicted in Figure 3, there are only three possible OPE limits. They are $\eta_3 \rightarrow \eta_4$, $\eta_5 \rightarrow \eta_6$, and $\eta_7 \rightarrow \eta_1$, respectively. With the invariance of I_7 under the symmetry group $H_{7|\text{snowflake}}^{\text{extended}} = \mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2)$, we only need to investigate I_7 under two OPE limits. Here, we choose $\eta_7 \rightarrow \eta_1$ (which is not related by symmetry to another pair) and $\eta_3 \rightarrow \eta_4$.

For the limit $\eta_7 \rightarrow \eta_1$, we have

$$I_{7(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3}, \Delta_{k_4})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_7}, \Delta_{i_1})} \Big|_{\text{extended snowflake}} \xrightarrow{\eta_7 \rightarrow \eta_1} (\eta_{17})^{-p_7} I_{6(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_1})} \Big|_{\text{snowflake}}, \quad (3.4)$$

as well as

$$\begin{aligned} L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}} &\rightarrow (\eta_{17})^{-p_7} L_6 \prod_{1 \leq a \leq 3} (u_a^6)^{\frac{\Delta_{k_a}}{2}}, \\ u_1^7 &\rightarrow u_1^6, & u_2^7 &\rightarrow u_2^6, & u_3^7 &\rightarrow u_3^6, & u_4^7 &\rightarrow 0, \\ v_{11}^7 &\rightarrow v_{11}^6, & v_{12}^7 &\rightarrow v_{12}^6, & v_{13}^7 &\rightarrow v_{13}^6, & v_{14}^7 &\rightarrow 1, \\ v_{22}^7 &\rightarrow v_{22}^6, & v_{23}^7 &\rightarrow v_{23}^6, & v_{24}^7 &\rightarrow v_{22}^6, & v_{33}^7 &\rightarrow v_{33}^6, \\ & & v_{34}^7 &\rightarrow v_{11}^6, & v_{44}^7 &\rightarrow v_{13}^6. \end{aligned}$$

Here, the quantities and the conformal dimensions on the RHS of the limits are the ones relevant for the six-point correlation functions in the snowflake channel, *i.e.* (2.11), (2.12), (2.13) and trivial substitutions for the vectors \mathbf{h} and \mathbf{p} .

As a consequence, the OPE limit (3.4) leads to the identity $G_{7|\eta_7 \rightarrow \eta_1} = G_6$, or more precisely

$$\begin{aligned} G_{7|\text{snowflake}}^{\text{extended}}(d, h_2, h_3, h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7) &(u_1^6, u_2^6, u_3^6, 0; v_{11}^6, v_{12}^6, v_{13}^6, 1, v_{22}^6, v_{23}^6, v_{22}^6, v_{33}^6, v_{11}^6, v_{13}^6) \\ &= G_{6|\text{snowflake}}(d, h_2, h_3, h_4, h_5; p_2, p_3, p_4, p_5, p_6) (u_1^6, u_2^6, u_3^6; v_{11}^6, v_{12}^6, v_{13}^6, v_{22}^6, v_{23}^6, v_{33}^6). \end{aligned}$$

Since $F_7 \rightarrow F_6$ in that limit, we have

$$\begin{aligned} G_{7|\eta_7 \rightarrow \eta_1} &= \sum_{m_a, m_{ab} \geq 0} \frac{(p_2 + h_3)_{m_1 + m_{23}} (p_3)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} (-h_3)_{m_1 + m_{11} + m_{13} + m_{22}}}{(p_2)_{2m_1 + m_{11} + m_{13} + m_{22} + m_{23}} (p_2 + 1 - d/2)_{m_1}} \\ &\times \frac{(p_3 - h_2 + h_4)_{m_2 + m_{11}} (p_2 + h_2)_{m_1 - m_2 + m_3 + m_{13} + m_{23}} (-h_4)_{m_2 + m_{12} + m_{22} + m_{33}}}{(p_3 - h_2)_{2m_2 + m_{11} + m_{12} + m_{22} + m_{33}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\ &\times \frac{(\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}} (-h_2)_{m_1 + m_2 - m_3 + m_{11} + m_{22}} (-h_5)_{m_3 + m_{13} + m_{23} + m_{33}}}{(\bar{p}_3 + h_2)_{2m_3 + m_{12} + m_{13} + m_{23} + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\ &\times \frac{(-h_6)_{m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m_{11} + m_{13} + m_{22} - m_{24} - m_{34} - m_{44}}}{(p_4 - h_3)_{m_{11} + m_{13} + m_{22}}} \\ &\times F_6 \frac{m_{11}! m_{13}! m_{22}!}{(m_{11} - m_{34})! (m_{13} - m_{44})! (m_{22} - m_{24})! m_{24}! m_{34}! m_{44}!} \end{aligned}$$

$$\times \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m_{ab}}}{m_{ab}!},$$

where we performed the following change of variables,

$$m_{11} \rightarrow m_{11} - m_{34}, \quad m_{13} \rightarrow m_{13} - m_{44}, \quad m_{22} \rightarrow m_{22} - m_{24}.$$

Evaluating the sums over m_{24} , m_{34} , and finally m_{44} using standard hypergeometric re-summation formula straightforwardly leads to $G_{7|\eta_7 \rightarrow \eta_1} = G_6$, proving (3.4).

Since the proof for the last independent OPE limit $\eta_3 \rightarrow \eta_4$ is more intricate, it is left for Appendix C.

3.3. Limit of Unit Operator

The limit of unit operator consists in setting one external scalar quasi-primary operator to the identity operator. As a consequence, a given M -point correlation function morphs into the appropriate $(M - 1)$ -point correlation function.

For scalar seven-point correlation functions in the extended snowflake channel, there are only three limits of unit operator to check, thanks to the symmetry group. They are $\mathcal{O}_{i_2}(\eta_2) \rightarrow \mathbb{1}$, $\mathcal{O}_{i_3}(\eta_3) \rightarrow \mathbb{1}$ and $\mathcal{O}_{i_7}(\eta_7) \rightarrow \mathbb{1}$, respectively.

Focusing first on the limit $\mathcal{O}_{i_7}(\eta_7) \rightarrow \mathbb{1}$, we have $\Delta_{i_7} = 0$, $\Delta_{k_4} = \Delta_{i_1}$, and

$$I_{7(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3}, \Delta_{k_4})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_7}, \Delta_{i_1})} \Big|_{\text{extended snowflake}} \xrightarrow{\mathcal{O}_{i_7}(\eta_7) \rightarrow \mathbb{1}} I_{6(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3})}^{(\Delta_{i_1}, \Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6})} \Big|_{\text{snowflake}}. \quad (3.5)$$

In this limit, it is straightforward to see that $h_6 = p_7 = 0$. Moreover, the remaining components of the vectors \mathbf{h} and \mathbf{p} (2.19) become the appropriate components of the scalar six-point correlation functions in the snowflake channel (2.14). Since the conformal cross-ratios transform as

$$u_a^7 \rightarrow u_a^6 \quad (1 \leq a \leq 3), \quad v_{ab}^7 \rightarrow v_{ab}^6, \quad (1 \leq a \leq b \leq 3),$$

and the remaining conformal cross-ratios disappear due to the sums over m_4 , m_{14} , m_{24} , m_{34} and m_{44} in (2.17) being trivial [thanks to the Pochhammer symbol $(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}$ forcing $m_4 = m_{14} = m_{24} = m_{34} = m_{44} = 0$], we obtain

$$L_7 \prod_{1 \leq a \leq 3} (u_a^7)^{\frac{\Delta_{k_a}}{2}} \rightarrow L_6 \prod_{1 \leq a \leq 3} (u_a^6)^{\frac{\Delta_{k_a}}{2}},$$

in the appropriate channels.

Furthermore, the limit of unit operator (3.5) leads to

$$F_{7|\mathcal{O}_{i_7} \rightarrow \mathbb{1}} = \frac{(-\bar{p}_3 - h_2 + d/2 - m_3)_{m_3} (-p_3 + h_2 + d/2 - m_2)_{m_2}}{(p_3)_{-m_1}} \sum_{r_1, r_2 \geq 0} \frac{(-m_2)_{t_1} (-m_3)_{t_2}}{t_1! t_2!} \\ \times \frac{(\bar{p}_3 - d/2)_{t_1+t_2}}{(\bar{p}_3 + h_2 + 1 - d/2)_{t_2} (p_3 - m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(p_3 - h_2 + 1 - d/2)_{t_1}},$$

which implies $F_7 \rightarrow F_6$ with the help of the standard ${}_3F_2$ -hypergeometric function identity (A.3). Taking into account that $C_7 \rightarrow C_6$ trivially, we thus prove that $G_7 \rightarrow G_6$ in the limit of unit operator (3.5).

The two remaining limits of unit operator being longer, their proofs are left for Appendix C.

4. Discussion and Conclusion

With the knowledge of the action of the OPE differential operator on a general product of conformal cross-ratios, the embedding space OPE formalism introduced in [9, 10] leads to explicit results for any conformal correlation function. Therefore, it is perfectly suited to investigate d -dimensional higher-point correlation functions, a nascent field of research in CFT. With the tools provided by the embedding space OPE formalism, we already determined scalar higher-point correlation functions in the comb channel in [16] and scalar six-point correlation functions in the snowflake channel in [17].

In this paper we computed the scalar seven-point conformal blocks in the extended snowflake channel, see (2.16), (2.17), (2.18), and (2.19). With that result and the scalar seven-point conformal blocks in the comb channel, the scalar conformal blocks for seven-point correlation functions are known in all topologies. We stress that all our results are derived from the OPE, as such they are exact and do not need to be verified *e.g.* from the Casimir equations. Nevertheless, to check the algebra, we did observe that all the symmetries [namely, the $\mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \times \mathbb{Z}_2)$ symmetry group of the extended snowflake topology] and appropriate limits (namely, the OPE limit where two embedding space coordinates coincide, and the limit of unit operator where one external quasi-primary operator is set to the identity operator) are satisfied.

As we did in prior work, we defined the scalar conformal blocks in terms of coefficients of the power series in the conformal cross-ratios. These coefficients are products of two functions, the C -function which is a product of terms, and the F -function that contains multiple sums, see (2.5).

In [16], we argued that using OPE limits and limits of unit operators repetitively allows one to construct the C -function. It seems this argument can be extended to all channels. Indeed, by rewriting the vectors \mathbf{h} and \mathbf{p} explicitly in terms of the conformal dimensions, it is clear that a pattern emerges. Focusing on the scalar M -point C -functions in the comb channel (2.8) and the scalar six- and seven-point C -functions in the snowflake and extended snowflake channels (2.12) and (2.17), we have

$$C_{M|\text{comb}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) = \frac{\left(\frac{\Delta_{k_1} - \Delta_{i_2} + \Delta_{i_3}}{2}\right)_{m_1 + \text{tr}_1 \mathbf{m}} \left(\frac{\Delta_{k_1} + \Delta_{i_2} - \Delta_{i_3}}{2}\right)_{m_1 + \text{tr}_0 \mathbf{m}}}{(\Delta_{k_1})_{2m_1 + \bar{m}_1 + \bar{\bar{m}}_1} (\Delta_{k_1} + 1 - d/2)_{m_1}}$$

$$\begin{aligned}
& \times \frac{\left(\frac{\Delta_{k_{M-3}} - \Delta_{i_1} + \Delta_{i_M}}{2}\right)_{m_{M-3} + \bar{m}_{M-4}} \left(\frac{\Delta_{k_{M-3}} + \Delta_{i_1} - \Delta_{i_M}}{2}\right)_{m_{M-3} + \bar{m}_{M-3} + \bar{\bar{m}}_{M-3}}}{(\Delta_{k_{M-3}})_{2m_{M-3} + \bar{m}_{M-4} + \bar{\bar{m}}_{M-3} + \bar{\bar{\bar{m}}}_{M-3}} (\Delta_{k_{M-3}} + 1 - d/2)_{m_{M-3}}} \\
& \times \frac{\left(\frac{\Delta_{k_2} - \Delta_{k_1} + \Delta_{i_4}}{2}\right)_{m_2 - m_1 + \text{tr}_2 \mathbf{m}} \left(\frac{\Delta_{k_2} + \Delta_{k_1} - \Delta_{i_4}}{2}\right)_{m_2 + m_1 + \bar{m}_1 + \bar{\bar{m}}_1}}{(\Delta_{k_2})_{2m_2 + \bar{m}_1 + \bar{\bar{m}}_2} (\Delta_{k_2} + 1 - d/2)_{m_2}} \\
& \quad \times \frac{\left(\frac{\Delta_{i_4} - \Delta_{k_2} + \Delta_{k_1}}{2}\right)_{m_1} \left(\frac{\Delta_{i_4} - \Delta_{k_2} + \Delta_{k_1}}{2}\right)_{m_1 - m_2}}{\left(\frac{\Delta_{i_4} + \Delta_{k_2} - \Delta_{k_1}}{2}\right)_{-m_1} \left(\frac{\Delta_{i_4} - \Delta_{k_2} + \Delta_{k_1}}{2}\right)_{m_1 - m_2}} \\
& \times \frac{\left(\frac{\Delta_{k_3} - \Delta_{k_2} + \Delta_{i_5}}{2}\right)_{m_3 - m_2 + \text{tr}_3 \mathbf{m}} \left(\frac{\Delta_{k_3} + \Delta_{k_2} - \Delta_{i_5}}{2}\right)_{m_3 + m_2 + \bar{m}_2 + \bar{\bar{m}}_2}}{(\Delta_{k_3})_{2m_3 + \bar{m}_2 + \bar{\bar{m}}_3} (\Delta_{k_3} + 1 - d/2)_{m_3}} \\
& \quad \times \frac{\left(\frac{\Delta_{i_5} - \Delta_{k_3} + \Delta_{k_2}}{2}\right)_{m_2} \left(\frac{\Delta_{i_5} - \Delta_{k_3} + \Delta_{k_2}}{2}\right)_{m_2 - m_3 + \bar{m}_1}}{\left(\frac{\Delta_{i_5} + \Delta_{k_3} - \Delta_{k_2}}{2}\right)_{-m_2} \left(\frac{\Delta_{i_5} - \Delta_{k_3} + \Delta_{k_2}}{2}\right)_{m_2 - m_3}} \\
& \quad \vdots \\
& \times \frac{\left(\frac{\Delta_{k_{M-3}} - \Delta_{k_{M-4}} + \Delta_{i_{M-1}}}{2}\right)_{m_{M-3} - m_{M-4} + \text{tr}_{M-3} \mathbf{m}} \left(\frac{\Delta_{k_{M-3}} + \Delta_{k_{M-4}} - \Delta_{i_{M-1}}}{2}\right)_{m_{M-3} + m_{M-4} + \bar{m}_{M-4} + \bar{\bar{m}}_{M-4}}}{(\Delta_{k_{M-4}})_{2m_{M-4} + \bar{m}_{M-5} + \bar{\bar{m}}_{M-4} + \bar{\bar{\bar{m}}}_{M-4}} (\Delta_{k_{M-4}} + 1 - d/2)_{m_{M-4}}} \\
& \quad \times \frac{\left(\frac{\Delta_{i_{M-1}} - \Delta_{k_{M-3}} + \Delta_{k_{M-4}}}{2}\right)_{m_{M-4}} \left(\frac{\Delta_{i_{M-1}} - \Delta_{k_{M-3}} + \Delta_{k_{M-4}}}{2}\right)_{m_{M-4} - m_{M-3} + \bar{m}_{M-5}}}{\left(\frac{\Delta_{i_{M-1}} + \Delta_{k_{M-3}} - \Delta_{k_{M-4}}}{2}\right)_{-m_{M-4}} \left(\frac{\Delta_{i_{M-1}} - \Delta_{k_{M-3}} + \Delta_{k_{M-4}}}{2}\right)_{m_{M-4} - m_{M-3}}},
\end{aligned}$$

as well as

$$\begin{aligned}
C_{6|\text{snowflake}}^{(d, \mathbf{h}; \mathbf{p})}(\mathbf{m}, \mathbf{m}) &= \frac{\left(\frac{\Delta_{k_1} - \Delta_{i_2} + \Delta_{i_3}}{2}\right)_{m_1 + m_{23}} \left(\frac{\Delta_{k_1} + \Delta_{i_2} - \Delta_{i_3}}{2}\right)_{m_1 + m_{11} + m_{22} + m_{13}}}{(\Delta_{k_1})_{2m_1 + m_{11} + m_{13} + m_{22} + m_{23}} (\Delta_{k_1} + 1 - d/2)_{m_1}} \\
& \times \frac{\left(\frac{\Delta_{k_2} - \Delta_{i_4} + \Delta_{i_5}}{2}\right)_{m_2 + m_{11}} \left(\frac{\Delta_{k_2} + \Delta_{i_4} - \Delta_{i_5}}{2}\right)_{m_2 + m_{12} + m_{22} + m_{33}}}{(\Delta_{k_2})_{2m_2 + m_{11} + m_{12} + m_{22} + m_{33}} (\Delta_{k_2} + 1 - d/2)_{m_2}} \\
& \times \frac{\left(\frac{\Delta_{k_3} - \Delta_{i_6} + \Delta_{i_1}}{2}\right)_{m_3 + m_{12}} \left(\frac{\Delta_{k_3} + \Delta_{i_6} - \Delta_{i_1}}{2}\right)_{m_3 + m_{13} + m_{23} + m_{33}}}{(\Delta_{k_3})_{2m_3 + m_{12} + m_{13} + m_{23} + m_{33}} (\Delta_{k_3} + 1 - d/2)_{m_3}} \\
& \times \left(\frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2}\right)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} \\
& \times \left(\frac{-\Delta_{k_2} + \Delta_{k_3} + \Delta_{k_1}}{2}\right)_{-m_2 + m_3 + m_1 + m_{13} + m_{23}} \\
& \times \left(\frac{-\Delta_{k_3} + \Delta_{k_1} + \Delta_{k_2}}{2}\right)_{-m_3 + m_1 + m_2 + m_{11} + m_{22}},
\end{aligned}$$

and

$$\begin{aligned}
C_{7\text{snowflake}}^{(d,h;p)}(\mathbf{m}, \mathbf{m}) &= \frac{\left(\frac{\Delta_{k_2}-\Delta_{i_3}+\Delta_{i_4}}{2}\right)_{m_2+m_{11}+m_{34}} \left(\frac{\Delta_{k_2}+\Delta_{i_3}-\Delta_{i_4}}{2}\right)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}}{(\Delta_{k_2})_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (\Delta_{k_2}+1-d/2)_{m_2}} \\
&\times \frac{\left(\frac{\Delta_{k_3}-\Delta_{i_5}+\Delta_{i_6}}{2}\right)_{m_3+m_{12}} \left(\frac{\Delta_{k_3}+\Delta_{i_5}-\Delta_{i_6}}{2}\right)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(\Delta_{k_3})_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}} (\Delta_{k_3}+1-d/2)_{m_3}} \\
&\times \frac{\left(\frac{\Delta_{k_4}-\Delta_{i_7}+\Delta_{i_1}}{2}\right)_{m_4+m_{11}+m_{13}+m_{22}} \left(\frac{\Delta_{k_4}+\Delta_{i_7}-\Delta_{i_1}}{2}\right)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}}{(\Delta_{k_4})_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (\Delta_{k_4}+1-d/2)_{m_4}} \\
&\times \frac{\left(\frac{\Delta_{k_1}-\Delta_{k_4}+\Delta_{i_2}}{2}\right)_{m_1-m_4+m_{23}} \left(\frac{\Delta_{k_1}+\Delta_{k_4}-\Delta_{i_2}}{2}\right)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}}{(\Delta_{k_1})_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}} (\Delta_{k_1}+1-d/2)_{m_1}} \\
&\times \frac{\left(\frac{\Delta_{i_2}+\Delta_{k_1}-\Delta_{k_4}}{2}\right)_{m_1} \left(\frac{\Delta_{i_2}-\Delta_{k_1}+\Delta_{k_4}}{2}\right)_{-m_1+m_4+m_{14}}}{\left(\frac{\Delta_{i_2}-\Delta_{k_1}+\Delta_{k_4}}{2}\right)_{-m_1} \left(\frac{\Delta_{i_2}+\Delta_{k_1}-\Delta_{k_4}}{2}\right)_{m_1-m_4}} \\
&\times \left(\frac{-\Delta_{k_1}+\Delta_{k_2}+\Delta_{k_3}}{2}\right)_{-m_1+m_2+m_3+m_{12}+m_{33}} \\
&\times \left(\frac{-\Delta_{k_2}+\Delta_{k_3}+\Delta_{k_1}}{2}\right)_{-m_2+m_3+m_1+m_{13}+m_{23}+m_{44}} \\
&\times \left(\frac{-\Delta_{k_3}+\Delta_{k_1}+\Delta_{k_2}}{2}\right)_{-m_3+m_1+m_2+m_{11}+m_{22}+m_{24}+m_{34}},
\end{aligned}$$

respectively. Hence, for a proper choice of \mathbf{u} conformal cross-ratios behaving accordingly under the OPE limit,⁴ the C -function is given by multiplying factors of

$$\frac{\left(\frac{\Delta_{k_a}-\Delta_i+\Delta_j}{2}\right)_{m_a+\dots} \left(\frac{\Delta_{k_a}+\Delta_i-\Delta_j}{2}\right)_{m_a+\dots}}{(\Delta_{k_a})_{2m_a+\dots} (\Delta_{k_a}+1-d/2)_{m_a}}, \quad (4.1)$$

for each pair of external quasi-primary operators \mathcal{O}_i and \mathcal{O}_j connected to an internal quasi-primary operator through the OPE,

$$\frac{\left(\frac{\Delta_{k_b}-\Delta_{k_a}+\Delta_i}{2}\right)_{m_b-m_a+\dots} \left(\frac{\Delta_{k_b}+\Delta_{k_a}-\Delta_i}{2}\right)_{m_b+m_a+\dots}}{(\Delta_{k_b})_{2m_b+\dots} (\Delta_{k_b}+1-d/2)_{m_b}} \frac{\left(\frac{\Delta_i-\Delta_{k_b}+\Delta_{k_a}}{2}\right)_{m_a} \left(\frac{\Delta_i-\Delta_{k_b}+\Delta_{k_a}}{2}\right)_{m_a-m_b+\dots}}{\left(\frac{\Delta_i+\Delta_{k_b}-\Delta_{k_a}}{2}\right)_{-m_a} \left(\frac{\Delta_i-\Delta_{k_b}+\Delta_{k_a}}{2}\right)_{m_a-m_b}}, \quad (4.2)$$

for each external quasi-primary operator \mathcal{O}_i connected to a pair of internal quasi-primary operators \mathcal{O}_{k_b} and \mathcal{O}_{k_a} , and

$$\begin{aligned}
&\left(\frac{-\Delta_{k_a}+\Delta_{k_b}+\Delta_{k_c}}{2}\right)_{-m_a+m_b+m_c+\dots} \left(\frac{-\Delta_{k_b}+\Delta_{k_c}+\Delta_{k_a}}{2}\right)_{-m_b+m_c+m_a+\dots} \\
&\times \left(\frac{-\Delta_{k_c}+\Delta_{k_a}+\Delta_{k_b}}{2}\right)_{-m_c+m_a+m_b+\dots}, \quad (4.3)
\end{aligned}$$

⁴The u conformal cross-ratios must be chosen such that only one of them vanishes for a given OPE limit.

for each internal OPE with three internal quasi-primary operators \mathcal{O}_{k_c} , \mathcal{O}_{k_b} , and \mathcal{O}_{k_a} . Here, internal quasi-primary operators appear only once and the second factor in the rule (4.2) is not completely fixed yet (as can be seen by comparing the comb to the extended snowflake). Moreover, the dependence on the matrix of indices \mathbf{m} (denoted by ellipses in the rules above) is not determined. For a proper choice of \mathbf{v} conformal cross-ratios, we conjecture that the remaining ambiguities can be fixed by the OPE limit and the limit of unit operator, although it is probable that a recurring process must be used [($M - 1$)-point correlation functions must be known to verify these limits for M -point correlation functions].

Moreover, we observed that the F -functions (2.9), (2.13), and (2.18), when expressed in terms of the conformal dimensions

$$\begin{aligned}
F_{M|\text{comb}}^{(d,\mathbf{h};\mathbf{p})}(\mathbf{m}) &= {}_3F_2 \left[\begin{matrix} -m_1, -m_2, -\Delta_{k_1} + d/2 - m_1 \\ \frac{\Delta_{k_2} - \Delta_{k_1} + \Delta_{i_4}}{2} - m_1, \frac{\Delta_{k_2} - \Delta_{k_1} - \Delta_{i_4}}{2} + 1 - m_1 \end{matrix} ; 1 \right] \\
&\times {}_3F_2 \left[\begin{matrix} -m_2, -m_3, -\Delta_{k_2} + d/2 - m_2 \\ \frac{\Delta_{k_3} - \Delta_{k_2} + \Delta_{i_5}}{2} - m_2, \frac{\Delta_{k_3} - \Delta_{k_2} - \Delta_{i_5}}{2} + 1 - m_2 \end{matrix} ; 1 \right] \\
&\quad \vdots \\
&\times {}_3F_2 \left[\begin{matrix} -m_{M-4}, -m_{M-3}, -\Delta_{k_{M-4}} + d/2 - m_{M-4} \\ \frac{\Delta_{k_{M-3}} - \Delta_{k_{M-4}} + \Delta_{i_{M-1}}}{2} - m_{M-4}, \frac{\Delta_{k_{M-3}} - \Delta_{k_{M-4}} - \Delta_{i_{M-1}}}{2} + 1 - m_{M-4} \end{matrix} ; 1 \right]
\end{aligned}$$

as well as

$$\begin{aligned}
F_{6|\text{snowflake}}^{(d,\mathbf{h};\mathbf{p})}(\mathbf{m}) &= \frac{1}{\left(\frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2}\right)_{-m_1} (-\Delta_{k_2} + d/2)_{-m_2} (-\Delta_{k_3} + d/2)_{-m_3}} \\
&\times F_{1,1,1}^{2,1,1} \left[\begin{matrix} \frac{\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3} - d}{2}, \frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2}; -m_2; -m_3 \\ \frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2} - m_1; \Delta_{k_2} + 1 - d/2; \Delta_{k_3} + 1 - d/2 \end{matrix} \middle| 1, 1 \right],
\end{aligned}$$

and finally

$$\begin{aligned}
F_{7|\text{extended snowflake}}^{(d,\mathbf{h};\mathbf{p})}(\mathbf{m}) &= \frac{1}{\left(\frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2}\right)_{-m_1} (-\Delta_{k_2} + d/2)_{-m_2} (-\Delta_{k_3} + d/2)_{-m_3}} \\
&\times F_{1,1,1}^{2,1,1} \left[\begin{matrix} \frac{\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3} - d}{2}, \frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2}; -m_2; -m_3 \\ \frac{-\Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3}}{2} - m_1; \Delta_{k_2} + 1 - d/2; \Delta_{k_3} + 1 - d/2 \end{matrix} \middle| 1, 1 \right] \\
&\times {}_3F_2 \left[\begin{matrix} -m_1, -m_4, -\Delta_{k_1} + d/2 - m_1 \\ \frac{\Delta_{k_4} - \Delta_{k_1} + \Delta_{i_2}}{2} - m_1, \frac{\Delta_{k_4} - \Delta_{k_1} - \Delta_{i_2}}{2} + 1 - m_1 \end{matrix} ; 1 \right],
\end{aligned}$$

satisfy interesting factorization properties. Indeed, in the comb channel it was already pointed out in [16] that the F -function is a product of hypergeometric functions. This can be stated schematically as $F_{M|\text{comb}} = (F_{5|\text{comb}})^{M-4}$ with $F_{5|\text{comb}}$ a building block for the F -function. For the snowflake channel, no such factorization occurs, implying that $F_{6|\text{snowflake}}$ is another building

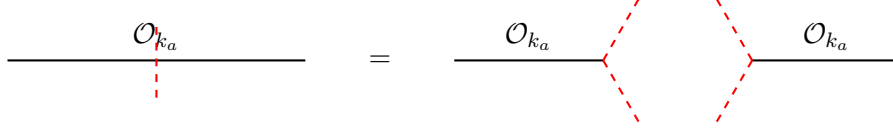


Fig. 5: Cutting procedure for the computation of the F -function. For a given topology, an internal line is cut and the remaining segments are both dressed as one OPE.

block for the F -function. We now argue that $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$, together with $F_{4|\text{comb}} = 1$, are the only building blocks necessary to construct the F -function in any topology, with the rules that the F -function is obtained by multiplication of factors of

$${}_3F_2 \left[\begin{matrix} -m_a, -m_b, -\Delta_{k_a} + d/2 - m_a \\ \frac{\Delta_{k_b} - \Delta_{k_a} + \Delta_i}{2} - m_a, \frac{\Delta_{k_b} - \Delta_{k_a} - \Delta_i}{2} + 1 - m_a \end{matrix} ; 1 \right], \quad (4.4)$$

for each external quasi-primary operator \mathcal{O}_i connected to two internal quasi-primary operators \mathcal{O}_{k_b} and \mathcal{O}_{k_a} , and factors of

$$\frac{1}{\left(\frac{-\Delta_{k_a} + \Delta_{k_b} + \Delta_{k_c}}{2} \right)_{-m_a} (-\Delta_{k_b} + d/2)_{-m_b} (-\Delta_{k_c} + d/2)_{-m_c}} \times F_{1,1,1}^{2,1,1} \left[\begin{matrix} \frac{\Delta_{k_a} + \Delta_{k_b} + \Delta_{k_c} - d}{2}, \frac{-\Delta_{k_a} + \Delta_{k_b} + \Delta_{k_c}}{2}; -m_b; -m_c \\ \frac{-\Delta_{k_a} + \Delta_{k_b} + \Delta_{k_c}}{2} - m_1; \Delta_{k_b} + 1 - d/2; \Delta_{k_c} + 1 - d/2 \end{matrix} \middle| 1, 1 \right], \quad (4.5)$$

for each internal OPE connecting three internal quasi-primary operators \mathcal{O}_{k_c} , \mathcal{O}_{k_b} and \mathcal{O}_{k_a} .

First, we observe that this rule is respected when we consider the extended snowflake for which $F_{7|\text{snowflake}}^{\text{extended}} = F_{6|\text{snowflake}} F_{5|\text{comb}}$. The main reason why this factorization should occur for the extended snowflake again originates in the OPE limit and the limit of unit operator. It can also be understood diagrammatically directly from the topology by cutting an internal line and replacing it by a double OPE as in Figure 5.

Indeed, concentrating on the extended snowflake, Figure 6 leads to several identities for its F -function. The two identities depicted in Figure 6 are

$$F_{7|\text{snowflake}}^{\text{extended}} = F_{6|\text{snowflake}} F_{5|\text{comb}} \quad \text{and} \quad F_{7|\text{snowflake}}^{\text{extended}} = F_{4|\text{comb}} F_{7|\text{snowflake}}^{\text{extended}},$$

where the second identity is trivial due to $F_{4|\text{comb}} = 1$. Obviously, there are two extra trivial identities when the cutting procedure is performed on the internal lines with \mathcal{O}_{k_3} and \mathcal{O}_{k_4} , respectively.

To verify that the cutting procedure of Figure 5 makes sense, we investigate its implications for the other topologies, starting from the building blocks $F_{4|\text{comb}} = 1$, $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$. In these cases, we get the results shown in Figure 7 which imply

$$F_{4|\text{comb}} = F_{4|\text{comb}} F_{4|\text{comb}}, \quad F_{5|\text{comb}} = F_{5|\text{comb}} F_{4|\text{comb}}, \quad F_{6|\text{snowflake}} = F_{6|\text{snowflake}} F_{4|\text{comb}}.$$

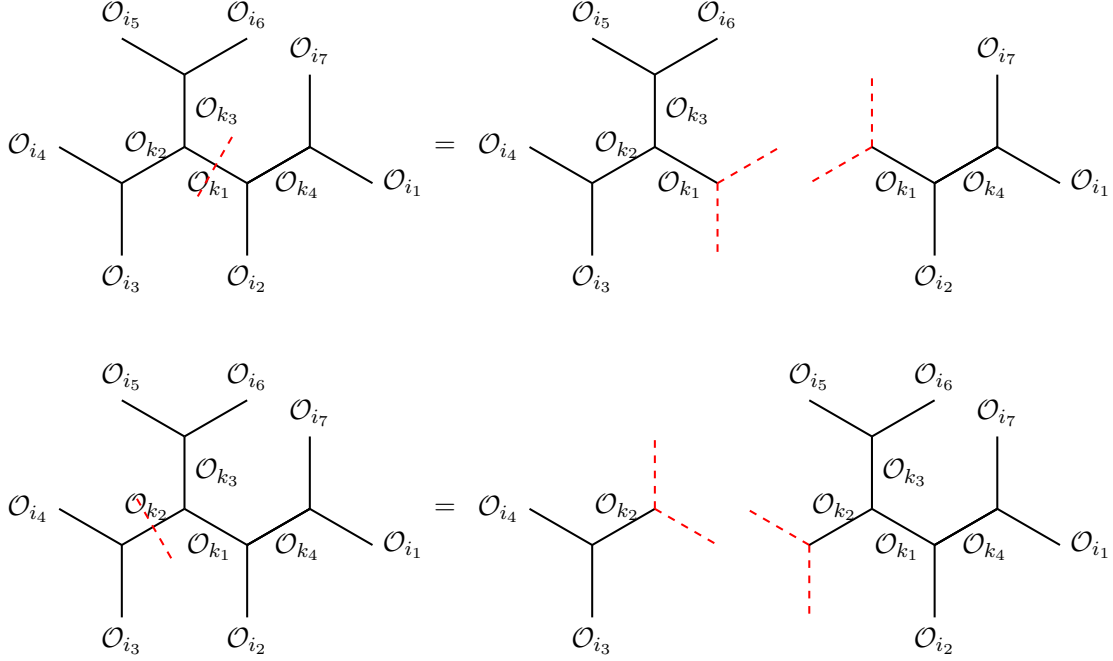


Fig. 6: Cutting procedure implemented on two different internal lines of the extended snowflake topology, leading to two factorization properties of its F -function.

These identities lead to $F_{4|\text{comb}} = 1$ as expected, and suggest that $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$ cannot be determined by the cutting procedure—they have to be computed independently, for example from the embedding space OPE formalism [16,17]. More complicated topologies always generate non-trivial factorization properties leading to the full determination of their F -functions from the building blocks (with the total number of extra sums appearing in F_M always fixed to $M - 4$).

For example, for $F_{M|\text{comb}}$, cutting the internal line with quasi-primary operator \mathcal{O}_{k_a} in the topology depicted in Figure 1 leads to $F_{M|\text{comb}} = F_{3+a|\text{comb}} F_{M+1-a|\text{comb}}$ for any $1 \leq a \leq M - 3$. From its definition (2.9), this identity is verified for any a . Moreover, repetitively cutting the comb topology to extract factors of $F_{5|\text{comb}}$ leads to the identity $F_{M|\text{comb}} = (F_{5|\text{comb}})^{M-4}$ mentioned above. Considering the ten-point correlation function in the topology of Figure 8 as another example, we expect schematically that

$$F_{10|\text{topology of Figure 8}} = (F_{5|\text{comb}} F_{6|\text{snowflake}})^2,$$

where the proper parameters are determined by the quasi-primary operators as in (4.4) and (4.5).

In general, from the counting of the number of extra sums, the factorization property should imply that

$$F_{M|\text{any topology}} = (F_{5|\text{comb}})^{M-4-2n} (F_{6|\text{snowflake}})^n, \quad (4.6)$$

for $0 \leq n \leq \lfloor \frac{M-4}{2} \rfloor$. From Figure 7, the factorization (4.6) should also be unique, meaning a given topology should have only one value of n in the factorization that depends on the topology.

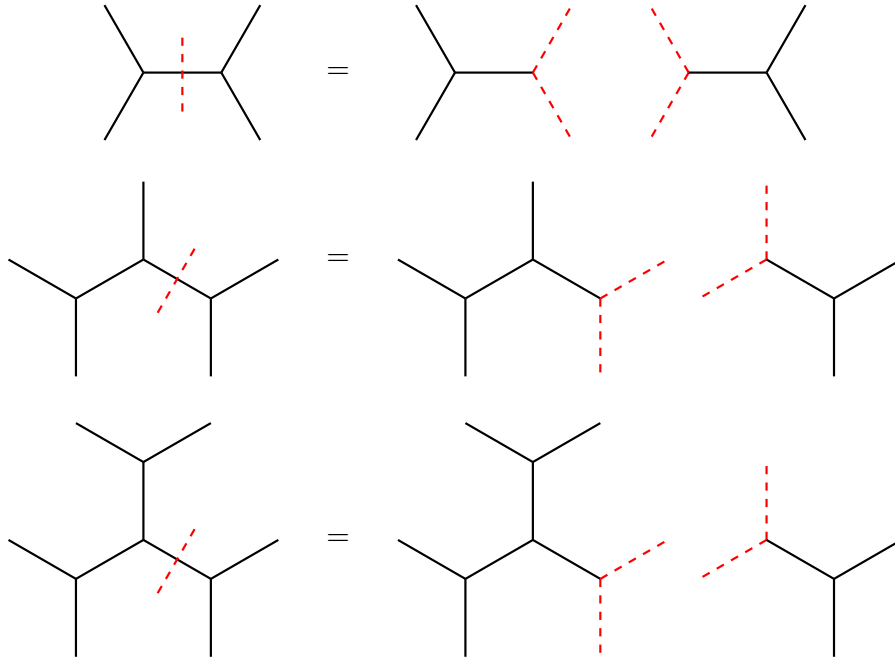


Fig. 7: Cutting procedure for the F -function building blocks.

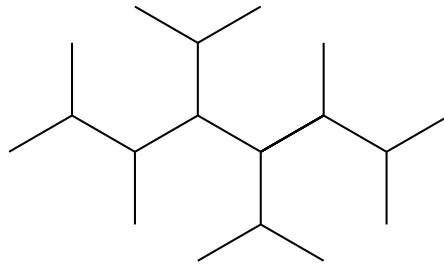


Fig. 8: Cutting procedure for one specific topology appearing in ten-point correlation functions.

Going the opposite way, one should be able to reconstruct the possible topologies by gluing the F -function building blocks $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$ ($F_{4|\text{comb}}$ being trivial, it can be discarded) following a gluing procedure analog to the opposite of the cutting procedure shown in Figure 5. Indeed, for a fixed M -point correlation function and n -factorization (4.6), the gluing procedure should lead to all inequivalent topologies for fixed M and n . Denoting the number of inequivalent M -point topologies with n -factorization as $T_0(M; n)$ and taking into account the uniqueness of the factorization, this observation implies that the number of unrooted binary trees with M unlabeled leaves should be expressible as

$$T_0(M) = \sum_{n \geq 0}^{\lfloor \frac{M-4}{2} \rfloor} T_0(M; n).$$

Hence, if $T_0(M; n)$ can be obtained by the gluing procedure, it should lead to an expression for $T_0(M)$.

Due to their symmetries, the building blocks $F_{5|\text{comb}}$ and $F_{6|\text{snowflake}}$ can be glued in only one independent way, however they can be glued at several different locations on an existing topology. For example, for $n = 0$, the factorization (4.6) leads to $(F_{5|\text{comb}})^{M-4}$ and since the $F_{5|\text{comb}}$ building blocks can only be glued together in one inequivalent topology associated to the comb, we find that $T_0(M;0) = 1$ corresponding to the comb channel. This is not the case when $n > 0$ since the $F_{6|\text{snowflake}}$ building blocks can be glued together in different topologies. Indeed, in the case of $M = 8$ we have $T_0(8;1) = 2$ since the two $F_{5|\text{comb}}$ building blocks can be glued in two inequivalent ways with $F_{6|\text{snowflake}}$, while for $M = 12$ we have $T_0(12;4) = 2$ because the four $F_{6|\text{snowflake}}$ building blocks can be glued in two inequivalent topologies. For $n = 1$, we find that $T_0(M;1)$ is given by the number of partitions of $M - 6$ with at most three parts, which is given by $\left[\frac{(M-3)^2}{12} \right]$ where $[\]$ means rounding to the nearest integer. It would be interesting to study higher n -factorizations in general.

To summarize, we conjecture that there exist rules as in (4.1), (4.2) and (4.3) to construct the C -function as well as (4.4) and (4.5) to construct the F -function, for any topology. These rules rely on the OPE limit and the limit of unit operator for consistency, and they necessitate a proper set of conformal cross-ratios which is not necessarily simple to find. We hope to verify and eventually prove such rules in future work.

Finally, other research avenues worth pursuing include the conformal bootstrap from higher-point correlation functions (where the knowledge of the correlation functions with external quasi-primary operators in scalar representations only can replace the usual conformal bootstrap of four-point correlation functions when fermions are discarded), higher-point correlation functions of quasi-primary operators in arbitrary irreducible representations, and their use in the AdS/CFT correspondence (for example with respect to geodesic Witten diagrams, see for instance [14]).

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A. Extended Snowflake and the OPE

This appendix presents the proof leading to the scalar seven-point conformal blocks in the extended snowflake channel, starting from the OPE acting on the scalar six-point conformal blocks in the snowflake channel found in [17].

The proof consists in several re-summations of the hypergeometric type. For example, we use

the binomial identity

$$(1-v)^{a+b} = \sum_{i \geq 0} (-1)^i \binom{a+b}{i} v^i = \sum_{i,j \geq 0} (-1)^{i+j} \binom{a}{i} \binom{b}{j} v^{i+j}, \quad (\text{A.1})$$

as well as

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z), \\ {}_2F_1 \left[\begin{matrix} -n, b \\ c \end{matrix}; 1 \right] &= \frac{(c-b)_n}{(c)_n}, \end{aligned} \quad (\text{A.2})$$

and

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} -n, b, c \\ d, e \end{matrix}; 1 \right] &= \frac{(d-b)_n}{(d)_n} {}_3F_2 \left[\begin{matrix} -n, b, e-c \\ b-d-n+1, e \end{matrix}; 1 \right], \\ {}_3F_2 \left[\begin{matrix} -n, b, c \\ d, 1+b+c-d-n \end{matrix}; 1 \right] &= \frac{(d-b)_n (d-c)_n}{(d)_n (d-b-c)_n}, \end{aligned} \quad (\text{A.3})$$

for n a non-negative integer.

A.1. Proof of the Extended Snowflake

In order to use (2.6), it is necessary to first shift the quasi-primary operators in the scalar six-point correlation functions (2.11), (2.12), and (2.13), such that $\mathcal{O}_{i_a}(\eta_a) \rightarrow \mathcal{O}_{i_{a-1}}(\eta_{a-1})$ with $\mathcal{O}_{i_0}(\eta_0) \equiv \mathcal{O}_{i_6}(\eta_6)$. Doing so, the legs and the conformal cross-ratios transform as

$$\begin{aligned} L_{6|\text{snowflake}}^{(\Delta_{i_2}, \dots, \Delta_{i_6}, \Delta_{k_4})} &= \left(\frac{\eta_{16}}{\eta_{12}\eta_{26}} \right)^{\frac{\Delta_{i_2}}{2}} \left(\frac{\eta_{24}}{\eta_{23}\eta_{34}} \right)^{\frac{\Delta_{i_3}}{2}} \left(\frac{\eta_{23}}{\eta_{24}\eta_{34}} \right)^{\frac{\Delta_{i_4}}{2}} \\ &\quad \times \left(\frac{\eta_{46}}{\eta_{45}\eta_{56}} \right)^{\frac{\Delta_{i_5}}{2}} \left(\frac{\eta_{45}}{\eta_{46}\eta_{56}} \right)^{\frac{\Delta_{i_6}}{2}} \left(\frac{\eta_{26}}{\eta_{12}\eta_{16}} \right)^{\frac{\Delta_{k_4}}{2}}, \\ u_1^6 &= \frac{\eta_{12}\eta_{46}}{\eta_{16}\eta_{24}}, & u_2^6 &= \frac{\eta_{26}\eta_{34}}{\eta_{23}\eta_{46}}, & u_3^6 &= \frac{\eta_{24}\eta_{56}}{\eta_{26}\eta_{45}}, \\ v_{11}^6 &= \frac{\eta_{14}\eta_{26}}{\eta_{16}\eta_{24}}, & v_{12}^6 &= \frac{\eta_{24}\eta_{36}}{\eta_{23}\eta_{46}}, & v_{13}^6 &= \frac{\eta_{15}\eta_{46}}{\eta_{16}\eta_{45}}, \\ v_{22}^6 &= \frac{\eta_{13}\eta_{26}}{\eta_{16}\eta_{23}}, & v_{23}^6 &= \frac{\eta_{25}\eta_{46}}{\eta_{26}\eta_{45}}, & v_{33}^6 &= \frac{\eta_{24}\eta_{35}}{\eta_{23}\eta_{45}}, \end{aligned} \quad (\text{A.4})$$

where we also changed $\mathcal{O}_{i_1}(\eta_1) \rightarrow \mathcal{O}_{k_4}(\eta_1)$ to implement the recurrence relation (2.6). The shift also implies that the vectors \mathbf{h} and \mathbf{p} become the vectors (2.19) when the new components h_6 and p_7 appearing in (2.6) are included. In addition, we note that C_6 (2.12) and F_6 (2.13) are the same functions but of the new vectors \mathbf{h} and \mathbf{p} (2.19), thus G_6 stays the same function but of the new vectors and conformal cross-ratios (A.4).

We can now use the recurrence relation (2.6) (choosing $k=5$, $l=6$, and $m=6$) to reach the scalar seven-point conformal blocks in the extended snowflake channel. To this end, we apply the

OPE differential operator following (2.3) on the conformal cross-ratios (2.2), *i.e.*

$$\begin{aligned}
x_6^7 &= \frac{\eta_{17}\eta_{56}}{\eta_{57}\eta_{16}}, & y_2^7 &= 1 - \frac{\eta_{12}\eta_{67}}{\eta_{16}\eta_{27}}, & y_3^7 &= 1 - \frac{\eta_{13}\eta_{67}}{\eta_{16}\eta_{37}}, & y_4^7 &= 1 - \frac{\eta_{14}\eta_{67}}{\eta_{16}\eta_{47}}, & y_5^7 &= 1 - \frac{\eta_{15}\eta_{67}}{\eta_{16}\eta_{57}}, \\
z_{23}^7 &= \frac{\eta_{57}\eta_{67}\eta_{23}}{\eta_{27}\eta_{37}\eta_{56}}, & z_{24}^7 &= \frac{\eta_{24}\eta_{57}\eta_{67}}{\eta_{27}\eta_{47}\eta_{56}}, & z_{25}^7 &= \frac{\eta_{25}\eta_{67}}{\eta_{27}\eta_{56}}, & z_{26}^7 &= \frac{\eta_{26}\eta_{57}}{\eta_{27}\eta_{56}}, \\
z_{34}^7 &= \frac{\eta_{34}\eta_{57}\eta_{67}}{\eta_{37}\eta_{47}\eta_{56}}, & z_{35}^7 &= \frac{\eta_{35}\eta_{67}}{\eta_{37}\eta_{56}}, & z_{36}^7 &= \frac{\eta_{36}\eta_{57}}{\eta_{37}\eta_{56}}, & z_{45}^7 &= \frac{\eta_{45}\eta_{67}}{\eta_{47}\eta_{56}}, & z_{46}^7 &= \frac{\eta_{46}\eta_{57}}{\eta_{47}\eta_{56}}.
\end{aligned} \tag{A.5}$$

Therefore, we must first re-express the conformal cross-ratios (A.4) in terms of the conformal cross-ratios (A.5), which leads to

$$u_1^6 = \frac{1 - y_2^7 z_{46}^7}{1 - y_6^7 z_{24}^7}, \quad v_{11}^6 = \frac{1 - y_4^7 z_{26}^7}{1 - y_6^7 z_{24}^7}, \quad v_{13}^6 = \frac{1 - y_5^7 z_{46}^7}{1 - y_6^7 z_{45}^7}, \quad v_{22}^6 = \frac{1 - y_3^7 z_{26}^7}{1 - y_6^7 z_{23}^7}.$$

After acting with the OPE differential operator, we then substitute the conformal cross-ratios (A.5) with the conformal cross-ratios (2.16) using

$$\begin{aligned}
x_6^7 &= \frac{u_1^7 u_3^7 u_4^7}{v_{44}^7}, & y_2^7 &= 1 - \frac{1}{v_{14}^7}, & y_3^7 &= 1 - \frac{v_{22}^7}{v_{24}^7}, & y_4^7 &= 1 - \frac{v_{11}^7}{v_{34}^7}, & y_5^7 &= 1 - \frac{v_{13}^7}{v_{44}^7}, \\
z_{23}^7 &= \frac{v_{44}^7}{v_{14}^7 v_{24}^7 u_1^7 u_3^7}, & z_{24}^7 &= \frac{v_{44}^7}{v_{14}^7 v_{34}^7 u_1^7 u_3^7}, & z_{25}^7 &= \frac{v_{23}^7}{v_{14}^7 u_1^7 u_3^7}, & z_{26}^7 &= \frac{v_{44}^7}{v_{14}^7 u_1^7 u_3^7}, \\
z_{34}^7 &= \frac{v_{44}^7 u_2^7}{v_{24}^7 v_{34}^7 u_3^7}, & z_{35}^7 &= \frac{v_{33}^7}{v_{24}^7 u_3^7}, & z_{36}^7 &= \frac{v_{12}^7 v_{44}^7}{v_{24}^7 u_3^7}, & z_{45}^7 &= \frac{1}{v_{34}^7 u_3^7}, & z_{46}^7 &= \frac{v_{44}^7}{v_{34}^7 u_3^7},
\end{aligned}$$

which results in

$$\begin{aligned}
G_7 &= \sum (-1)^{l_2+l_3+l_4+l_5+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}+k_{12}+k_{23}+k_{33}+s_{11}+s_{22}+s_{13}} \\
&\times \frac{(-h_6)_{m_4} (-h_3 + n_1 + s_{11} + s_{22} + s_{13})_{\bar{r}_6} (p_4 - h_3 + h_6)_{m_4 + \sum_{a=2}^5 \sigma_a}}{(p_4 - h_3)_{2m_4 + \sum_{a=2}^5 \sigma_a} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
&\times \frac{(p_4 - n_1)_{\sigma_2 + r_{26} + \bar{r}_2} (-s_{22})_{\sigma_3 + r_{36} + \bar{r}_3} (-s_{11})_{\sigma_4 + r_{46} + \bar{r}_4} (-s_{13})_{\sigma_5 + r_{56} + \bar{r}_5}}{\sigma_2! \sigma_3!} \frac{(-s_{11})_{\sigma_4 + r_{46} + \bar{r}_4} (-s_{13})_{\sigma_5 + r_{56} + \bar{r}_5}}{\sigma_4! \sigma_5!} \binom{n_{11}}{s_{11}} \binom{n_{22}}{s_{22}} \binom{n_{13}}{s_{13}} \\
&\times \binom{\sigma_2}{l_2} \binom{\sigma_3}{l_3} \binom{\sigma_4}{l_4} \binom{\sigma_5}{l_5} \binom{l_5}{m_{13}} \binom{-p_4 + n_1 - r_{26} - \bar{r}_2 - l_2}{m_{14}} \binom{l_4}{m_{11}} \binom{r_{36}}{k_{12}} \\
&\times \binom{l_3}{m_{22}} \binom{r_{25}}{k_{23}} \binom{r_{35}}{k_{33}} \binom{s_{22} - r_{36} - \bar{r}_3 - l_3}{m_{24}} \binom{s_{11} - r_{46} - \bar{r}_4 - l_4}{m_{34}} \binom{s_{13} - r_{56} - \bar{r}_5 - l_5}{m_{44}} \\
&\times \frac{(u_1^7)^{n_1 + \sum_{a,b \neq 2} r_{ab}} (u_2^7)^{n_2 + r_{34}} (u_3^7)^{n_3 + r_{56}} (u_4^7)^{\bar{r}_6 + \bar{r}} \prod_{1 \leq a < b \leq 4} (1 - v_{ab}^7)^{m_{ab}}}{r_{23}! r_{24}! r_{25}! r_{26}! r_{34}! r_{35}! r_{36}! r_{45}! r_{46}! r_{56}! n_1! n_2! n_3! n_{11}! m_{12}! n_{13}! n_{22}! m_{23}! n_{33}!} C_6 F_6,
\end{aligned} \tag{A.6}$$

with the appropriate legs (2.16) and also $n_{12} = m_{12}$, $n_{23} = m_{23}$, $m_{12} = n_{12} + k_{12}$, $m_{23} = n_{23} + k_{23}$, and $n_{33} + k_{33} = m_{33}$ after some simple manipulations (here the new indices of summation originating from the OPE are denoted by r). In (A.6) and most of the appendixes, we omit the indices under the summation sign to avoid cluttering the equations. The scalar seven-point

conformal blocks in the extended snowflake channel are thus given explicitly by (A.6), and the rest of this appendix is dedicated to re-summing the extra sums [using (A.1), (A.2) and (A.3) repetitively] to reach the results stated in (2.17) and (2.18).

We first evaluate the sums over l_a and σ_a by changing variables to

$$\begin{aligned} l_3 &\rightarrow l_3 + m_{22}, & l_4 &\rightarrow l_4 + m_{11}, & l_5 &\rightarrow l_5 + m_{13}, \\ \sigma_3 &\rightarrow \sigma_3 + m_{22}, & \sigma_4 &\rightarrow \sigma_4 + m_{11}, & \sigma_5 &\rightarrow \sigma_5 + m_{13}, \end{aligned}$$

which results in

$$\begin{aligned} G_7 &= \sum (-1)^{k_{12}+k_{23}+k_{33}+s_{11}+s_{22}+s_{13}} \binom{n_{11}}{s_{11}} \binom{n_{22}}{s_{22}} \binom{n_{13}}{s_{13}} \binom{r_{36}}{k_{12}} \binom{r_{25}}{k_{23}} \binom{r_{35}}{k_{33}} \\ &\times \frac{(p_4 - n_1)_{m_{14}+r_{26}+\bar{r}_2} (-s_{22})_{m_{22}+m_{24}+r_{36}+\bar{r}_3} m_{33}! (-s_{11})_{m_{11}+m_{34}+r_{46}+\bar{r}_4} (-s_{13})_{m_{13}+m_{44}+r_{56}+\bar{r}_5}}{r_{23}! r_{24}! r_{25}! r_{26}! r_{34}! r_{35}! r_{36}! r_{45}! r_{46}! r_{56}!} \frac{1}{n_{11}! n_{13}! n_{22}! n_{33}!} \\ &\times \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}} (-h_3 + n_1 + s_{11} + s_{22} + s_{13})_{\bar{r}_6} (p_4 - h_3 + h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\ &\times \frac{(u_1^7)^{n_1+\sum_{a,b \neq 2} r_{ab}} (u_2^7)^{n_2+r_{34}} (u_3^7)^{n_3+r_{56}} (u_4^7)^{\bar{r}_6+\bar{r}}}{n_1! n_2! n_3!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)^{m_{ab}}}{m_{ab}!} C_6 F_6. \end{aligned}$$

We then change variables as

$$\begin{aligned} s_{11} &\rightarrow s_{11} + m_{11} + m_{34} + r_{46} + \bar{r}_4, \\ s_{22} &\rightarrow s_{22} + m_{22} + m_{24} + r_{36} + \bar{r}_3, \\ s_{13} &\rightarrow s_{13} + m_{13} + m_{44} + r_{56} + \bar{r}_5, \end{aligned}$$

and define a new set of summation indices by

$$\begin{aligned} r_{23} &= r_{23}, & r_{34} &= r_{34}, & r_{46} &= r_{46}, & r_{56} &= r_{56}, \\ r_1^* &= r_{23} + r_{24}, & r_2^* &= r_1^* + r_{25}, & r_1 &= r_{34} + r_{35}, & r_2 &= r_1 + r_{45}, \\ r_{26} &= m_4 - r - r_2^*, & r_{36} &= r - r_{46} - r_{56} - r_2, \end{aligned}$$

such that

$$\begin{aligned} G_7 &= \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\ &\times \frac{(p_4 - n_1)_{m_4+m_{14}-r}}{r_{23}! (r_1^* - r_{23})! (r_2^* - r_1^*)! (m_4 - r - r_2^*)! r_{34}! (r_1 - r_{34})! (r - r_{46} - r_{56} - r_2)! (r_2 - r_1)! r_{46}! r_{56}!} \\ &\times \frac{m_{33}! (-1)^{k_{12}+k_{23}+k_{33}+s_{11}+s_{22}+s_{13}}}{(n_{13} - s_{13} - m_{13} - m_{44} - r_{56} - r_2 - r_2^* + r_1^* + r_{34})! n_{33}!} \binom{r - r_{46} - r_{56} - r_2}{k_{12}} \binom{r_2^* - r_1^*}{k_{23}} \binom{r_1 - r_{34}}{k_{33}} \\ &\times \frac{(-h_3 + n_1 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44} + s_{11} + s_{22} + s_{13} + r + r_2 + r_2^*)_{m_4-r_2-r_2^*}}{(n_{22} - s_{22} - m_{22} - m_{24} - r - r_1 - r_{23} + r_2 + r_{46} + r_{56})!} \\ &\times \frac{1}{(n_{11} - s_{11} - m_{11} - m_{34} - r_{46} - r_1^* - r_2 - r_{34} + r_1 + r_{23})! s_{11}! s_{13}! s_{22}!} \end{aligned}$$

$$\times \frac{(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!} C_6 F_6.$$

We can now perform the summations over r_{23} , r_1^* , r_2^* , r_{46} , r_1 , r_2 , and finally s_{13} ,⁵ leading to

$$\begin{aligned} G_7 = & \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}} \\ & \times \frac{(-h_3+n_1+n_{11}+n_{13}+n_{22})_{m_4-r-k_{23}}(p_4-n_1)_{m_4+m_{14}-r}}{(m_4-r-k_{23})!r_{34}!(r-r_{56}-r_{34}-k_{12}-k_{33})!r_{56}!k_{12}!k_{23}!k_{33}!} \\ & \times \frac{m_{33}!(-1)^{r-r_{34}-r_{56}+k_{23}+s_{11}+s_{22}}(-r_{56}-k_{12})_{n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33}}}{(n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33})!n_{33}!} \\ & \times \frac{(-h_3+n_1+n_{13}+m_4+m_{11}+m_{22}+m_{24}+m_{34}+s_{11}+s_{22}+r_{34}-r_{56}-k_{23})_{m_{13}+m_{44}-n_{13}+r+r_{56}+k_{23}-r_{34}}}{(n_{22}-s_{22}-m_{22}-m_{24}-r_{34}-k_{12}-k_{33})!} \\ & \times \frac{(-n_{11}-n_{22}+s_{11}+s_{22}+m_{11}+m_{22}+m_{24}+m_{34}+2r_{34}+k_{12}+k_{33})_{r-r_{34}-r_{56}-k_{12}-k_{33}}}{(n_{11}-s_{11}-m_{11}-m_{34}-r_{34})!s_{11}!s_{22}!} \\ & \times \frac{(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!} C_6 F_6. \end{aligned}$$

After defining $s_{22} = s - s_{11}$, we evaluate the sum over s_{11} and s , giving

$$\begin{aligned} G_7 = & \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}} \\ & \times \frac{(-m_{13}-m_{44}+n_{13}+r_{34}-r-r_{56}-k_{23})_{n_{11}+n_{22}-m_{11}-m_{22}-m_{24}-m_{34}-r-r_{34}+r_{56}}(p_4-n_1)_{m_4+m_{14}-r}}{(m_4-r-k_{23})!r_{34}!(r-r_{56}-r_{34}-k_{12}-k_{33})!r_{56}!k_{12}!k_{23}!k_{33}!} \\ & \times \frac{m_{33}!(-1)^{r-r_{34}-r_{56}+k_{23}}(-r_{56}-k_{12})_{n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33}}}{(n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33})!(n_{22}-m_{22}-m_{24}-r_{34}-k_{12}-k_{33})!n_{33}!} \\ & \times \frac{(-n_{11}-n_{22}+m_{11}+m_{22}+m_{24}+m_{34}+2r_{34}+k_{12}+k_{33})_{r-r_{34}-r_{56}-k_{12}-k_{33}}}{(n_{11}-m_{11}-m_{34}-r_{34})!} \\ & \times \frac{(p_2+h_3)_{n_1+n_{23}}(p_3-h_2+h_4)_{n_2+n_{11}}(\bar{p}_3+h_2+h_5)_{n_3+n_{12}}}{(\bar{p}_3+h_2)_{2n_3+n_{12}+n_{13}+n_{23}+n_{33}}(\bar{p}_3+h_2+1-d/2)_{n_3}} \\ & \times \frac{(p_3)_{n_2+n_3-n_1+n_{12}+n_{33}}(p_2+h_2)_{n_1-n_2+n_3+n_{13}+n_{23}}(-h_2)_{n_1+n_2-n_3+n_{11}+n_{22}}}{(p_2)_{2n_1+n_{11}+n_{22}+n_{13}+n_{23}}(p_2+1-d/2)_{n_1}} \\ & \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{n_2+n_{12}+n_{22}+n_{33}}(-h_5)_{n_3+n_{13}+n_{23}+n_{33}}}{(p_3-h_2)_{2n_2+n_{11}+n_{12}+n_{22}+n_{33}}(p_3-h_2+1-d/2)_{n_2}} \\ & \times \frac{(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!} F_6, \end{aligned}$$

where we explicitly expanded C_6 (2.12).

⁵To evaluate the sums over r_2^* , r_1 and r_2 , we first change variables by

$$r_2^* \rightarrow r_2^* + k_{23}, \quad r_1 \rightarrow r_1 + r_{34} + k_{33}, \quad r_2 \rightarrow r_2 + r_{34} + k_{33}.$$

We then define $n_{22} = n - n_{11}$ and change variables by $n_{11} \rightarrow n_{11} + m_{11} + m_{34} + r_{34}$. The sum over n_{11} thus leads to

$$\begin{aligned}
G_7 = & \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}} \\
& \times \frac{(-m_{13}-m_{44}+n_{13}+r_{34}-r-r_{56}-k_{23})_{n-m_{11}-m_{22}-m_{24}-m_{34}-r-r_{34}+r_{56}}(p_4-n_1)_{m_4+m_{14}-r}}{(m_4-r-k_{23})!r_{34}!(r-r_{56}-r_{34}-k_{12}-k_{33})!r_{56}!k_{12}!k_{23}!k_{33}!} \\
& \times \frac{m_{33}!(-1)^{r-r_{34}-r_{56}+k_{23}}(-r_{56}-k_{12})_{n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33}}}{(n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33})!n_{33}!} \\
& \times \frac{(-n+m_{11}+m_{22}+m_{24}+m_{34}+2r_{34}+k_{12}+k_{33})_{r-r_{34}-r_{56}-k_{12}-k_{33}}}{(n-m_{11}-m_{34}-m_{22}-m_{24}-2r_{34}-k_{12}-k_{33})!} \\
& \times \frac{(p_2+h_3)_{n_1+n_{23}}(p_3-h_2+h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{n_3+n_{12}}}{(\bar{p}_3+h_2)_{2n_3+n_{12}+n_{13}+n_{23}+n_{33}}(\bar{p}_3+h_2+1-d/2)_{n_3}} \\
& \times \frac{(p_3)_{n_2+n_3-n_1+n_{12}+n_{33}}(p_2+h_2)_{n_1-n_2+n_3+n_{13}+n_{23}}(-h_2)_{n_1+n_2-n_3+n}}{(p_2)_{2n_1+n+n_{13}+n_{23}}(p_2+1-d/2)_{n_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{n_3+n_{13}+n_{23}+n_{33}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{n_2}} \\
& \times \frac{(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!} F_6.
\end{aligned}$$

After shifting n such that $n \rightarrow n + m_{11} + m_{22} + m_{24} + m_{34} + r + r_{34} - r_{56}$, we can evaluate the sum over n , giving

$$\begin{aligned}
G_7 = & \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}} \\
& \times \frac{(p_4-n_1)_{m_4+m_{14}-r}}{(m_4-r-k_{23})!r_{34}!(r-r_{56}-r_{34}-k_{12}-k_{33})!r_{56}!k_{12}!k_{23}!k_{33}!} \\
& \times \frac{m_{33}!(-1)^{k_{12}+k_{23}+k_{33}}(-r_{56}-k_{12})_{n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33}}}{(n_{13}-m_{13}-m_{44}-r_{56}-k_{23}-k_{33})!n_{33}!} \\
& \times \frac{(p_2+h_3)_{n_1+n_{23}}(p_3-h_2+h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{n_3+n_{12}}}{(\bar{p}_3+h_2)_{2n_3+n_{12}+n_{13}+n_{23}+n_{33}}(\bar{p}_3+h_2+1-d/2)_{n_3}} \\
& \times \frac{(p_3)_{n_2+n_3-n_1+n_{12}+n_{33}}(p_2+h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2+1-d/2)_{n_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{n_3+n_{13}+n_{23}+n_{33}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{n_2}} \\
& \times \frac{(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!} F_6.
\end{aligned}$$

We then change variables again, this time $n_{13} \rightarrow n_{13} + m_{13} + m_{44} + r_{56} + k_{23} + k_{33}$, and sum over n_{13} , which leads to

$$G_7 = \sum \frac{(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}}$$

$$\begin{aligned}
& \times \frac{(p_4 - n_1)_{m_4+m_{14}-r}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(m_4 - r - k_{23})!r_{34}!(r - r_{56} - r_{34} - k_{12} - k_{33})!r_{56}!k_{12}!k_{23}!k_{33}!} \\
& \times \frac{(p_2 + h_3)_{n_1+n_{23}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{n_3}} \\
& \times \frac{(p_3)_{n_2+n_3-n_1+n_{12}+n_{33}}(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{n_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{n_2}} \\
& \times \frac{m_{12}!m_{23}!m_{33}!(-1)^{k_{12}+k_{23}+k_{33}}(u_1^7)^{n_1+r}(u_2^7)^{n_2+r_{34}}(u_3^7)^{n_3+r_{56}}(u_4^7)^{m_4}}{n_{12}!n_{23}!n_{33}!n_1!n_2!n_3!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m_{ab}}}{m_{ab}!} F_6.
\end{aligned}$$

Using the fact that $m_{12} = n_{12} + k_{12}$, $m_{23} = n_{13} + k_{23}$, and $m_{33} = k_{33} + n_{33}$, we evaluate the sums over k_{12} , k_{23} , and k_{33} , to obtain

$$\begin{aligned}
G_7 &= \sum \frac{(p_4 - m_1)_{m_4+m_{14}}(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m_1)_{-m_4}(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1-m_4+m_{23}}(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times \frac{(-m_1)_r(-m_4)_r(-p_2 + d/2 - m_1)_r(p_3 - m_1 + m_2 + m_3)_{r-r_{34}-r_{56}}}{(1 - p_2 - h_3 - m_1)_r(p_4 - m_1)_r(r - r_{56} - r_{34})!} \\
& \times \frac{(-m_2)_{r_{34}}(-m_3)_{r_{56}}(-\bar{p}_3 + h_2 + d/2 - m_2)_{r_{34}}(-\bar{p}_3 - h_2 + d/2 - m_3)_{r_{56}}}{r_{34}!r_{56}!} \\
& \times F_6 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m_{ab}}}{m_{ab}!}.
\end{aligned}$$

Expanding F_6 with the help of the first result in (2.13) leads to

$$\begin{aligned}
G_7 &= \sum \frac{(p_4 - m_1)_{m_4+m_{14}}(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m_1)_{-m_4}(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1-m_4+m_{23}}(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m_{ab}}}{m_{ab}!},
\end{aligned}$$

which results in the appropriate C_7 presented in (2.17) and implies that F_7 is given by

$$\begin{aligned}
F_7 &= \frac{(-p_3 + h_2 + d/2 - m_2)_{m_2}}{(p_3)_{-m_1}(-h_2)_{-m_3}} \sum \frac{(-m_2)_{r_{34}+t_1}(-m_3)_{r_{56}+t_2}(-\bar{p}_3 - h_2 + d/2 - m_3)_{r_{56}}}{(r - r_{56} - r_{34})!r_{34}!r_{56}!} \\
&\times \frac{(-m_1)_{r+t_2}(-m_4)_r(-p_2 + d/2 - m_1)_r(p_3 - m_1 + m_2 + m_3)_{r-r_{56}-r_{34}}}{(1 - p_2 - h_3 - m_1)_r(p_4 - m_1)_r} \\
&\times \frac{(\bar{p}_3 - d/2)_{t_1+t_2}}{(-h_2 - m_3)_{r_{56}+t_1+t_2}(p_3 - m_1)_{r+t_1+t_2}} \frac{(-h_2)_{t_1}(p_3)_{t_1}}{(p_3 - h_2 + 1 - d/2)_{t_1}t_1!t_2!}.
\end{aligned}$$

To reach the solution introduced in (2.18), we first evaluate the sum over r_{34} , leading to

$$\begin{aligned}
F_7 &= \frac{(-p_3 + h_2 + d/2 - m_2)_{m_2}}{(p_3)_{-m_1}(-h_2)_{-m_3}} \sum \frac{(-m_2)_{t_1}(-m_3)_{r_{56}+t_2}(-\bar{p}_3 - h_2 + d/2 - m_3)_{r_{56}}}{(r - r_{56})!r_{56}!} \\
&\times \frac{(-m_1)_{r+t_2}(-m_4)_r(-p_2 + d/2 - m_1)_r(p_3 - m_1 + m_3 + t_1)_{r-r_{56}}}{(1 - p_2 - h_3 - m_1)_r(p_4 - m_1)_r} \\
&\times \frac{(\bar{p}_3 - d/2)_{t_1+t_2}}{(-h_2 - m_3)_{r_{56}+t_1+t_2}(p_3 - m_1)_{r+t_1+t_2}} \frac{(-h_2)_{t_1}(p_3)_{t_1}}{(p_3 - h_2 + 1 - d/2)_{t_1}t_1!t_2!}.
\end{aligned}$$

We then use the first identity in (A.3) and get

$$\begin{aligned}
F_7 &= \frac{(-p_3 + h_2 + d/2 - m_2)_{m_2}}{(p_3)_{-m_1}(-h_2)_{-m_3}} \sum \frac{(-m_2)_{t_1}(-m_3)_{r_{56}+t_2}(-\bar{p}_3 - h_2 + d/2 - m_3)_{m_3}}{(r - r_{56})!r_{56}!} \\
&\times \frac{(-m_1)_r(-m_4)_r(-p_2 + d/2 - m_1)_r(p_3 - m_1 + m_3 + t_1)_{r-r_{56}}}{(1 - p_2 - h_3 - m_1)_r(p_4 - m_1)_r(\bar{p}_3 + h_2 + 1 - d/2)_{t_2}} \\
&\times \frac{(\bar{p}_3 - d/2)_{t_1+t_2}}{(-h_2 - m_3)_{m_3+t_1}(p_3 - m_1)_{r+t_1+t_2}} \frac{(-h_2)_{t_1}(p_3)_{t_1+t_2}}{(p_3 - h_2 + 1 - d/2)_{t_1}t_1!t_2!}.
\end{aligned}$$

We finally sum over r_{56} , leading to (2.18) with the second equality for F_6 in (2.13), which concludes our proof of the scalar seven-point conformal blocks in the extended snowflake channel.

B. Symmetry Properties

This appendix is dedicated to the proofs of the symmetry properties of the scalar seven-point conformal blocks in the extended snowflake channel. The three generators to investigate are the reflections as well as the dendrite permutations of the first and second kinds introduced in (3.1), (3.2), and (3.3), respectively.

B.1. Reflections

We aim to prove (3.1), which corresponds to the invariance of the scalar seven-point conformal blocks under reflections. For simplicity, we denote (3.1) by $G_7 = G_{7R}$. We first rewrite the conformal cross-ratios in (3.1) in terms of the original conformal cross-ratios and then re-sum the extra sums to obtain G_7 back, taking into account that F_7 is invariant under reflections as dictated by (2.18).

We therefore start with

$$\begin{aligned}
G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m_{14}} (-h_6)_{m'_4+m_{14}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + h_6)_{m'_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
& \times \frac{(p_2 + h_3)_{m'_1-m'_4+m_{23}} (p_3)_{-m'_1+m'_2+m'_3+m_{12}+m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3+m_{11}+m_{34}} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m_{13}+m_{23}+m_{44}} (p_2 + h_2)_{m'_1-m'_2+m'_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m'_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(-h_3)_{m'_1+m'_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}} (-h_5)_{m'_3+m_{12}+m_{22}+m_{24}+m_{33}} (-h_4)_{m'_2+m_{13}+m_{23}+m_{33}+m_{44}}}{(\bar{p}_3 + h_2)_{2m'_3+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
& \times \prod_{1 \leq a \leq b \leq 4} \binom{m_{ab}}{k_{ab}} \binom{h_3 - h_6 - m'_1 - k_{11} - k_{13} + k_{14} - k_{22}}{m'_{11}} \binom{h_4 - m'_2 - k_{13} - k_{23} - k_{33} - k_{44}}{m'_{12}} \\
& \times \binom{k_{22}}{m'_{13}} \binom{k_{14}}{m'_{14}} \binom{k_{13}}{m'_{22}} \binom{h_5 - m'_3 - k_{12} - k_{22} - k_{24} - k_{33}}{m'_{23}} \binom{k_{44}}{m'_{24}} \binom{k_{33}}{m'_{33}} \\
& \times \binom{h_6 - m'_4 - k_{14} - k_{24} - k_{34} - k_{44}}{m'_{34}} \binom{k_{24}}{m'_{44}} (-1)^{\sum_{1 \leq a \leq b \leq 4} (k_{ab} + m'_{ab})} \\
& \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)_{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)_{m'_{ab}}}{m_{ab}!},
\end{aligned}$$

and evaluate the sums over k_{11} , k_{12} , k_{23} and k_{34} to get

$$\begin{aligned}
G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m_{14}} (-h_6)_{m'_4+m_{14}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + h_6)_{m'_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
& \times \frac{(p_2 + h_3)_{m'_1-m'_4+m_{23}} (p_3)_{-m'_1+m'_2+m'_3+m_{12}+m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3+m_{11}+m_{34}} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m_{13}+m_{23}+m_{44}} (p_2 + h_2)_{m'_1-m'_2+m'_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m'_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(-h_3)_{m'_1+m'_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}} (-h_5)_{m'_3+m_{12}+m_{22}+m_{24}+m_{33}} (-h_4)_{m'_2+m_{13}+m_{23}+m_{33}+m_{44}}}{(\bar{p}_3 + h_2)_{2m'_3+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
& \times \binom{m_{13}}{k_{13}} \binom{m_{14}}{k_{14}} \binom{m_{22}}{k_{22}} \binom{m_{24}}{k_{24}} \binom{m_{33}}{k_{33}} \binom{m_{44}}{k_{44}} \binom{k_{22}}{m'_{13}} \binom{k_{14}}{m'_{14}} \binom{k_{13}}{m'_{22}} \binom{k_{44}}{m'_{24}} \binom{k_{33}}{m'_{33}} \binom{k_{24}}{m'_{44}} \\
& \times (-1)^{k_{13}+k_{14}+k_{22}+k_{24}+k_{33}+k_{44}+m'_{13}+m'_{14}+m'_{22}+m'_{24}+m'_{33}+m'_{44}} \\
& \times \frac{(-m'_{11})_{m_{11}} (-h_3 + h_6 + m'_1 + m_{11} + k_{13} - k_{14} + k_{22})_{m'_{11}-m_{11}}}{m'_{11}!} \\
& \times \frac{(-m'_{23})_{m_{12}} (-h_5 + m'_3 + m_{12} + k_{22} + k_{24} + k_{33})_{m'_{23}-m_{12}}}{m'_{23}!} \\
& \times \frac{(-m'_{12})_{m_{23}} (-h_4 + m'_2 + m_{23} + k_{13} + k_{33} + k_{44})_{m'_{12}-m_{23}}}{m'_{12}!}
\end{aligned}$$

$$\begin{aligned} & \times \frac{(-m'_{34})_{m_{34}}(-h_6 + m'_4 + m_{34} + k_{14} + k_{24} + k_{44})_{m'_{34} - m_{34}}}{m'_{34}!} \\ & \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)_{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)_{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

We then change variables by

$$\begin{aligned} k_{13} &\rightarrow k_{13} + m'_{22}, & k_{14} &\rightarrow k_{14} + m'_{14}, & k_{22} &\rightarrow k_{22} + m'_{13}, \\ k_{24} &\rightarrow k_{24} + m'_{44}, & k_{33} &\rightarrow k_{33} + m'_{33}, & k_{44} &\rightarrow k_{44} + m'_{24}, \end{aligned}$$

and use the following identity

$$\begin{aligned} & (-h_3 + h_6 + m'_1 + m'_{13} - m'_{14} + m'_{22} + m_{11} + k_{13} - k_{14} + k_{22})_{m'_{11} - m_{11}} \\ & = \sum_{j_1 \geq 0} \binom{m'_{11} - m_{11}}{j_1} (-m_{13} + m'_{22} + k_{13})_{j_1} \\ & \quad \times (-h_3 + h_6 + m'_1 + m'_{13} - m'_{14} + m_{11} + m_{13} - k_{14} + k_{22})_{m'_{11} - m_{11} - j_1}, \end{aligned}$$

to compute the sum over k_{13} . With the help of a similar trick, we evaluate the sums over all other k_{ab} 's except k_{14} , to get

$$\begin{aligned} G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m_{14}} (-h_6)_{m'_4 + m_{14} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m'_4 + m_{11} + m_{13} + m_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\ & \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3 + m_{11} + m_{34}} (p_3 - h_2 + h_4)_{m'_2 + m_{12}}}{(p_3 - h_2)_{2m'_2 + m_{12} + m_{13} + m_{23} + m_{33} + m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\ & \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m_{13} + m_{23} + m_{44}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m'_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\ & \times \frac{(-h_3)_{m'_1 + m'_4 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m_{33} - j_4}}{(\bar{p}_3 + h_2)_{2m'_3 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\ & \times \frac{(-m'_{11})_{m_{11} + j_1 + j_2} (-h_3 + h_6 + m'_1 - m'_{14} + m_{11} + m_{13} + m_{22} - k_{14})_{m'_{11} - m_{11} - j_1 - j_2}}{(m_{22} - m'_{13} - j_2)! (m_{13} - m'_{22} - j_1)!} \\ & \times \frac{(-m'_{23})_{m_{12} + m_{22} + m_{24} + m_{33} - m'_{13} - m'_{33} - m'_{44} - j_2 - j_3 - j_4}}{(-h_5 + m'_3 + m_{12} + m_{22} + m_{24} + m_{33})_{-j_2 - j_3 - j_4} (m_{24} - m'_{44} - j_3)! (m_{44} - m'_{24} - j_5)!} \\ & \times \frac{(-m'_{34})_{m_{34} + j_3 + j_5} (-m'_{12})_{m_{13} + m_{23} + m_{44} - m'_{22} - m'_{24} - j_1 + j_4 - j_5}}{(-h_4 + m'_2 + m_{13} + m_{23} + m_{33} + m_{44})_{-j_1 - j_5} j_1! j_2! j_3! j_4! j_5! (m_{33} - m'_{33} - j_4)!} \\ & \times (-1)^{k_{14}} \frac{(-h_6 + m'_4 + m'_{14} + m_{24} + m_{34} + m_{44} + k_{14})_{m'_{34} - m_{34} - j_3 - j_5}}{(m_{14} - m'_{14} - k_{14})! k_{14}! m_{11}! m_{12}! m_{23}! m_{34}!} \\ & \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)_{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)_{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

We now use the identity

$$(-h_3 + h_6 + m'_1 - m'_{14} + m_{11} + m_{13} + m_{22} - k_{14})_{m'_{11} - m_{11} - j_1 - j_2}$$

$$\begin{aligned}
&= \sum_{j_6 \geq 0} \binom{m'_{11} - m_{11} - j_1 - j_2}{j_6} (-k_{14})_{j_6} \\
&\quad \times (-h_3 + h_6 + m'_1 - m'_{14} + m_{11} + m_{13} + m_{22})_{m'_{11} - m_{11} - j_1 - j_2 - j_6},
\end{aligned}$$

and then change k_{14} by $k_{14} \rightarrow k_{14} + j_6$ to re-sum over k_{14} . This procedure leads to

$$\begin{aligned}
G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m_{14}} (-h_6)_{m'_4 + m'_{14} + m_{24} + m'_{34} + m_{44} - j_3 - j_5 + j_6} (p_4 - h_3 + h_6)_{m'_4 + m_{11} + m_{13} + m_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
&\quad \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3 + m_{11} + m_{34}} (p_3 - h_2 + h_4)_{m'_2 + m_{12}}}{(p_3 - h_2)_{2m'_2 + m_{12} + m_{13} + m_{23} + m_{33} + m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
&\quad \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m_{13} + m_{23} + m_{44}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m'_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\
&\quad \times \frac{(-h_3)_{m'_1 + m'_4 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m_{33} - j_4}}{(\bar{p}_3 + h_2)_{2m'_3 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
&\quad \times \frac{(-m'_{11})_{m_{11} + j_1 + j_2 + j_6} (-h_3 + h_6 + m'_1 - m'_{14} + m_{11} + m_{13} + m_{22})_{m'_{11} - m_{11} - j_1 - j_2 - j_6}}{(m_{22} - m'_{13} - j_2)! (m_{13} - m'_{22} - j_1)!} \\
&\quad \times \frac{(-m'_{23})_{m_{12} + m_{22} + m_{24} + m_{33} - m'_{13} - m'_{33} - m'_{44} - j_2 - j_3 - j_4}}{(-h_5 + m'_3 + m_{12} + m_{22} + m_{24} + m_{33})_{-j_2 - j_3 - j_4} (m_{24} - m'_{44} - j_3)! (m_{44} - m'_{24} - j_5)!} \\
&\quad \times \frac{(-m'_{34})_{m_{14} + m_{34} - m'_{14} + j_3 + j_5 - j_6} (-m'_{12})_{m_{13} + m_{23} + m_{44} - m'_{22} - m'_{24} - j_1 + j_4 - j_5}}{(-h_4 + m'_2 + m_{13} + m_{23} + m_{33} + m_{44})_{-j_1 - j_5} j_1! j_2! j_3! j_4! j_5! j_6! (m_{33} - m'_{33} - j_4)!} \\
&\quad \times \frac{(-1)^{j_6}}{(m_{14} - m'_{14} - j_6)! m_{11}! m_{12}! m_{23}! m_{34}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

It is thus possible to evaluate the sum over m_{14} (after shifting m_{14} by $m_{14} + m'_{14} + j_6$) such that

$$\begin{aligned}
G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m'_{14} + j_6} (-h_6)_{m'_4 + m'_{14} + m_{24} + m'_{34} + m_{44} - j_3 - j_5 + j_6} (p_4 - h_3 + h_6)_{m'_4 + m_{11} + m_{13} + m_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m_{11} + m_{13} + m'_{14} + m_{22} + m_{24} + m'_{34} + m_{44} - j_3 - j_5 + j_6} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
&\quad \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3 + m_{11} + m_{34}} (p_3 - h_2 + h_4)_{m'_2 + m_{12}}}{(p_3 - h_2)_{2m'_2 + m_{12} + m_{13} + m_{23} + m_{33} + m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
&\quad \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m_{13} + m_{23} + m_{44}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m'_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\
&\quad \times \frac{(-h_3)_{m'_1 + m'_4 + m_{11} + m_{13} + m_{22} + m_{24} + m'_{34} + m_{44} - j_3 - j_5} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m_{33} - j_4}}{(\bar{p}_3 + h_2)_{2m'_3 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
&\quad \times \frac{(-m'_{11})_{m_{11} + j_1 + j_2 + j_6} (-h_3 + h_6 + m'_1 - m'_{14} + m_{11} + m_{13} + m_{22})_{m'_{11} - m_{11} - j_1 - j_2 - j_6}}{(m_{22} - m'_{13} - j_2)! (m_{13} - m'_{22} - j_1)!} \\
&\quad \times \frac{(-m'_{23})_{m_{12} + m_{22} + m_{24} + m_{33} - m'_{13} - m'_{33} - m'_{44} - j_2 - j_3 - j_4}}{(-h_5 + m'_3 + m_{12} + m_{22} + m_{24} + m_{33})_{-j_2 - j_3 - j_4} (m_{24} - m'_{44} - j_3)! (m_{44} - m'_{24} - j_5)!} \\
&\quad \times \frac{(-m'_{34})_{m_{34} + j_3 + j_5} (-m'_{12})_{m_{13} + m_{23} + m_{44} - m'_{22} - m'_{24} - j_1 + j_4 - j_5}}{(-h_4 + m'_2 + m_{13} + m_{23} + m_{33} + m_{44})_{-j_1 - j_5} j_1! j_2! j_3! j_4! j_5! j_6! (m_{33} - m'_{33} - j_4)!}
\end{aligned}$$

$$\times \frac{(-1)^{j_6}}{m_{11}!m_{12}!m_{23}!m_{34}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.$$

With the help of the second identity in (A.3), we now eliminate the sum over j_6 and get

$$\begin{aligned} G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m'_1} (-h_6)_{m'_4+m'_1} (p_4 - h_3 + h_6)_{m'_4+m'_1} (p_4 - h_3 + 1 - d/2)_{m'_4}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'_1} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\ & \times \frac{(p_2 + h_3)_{m'_1-m'_4} (p_3)_{-m'_1+m'_2+m'_3} (\bar{p}_3 + h_2 + h_5)_{m'_3+m_{11}+m_{34}} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\ & \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m_{13}+m_{23}+m_{44}} (p_2 + h_2)_{m'_1-m'_2+m'_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m'_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}} (p_2 + 1 - d/2)_{m'_1}} \\ & \times \frac{(-h_3)_{m'_1+m'_4+m'_{11}+m_{13}+m_{22}+m_{24}+m'_{34}+m_{44}-j_1-j_2-j_3-j_5} (-h_5)_{m'_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\ & \times \frac{(-h_4)_{m'_2+m'_{12}+m'_{22}+m'_{24}+m_{33}-j_4} (-m'_{23})_{m_{12}+m_{22}+m_{24}+m_{33}-m'_{13}-m'_{33}-m'_{44}-j_2-j_3-j_4}}{(-h_5 + m'_3 + m_{12} + m_{22} + m_{24} + m_{33})_{-j_2-j_3-j_4} (m_{24} - m'_{44} - j_3)! (m_{44} - m'_{24} - j_5)!} \\ & \times \frac{(-m'_{34})_{m_{34}+j_3+j_5} (-m'_{12})_{m_{13}+m_{23}+m_{44}-m'_{22}-m'_{24}-j_1+j_4-j_5}}{(-h_4 + m'_2 + m_{13} + m_{23} + m_{33} + m_{44})_{-j_1-j_5} j_1! j_2! j_3! j_4! j_5! (m_{22} - m'_{13} - j_2)! (m_{33} - m'_{33} - j_4)!} \\ & \times \frac{(-m'_{11})_{m_{11}+j_1+j_2}}{(m_{13} - m'_{22} - j_1)! m_{11}! m_{12}! m_{23}! m_{34}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

At this point, we introduce $m_{44} = m - m_{13}$ and then change m_{13} by $m_{13} \rightarrow m_{13} + m'_{22} + j_1$.

This allows us to sum over m_{13} ,

$$\begin{aligned} G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m'_1} (-h_6)_{m'_4+m'_1} (p_4 - h_3 + h_6)_{m'_4+m'_1} (p_4 - h_3 + 1 - d/2)_{m'_4}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'_1} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\ & \times \frac{(p_2 + h_3)_{m'_1-m'_4} (p_3)_{-m'_1+m'_2+m'_3} (\bar{p}_3 + h_2 + h_5)_{m'_3+m_{11}+m_{34}} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m+m_{23}+m_{33}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\ & \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m+m_{23}} (p_2 + h_2)_{m'_1-m'_2+m'_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m'_1+m_{11}+m+m_{22}+m_{23}+m_{24}+m_{34}} (p_2 + 1 - d/2)_{m'_1}} \\ & \times \frac{(-h_3)_{m'_1+m'_4+m'_{11}+m+m_{22}+m_{24}+m'_{34}-j_1-j_2-j_3-j_5} (-h_5)_{m'_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\ & \times \frac{(-h_4)_{m'_2+m'_{12}+m'_{22}+m'_{24}+m_{33}-j_4} (-m'_{23})_{m_{12}+m_{22}+m_{24}+m_{33}-m'_{13}-m'_{33}-m'_{44}-j_2-j_3-j_4}}{(-h_5 + m'_3 + m_{12} + m_{22} + m_{24} + m_{33})_{-j_2-j_3-j_4} (m_{24} - m'_{44} - j_3)! (m - m'_{22} - m'_{24} - j_1 - j_5)!} \\ & \times \frac{(-m'_{34})_{m_{34}+j_3+j_5} (-m'_{12})_{m+m_{23}-m'_{22}-m'_{24}-j_1+j_4-j_5}}{(-h_4 + m'_2 + m + m_{23} + m_{33})_{-j_1-j_5} j_1! j_2! j_3! j_4! j_5! (m_{22} - m'_{13} - j_2)! (m_{33} - m'_{33} - j_4)!} \\ & \times \frac{(-m'_{11})_{m_{11}+j_1+j_2}}{m_{11}! m_{12}! m_{23}! m_{34}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

In the same spirit, we define $m_{24} = n - m_{22}$ and change m_{22} by $m_{22} \rightarrow m_{22} + m'_{13} + j_2$ to

evaluate the sum over m_{22} , which leads to

$$\begin{aligned}
G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m'_{14}} (-h_6)_{m'_4+m'_{14}+m'_{24}+m'_{34}+m'_{44}} (p_4 - h_3 + h_6)_{m'_4+m'_{11}+m'_{13}+m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'_{11}+m'_{13}+m'_{14}+m'_{22}+m'_{24}+m'_{34}+m'_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
& \times \frac{(p_2 + h_3)_{m'_1-m'_4+m_{23}} (p_3)_{-m'_1+m'_2+m'_3+m_{12}+m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3+m_{11}+m_{34}} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m+m_{23}+m_{33}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m+m_{23}} (p_2 + h_2)_{m'_1-m'_2+m'_3+m_{11}+n+m_{34}}}{(p_2)_{2m'_1+m_{11}+m+n+m_{23}+m_{34}}} (p_2 + 1 - d/2)_{m'_1} \\
& \times \frac{(-h_3)_{m'_1+m'_4+m'_{11}+m+n+m'_{34}-j_1-j_2-j_3-j_5} (-h_5)_{m'_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}} (-h_4)_{m'_2+m'_{12}+m'_{22}+m'_{24}+m_{33}-j_4}}{(\bar{p}_3 + h_2)_{2m'_3+m_{11}+m_{12}+n+m_{33}+m_{34}}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3} \\
& \times \frac{(-m'_{23})_{m_{12}+n+m_{33}-m'_{13}-m'_{33}-m'_{44}-j_2-j_3-j_4}}{(-h_5 + m'_3 + m_{12} + n + m_{33})_{-j_2-j_3-j_4} (n - m'_{13} - m'_{44} - j_2 - j_3)! (m - m'_{22} - m'_{24} - j_1 - j_5)!} \\
& \times \frac{(-m'_{34})_{m_{34}+j_3+j_5} (-m'_{12})_{m+m_{23}-m'_{22}-m'_{24}-j_1+j_4-j_5}}{(-h_4 + m'_2 + m + m_{23} + m_{33})_{-j_1-j_5} j_1! j_2! j_3! j_4! j_5! (m_{33} - m'_{33} - j_4)!} \\
& \times \frac{(-m'_{11})_{m_{11}+j_1+j_2}}{m_{11}! m_{12}! m_{23}! m_{34}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now redefine $m_{34} = k_1 - m_{11}$, $j_5 = k_2 - j_1$, $j_3 = k_3 - j_2$, and compute the sums over j_1 , m_{11} , and j_2 , to obtain

$$\begin{aligned}
G_{7R} = & \sum \frac{(p_4 - m'_1)_{m'_4+m'_{14}} (-h_6)_{m'_4+m'_{14}+m'_{24}+m'_{34}+m'_{44}} (p_4 - h_3 + h_6)_{m'_4+m'_{11}+m'_{13}+m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'_{11}+m'_{13}+m'_{14}+m'_{22}+m'_{24}+m'_{34}+m'_{44}}} (p_4 - h_3 + 1 - d/2)_{m'_4} \\
& \times \frac{(p_2 + h_3)_{m'_1-m'_4+m_{23}} (p_3)_{-m'_1+m'_2+m'_3+m_{12}+m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3+k_1} (p_3 - h_2 + h_4)_{m'_2+m_{12}}}{(p_3 - h_2)_{2m'_2+m_{12}+m+m_{23}+m_{33}}} (p_3 - h_2 + 1 - d/2)_{m'_2} \\
& \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m+m_{23}} (p_2 + h_2)_{m'_1-m'_2+m'_3+k_1+n}}{(p_2)_{2m'_1+k_1+m+n+m_{23}}} (p_2 + 1 - d/2)_{m'_1} \\
& \times \frac{(-h_3)_{m'_1+m'_4+m'_{11}+m+n+m'_{34}-k_2-k_3} (-h_5)_{m'_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}} (-h_4)_{m'_2+m'_{12}+m'_{22}+m'_{24}+m_{33}-j_4}}{(\bar{p}_3 + h_2)_{2m'_3+k_1+m_{12}+n+m_{33}}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3} \\
& \times \frac{(-m'_{23})_{m_{12}+n+m_{33}-m'_{13}-m'_{33}-m'_{44}-k_3-j_4}}{(-h_5 + m'_3 + m_{12} + n + m_{33})_{-k_3-j_4} (n - m'_{13} - m'_{44} - k_3)! (m - m'_{22} - m'_{24} - k_2)!} \\
& \times \frac{(-m'_{12})_{m+m_{23}-m'_{22}-m'_{24}-k_2+j_4}}{(-h_4 + m'_2 + m + m_{23} + m_{33})_{-k_2} k_1! k_2! k_3! j_4! (m_{33} - m'_{33} - j_4)!} \\
& \times \frac{(-m'_{11} - m'_{34})_{k_1+k_2+k_3}}{m_{12}! m_{23}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We then shift m by $m \rightarrow m + m'_{22} + m'_{24} + k_2$ and use the first identity in (A.3) to rewrite the sum over k_2 , leading to

$$G_{7R} = \sum \frac{(p_4 - m'_1)_{m'_4+m'_{14}} (-h_6)_{m'_4+m'_{14}+m'_{24}+m'_{34}+m'_{44}} (p_4 - h_3 + h_6)_{m'_4+m'_{11}+m'_{13}+m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'_{11}+m'_{13}+m'_{14}+m'_{22}+m'_{24}+m'_{34}+m'_{44}}} (p_4 - h_3 + 1 - d/2)_{m'_4}$$

$$\begin{aligned}
& \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3 + k_1} (p_3 - h_2 + h_4)_{m'_2 + m_{12} + k_2}}{(p_3 - h_2)_{2m'_2 + m_{12} + m'_{22} + m'_{24} + m + m_{23} + m_{33} + k_2} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m'_{22} + m'_{24} + m + m_{23} + k_2} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m'_{11} + m'_{34} - k_2 - k_3 + n}}{(p_2)_{2m'_1 + m'_{11} + m'_{22} + m'_{24} + m'_{34} + m + n + m_{23} - k_3} (p_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(-h_3)_{m'_1 + m'_4 + m'_{11} + m'_{22} + m'_{24} + m + n + m'_{34} - k_3} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3 + k_1 + m_{12} + n + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
& \times \frac{(-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m_{33} - j_4} (-m'_{23})_{m_{12} + n + m_{33} - m'_{13} - m'_{33} - m'_{44} - k_3 - j_4}}{(-h_5 + m'_3 + m_{12} + n + m_{33})_{-k_3 - j_4} (n - m'_{13} - m'_{44} - k_3)! m!} \\
& \times \frac{(-1)^{k_2} (-m'_{12})_{m + m_{23} + j_4}}{k_1! k_2! k_3! j_4! (m_{33} - m'_{33} - j_4)!} \frac{(-m'_{11} - m'_{34})_{k_1 + k_2 + k_3}}{m_{12}! m_{23}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

With these modifications, we can complete the sum over k_1 and obtain

$$\begin{aligned}
G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m'_{14}} (-h_6)_{m'_4 + m'_{14} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + h_6)_{m'_4 + m'_{11} + m'_{13} + m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m'_{11} + m'_{13} + m'_{14} + m'_{22} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
& \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3} (p_3 - h_2 + h_4)_{m'_2 + m_{12} + k_2}}{(p_3 - h_2)_{2m'_2 + m_{12} + m'_{22} + m'_{24} + m + m_{23} + m_{33} + k_2} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m'_{22} + m'_{24} + m + m_{23} + k_2} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m'_{11} + m'_{34} - k_2 - k_3 + n}}{(p_2)_{2m'_1 + m'_{11} + m'_{22} + m'_{24} + m'_{34} + m + n + m_{23} - k_3} (p_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(-h_3)_{m'_1 + m'_4 + m'_{11} + m'_{22} + m'_{24} + m + n + m'_{34} - k_3} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m_{33} - j_4}}{(\bar{p}_3 + h_2)_{2m'_3 + m'_{11} + m'_{34} + m_{12} + n + m_{33} - k_2 - k_3} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
& \times \frac{(-h_5 + m'_3 + m_{12} + n + m_{33})_{m'_{11} + m'_{34} - k_2 - k_3} (-m'_{23})_{m_{12} + n + m_{33} - m'_{13} - m'_{33} - m'_{44} - k_3 - j_4}}{(-h_5 + m'_3 + m_{12} + n + m_{33})_{-k_3 - j_4} (n - m'_{13} - m'_{44} - k_3)! m!} \\
& \times \frac{(-1)^{k_2} (-m'_{12})_{m + m_{23} + j_4}}{k_2! k_3! j_4! (m_{33} - m'_{33} - j_4)!} \frac{(-m'_{11} - m'_{34})_{k_2 + k_3}}{m_{12}! m_{23}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

It is then possible to shift n by $n \rightarrow n + m'_{13} + m'_{44} + k_3$ and evaluate the sum over k_3 ,⁶ leading to

$$\begin{aligned}
G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m'_{14}} (-h_6)_{m'_4 + m'_{14} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + h_6)_{m'_4 + m'_{11} + m'_{13} + m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m'_{11} + m'_{13} + m'_{14} + m'_{22} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\
& \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m_{12} + m_{33}} (\bar{p}_3 + h_2 + h_5)_{m'_3} (p_3 - h_2 + h_4)_{m'_2 + m'_{11} + m'_{34} + m_{12}}}{(p_3 - h_2)_{2m'_2 + m_{12} + m'_{11} + m'_{22} + m'_{24} + m'_{34} + m + m_{23} + m_{33}} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m'_{11} + m'_{22} + m'_{24} + m'_{34} + m + m_{23}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m'_{13} + m'_{44} + n}}{(p_2)_{2m'_1 + m'_{11} + m'_{13} + m'_{22} + m'_{24} + m'_{34} + m'_{44} + m + n + m_{23}} (p_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(-h_3)_{m'_1 + m'_4 + m'_{11} + m'_{13} + m'_{22} + m'_{24} + m'_{44} + m + n + m'_{34}} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3 + m'_{13} + m'_{44} + m_{12} + n + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}}
\end{aligned}$$

⁶Using the binomial identity as in (A.1) with $v = 1$, the sum over k_3 forces $k_2 = m'_{11} + m'_{34}$.

$$\begin{aligned} & \times \frac{(-h_4)m'_2+m'_12+m'_22+m'_24+m'33-j_4(-m'23)_{m12+n+m33-m'33-j_4}}{(-h_5+m'_3+m'13+m'44+m12+n+m33)_{-j_4}n!m!} \\ & \times \frac{(-m'12)_{m+m23+j_4}}{j_4!(m33-m'33-j_4)!m12!m23!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

As before, we shift m_{33} by $m_{33} \rightarrow m_{33} + m'_{33} + j_4$ and rewrite the sum over m_{23} with the help of the first identity in (A.3). This allows us to express G_{7R} as

$$\begin{aligned} G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4+m'14} (-h_6)_{m'_4+m'14+m'24+m'34+m'44} (p_4 - h_3 + h_6)_{m'_4+m'11+m'13+m'22}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'11+m'13+m'14+m'22+m'24+m'34+m'44} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\ & \times \frac{(p_2 + h_3)_{m'_1-m'_4} (p_3)_{-m'_1+m'_2+m'_3+m'12+m12+m'33+m33-m23-m(\bar{p}3+h2+h5)_{m'_3}}}{(p_3 - h_2)_{2m'_2+m12+m'11+m'12+m'22+m'24+m'33+m'34+m33} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\ & \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m'11+m'22+m'24+m'34+m+m23} (p_2 + h_2)_{m'_1-m'_2+m'_3+m'13+m'44+n}}{(p_2)_{2m'_1+m'11+m'13+m'22+m'24+m'34+m'44+m+n+m23} (p_2 + 1 - d/2)_{m'_1}} \\ & \times \frac{(-h_3)_{m'_1+m'_4+m'11+m'13+m'22+m'24+m'44+m'34+m23+m+n} (-h_5)_{m'_3+m'13+m'23+m'33+m'44}}{(\bar{p}3 + h_2)_{2m'_3+m'13+m'33+m'44+m12+n+m33+j_4} (\bar{p}3 + h_2 + 1 - d/2)_{m'_3}} \\ & \times \frac{(p_3 - h_2 + h_4)_{m'_2+m'11+m'34+m12} (-h_4)_{m'_2+m'12+m'22+m'24+m'33+m33} (-m'23)_{m12+n+m33}}{n!m!} \\ & \times \frac{(-1)^{m23} (-h_5 + m'_3 + m'13 + m'33 + m'44 + m12 + n + m33)_{j_4} (-m'12)_{m+m23+j_4}}{j_4!m33!m12!m23!} \\ & \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

At this point, it is trivial to see that the sum over j_4 gives

$$\begin{aligned} G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4+m'14} (-h_6)_{m'_4+m'14+m'24+m'34+m'44} (p_4 - h_3 + h_6)_{m'_4+m'11+m'13+m'22}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'11+m'13+m'14+m'22+m'24+m'34+m'44} (p_4 - h_3 + 1 - d/2)_{m'_4}} \\ & \times \frac{(p_2 + h_3)_{m'_1-m'_4} (p_3)_{-m'_1+m'_2+m'_3+m'12+m12+m'33+m33-m23-m(\bar{p}3+h2+h5)_{m'_3+m'12-m-m23}}}{(p_3 - h_2)_{2m'_2+m12+m'11+m'12+m'22+m'24+m'33+m'34+m33} (p_3 - h_2 + 1 - d/2)_{m'_2}} \\ & \times \frac{(-h_2)_{m'_1+m'_2-m'_3+m'11+m'22+m'24+m'34+m+m23} (p_2 + h_2)_{m'_1-m'_2+m'_3+m'13+m'44+n}}{(p_2)_{2m'_1+m'11+m'13+m'22+m'24+m'34+m'44+m+n+m23} (p_2 + 1 - d/2)_{m'_1}} \\ & \times \frac{(-h_3)_{m'_1+m'_4+m'11+m'13+m'22+m'24+m'44+m'34+m23+m+n} (-h_5)_{m'_3+m'13+m'23+m'33+m'44}}{(\bar{p}3 + h_2)_{2m'_3+m'12+m'13+m'33+m'44+m12+n+m33-m23-m(\bar{p}3+h2+1-d/2)_{m'_3}} \\ & \times \frac{(p_3 - h_2 + h_4)_{m'_2+m'11+m'34+m12} (-h_4)_{m'_2+m'12+m'22+m'24+m'33+m33} (-m'23)_{m12+n+m33}}{n!m!} \\ & \times \frac{(-1)^{m23} (-m'12)_{m+m23}}{m33!m12!m23!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}. \end{aligned}$$

After redefining $m_{33} = k - m_{12}$, we can compute the sums over m_{12} , k , and n , leading to

$$G_{7R} = \sum \frac{(p_4 - m'_1)_{m'_4+m'14} (-h_6)_{m'_4+m'14+m'24+m'34+m'44} (p_4 - h_3 + h_6)_{m'_4+m'11+m'13+m'22}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4+m'11+m'13+m'14+m'22+m'24+m'34+m'44} (p_4 - h_3 + 1 - d/2)_{m'_4}}$$

$$\begin{aligned}
& \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m'_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m'_{12} + m'_{33} - m_{23} - m} (\bar{p}_3 + h_2 + h_5)_{m'_3 + m'_{12} - m - m_{23}}}{(p_3 - h_2)_{2m'_2 + m'_{11} + m'_{12} + m'_{22} + m'_{24} + m'_{33} + m'_{34}} (p_3 - h_2 + 1 - d/2)_{m'_2} \\
& \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m'_{11} + m'_{22} + m'_{24} + m'_{34} + m + m_{23}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m'_{13} + m'_{23} + m'_{44}}}{(p_2)_{2m'_1 + m'_{11} + m'_{13} + m'_{22} + m'_{23} + m'_{24} + m'_{34} + m'_{44} + m + m_{23}} (p_2 + 1 - d/2)_{m'_1} \\
& \times \frac{(-h_3)_{m'_1 + m'_4 + m'_{11} + m'_{13} + m'_{22} + m'_{24} + m'_{44} + m'_{34} + m_{23} + m} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3 + m'_{12} + m'_{13} + m'_{23} + m'_{33} + m'_{44} - m_{23} - m}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3} \\
& \times \frac{(p_3 - h_2 + h_4)_{m'_2 + m'_{11} + m'_{34}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m'_{33}}}{m!} \\
& \times \frac{(-1)^{m_{23}} (-m'_{12})_{m + m_{23}}}{m_{23}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)_{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)_{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Finally, we introduce $m_{23} = r - m$ and evaluate the sums over m and r ,⁷ such that

$$\begin{aligned}
G_{7R} &= \sum \frac{(p_4 - m'_1)_{m'_4 + m'_{14}} (-h_6)_{m'_4 + m'_{14} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + h_6)_{m'_4 + m'_{11} + m'_{13} + m'_{22}}}{(p_2 + h_3 + m'_1)_{-m'_4} (p_4 - h_3)_{2m'_4 + m'_{11} + m'_{13} + m'_{14} + m'_{22} + m'_{24} + m'_{34} + m'_{44}} (p_4 - h_3 + 1 - d/2)_{m'_4} \\
& \times \frac{(p_2 + h_3)_{m'_1 - m'_4 + m'_{23}} (p_3)_{-m'_1 + m'_2 + m'_3 + m'_{12} + m'_{33}} (p_3 - h_2 + h_4)_{m'_2 + m'_{11} + m'_{34}} (\bar{p}_3 + h_2 + h_5)_{m'_3 + m'_{12}}}{(p_3 - h_2)_{2m'_2 + m'_{11} + m'_{12} + m'_{22} + m'_{24} + m'_{33} + m'_{34}} (p_3 - h_2 + 1 - d/2)_{m'_2} \\
& \times \frac{(-h_2)_{m'_1 + m'_2 - m'_3 + m'_{11} + m'_{22} + m'_{24} + m'_{34}} (p_2 + h_2)_{m'_1 - m'_2 + m'_3 + m'_{13} + m'_{23} + m'_{44}}}{(p_2)_{2m'_1 + m'_{11} + m'_{13} + m'_{22} + m'_{23} + m'_{24} + m'_{34} + m'_{44}} (p_2 + 1 - d/2)_{m'_1} \\
& \times \frac{(-h_3)_{m'_1 + m'_4 + m'_{11} + m'_{13} + m'_{22} + m'_{24} + m'_{34} + m'_{44}} (\bar{p}_3 + h_2)_{2m'_3 + m'_{12} + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (-h_4)_{m'_2 + m'_{12} + m'_{22} + m'_{24} + m'_{33}} (-h_5)_{m'_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44}}}{(\bar{p}_3 + h_2)_{2m'_3 + m'_{12} + m'_{13} + m'_{23} + m'_{33} + m'_{44}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_3}} \\
& \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)_{m'_a}}{m'_a!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)_{m'_{ab}}}{m'_{ab}!} \\
& = G_7.
\end{aligned}$$

This ends our proof of the invariance of the scalar seven-point correlation functions under reflections (3.1).

B.2. Dendrite Permutations of the First Kind

Invariance under dendrite permutations of the first kind implies the identity (3.2), which we rewrite as $G_7 = G_{7P_1}$ to simplify the notation. Once again, F_7 is invariant under the generator, as can be seen from (2.18). Expressing G_{7P_1} in terms of the original conformal cross-ratios and expanding, we find

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4 - m_1)_{m_4 + m_{14}} (-h_6)_{m_4 + m_{14} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m_4 + m_{11} + m_{13} + m_{22}}}{(p_4 - h_3)_{2m_4 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1 - m_4 + m_{23}} (-h_4)_{m_2 + m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3 + m_{13} + m_{44} + m_{12} + m_{23} + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}}
\end{aligned}$$

⁷Again, the binomial identity (A.1) with $v = 1$ implies that the sum over m forces $r = 0$.

$$\begin{aligned}
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_2+h_2)_{m_1-m_2+m_3+m_{13}+m_{44}+m_{23}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(p_3-h_2+h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \binom{m_{12}}{k_{12}} \binom{m_{13}}{k_{13}} \binom{m_{23}}{k_{23}} \binom{m_{33}}{k_{33}} \binom{m_{44}}{k_{44}} \binom{-p_3-h_5+m_1-m_2-k_{12}+k_{13}+k_{23}+k_{44}}{m'_{12}} \\
& \times \binom{k_{13}}{m'_{13}} \binom{k_{23}}{m'_{23}} \binom{h_5-m_3-k_{13}-k_{23}-k_{33}-k_{44}}{m'_{33}} \binom{k_{44}}{m'_{44}} \\
& \times (-1)^{k_{12}+k_{13}+k_{23}+k_{33}+k_{44}+m'_{12}+m'_{13}+m'_{23}+m'_{33}+m'_{44}} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!},
\end{aligned}$$

and we must now re-sum the extra sums to rewrite the result as G_7 .⁸

We first evaluate the sums over k_{12} and k_{33} , leading to

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4-m_1)_{m_4+m_{14}}(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}} \\
& \times \frac{(p_2+h_3)_{m_1-m_4+m_{23}}(-h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(p_2+h_3+m_1)_{-m_4}(\bar{p}_3+h_2)_{2m_3+m_{13}+m_{44}+m_{12}+m_{23}+m_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_2+h_2)_{m_1-m_2+m_3+m_{13}+m_{44}+m_{23}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(p_3-h_2+h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \binom{m_{13}}{k_{13}} \binom{m_{23}}{k_{23}} \binom{m_{44}}{k_{44}} \frac{(-m'_{12})_{m_{12}}(p_3+h_5-m_1+m_2+m_{12}-k_{13}-k_{23}-k_{44})_{m'_{12}-m_{12}}}{m'_{12}!} \\
& \times \binom{k_{13}}{m'_{13}} \binom{k_{23}}{m'_{23}} \binom{k_{44}}{m'_{44}} \frac{(-m'_{33})_{m_{33}}(-h_5+m_3+m_{33}+k_{13}+k_{23}+k_{44})_{m'_{33}-m_{33}}}{m'_{33}!} \\
& \times (-1)^{k_{13}+k_{23}+k_{44}+m'_{13}+m'_{23}+m'_{44}} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m_{ab}}}{m_{ab}!}.
\end{aligned}$$

Using the following identity

$$\begin{aligned}
& (p_3+h_5-m_1+m_2+m_{12}-k_{13}-k_{23}-k_{44})_{m'_{12}-m_{12}} \\
&= \sum_{j_1 \geq 0} \binom{m'_{12}-m_{12}}{j_1} (p_3+h_5-m_1+m_2+m_{12}-m'_{13}-k_{23}-k_{44})_{m'_{12}-m_{12}-j_1} (-k_{13}+m'_{13})_{j_1},
\end{aligned}$$

and then changing the variable such that $k_{13} \rightarrow k_{13} + m'_{13} + j_1$, we can then evaluate the sum over k_{13} to obtain

$$G_{7P_1} = \sum \frac{(p_4-m_1)_{m_4+m_{14}}(-h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(p_4-h_3+h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_4-h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4-h_3+1-d/2)_{m_4}}$$

⁸For simplicity, we wrote the product over the v_{ab} cross-ratios although some of the cross-ratios are mixed as per (3.2). They will be unmixed at the end of the proof.

$$\begin{aligned}
& \times \frac{(p_2 + h_3)_{m_1 - m_4 + m_{23}} (-h_4)_{m_2 + m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3 + m_{13} + m_{44} + m_{12} + m_{23} + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} (p_2 + h_2)_{m_1 - m_2 + m_3 + m_{13} + m_{44} + m_{23}} (-h_2)_{m_1 + m_2 - m_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1 + m_4 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (p_3 - h_2 + h_4)_{m_2 + m_{12} + m_{22} + m_{24} + m_{33}} (-h_5)_{m_3 + m_{13} + m_{23} + m_{33} + m_{44}}}{(p_3 - h_2)_{2m_2 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times \frac{\binom{m_{23}}{k_{23}} \binom{m_{44}}{k_{44}} m_{13}! (-m'_{12})_{j_1 + m_{12}} (p_3 + h_5 - m_1 + m_2 + m_{12} - m'_{13} - k_{23} - k_{44})_{m'_{12} - m_{12} - j_1}}{\binom{m'_{12}}{k_{23}} \binom{m'_{13}}{k_{44}} m'_{12}! m'_{13}! (m_{13} - m'_{13} - j_1)!} \\
& \times \frac{\binom{k_{23}}{m'_{23}} \binom{k_{44}}{m'_{44}} (-m'_{33})_{m_{13} + m_{33} - m'_{13} - j_1} (-h_5 + m_3 + m_{13} + m_{33} + k_{23} + k_{44})_{j_1 + m'_{13} + m'_{33} - m_{13} - m_{33}}}{\binom{m'_{23}}{m'_{23}} \binom{m'_{44}}{m'_{44}} j_1! m'_{33}!} \\
& \times (-1)^{j_1 + k_{23} + k_{44} + m'_{23} + m'_{44}} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m_{ab}}}{m_{ab}!}.
\end{aligned}$$

In a similar way, we compute the sums over k_{23} and k_{44} , giving us

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4 - m_1)_{m_4 + m_{14}} (-h_6)_{m_4 + m_{14} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m_4 + m_{11} + m_{13} + m_{22}}}{(p_4 - h_3)_{2m_4 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1 - m_4 + m_{23}} (-h_4)_{m_2 + m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3 + m_{13} + m_{44} + m_{12} + m_{23} + m_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} (p_2 + h_2)_{m_1 - m_2 + m_3 + m_{13} + m_{44} + m_{23}} (-h_2)_{m_1 + m_2 - m_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m_1 + m_{11} + m_{13} + m_{22} + m_{23} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1 + m_4 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (p_3 - h_2 + h_4)_{m_2 + m_{12} + m_{22} + m_{24} + m_{33}}}{(p_3 - h_2)_{2m_2 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times \frac{(-m'_{12})_{j_1 + j_2 + j_3 + m_{12}} (p_3 + h_5 - m_1 + m_2 + m_{12} - m'_{13} - m'_{23} - m'_{44})_{m'_{12} - m_{12} - j_1 - j_2 - j_3}}{(m_{13} - m'_{13} - j_1)! (m_{23} - m'_{23} - j_2)!} \\
& \times \frac{(-h_5)_{m_3 + m'_{13} + m'_{23} + m'_{33} + m'_{44} + j_1 + j_2 + j_3} (-m'_{33})_{m_{13} + m_{23} + m_{33} + m_{44} - m'_{13} - m'_{23} - m'_{44} - j_1 - j_2 - j_3}}{j_1! j_2! j_3! m_{12}! m_{33}! (m_{44} - m'_{44} - j_3)!} \\
& \times (-1)^{j_1 + j_2 + j_3} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We then define $m_{13} = \bar{m} - m_{44}$ and shift m_{44} by $m_{44} + m'_{44} + j_3$. This allows us to sum over m_{44} , leading to

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4 - m_1)_{m_4 + m_{14}} (-h_6)_{m_4 + m'_{44} + m_{14} + m_{24} + m_{34} + j_3} (p_4 - h_3 + h_6)_{m_4 + m'_{13} + m_{11} + m_{22} + j_1}}{(p_4 - h_3)_{2m_4 + m'_{13} + m'_{44} + m_{11} + m_{14} + m_{22} + m_{24} + m_{34} + j_1 + j_3} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1 - m_4 + m_{23}} (-h_4)_{m_2 + m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m_3 + m_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3 + m_{12} + m_{23} + m_{33} + m} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1 + m_2 + m_3 + m_{12} + m_{33}} (p_2 + h_2)_{m_1 - m_2 + m_3 + m_{23} + m} (-h_2)_{m_1 + m_2 - m_3 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m_1 + m_{11} + m_{22} + m_{23} + m_{24} + m_{34} + m} (p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1 + m_4 + m_{11} + m_{22} + m_{24} + m_{34} + m} (p_3 - h_2 + h_4)_{m_2 + m_{12} + m_{22} + m_{24} + m_{33}}}{(p_3 - h_2)_{2m_2 + m_{11} + m_{12} + m_{22} + m_{24} + m_{33} + m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{(-m'_{12})_{j_1+j_2+j_3+m_{12}}(p_3+h_5-m_1+m_2+m_{12}-m'_{13}-m'_{23}-m'_{44})_{m'_{12}-m_{12}-j_1-j_2-j_3}}{(m-m'_{13}-m'_{44}-j_1-j_3)!(m_{23}-m'_{23}-j_2)!} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j_1+j_2+j_3}(-m'_{33})_{m_{23}+m_{33}+m-m'_{13}-m'_{23}-m'_{44}-j_1-j_2-j_3}}{j_1!j_2!j_3!m_{12}!m_{33}!} \\
& \times (-1)^{j_1+j_2+j_3} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Defining $j_3 = j - j_1$, the sum over j_1 thus gives

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4-m_1)_{m_4+m_{14}}(-h_6)_{m_4+m'_{44}+m_{14}+m_{24}+m_{34}}(p_4-h_3+h_6)_{m_4+m'_{13}+m_{11}+m_{22}}}{(p_4-h_3)_{2m_4+m'_{13}+m'_{44}+m_{11}+m_{14}+m_{22}+m_{24}+m_{34}}(p_4-h_3+1-d/2)_{m_4}} \\
& \times \frac{(p_2+h_3)_{m_1-m_4+m_{23}}(-h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(p_2+h_3+m_1)_{-m_4}(\bar{p}_3+h_2)_{2m_3+m_{12}+m_{23}+m_{33}+m}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_2+h_2)_{m_1-m_2+m_3+m_{23}+m}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{22}+m_{23}+m_{24}+m_{34}+m}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{22}+m_{24}+m_{34}+m}(p_3-h_2+h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-m'_{12})_{j+j_2+m_{12}}(p_3+h_5-m_1+m_2+m_{12}-m'_{13}-m'_{23}-m'_{44})_{m'_{12}-m_{12}-j-j_2}}{(m-m'_{13}-m'_{44}-j)!(m_{23}-m'_{23}-j_2)!} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j+j_2}(-m'_{33})_{m_{23}+m_{33}+m-m'_{13}-m'_{23}-m'_{44}-j-j_2}}{j!j_2!m_{12}!m_{33}!} \\
& \times (-1)^{j+j_2} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now define $m_{23} = k - m$ and then change m by $m \rightarrow m + m'_{13} + m'_{44} + j$. The sums over m and k (with the extra change $k \rightarrow k + m'_{13} + m'_{23} + m'_{44} + j + j_2$) can be performed and lead to

$$\begin{aligned}
G_{7P_1} &= \sum \frac{(p_4-m_1)_{m_4+m_{14}}(-h_6)_{m_4+m'_{44}+m_{14}+m_{24}+m_{34}}(p_4-h_3+h_6)_{m_4+m'_{13}+m_{11}+m_{22}}}{(p_4-h_3)_{2m_4+m'_{13}+m'_{44}+m_{11}+m_{14}+m_{22}+m_{24}+m_{34}}(p_4-h_3+1-d/2)_{m_4}} \\
& \times \frac{(p_2+h_3)_{m_1-m_4+m'_{23}+j_2}(-h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(p_2+h_3+m_1)_{-m_4}(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j+j_2}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m'_{33}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}+m'_{44}+j+j_2}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m'_{13}+m'_{23}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}+j+j_2}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m'_{13}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}+j}(p_3-h_2+h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-m'_{33})_{m_{33}}(-m'_{12})_{j+j_2+m_{12}}(p_3+h_5-m_1+m_2+m_{12}-m'_{13}-m'_{23}-m'_{44})_{m'_{12}-m_{12}-j-j_2}}{j!j_2!m_{12}!m_{33}!} \\
& \times (-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j+j_2} (-1)^{j+j_2} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1-v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

It is now possible to re-sum over m_{33} to get

$$\begin{aligned}
G_{7P_1} = & \sum \frac{(p_4 - m_1)_{m_4+m_{14}} (-h_6)_{m_4+m'_{44}+m_{14}+m_{24}+m_{34}} (p_4 - h_3 + h_6)_{m_4+m'_{13}+m_{11}+m_{22}}}{(p_4 - h_3)_{2m_4+m'_{13}+m'_{44}+m_{11}+m_{14}+m_{22}+m_{24}+m_{34}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1-m_4+m'_{23}+j_2} (-h_4)_{m_2+m_{11}+m_{34}+m'_{33}} (\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3+m_{12}+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j+j_2} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m'_{33}} (p_2 + h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}+m'_{44}+j+j_2} (-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m'_{13}+m'_{23}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}+j+j_2} (p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m'_{13}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}+j} (p_3 - h_2 + h_4)_{m_2+m_{12}+m_{22}+m_{24}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m'_{33}+m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times \frac{(-m'_{12})_{j+j_2+m_{12}} (p_3 + h_5 - m_1 + m_2 + m_{12} - m'_{13} - m'_{23} - m'_{44})_{m'_{12}-m_{12}-j-j_2}}{j!j_2!m_{12}!} \\
& \times (-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}+j+j_2} (-1)^{j+j_2} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We finally shift j by $j \rightarrow j - j_2$ and evaluate the sums over j_2 , j , and m_{12} ,⁹ to obtain

$$\begin{aligned}
G_{7P_1} = & \sum \frac{(p_4 - m_1)_{m_4+m_{14}} (-h_6)_{m_4+m'_{44}+m_{14}+m_{24}+m_{34}} (p_4 - h_3 + h_6)_{m_4+m'_{13}+m_{11}+m_{22}}}{(p_4 - h_3)_{2m_4+m'_{13}+m'_{44}+m_{11}+m_{14}+m_{22}+m_{24}+m_{34}} (p_4 - h_3 + 1 - d/2)_{m_4}} \\
& \times \frac{(p_2 + h_3)_{m_1-m_4+m'_{23}} (-h_4)_{m_2+m_{11}+m_{34}+m'_{12}+m'_{33}} (\bar{p}_3 + h_2 + h_5)_{m_3+m'_{12}}}{(p_2 + h_3 + m_1)_{-m_4} (\bar{p}_3 + h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}+m'_{44}} (\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+m'_{33}} (p_2 + h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}+m'_{44}} (-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m'_{13}+m'_{23}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}} (p_2 + 1 - d/2)_{m_1}} \\
& \times \frac{(-h_3)_{m_1+m_4+m'_{13}+m'_{44}+m_{11}+m_{22}+m_{24}+m_{34}} (p_3 - h_2 + h_4)_{m_2+m_{22}+m_{24}} (-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}+m'_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m'_{12}+m_{22}+m_{24}+m'_{33}+m_{34}} (p_3 - h_2 + 1 - d/2)_{m_2}} \\
& \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Using the fact that $m_{11} = m'_{22}$, $m_{22} = m'_{11}$, $m_{14} = m'_{14}$, $m_{24} = m'_{34}$ and $m_{34} = m'_{24}$, we find that $G_{7P} = G_7$ as expected from (3.2), proving invariance of the scalar seven-point conformal blocks in the extended snowflake channel under dendrite permutations of the first kind.

B.3. Dendrite Permutations of the Second Kind

We finally focus on the proof of the invariance of the scalar seven-point correlation functions under dendrite permutations of the second kind (3.3), which we write as $G_7 = G_{7P_2}$ for notational simplicity.

⁹Before proceeding with the sum over j , we use the first identity in (A.3).

Taking into account the fact that F_7 is invariant under dendrite permutations of the second kind, we find that

$$\begin{aligned}
G_{7P_2} &= \sum \frac{(p_4 - m_1)_{m_4+m_{14}}(p_4 - h_3 + h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(-h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m_1)_{-m_4}(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m_4}} \\
&\times \frac{(p_2 + h_3)_{m_1-m_4+m_{23}}(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
&\times \frac{(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{m_1}} \\
&\times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{m_2}} \\
&\times (-1)^{k_{14}+m'_{14}} \frac{m'_{14}!}{m_{14}!} \binom{m_{14}}{k_{14}} \binom{-p_4 + m_1 - m_4 - k_{14}}{m'_{14}} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!},
\end{aligned}$$

with

$$\begin{aligned}
m_{11} &= m'_{34}, & m_{12} &= m'_{12}, & m_{22} &= m'_{24}, \\
m_{13} &= m'_{44}, & m_{23} &= m'_{23}, & m_{33} &= m'_{33}, \\
m_{24} &= m'_{22}, & m_{34} &= m'_{11}, & m_{44} &= m'_{13}.
\end{aligned}$$

Computing the sum over k_{14} leads to

$$\begin{aligned}
G_{7P_2} &= \sum \frac{(p_4 - m_1)_{m_4+m'_{14}}(p_4 - h_3 + h_6)_{m_4+m_{14}+m_{24}+m_{34}+m_{44}}(-h_6)_{m_4+m_{11}+m_{13}+m_{22}}}{(p_2 + h_3 + m_1)_{-m_4}(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m_4}} \\
&\times \frac{(p_2 + h_3)_{m_1-m_4+m_{23}}(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
&\times \frac{(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{m_1}} \\
&\times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{m_2}} \\
&\times \frac{(-m'_{14})_{m_{14}}}{m_{14}!} F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a < b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We then evaluate the sum over m_{14} , giving

$$\begin{aligned}
G_{7P_2} &= \sum \frac{(p_4 - m_1)_{m_4+m'_{14}}(p_4 - h_3 + h_6)_{m_4+m_{24}+m_{34}+m_{44}}(-h_6)_{m_4+m_{11}+m_{13}+m'_{14}+m_{22}}}{(p_2 + h_3 + m_1)_{-m_4}(p_4 - h_3)_{2m_4+m_{11}+m_{13}+m'_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m_4}} \\
&\times \frac{(p_2 + h_3)_{m_1-m_4+m_{23}}(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3 - h_2 + h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m_3+m_{12}}}{(\bar{p}_3 + h_2)_{2m_3+m_{12}+m_{13}+m_{23}+m_{33}+m_{44}}(\bar{p}_3 + h_2 + 1 - d/2)_{m_3}} \\
&\times \frac{(p_2 + h_2)_{m_1-m_2+m_3+m_{13}+m_{23}+m_{44}}(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{23}+m_{24}+m_{34}+m_{44}}(p_2 + 1 - d/2)_{m_1}} \\
&\times \frac{(-h_3)_{m_1+m_4+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m_{13}+m_{23}+m_{33}+m_{44}}}{(p_3 - h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3 - h_2 + 1 - d/2)_{m_2}}
\end{aligned}$$

$$\begin{aligned}
& \times F_7 \prod_{1 \leq a \leq 4} \frac{(u_a^7)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 4} \frac{(1 - v_{ab}^7)^{m'_{ab}}}{m'_{ab}!} \\
& = G_7,
\end{aligned}$$

which ends the proof of the invariance of the scalar seven-point correlation functions under den-drite permutations of the second kind (3.3).

C. OPE Limit and Limit of Unit Operator

This appendix presents the remaining proofs for the OPE limit as well as the limit of unit operator. As usual, all re-summations are performed with the help of (A.1), (A.2) and (A.3).

C.1. OPE Limit

In the OPE limit $\eta_3 \rightarrow \eta_4$, we expect from the topologies that

$$I_{7(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3}, \Delta_{k_4})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_7}, \Delta_{i_1})} \Big|_{\text{extended snowflake}} \xrightarrow{\eta_3 \rightarrow \eta_4} (\eta_{34})^{-p_5} I_{6(\Delta_{k_3}, \Delta_{k_1}, \Delta_{k_4})}^{(\Delta_{i_5}, \Delta_{i_6}, \Delta_{i_4}, \Delta_{i_2}, \Delta_{i_7}, \Delta_{i_1})} \Big|_{\text{comb}}. \quad (\text{C.1})$$

Defining all quantities in the vectors \mathbf{h} and \mathbf{p} on the RHS of (C.1) with primes, this leads to

$$\begin{aligned}
L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{ka}}{2}} & \rightarrow (v_{23}^6)^{-\bar{p}'_4 - \bar{h}'_5} (v_{12}^6)^{h'_5} L_6 \prod_{1 \leq a \leq 3} (u_a^6)^{\frac{\Delta'_{ka}}{2}}, \\
u_1^7 & \rightarrow \frac{u_2^6}{v_{23}^6}, & u_2^7 & \rightarrow 0, & u_3^7 & \rightarrow u_1^6, & u_4^7 & \rightarrow \frac{u_3^6}{v_{12}^6}, \\
v_{11}^7 & \rightarrow \frac{v_{13}^6}{v_{23}^6}, & v_{12}^7 & \rightarrow 1, & v_{13}^7 & \rightarrow \frac{v_{33}^6}{v_{23}^6}, & v_{14}^7 & \rightarrow \frac{v_{23}^6}{v_{12}^6}, \\
v_{22}^7 & \rightarrow \frac{v_{13}^6}{v_{23}^6}, & v_{23}^7 & \rightarrow v_{11}^6, & v_{24}^7 & \rightarrow \frac{1}{v_{12}^6}, & v_{33}^7 & \rightarrow 1, \\
v_{34}^7 & \rightarrow \frac{1}{v_{12}^6}, & v_{44}^7 & \rightarrow \frac{v_{22}^6}{v_{12}^6},
\end{aligned}$$

which implies the identity

$$\begin{aligned}
& G_{7|\text{snowflake}}^{(d, h_2, h_3, h_4, h_5, h_6; p_2, p_3, p_4, p_5, p_6, p_7)} \left(\frac{u_2^6}{v_{23}^6}, 0, u_1^6, \frac{u_3^6}{v_{12}^6}, \frac{v_{13}^6}{v_{23}^6}, 1, \frac{v_{33}^6}{v_{23}^6}, \frac{v_{23}^6}{v_{12}^6}, \frac{v_{13}^6}{v_{23}^6}, v_{11}^6, \frac{1}{v_{12}^6}, 1, \frac{1}{v_{12}^6}, \frac{v_{22}^6}{v_{12}^6} \right) \\
& = (v_{23}^6)^{\bar{p}'_4 + \bar{h}'_5} (v_{12}^6)^{-h'_5} G_{6|\text{comb}}^{(d, h'_2, h'_3, h'_4, h'_5; p'_2, p'_3, p'_4, p'_5, p'_6)} (u_1^6, u_2^6, u_3^6, v_{11}^6, v_{12}^6, v_{13}^6, v_{22}^6, v_{23}^6, v_{33}^6).
\end{aligned}$$

Here the scalar six-point conformal blocks in the comb channel is given by

$$\begin{aligned}
& G_{6|\text{comb}}^{(d, h'_2, h'_3, h'_4, h'_5; p'_2, p'_3, p'_4, p'_5, p'_6)} \\
& = \sum \frac{(-h'_4 + m_2 - m_3)_{m_{11}} (p'_5 - m_2)_{m_3} (\bar{p}'_5 + \bar{h}'_5)_{m_3 + m_{13} + m_{23} + m_{33}} (-h'_5)_{m_3 + m_{12} + m_{22}}}{(\bar{p}'_5 + \bar{h}'_4)_{2m_3 + m_{12} + m_{22} + m_{13} + m_{23} + m_{33}} (\bar{p}'_5 + \bar{h}'_4 + 1 - d/2)_{m_3}}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{(p'_4 - m_1)_{m_2+m_{13}}(p'_3)_{m_1+m_{11}+m_{22}+m_{33}}(-h'_3)_{m_1}(p'_2 + h'_2)_{m_1+m_{23}+m_{12}}(-h'_4)_{m_2}}{(\bar{p}'_3 + h'_2)_{2m_1+m_{11}+m_{23}+m_{12}+m_{22}+m_{33}}(\bar{p}'_3 + h'_2 + 1 - d/2)_{m_1}(\bar{p}'_4 + \bar{h}'_3 + 1 - d/2)_{m_2}} \\
& \times \frac{(p'_3 + \bar{h}'_3)_{m_1+m_2+m_{11}+m_{23}+m_{12}+m_{22}+m_{33}}(\bar{p}'_4 + \bar{h}'_4)_{m_2+m_3+m_{12}+m_{22}+m_{13}+m_{23}+m_{33}}}{(\bar{p}'_4 + \bar{h}'_3)_{2m_2+m_{11}+m_{12}+m_{22}+m_{13}+m_{23}+m_{33}}} \\
& \times F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{(m_a)!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m_{ab}}}{(m_{ab})!},
\end{aligned}$$

with

$$F_6 = {}_3F_2 \left[\begin{matrix} -m_1, -m_2, -\bar{p}'_3 - \bar{h}'_2 + d/2 - m_1; \\ p'_4 - m_1, h'_3 + 1 - m_1 \end{matrix} ; 1 \right] {}_3F_2 \left[\begin{matrix} -m_2, -m_3, -\bar{p}'_4 - \bar{h}'_3 + d/2 - m_2; \\ p'_5 - m_2, h'_4 + 1 - m_2 \end{matrix} ; 1 \right],$$

and

$$\begin{aligned}
h_2 & \rightarrow -p'_4, & h_3 & \rightarrow -\bar{p}'_4 - \bar{h}'_4, & h_5 & \rightarrow -p'_3, & h_6 & \rightarrow h'_5, \\
p_2 & \rightarrow \bar{p}'_4 + \bar{h}'_3, & p_3 & \rightarrow -h'_3, & p_4 & \rightarrow p'_5, \\
p_2 + h_3 & \rightarrow -h'_4, & \bar{p}_3 + h_2 + h_5 & \rightarrow p'_2 + h'_2, & p_4 - h_3 + h_6 & \rightarrow \bar{p}'_5 + \bar{h}'_5, \\
p_2 + h_2 & \rightarrow \bar{p}'_3 + \bar{h}'_3, & \bar{p}_3 + \bar{h}_2 & \rightarrow \bar{p}'_3 + \bar{h}'_2, & p_4 - h_3 & \rightarrow \bar{p}'_5 + \bar{h}'_4.
\end{aligned}$$

To prove (C.1), which we rewrite as $G_{7|\eta_3 \rightarrow \eta_4} = G_6$ for simplicity, we first note that

$$\begin{aligned}
F_7 & = \frac{(-\bar{p}_3 - h_2 + d/2 - m'_1)_{m'_1}}{(p_3)_{-m'_2}} \sum \frac{(-m'_1)_{t_2}(\bar{p}_3 - d/2)_{t_2}(p_3)_{t_2}}{t_2!(p_3 - m'_2)_{t_2}(\bar{p}_3 + h_2 + 1 - d/2)_{t_2}} \\
& \quad \times \frac{(-m'_2)_{t_3}(-m'_3)_{t_3}(-p_2 + d/2 - m'_2)_{t_3}}{t_3!(1 - p_2 - h_3 - m'_2)_{t_3}(p_4 - m'_2)_{t_3}} \\
& = \frac{(-h_2 - m'_1)_{m'_1}}{(p_3)_{-m'_2}} \sum \frac{(-m'_1)_{t_2}(-m'_2)_{t_2}(\bar{p}_3 - d/2)_{t_2}}{t_2!(p_3 - m'_2)_{t_2}(-h_2 - m'_1)_{t_2}} \\
& \quad \times \frac{(-m'_2)_{t_3}(-m'_3)_{t_3}(-p_2 + d/2 - m'_2)_{t_3}}{t_3!(1 - p_2 - h_3 - m'_2)_{t_3}(p_4 - m'_2)_{t_3}} \\
& = \frac{(-h_2 - m'_1)_{m'_1}(p_3)_{m'_1}}{(p_3)_{m'_1 - m'_2}} \sum \frac{(-m'_1)_{t_2}(-m'_2)_{t_2}(-\bar{p}_3 - h_2 + d/2 - m'_1)_{t_2}}{t_2!(1 - p_3 - m'_1)_{t_2}(-h_2 - m'_1)_{t_2}} \\
& \quad \times \frac{(-m'_2)_{t_3}(-m'_3)_{t_3}(-p_2 + d/2 - m'_2)_{t_3}}{t_3!(1 - p_2 - h_3 - m'_2)_{t_3}(p_4 - m'_2)_{t_3}} \\
& = \frac{(-h_2 - m'_1)_{m'_1}(p_3)_{m'_1}}{(p_3)_{m'_1 - m'_2}} F_6.
\end{aligned}$$

Thus, multiplying G_7 by $(v_{23}^6)^{h_3 - h_6}(v_{12}^6)^{h_6}$ and taking the OPE limit $\eta_3 \rightarrow \eta_4$, we need to recover G_6 from

$$\begin{aligned}
G_{7|\eta_3 \rightarrow \eta_4} & = \sum \frac{(p_4 - m'_2)_{m'_3+m_{14}}(-h_6)_{m'_3+m_{14}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + h_6)_{m'_3+m_{11}+m_{13}+m_{22}}}{(p_4 - h_3)_{2m'_3+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}}(p_4 - h_3 + 1 - d/2)_{m'_3}} \\
& \quad \times \frac{(p_2 + h_3)_{m'_2 - m'_3 + m'_{11}}(p_3 - h_2 + h_4)_{m_{11}+m_{34}}(\bar{p}_3 + h_2 + h_5)_{m'_1}}{(p_2 + h_3 + m'_2)_{-m'_3}(\bar{p}_3 + h_2)_{2m'_1+m_{13}+m_{44}+m'_{11}}(\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_2 + m'_1 + m_{13} + m_{44} + m'_{11}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_2 - m'_1 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m'_2 + m_{11} + m_{13} + m_{22} + m'_{11} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_3)_{m'_2 + m'_3 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (-h_4)_{m_{22} + m_{24}} (-h_5)_{m'_1 + m_{13} + m'_{11} + m_{44}}}{(p_3 - h_2)_{m_{11} + m_{22} + m_{24} + m_{34}}} \\
& \times \binom{m_{11}}{k_{11}} \binom{m_{13}}{k_{13}} \binom{m_{14}}{k_{14}} \binom{m_{22}}{k_{22}} \binom{m_{24}}{k_{24}} \binom{m_{34}}{k_{34}} \binom{m_{44}}{k_{44}} \binom{h_6 - m'_3 - k_{14} - k_{24} - k_{34} - k_{44}}{m'_{12}} \\
& \times \binom{k_{11}}{l_{13}} \binom{k_{22}}{m'_{13} - l_{13}} \binom{k_{44}}{m'_{22}} \binom{h_3 - h_6 - m'_2 - k_{11} - k_{13} - k_{22} + k_{14}}{m'_{23}} \binom{k_{13}}{m'_{33}} \\
& \times (-1)^{k_{11} + k_{13} + k_{14} + k_{22} + k_{24} + k_{34} + k_{44} + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}} \\
& \times F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \frac{m_{12}! m_{33}! \prod_{1 \leq a \leq b \leq 3} (1 - v_{ab}^6)^{m'_{ab}}}{\prod_{1 \leq a \leq b \leq 4} m_{ab}!}.
\end{aligned}$$

To proceed, we first evaluate the sums over k_{13} , k_{22} , k_{24} , k_{34} , and finally k_{44} ,¹⁰ which gives

$$\begin{aligned}
G_{7|\eta_3 \rightarrow \eta_4} &= \sum \frac{(p_4 - m'_2)_{m'_3 + m_{14}} (-h_6)_{m'_3 + m_{14} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m'_3 + m_{11} + m_{13} + m_{22}}}{(p_4 - h_3)_{2m'_3 + m_{11} + m_{13} + m_{14} + m_{22} + m_{24} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_3}} \\
& \times \frac{(p_2 + h_3)_{m'_2 - m'_3 + m'_{11}} (p_3 - h_2 + h_4)_{m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m'_1}}{(p_2 + h_3 + m'_2)_{-m'_3} (\bar{p}_3 + h_2)_{2m'_1 + m_{13} + m_{44} + m'_{11}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_2 + m'_1 + m_{13} + m_{44} + m'_{11}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_2 - m'_1 + m_{11} + m_{22} + m_{24} + m_{34}}}{(p_2)_{2m'_2 + m_{11} + m_{13} + m_{22} + m'_{11} + m_{24} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_3)_{m'_2 + m'_3 + m_{11} + m_{13} + m_{22} + m_{24} + m_{34} + m_{44}} (-h_4)_{m_{22} + m_{24}} (-h_5)_{m'_1 + m_{13} + m'_{11} + m_{44}}}{(p_3 - h_2)_{m_{11} + m_{22} + m_{24} + m_{34}}} \\
& \times \binom{m_{14}}{k_{14}} (-1)^{k_{14}} \frac{m'_{13}! (-m'_{23})_{m_{11} + m_{13} + m_{22} - m'_{13} - m'_{33}} (-m'_{12})_{m_{24} + m_{34} + m_{44} - m'_{22}}}{(m_{11} - l_{13})! (m_{22} - m'_{13} + l_{13})!} \\
& \times \frac{(-h_3 + h_6 + m'_2 + m_{11} + m_{13} + m_{22} - k_{14})_{m'_{13} + m'_{23} + m'_{33} - m_{11} - m_{13} - m_{22}}}{(m_{13} - m'_{33})!} \\
& \times \frac{(-h_6 + m'_3 + m_{24} + m_{34} + m_{44} + k_{14})_{m'_{12} + m'_{22} - m_{24} - m_{34} - m_{44}}}{l_{13}! (m'_{13} - l_{13})! (m_{44} - m'_{22})! m_{14}! m_{24}! m_{34}!} \\
& \times F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

To eliminate the sum over k_{14} , we now use the following identity

$$\begin{aligned}
& (-h_3 + h_6 + m'_2 + m_{11} + m_{13} + m_{22} - k_{14})_{m'_{13} + m'_{23} + m'_{33} - m_{11} - m_{13} - m_{22}} \\
&= \sum_j \binom{m'_{13} + m'_{23} + m'_{33} - m_{11} - m_{13} - m_{22}}{j} \\
& \quad \times (-h_3 + h_6 + m'_2 + m_{11} + m_{13} + m_{22})_{m'_{13} + m'_{23} + m'_{33} - m_{11} - m_{13} - m_{22} - j} (-k_{14})_j,
\end{aligned}$$

¹⁰We first change variables such that $k_{13} \rightarrow k_{13} + m'_{33}$ and $k_{44} \rightarrow k_{44} + m'_{22}$.

and then change the variable by $k_{14} \rightarrow k_{14} + j$. The sum over k_{14} can be performed, leading to

$$\begin{aligned}
G_{7|\eta_3 \rightarrow \eta_4} &= \sum \frac{(p_4 - m'_2)_{m'_3+m_{14}} (-h_6)_{m'_3+m'_{12}+m'_{22}+j} (p_4 - h_3 + h_6)_{m'_3+m_{11}+m_{13}+m_{22}}}{(p_4 - h_3)_{2m'_3+m_{11}+m_{13}+m_{14}+m_{22}+m_{24}+m_{34}+m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_3}} \\
&\times \frac{(p_2 + h_3)_{m'_2-m'_3+m'_{11}} (p_3 - h_2 + h_4)_{m_{11}+m_{34}} (\bar{p}_3 + h_2 + h_5)_{m'_1}}{(p_2 + h_3 + m'_2)_{-m'_3} (\bar{p}_3 + h_2)_{2m'_1+m_{13}+m_{44}+m'_{11}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}} \\
&\times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_2+m'_1+m_{13}+m_{44}+m'_{11}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_2-m'_1+m_{11}+m_{22}+m_{24}+m_{34}}}{(p_2)_{2m'_2+m_{11}+m_{13}+m_{22}+m'_{11}+m_{24}+m_{34}+m_{44}} (p_2 + 1 - d/2)_{m'_2}} \\
&\times \frac{(-h_3)_{m'_2+m'_3+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m_{44}} (-h_4)_{m_{22}+m_{24}} (-h_5)_{m'_1+m_{13}+m'_{11}+m_{44}}}{(p_3 - h_2)_{m_{11}+m_{22}+m_{24}+m_{34}}} \\
&\times \frac{m'_{13}! (-m'_{23})_{j+m_{11}+m_{13}+m_{22}-m'_{13}-m'_{33}} (-m'_{12})_{m_{14}+m_{24}+m_{34}+m_{44}-m'_{22}-j}}{(m_{11} - l_{13})! (m_{22} - m'_{13} + l_{13})!} \\
&\times \frac{(-h_3 + h_6 + m'_2 + m_{11} + m_{13} + m_{22})_{m'_{13}+m'_{23}+m'_{33}-m_{11}-m_{13}-m_{22}-j}}{l_{13}! (m'_{13} - l_{13})! (m_{13} - m'_{33})!} \\
&\times \frac{(-1)^j}{(m_{44} - m'_{22})! (m_{14} - j)! j! m_{24}! m_{34}!} F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now define $m_{34} = m - m_{24}$ to evaluate the sums over m_{24} , m_{14} , m , and finally m_{44} ,¹¹ leading to

$$\begin{aligned}
G_{7|\eta_3 \rightarrow \eta_4} &= \sum \frac{(p_4 - m'_2)_{m'_3+j} (-h_6)_{m'_3+m'_{12}+m'_{22}+j} (p_4 - h_3 + h_6)_{m'_3+m_{11}+m_{13}+m_{22}}}{(p_4 - h_3)_{2m'_3+m'_{12}+m'_{22}+m_{11}+m_{13}+m_{22}+j} (p_4 - h_3 + 1 - d/2)_{m'_3}} \\
&\times \frac{(p_2 + h_3)_{m'_2-m'_3+m'_{11}} (p_3 - h_2 + h_4)_{m_{11}} (\bar{p}_3 + h_2 + h_5)_{m'_1+m'_{12}}}{(p_2 + h_3 + m'_2)_{-m'_3} (\bar{p}_3 + h_2)_{2m'_1+m'_{11}+m'_{12}+m'_{22}+m_{13}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}} \\
&\times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_1+m'_2+m'_{11}+m'_{12}+m'_{22}+m_{13}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_2-m'_1+m_{11}+m_{22}}}{(p_2)_{2m'_2+m'_{11}+m'_{12}+m'_{22}+m_{11}+m_{13}+m_{22}} (p_2 + 1 - d/2)_{m'_2}} \\
&\times \frac{(-h_3)_{m'_2+m'_3+m'_{12}+m'_{22}+m_{11}+m_{13}+m_{22}} (-h_4)_{m_{22}} (-h_5)_{m'_1+m'_{11}+m'_{22}+m_{13}}}{(p_3 - h_2)_{m_{11}+m_{22}}} \\
&\times \frac{(-h_3 + h_6 + m'_2 + m_{11} + m_{13} + m_{22})_{m'_{13}+m'_{23}+m'_{33}-m_{11}-m_{13}-m_{22}-j}}{(m'_{13} - l_{13})! (m_{13} - m'_{33})!} \\
&\times \frac{m'_{13}! (-m'_{23})_{j+m_{11}+m_{13}+m_{22}-m'_{13}-m'_{33}} (-1)^j}{l_{13}! (m_{11} - l_{13})! (m_{22} - m'_{13} + l_{13})! j!} F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

With the help of the first identity involving ${}_3F_2$ shown in (A.3), we evaluate the sum over j and get

$$G_{7|\eta_3 \rightarrow \eta_4} = \sum \frac{(p_4 - m'_2)_{m'_3} (-h_6)_{m'_3+m'_{12}+m'_{22}} (p_4 - h_3 + h_6)_{m'_3+m'_{13}+m'_{23}+m'_{33}}}{(p_4 - h_3)_{2m'_3+m'_{12}+m'_{13}+m'_{22}+m'_{23}+m'_{33}} (p_4 - h_3 + 1 - d/2)_{m'_3}}$$

¹¹We first shift $m_{14} \rightarrow m_{14} + j$ and $m_{44} \rightarrow m_{44} + m'_{22}$.

$$\begin{aligned}
& \times \frac{(p_2 + h_3)_{m'_2 - m'_3 + m'_{11}} (p_3 - h_2 + h_4)_{m_{11}} (\bar{p}_3 + h_2 + h_5)_{m'_1 + m'_{12}}}{(p_2 + h_3 + m'_2)_{-m'_3} (\bar{p}_3 + h_2)_{2m'_1 + m'_{11} + m'_{12} + m'_{22} + m_{13}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_1 + m'_2 + m'_{11} + m'_{12} + m'_{22} + m_{13}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_2 - m'_1 + m_{11} + m_{22}}}{(p_2)_{2m'_2 + m'_{11} + m'_{12} + m'_{22} + m_{11} + m_{13} + m_{22}} (p_2 + 1 - d/2)_{m'_2}} \\
& \times \frac{(-h_3)_{m'_2 + m'_3 + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}} (-h_4)_{m_{22}} (-h_5)_{m'_1 + m'_{11} + m'_{22} + m_{13}}}{(p_3 - h_2)_{m_{11} + m_{22}}} \\
& \times \frac{m'_{13}! (-m'_{23})_{m_{11} + m_{13} + m_{22} - m'_{13} - m'_{33}}}{l_{13}! (m'_{13} - l_{13})! (m_{13} - m'_{33})! (m_{11} - l_{13})! (m_{22} - m'_{13} + l_{13})!} \\
& \times F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We finally change the variables by

$$m_{11} \rightarrow m_{11} + l_{13}, \quad m_{22} \rightarrow m_{22} + m'_{13} - l_{13},$$

and then define $m_{22} = n - m_{11}$. We can thus evaluate the sums over m_{11} , n , l_{13} , and m_{13} after shifting $m_{13} \rightarrow m_{13} + m'_{33}$, resulting in

$$\begin{aligned}
G_{7|\eta_3 \rightarrow \eta_4} &= \sum \frac{(p_4 - m'_2)_{m'_3} (-h_6)_{m'_3 + m'_{12} + m'_{22}} (p_4 - h_3 + h_6)_{m'_3 + m'_{13} + m'_{23} + m'_{33}}}{(p_4 - h_3)_{2m'_3 + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}} (p_4 - h_3 + 1 - d/2)_{m'_3}} \\
& \times \frac{(p_2 + h_3)_{m'_2 - m'_3 + m'_{11}} (\bar{p}_3 + h_2 + h_5)_{m'_1 + m'_{12} + m'_{23}}}{(p_2 + h_3 + m'_2)_{-m'_3} (\bar{p}_3 + h_2)_{2m'_1 + m'_{11} + m'_{12} + m'_{22} + m'_{23} + m'_{33}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}} \\
& \times \frac{(p_3)_{m'_1} (p_2 + h_2)_{m'_1 + m'_2 + m'_{11} + m'_{12} + m'_{22} + m'_{23} + m'_{33}} (-h_2 - m'_1)_{m'_2 + m'_{13}}}{(p_2)_{2m'_2 + m'_{11} + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}}} \\
& \times \frac{(-h_3)_{m'_2 + m'_3 + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}} (-h_5)_{m'_1 + m'_{11} + m'_{22} + m'_{33}}}{(p_2 + 1 - d/2)_{m'_2}} \\
& \times F_6 \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1 - v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Re-expressing the unprimed variables in terms of the primed variables leads to G_6 which completes our proof of the OPE limit $\eta_3 \rightarrow \eta_4$ (C.1).

C.2. Limit of Unit Operator

We focus first on the limit of unit operator given by $\mathcal{O}_{i_2}(\eta_2) \rightarrow \mathbb{1}$, for which we have $\Delta_{i_2} = 0$ as well as $\Delta_{k_4} = \Delta_{k_1}$. It implies

$$I_{7(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3}, \Delta_{k_4})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_7}, \Delta_{i_1})} \Big|_{\text{extended snowflake}} \xrightarrow{\mathcal{O}_{i_2}(\eta_2) \rightarrow \mathbb{1}} I_{6(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3})}^{(\Delta_{i_7}, \Delta_{i_1}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6})} \Big|_{\text{snowflake}}, \quad (\text{C.2})$$

with $p_4 = p_2 + h_3 = 0$. Moreover, in this limit the legs and conformal cross-ratios of the six- and seven-point correlation functions are related by

$$L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{ka}}{2}} = (v_{24}^7)^{-h_4} (v_{34}^7)^{h_4 - h_2} L_6 \prod_{1 \leq a \leq 3} (u_a^6)^{\frac{\Delta_{ka}}{2}},$$

$$u_1^6 = \frac{u_1^7 u_4^7}{v_{34}^7}, \quad u_2^6 = \frac{u_2^7}{v_{24}^7}, \quad u_3^6 = u_3^7 v_{34}^7,$$

$$v_{11}^6 = \frac{v_{11}^7}{v_{34}^7}, \quad v_{12}^6 = \frac{v_{12}^7 v_{34}^7}{v_{24}^7}, \quad v_{13}^6 = v_{13}^7,$$

$$v_{22}^6 = \frac{v_{22}^7}{v_{24}^7}, \quad v_{23}^6 = v_{44}^7, \quad v_{33}^6 = \frac{v_{33}^7 v_{34}^7}{v_{24}^7}.$$

These observations lead to the identity $G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} = (v_{24}^7)^{h_4} (v_{34}^7)^{h_2 - h_4} G_6$ that we now prove.

From the vanishing of p_4 and $p_2 + h_3$, we have

$$m_1 = m_4 = r, \quad m_{14} = m_{23} = 0.$$

As a result, we find that

$$G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} = \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}} (p_2+h_6)_{m_1+m_{11}+m_{13}+m_{22}} (p_2+h_2)_{m_1-m_2+m_3+m_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m_{13}+m_{22}+m_{24}+m_{34}+m'_{23}} (p_2+1-d/2)_{m_1}}$$

$$\times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}} (p_3-h_2+h_4)_{m_2+m_{11}+m_{34}} (\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m_{13}+m_{33}+m'_{23}} (\bar{p}_3+h_2+1-d/2)_{m_3}}$$

$$\times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}} (-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}} (-h_5)_{m_3+m_{13}+m_{33}+m'_{23}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (p_3-h_2+1-d/2)_{m_2}}$$

$$\times \frac{(-\bar{p}_3-h_2+d/2-m_3)_{m_3} (-p_3+h_2+d/2-m_2)_{m_2} (-m_2)_{t_1} (-m_3)_{t_2}}{(p_3)_{-m_1} t_1! t_2!}$$

$$\times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2} (p_3-h_2+1-d/2)_{t_1}}$$

$$\times \binom{m_{11}}{k_{11}} \binom{m_{12}}{k_{12}} \binom{m_{22}}{k_{22}} \binom{m_{33}}{k_{33}} \binom{k_{11}}{m'_{11}} \binom{k_{12}}{m'_{12}} \binom{k_{22}}{m'_{22}} \binom{k_{33}}{m'_{33}}$$

$$\times (v_{24}^7)^{m_2+k_{12}+k_{22}+k_{33}} (v_{34}^7)^{m_1-m_3+k_{11}-k_{12}-k_{33}} \frac{(1-v_{24}^7)^{m_{24}}}{m_{24}!} \frac{(1-v_{34}^7)^{m_{34}}}{m_{34}!}$$

$$\times (-1)^{k_{11}+k_{12}+k_{22}+k_{33}+m'_{11}+m'_{12}+m'_{22}+m'_{33}} \frac{m_{23}!}{m'_{23}!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m_{ab}!},$$

where we defined $m'_{23} = m_{44}$.

We then shift all of k_{ab} by $k_{ab} \rightarrow k_{ab} + m'_{ab}$ and use the fact that

$$\frac{1}{m_{ab}!} \binom{m_{ab}}{k_{ab} + m'_{ab}} \binom{k_{ab} + m'_{ab}}{m'_{ab}} = \frac{1}{m'_{ab}! (m_{ab} - m'_{ab})!} \binom{m_{ab} - m'_{ab}}{k_{ab}},$$

to write

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}} (p_2+h_6)_{m_1+m_{11}+m'_{13}+m_{22}} (p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m'_{13}+m_{22}+m_{24}+m_{34}+m'_{23}} (p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}} (p_3-h_2+h_4)_{m_2+m_{11}+m_{34}} (\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{13}+m_{33}+m'_{23}} (\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}} (-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}} (-h_5)_{m_3+m'_{13}+m_{33}+m'_{23}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-\bar{p}_3-h_2+d/2-m_3)_{m_3} (-p_3+h_2+d/2-m_2)_{m_2} (-m_2)_{t_1} (-m_3)_{t_2}}{(p_3)_{-m_1} t_1! t_2!} \\
&\times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2} (p_3-h_2+1-d/2)_{t_1}} \\
&\times \binom{m_{11}-m'_{11}}{k_{11}} \binom{m_{12}-m'_{12}}{k_{12}} \binom{m_{22}-m'_{22}}{k_{22}} \binom{m_{33}-m'_{33}}{k_{33}} \\
&\times \frac{(v_{24}^7)^{m_2+k_{12}+k_{22}+k_{33}+m'_{12}+m'_{22}+m'_{33}} (v_{34}^7)^{m_1-m_3+k_{11}-k_{12}-k_{33}+m'_{11}-m'_{12}-m'_{33}}}{(m_{11}-m'_{11})! (m_{12}-m'_{12})! (m_{22}-m'_{22})! (m_{33}-m'_{33})!} \\
&\times \frac{(1-v_{24}^7)^{m_{24}} (1-v_{34}^7)^{m_{34}}}{m_{24}! m_{34}!} (-1)^{k_{11}+k_{12}+k_{22}+k_{33}} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!},
\end{aligned}$$

where we defined $m'_{13} = m_{13}$.

Using the identity (A.1), we can compute the sums over k_{11} and k_{22} , leading to

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}} (p_2+h_6)_{m_1+m_{11}+m'_{13}+m_{22}} (p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m'_{13}+m_{22}+m_{24}+m_{34}+m'_{23}} (p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}} (p_3-h_2+h_4)_{m_2+m_{11}+m_{34}} (\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{13}+m_{33}+m'_{23}} (\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}} (-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}} (-h_5)_{m_3+m'_{13}+m_{33}+m'_{23}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}} (p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-\bar{p}_3-h_2+d/2-m_3)_{m_3} (-p_3+h_2+d/2-m_2)_{m_2} (-m_2)_{t_1} (-m_3)_{t_2}}{(p_3)_{-m_1} t_1! t_2!} \\
&\times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2} (p_3-h_2+1-d/2)_{t_1}} \\
&\times \binom{m_{12}-m'_{12}}{k_{12}} \binom{m_{33}-m'_{33}}{k_{33}} \binom{k_{12}}{r_{12}} \binom{k_{33}}{r_{33}} \\
&\times \frac{(v_{24}^7)^{m_2+m'_{12}+m'_{22}+m'_{33}} (v_{34}^7)^{m_1-m_3-k_{12}-k_{33}+m'_{11}-m'_{12}-m'_{33}} (1-v_{24}^7)^{r_{12}+r_{33}+m_{22}+m_{24}-m'_{22}}}{(m_{11}-m'_{11})! (m_{12}-m'_{12})! (m_{22}-m'_{22})! (m_{33}-m'_{33})! m_{24}!} \\
&\times \frac{(1-v_{34}^7)^{m_{11}+m_{34}-m'_{11}}}{m_{34}!} (-1)^{r_{12}+r_{33}+k_{12}+k_{33}} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now change k_{12} and k_{33} by $k_{12} \rightarrow k_{12} + r_{12}$ and $k_{33} \rightarrow k_{33} + r_{33}$, respectively, giving us

$$G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} = \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}} (p_2+h_6)_{m_1+m_{11}+m'_{13}+m_{22}} (p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m'_{13}+m_{22}+m_{24}+m_{34}+m'_{23}} (p_2+1-d/2)_{m_1}}$$

$$\begin{aligned}
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3-h_2+h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{13}+m_{33}+m'_{23}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m'_{13}+m_{33}+m'_{23}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \frac{(-m_2)_{t_1}(-m_3)_{t_2}}{t_1!t_2!} \\
& \times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \binom{m_{12}-m'_{12}-r_{12}}{k_{12}} \binom{m_{33}-m'_{33}-r_{33}}{k_{33}} \\
& \times \frac{(v_{24}^7)_{m_2+m'_{12}+m'_{22}+m'_{33}}(v_{34}^7)_{m_1-m_3-r_{12}-r_{33}-k_{12}-k_{33}+m'_{11}-m'_{12}-m'_{33}}}{r_{12}!r_{33}!(m_{11}-m'_{11})!(m_{12}-m'_{12}-r_{12})!(m_{22}-m'_{22})!(m_{33}-m'_{33}-r_{33})!} \\
& \times \frac{(1-v_{24}^7)^{r_{12}+r_{33}+m_{22}+m_{24}-m'_{22}}(1-v_{34}^7)^{m_{11}+m_{34}-m'_{11}}}{m_{24}!m_{34}!} \\
& \times (-1)^{k_{12}+k_{33}} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

The sums over k_{12} and k_{33} then lead to

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}}(p_2+h_6)_{m_1+m_{11}+m'_{13}+m_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m'_{13}+m_{22}+m_{24}+m_{34}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m_{33}}(p_3-h_2+h_4)_{m_2+m_{11}+m_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m_{12}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{13}+m_{33}+m'_{23}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m_{22}+m_{24}+m_{34}}(-h_4)_{m_2+m_{12}+m_{22}+m_{24}+m_{33}}(-h_5)_{m_3+m'_{13}+m_{33}+m'_{23}}}{(p_3-h_2)_{2m_2+m_{11}+m_{12}+m_{22}+m_{24}+m_{33}+m_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \frac{(-m_2)_{t_1}(-m_3)_{t_2}}{t_1!t_2!} \\
& \times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)_{m_2+m'_{12}+m'_{22}+m'_{33}}(v_{34}^7)_{m_1-m_3-r_{12}-r_{33}+m'_{11}-m'_{12}-m'_{33}}}{r_{12}!r_{33}!(m_{11}-m'_{11})!(m_{12}-m'_{12}-r_{12})!(m_{22}-m'_{22})!(m_{33}-m'_{33}-r_{33})!} \\
& \times \frac{(1-v_{24}^7)^{r_{12}+r_{33}+m_{22}+m_{24}-m'_{22}}(1-\frac{1}{v_{34}^7})^{m_{12}+m_{33}-m'_{12}-m'_{33}-r_{12}-r_{33}}(1-v_{34}^7)^{m_{11}+m_{34}-m'_{11}}}{m_{24}!m_{34}!} \\
& \times \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Changing variables such that

$$m_{11} \rightarrow m_{11} + m'_{11}, \quad m_{12} \rightarrow m_{12} + m'_{11} + r_{12}, \quad m_{22} \rightarrow m_{22} + m'_{22}, \quad m_{33} \rightarrow m_{33} + m'_{33} + r_{33},$$

we obtain

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m_{24}+m_{34}+m'_{23}} (p_2+h_6)_{m_1+m_{11}+m'_{11}+m'_{13}+m_{22}+m'_{22}} (p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m_{11}+m'_{11}+m'_{13}+m_{22}+m'_{22}+m_{24}+m_{34}+m'_{23}} (p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m_{12}+m'_{12}+r_{12}+m_{33}+m'_{33}+r_{33}} (p_3-h_2+h_4)_{m_2+m_{11}+m'_{11}+m_{34}}}{(\bar{p}_3+h_2)_{2m_3+m_{12}+m'_{12}+r_{12}+m'_{13}+m_{33}+m'_{33}+r_{33}+m'_{23}} (\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m_{11}+m'_{11}+m_{22}+m'_{22}+m_{24}+m_{34}} (-h_4)_{m_2+m_{12}+m'_{12}+r_{12}+m_{22}+m'_{22}+m_{24}+m_{33}+m'_{33}+r_{33}}}{(p_3-h_2)_{2m_2+m_{11}+m'_{11}+m_{12}+m'_{12}+r_{12}+m_{22}+m'_{22}+m_{24}+m_{33}+m'_{33}+r_{33}+m_{34}} (p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-h_5)_{m_3+m'_{13}+m_{33}+m'_{33}+r_{33}+m'_{23}} (-\bar{p}_3-h_2+d/2-m_3)_{m_3} (-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
&\times \frac{(-m_2)_{t_1} (-m_3)_{t_2} (\bar{p}_3-d/2)_{t_1+t_2}}{t_1! t_2!} \frac{(p_3)_{t_1+t_2} (\bar{p}_3+h_2+h_5)_{m_3+m_{12}+m'_{12}+r_{12}}}{(\bar{p}_3-h_2+1-d/2)_{t_2} (p_3-h_2+1-d/2)_{t_1}} \\
&\times \frac{(v_{24}^7)_{m_2+m'_{12}+m'_{22}+m'_{33}} (v_{34}^7)_{m_1-m_3-r_{12}-r_{33}+m'_{11}-m'_{12}-m'_{33}}}{r_{12}! r_{33}! m_{11}! m_{12}! m_{22}! m_{33}!} \\
&\times \frac{(1-v_{24}^7)_{r_{12}+r_{33}+m_{22}+m_{24}} (1-\frac{1}{v_{34}^7})_{m_{12}+m_{33}} (1-v_{34}^7)_{m_{11}+m_{34}}}{m_{24}! m_{34}!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)_{m_a}}{m_a!} \prod_{1 \leq a < b \leq 3} \frac{(1-v_{ab}^6)_{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now define

$$m_{24} = n_{24} - m_{22} - r_{12} - r_{33}, \quad m_{34} = n_{34} - m_{11}, \quad m_{33} = s_{34} - m_{12},$$

such that

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+n_{24}+n_{34}+m'_{23}-m_{11}-m_{22}-r_{12}-r_{33}} (p_2+h_6)_{m_1+m_{11}+m'_{11}+m'_{13}+m_{22}+m'_{22}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+n_{24}+n_{34}-r_{12}-r_{33}+m'_{23}} (p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+r_{12}+s_{34}+m'_{33}+r_{33}} (p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}} (\bar{p}_3+h_2+h_5)_{m_3+m_{12}+m'_{12}+r_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+r_{12}+m'_{13}+s_{34}+m'_{33}+r_{33}+m'_{23}} (\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{24}+n_{34}-r_{12}-r_{33}} (-h_4)_{m_2+m'_{12}+m'_{22}+n_{24}+s_{34}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+n_{24}+n_{34}+s_{34}} (p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-h_5)_{m_3+m'_{13}+s_{34}-m_{12}+m'_{33}+r_{33}+m'_{23}} (-\bar{p}_3-h_2+d/2-m_3)_{m_3} (-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
&\times \frac{(-m_2)_{t_1} (-m_3)_{t_2} (\bar{p}_3-d/2)_{t_1+t_2}}{t_1! t_2!} \frac{(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}} (p_3)_{t_1+t_2}}{(\bar{p}_3-h_2+1-d/2)_{t_2} (p_3-h_2+1-d/2)_{t_1}} \\
&\times \frac{(v_{24}^7)_{m_2+m'_{12}+m'_{22}+m'_{33}} (v_{34}^7)_{m_1-m_3-r_{12}-r_{33}+m'_{11}-m'_{12}-m'_{33}}}{r_{12}! r_{33}! m_{11}! m_{12}! m_{22}! (s_{34}-m_{12})!} \\
&\times \frac{(1-v_{24}^7)_{n_{24}} (1-\frac{1}{v_{34}^7})_{s_{34}} (1-v_{34}^7)_{n_{34}}}{(n_{24}-m_{22}-r_{12}-r_{33})! (n_{34}-m_{11})!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)_{m_a}}{m_a!} \prod_{1 \leq a < b \leq 3} \frac{(1-v_{ab}^6)_{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

The sums over m_{11} , m_{12} , and m_{22} can be performed, leading to

$$G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} = \sum \frac{(-h_6)_{m_1+m'_{23}} (p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}} (p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}} (p_2+1-d/2)_{m_1}}$$

$$\begin{aligned}
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+r_{12}+s_{34}+m'_{33}+r_{33}}(p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}+r_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+r_{12}+m'_{13}+m'_{33}+r_{33}+m'_{23}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{24}+n_{34}-r_{12}-r_{33}}(-h_4)_{m_2+m'_{12}+m'_{22}+n_{24}+s_{34}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+n_{24}+n_{34}+s_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{33}+r_{33}+m'_{23}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
& \times \frac{(-m_2)_{t_1}(-m_3)_{t_2}(\bar{p}_3-d/2)_{t_1+t_2}}{t_1!t_2!} \frac{(p_3)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)^{m_2+m'_{12}+m'_{22}+m'_{33}}(v_{34}^7)^{m_1-m_3-r_{12}-r_{33}+m'_{11}-m'_{12}-m'_{33}}}{r_{12}!r_{33}!s_{34}!} \\
& \times \frac{(1-v_{24}^7)^{n_{24}}}{(n_{24}-r_{12}-r_{33})!} \frac{(1-\frac{1}{v_{34}^7})^{s_{34}}(1-v_{34}^7)^{n_{34}}}{n_{34}!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now redefine $r_{33} = r - r_{12}$ to evaluate the sum over r_{12} and get

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+s_{34}+m'_{33}+r}(p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{24}+n_{34}-r}(-h_4)_{m_2+m'_{12}+m'_{22}+n_{24}+s_{34}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+n_{24}+n_{34}+s_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
& \times \frac{(-m_2)_{t_1}(-m_3)_{t_2}(\bar{p}_3-d/2)_{t_1+t_2}}{t_1!t_2!} \frac{(p_3)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)^{m_2+m'_{12}+m'_{22}+m'_{33}}(v_{34}^7)^{m_1-m_3+m'_{11}-m'_{12}-m'_{33}}}{r!s_{34}!} \\
& \times \binom{r}{j} \frac{(1-v_{24}^7)^{n_{24}}}{(n_{24}-r)!} \frac{(1-\frac{1}{v_{34}^7})^{j+s_{34}}(1-v_{34}^7)^{n_{34}}}{n_{34}!} (-1)^j \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!},
\end{aligned}$$

where we also expanded $(v_{34}^7)^{-r}$ in a power series in $1-v_{34}^7$.

After shifting r by $r \rightarrow r+j$, we evaluate the sum over r , leading to

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+s_{34}+m'_{33}+j}(p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{34}}(-h_4)_{m_2+m'_{12}+m'_{22}+n_{24}+s_{34}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+j+n_{34}+s_{34}}(p_3-h_2+1-d/2)_{m_2}}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
& \times \frac{(-m_2)_{t_1}(-m_3)_{t_2}(\bar{p}_3-d/2)_{t_1+t_2}}{t_1!t_2!} \frac{(p_3)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)^{m_2+m'_{12}+m'_{22}+m'_{33}}(v_{34}^7)^{m_1-m_3+m'_{11}-m'_{12}-m'_{33}}}{j!s_{34}!} \\
& \times \frac{(1-v_{24}^7)^{n_{24}}(1-\frac{1}{v_{34}^7})^{j+s_{34}}(1-v_{34}^7)^{n_{34}}}{(n_{24}-j)!n_{34}!}(-1)^j \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

Changing n_{24} by $n_{24} + j$, the sum over n_{24} gives

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow 1} &= \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+s_{34}+m'_{33}+j}(p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{34}}(-h_4)_{m_2+m'_{12}+m'_{22}+j+s_{34}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+j+n_{34}+s_{34}}(p_3-h_2+1-d/2)_{m_2}} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \\
& \times \frac{(-m_2)_{t_1}(-m_3)_{t_2}(\bar{p}_3-d/2)_{t_1+t_2}}{t_1!t_2!} \frac{(p_3)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)^{h_4-j-s_{34}}(v_{34}^7)^{m_1-m_3+m'_{11}-m'_{12}-m'_{33}}}{s_{34}!} \\
& \times \frac{(1-v_{24}^7)^j(1-\frac{1}{v_{34}^7})^{j+s_{34}}(1-v_{34}^7)^{n_{34}}}{j!n_{34}!}(-1)^j \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We now define $s_{34} = s - j$ and evaluate the sum over j to reach

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow 1} &= \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
& \times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+s+m'_{33}}(p_3-h_2+h_4)_{m_2+m'_{11}+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
& \times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+n_{34}}(-h_4)_{m_2+m'_{12}+m'_{22}+s+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+n_{34}+s}}(p_3-h_2+1-d/2)_{m_2} \\
& \times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}}{(p_3)_{-m_1}} \frac{(-m_2)_{t_1}(-m_3)_{t_2}}{t_1!t_2!} \\
& \times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
& \times \frac{(v_{24}^7)^{h_4}(v_{34}^7)^{m_1-m_3+m'_{11}-m'_{12}-m'_{33}}}{s!} \frac{(1-\frac{1}{v_{34}^7})^s(1-v_{34}^7)^{n_{34}}}{n_{34}!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a \leq b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!}.
\end{aligned}$$

We then express the sum over s in terms of a hypergeometric function and use the first identity in (A.2) to rewrite the summation over s , leading to

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+m'_{33}}(p_3-h_2+h_4)_{m_2+m'_{11}+s+n_{34}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}+s+n_{34}}(-h_4)_{m_2+m'_{12}+m'_{22}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}+n_{34}+s}(p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}(-m_2)_{t_1}(-m_3)_{t_2}}{(p_3)_{-m_1} t_1! t_2!} \\
&\times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \\
&\times (-1)^s \frac{(v_{24}^7)^{h_4}(v_{34}^7)^{h_2-h_4-s-n_{34}}(1-v_{34}^7)^{s+n_{34}}}{s! n_{34}!} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a < b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!},
\end{aligned}$$

where we used $(1-1/v_{34}^7)^s = (-1)^s (v_{34}^7)^{-s} (1-v_{34}^7)^s$.

After defining $n_{34} = m - s$, we evaluate the sum over s to get

$$\begin{aligned}
G_{7|\mathcal{O}_{i_2} \rightarrow \mathbb{1}} &= (v_{24}^7)^{h_4} (v_{34}^7)^{h_2-h_4} \\
&\times \sum \frac{(-h_6)_{m_1+m'_{23}}(p_2+h_6)_{m_1+m'_{11}+m'_{13}+m'_{22}}(p_2+h_2)_{m_1-m_2+m_3+m'_{13}+m'_{23}}}{(p_2)_{2m_1+m'_{11}+m'_{13}+m'_{22}+m'_{23}}(p_2+1-d/2)_{m_1}} \\
&\times \frac{(p_3)_{-m_1+m_2+m_3+m'_{12}+m'_{33}}(p_3-h_2+h_4)_{m_2+m'_{11}}(\bar{p}_3+h_2+h_5)_{m_3+m'_{12}}}{(\bar{p}_3+h_2)_{2m_3+m'_{12}+m'_{13}+m'_{23}+m'_{33}}(\bar{p}_3+h_2+1-d/2)_{m_3}} \\
&\times \frac{(-h_2)_{m_1+m_2-m_3+m'_{11}+m'_{22}}(-h_4)_{m_2+m'_{12}+m'_{22}+m'_{33}}}{(p_3-h_2)_{2m_2+m'_{11}+m'_{12}+m'_{22}+m'_{33}}(p_3-h_2+1-d/2)_{m_2}} \\
&\times \frac{(-h_5)_{m_3+m'_{13}+m'_{23}+m'_{33}}(-\bar{p}_3-h_2+d/2-m_3)_{m_3}(-p_3+h_2+d/2-m_2)_{m_2}(-m_2)_{t_1}(-m_3)_{t_2}}{(p_3)_{-m_1} t_1! t_2!} \\
&\times \frac{(\bar{p}_3-d/2)_{t_1+t_2}}{(p_3-m_1)_{t_1+t_2}} \frac{(p_3)_{t_1+t_2}}{(\bar{p}_3+h_2+1-d/2)_{t_2}(p_3-h_2+1-d/2)_{t_1}} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m_a}}{m_a!} \prod_{1 \leq a < b \leq 3} \frac{(1-v_{ab}^6)^{m'_{ab}}}{m'_{ab}!} \\
&= (v_{24}^7)^{h_4} (v_{34}^7)^{h_2-h_4} G_6,
\end{aligned}$$

which complete our proof for the limit of unit operator $\mathcal{O}_2(\eta_2) \rightarrow \mathbb{1}$ (C.2).

The last limit of unit operator to verify is $\mathcal{O}_{i_3}(\eta_3) = \mathbb{1}$ which leads to

$$\left. I_{7(\Delta_{k_1}, \Delta_{k_2}, \Delta_{k_3}, \Delta_{k_4})}^{(\Delta_{i_2}, \Delta_{i_3}, \Delta_{i_4}, \Delta_{i_5}, \Delta_{i_6}, \Delta_{i_7}, \Delta_{i_1})} \right|_{\text{extended snowflake}} \xrightarrow{\mathcal{O}_{i_2}(\eta_2) \rightarrow \mathbb{1}} \left. I_{6(\Delta_{k_3}, \Delta_{k_1}, \Delta_{k_4})}^{(\Delta_{i_5}, \Delta_{i_6}, \Delta_{i_4}, \Delta_{i_2}, \Delta_{i_7}, \Delta_{i_1})} \right|_{\text{comb}}. \quad (\text{C.3})$$

Moreover, in this limit we have $\Delta_{i_3} = 0$, $\Delta_{k_2} = \Delta_{i_4}$ as well as

$$L_7 \prod_{1 \leq a \leq 4} (u_a^7)^{\frac{\Delta_{k_a}}{2}} = (v_{23}^6)^{h_3-h_6} (v_{12}^6)^{h_6} L_6 \prod_{1 \leq a \leq 3} (u_i^6)^{\frac{\Delta_{k'_i}}{2}},$$

$$u_1^6 = u_3^7, \quad u_2^6 = \frac{u_1^7 v_{14}^7}{v_{34}^7}, \quad u_3^6 = \frac{u_4^7}{v_{34}^7},$$

$$v_{11}^6 = v_{23}^7, \quad v_{12}^6 = \frac{1}{v_{34}^7}, \quad v_{13}^6 = \frac{v_{11}^7 v_{14}^7}{v_{34}^7},$$

$$v_{22}^6 = \frac{v_{44}^7}{v_{34}^7}, \quad v_{23}^6 = \frac{v_{14}^7}{v_{34}^7}, \quad v_{33}^6 = \frac{v_{13}^7 v_{14}^7}{v_{34}^7},$$

for the legs and conformal cross-ratios. Here, the primed parameters correspond to the parameters relevant for the RHS of (C.3). Considering that $h_4 = p_5 = p_2 + h_3 = 0$, the limit of unit operator (C.3) implies the identity $G_{7|\mathcal{O}_{i_3} \rightarrow \mathbf{1}} \equiv (v_{23}^6)^{h_3-h_6} (v_{12}^6)^{h_6} G_7 = G_6$.

From the vanishing components of \mathbf{h} and \mathbf{p} , we have

$$m_2 = m_{12} = m_{22} = m_{24} = m_{33} = 0.$$

As a result, we find that

$$F_7 = \frac{(-\bar{p}_3 - h_2 + d/2 - m_3)_{m_3}}{(p_3)_{-m_1}} \sum \frac{(-m_3)_{t_2} (\bar{p}_3 - d/2)_{t_2}}{t_2! (p_3 - m_1)_{t_2} (p_3)_{t_2} (\bar{p}_3 + h_2 + 1 - d/2)_{t_2}}$$

$$\times \frac{(-m_1)_r (-m_4)_r (-p_2 + d/2 - m_1)_r}{r! (1 - p_2 - h_3 - m_1)_r (p_4 - m_1)_r}$$

$$= \frac{(-h_2 - m'_1)_{m'_1} (p_3)_{m'_1}}{(p_3)_{m'_1 - m'_2}} F_{6|\text{comb}}.$$

Thus, $G_{7|\mathcal{O}_{i_3} \rightarrow \mathbf{1}}$ becomes

$$G_{7|\mathcal{O}_{i_3} \rightarrow \mathbf{1}} = \sum \frac{(p_4 - m'_2)_{m'_3 + m_{14}} (-h_6)_{m'_3 + m_{14} + m_{34} + m_{44}} (p_4 - h_3 + h_6)_{m'_3 + m_{11} + m_{13}}}{(p_2 + h_3 + m'_2)_{-m'_3} (p_4 - h_3)_{2m'_3 + m_{11} + m_{13} + m_{14} + m_{34} + m_{44}} (p_4 - h_3 + 1 - d/2)_{m'_3}}$$

$$\times \frac{(p_2 + h_3)_{m'_2 - m'_3 + m'_{11}} (p_3)_{m'_1} (p_3 - h_2)_{m_{11} + m_{34}} (\bar{p}_3 + h_2 + h_5)_{m'_1}}{(\bar{p}_3 + h_2)_{2m'_1 + m_{13} + m'_{11} + m_{44}} (\bar{p}_3 + h_2 + 1 - d/2)_{m'_1}}$$

$$\times \frac{(p_2 + h_2)_{m'_1 + m'_2 + m_{13} + m'_{11} + m_{44}} (-h_2 - m'_1)_{m'_1} (-h_2)_{m'_1 - m'_2 + m_{11} + m_{34}}}{(p_2)_{2m'_2 + m_{11} + m_{13} + m'_{11} + m_{34} + m_{44}} (p_2 + 1 - d/2)_{m'_2}}$$

$$\times \frac{(-h_3)_{m'_2 + m'_3 + m_{11} + m_{13} + m_{34} + m_{44}} (-h_5)_{m'_2 + m_{13} + m'_{11} + m_{44}}}{(p_3 - h_2)_{m_{11} + m_{34}}}$$

$$\times \binom{m_{11}}{k_{11}} \binom{m_{13}}{k_{13}} \binom{m_{14}}{k_{14}} \binom{m_{34}}{k_{34}} \binom{m_{44}}{k_{44}} \binom{h_6 - m'_3 - k_{14} - k_{34} - k_{44}}{m'_{12}} \binom{k_{11}}{m'_{13}} \binom{k_{44}}{m'_{22}}$$

$$\times \binom{h_3 - h_6 - m'_2 - k_{11} - k_{13} + k_{14}}{m'_{23}} \binom{k_{13}}{m'_{33}} (-1)^{k_{11} + k_{13} + k_{14} + k_{34} + k_{44} + m'_{12} + m'_{13} + m'_{22} + m'_{23} + m'_{33}}$$

$$\times \frac{(1 - v_{13}^6)_{m'_{13}} (1 - v_{33}^6)_{m'_{33}} (1 - v_{11}^6)_{m'_{11}} (1 - v_{22}^6)_{m'_{22}} (1 - v_{23}^6)_{m'_{23}} (1 - v_{12}^6)_{m'_{12}}}{m_{11}! m_{13}! m'_{11}! m_{44}! m_{14}! m_{34}!}$$

$$\times F_{6|\text{comb}} \prod_{1 \leq a \leq 3} \frac{(u_a^6)^{m'_a}}{m'_a!}.$$

At this point, the re-summations mirror the ones of the previous limit of unit operator. As such, they are left for the interested reader.

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