DERIVATIVES, EULERIAN POLYNOMIALS AND THE g-INDEXES OF YOUNG TABLEAUX

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ABSTRACT. In this paper we first present summation formulas for k-order Eulerian polynomials and 1/k-Eulerian polynomials. We then present combinatorial expansions of $(c(x)D)^n$ in terms of inversion sequences as well as k-Young tableaux, where c(x) is a differentiable function in the indeterminate x and D is the derivative with respect to x. We define the g-indexes of k-Young tableaux and Young tableaux, which have important applications in combinatorics. By establishing some relations between k-Young tableaux and standard Young tableaux, we express Eulerian polynomials, second-order Eulerian polynomials, André polynomials and the generating polynomials of gamma coefficients of Eulerian polynomials in terms of standard Young tableaux, which imply a deep connection among these polynomials.

1. INTRODUCTION

Let \mathfrak{S}_n be the symmetric group on the set $[n] = \{1, 2, ..., n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* of π is an index $i \in [n]$ such that $\pi(i) > \pi(i+1)$ or i = n. Let des (π) be the number of descents of π . The number $\langle i \rangle = \{\pi \in \mathfrak{S}_n : \operatorname{des}(\pi) = i\}$ is called the Eulerian number, and the polynomial

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)}$$

is called the *Eulerian polynomial*. The historical origin of Eulerian polynomial is the following summation formula (see [29]):

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$
(1)

In the past decades, there has been much work on Eulerian polynomial and its generalizations (see [18, 21, 30, 37] for instance). For example, by using a kind of first-order differential equation, Rządkowski and Urlińska [30] considered a unified generalization of Eulerian polynomials and second-order Eulerian polynomials. In the following we first recall the definitions of k-order Eulerian polynomials and 1/k-Eulerian polynomials, and then we present summation formulas for these polynomials.

A k-Stirling permutation of order n is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ such that for each $i, 1 \leq i \leq n$, all entries between any two occurrences of i are at least i. When k = 2, the k-Stirling permutation is reduced to the ordinary Stirling permutation ([19]). We say that

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an index $i \in [kn]$ is a descent of σ if $\sigma_i > \sigma_{i+1}$ or i = kn. Let $\mathcal{Q}_n(k)$ be the set of k-Stirling permutations of order n. The k-order Eulerian polynomials are defined by

$$C_n(x;k) = \sum_{\sigma \in Q_n(k)} x^{\text{des}\,(\pi)}, \ C_0(x;k) = 1.$$

Following [14, Lemma 1], the polynomials $C_n(x;k)$ satisfy the recurrence relation

$$C_{n+1}(x;k) = (kn+1)xC_n(x;k) + x(1-x)C'_n(x;k),$$
(2)

In particular, $C_n(x;1) = A_n(x)$. When k = 2, the polynomial $C_n(x;k)$ is reduced to the second-order Eulerian polynomial $C_n(x)$, i.e., $C_n(x;2) = C_n(x)$. Stirling permutations and the second-order Eulerian polynomial were defined by Gessel and Stanley [19], and they proved that

$$\sum_{k=0}^{\infty} {\binom{k+n}{k}} x^k = \frac{C_n(x)}{(1-x)^{2n+1}}$$

where $\binom{n}{k}$ is the Stirling number of the second kind, i.e., the number of ways to partition the set [n] into k non-empty subsets. The second-order Eulerian polynomials have been extensively studied in recent years, see [20, 21, 26] and references therein.

Let $s = \{s_i\}_{i\geq 1}$ be a sequence of positive integers. A geometric interpretation of Eulerian polynomials is obtained by considering the s-lecture hall polytope $\mathcal{P}_n^{(s)}$, which is defined by

$$\mathcal{P}_n^{(s)} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n \mid 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \dots \le \frac{\lambda_n}{s_n} \le 1 \right\}$$

Set $e_0 = 0$ and $s_0 = 1$. Let $I_n^{(s)} = \{ \mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n \mid 0 \le e_i < s_i \text{ for } 1 \le i \le n \}$ be the set of *n*-dimensional s-inversion sequences. The polynomial

$$E_n^{(s)}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(s)}} x^{\operatorname{asc}(\mathbf{e})}$$

is known as the s-Eulerian polynomial, where $\operatorname{asc}(\mathbf{e}) = \#\left\{i \in \{0, 1, 2, \dots, n-1\} | \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}\right\}$. In particular, we have

$$E_n^{(1,2,\dots,n)}(x) = A_n(x)/x.$$

Let k be a fixed positive integer. The 1/k-Eulerian polynomials $A_n^{(k)}(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} A_n^{(k)}(x) \frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)}-x}\right)^{\frac{1}{k}}$$

Savage and Viswanathan [31] showed that

$$A_n^{(k)}(x) = E_n^{(1,k+1,2k+1\dots,(n-1)k+1)}(x)$$

For $\pi \in \mathfrak{S}_n$, an excedance of π is an index $i \in [n]$ such that $\pi(i) > i$. Let $\exp(\pi)$ (resp. $\exp(\pi)$) be the number of excedances (resp. cycles) of π . It follows from [6, Proposition 7.3] that

$$A_n^{(k)}(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{exc}(\pi)} k^{n - \operatorname{cyc}(\pi)}.$$

Another combinatorial interpretation of $A_n^{(k)}(x)$ is given as follows:

$$A_n^{(k)}(x) = \sum_{\sigma \in \mathcal{Q}_n(k)} x^{\operatorname{ap}(\sigma)}$$

where ap (σ) is the number of the longest ascent plateaus of σ , i.e., the number of indexes $i \in \{2, 3, \ldots, nk - k + 1\}$ such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+k-1}$ (see [23, Theorem 2]). The polynomials $A_n^{(k)}(x)$ satisfy the recurrence relation

$$A_{n+1}^{(k)}(x) = (1 + knx)A_n^{(k)}(x) + kx(1 - x)\frac{d}{dx}A_n^{(k)}(x),$$
(3)

with the initial conditions $A_0^{(k)}(x) = A_1^{(k)}(x) = 1$ (see [23, Eq. (6)]). Set $M_n(x) = A_n^{(2)}(x)$. Let $N_n(x) = \sum x^{\operatorname{lap}(\sigma)}$

$$N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{iap}\,(\sigma)}$$

be the *left ascent plateau polynomial*, where lap (σ) is the number of the left ascent plateaux of σ , i.e., the number of indices $i \in \{1, 2, 3, ..., 2n - 1\}$ such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$ and $\sigma(0) = 0$ (see [23, Theorem 3]). From [25, p. 2], we see that $N_n(x) = x^n M_n(1/x) = x^n A_n^{(2)}(1/x)$. Let $B_n(x)$ be the type *B* Eulerian polynomial. According to [25, Proposition 1], we have

$$2^{n}A_{n}(x) = \sum_{i=0}^{n} \binom{n}{i} N_{i}(x)N_{n-i}(x), \ B_{n}(x) = \sum_{i=0}^{n} \binom{n}{i} N_{i}(x)M_{n-i}(x)$$

Let $F_n := F_n(x; \alpha, \beta, a, b, c)$ be the polynomials defined by the following relation:

$$\left(\frac{a+bx+cx^2}{(1-x)^{\alpha}}\frac{d}{dx}\right)^n \frac{1}{(1-x)^{\beta}} = \frac{F_n}{(1-x)^{n+n\alpha+\beta}}.$$

Then $F_0 = 1$ and it is routine to verify that the polynomials F_n satisfy the recurrence relation

$$F_{n+1} = (n + n\alpha + \beta)(a + bx + cx^2)F_n + (a + bx + cx^2)(1 - x)F'_n.$$
(4)

Comparing (2) and (3) with (4), it is routine to verify the first main result of this paper.

Theorem 1. Let k be a positive integer. For $n \ge 1$, we have

$$\left(\frac{x}{(1-x)^k}\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{C_n(x;k+1)}{(1-x)^{n+kn+1}}$$
$$\left(kx\frac{d}{dx}\right)^n \frac{1}{(1-x)^{1/k}} = \frac{x^n A_n^{(k)}(1/x)}{(1-x)^{n+\frac{1}{k}}}.$$

In particular, we have

$$\left(\frac{x}{1-x}\frac{d}{dx}\right)^{n}\frac{1}{1-x} = \frac{C_{n}(x)}{(1-x)^{2n+1}},$$

$$\left(2x\frac{d}{dx}\right)^{n}\frac{1}{\sqrt{1-x}} = \frac{N_{n}(x)}{(1-x)^{n}\sqrt{1-x}}.$$
(5)

Throughout this paper, we always let c := c(x) and f := f(x) be two differentiable functions in the indeterminate x, and let $D = \frac{d}{dx}$. Motivated by Theorem 1, we shall consider expansions of $(cD)^n f$. The paper is organized as follows. In the next section, we collect the definitions, notation and preliminary results. In Section 3, we express $(cD)^n f$ in terms of inversion sequences as well as k-Young tableaux. In particular, we define the g-indexes of k-Young tableau and Young tableau, which have important applications. Also, several main results including Theorems 14 and 17 are stated in that section. In Sections 4, 5 and 6, we respectively prove three main results, i.e., second-order Eulerian polynomials, Eulerian polynomials and André polynomials can be expressed in terms of standard Young tableaux.

2. Preliminary

The expansions of $(cD)^n f$ have been studied as early as 1823 by Scherk [32]. An illustration of the correspondence between Scherk's expansion of $(cD)^n f$ and forests of trees can be found in Appendix A of [2]. In particular, Scherk [32, p. 6] found that

$$(xD)^n = \sum_{k=0}^n {n \\ k} x^k D^k.$$
(6)

Many generalizations and variations of (6) frequently appeared in combinatorics and normal ordering problems (see [7, 15, 27] for instance).

It will be convenient in the sequel to adopt the convention that $\mathbf{f}_k = D^k f$ and $c_k = D^k c$. In particular, $\mathbf{f}_0 = f$ and $c_0 = c$. The first few $(cD)^n f$ are given as follows:

$$\begin{aligned} (cD)f &= (c)\mathbf{f}_{1}, \\ (cD)^{2}f &= (cc_{1})\mathbf{f}_{1} + (c^{2})\mathbf{f}_{2}, \\ (cD)^{3}f &= (cc_{1}^{2} + c^{2}c_{2})\mathbf{f}_{1} + (3c^{2}c_{1})\mathbf{f}_{2} + (c^{3})\mathbf{f}_{3}, \\ (cD)^{4}f &= (cc_{1}^{3} + 4c^{2}c_{1}c_{2} + c^{3}c_{3})\mathbf{f}_{1} + (7c^{2}c_{1}^{2} + 4c^{3}c_{2})\mathbf{f}_{2} + (6c^{3}c_{1})\mathbf{f}_{3} + (c^{4})\mathbf{f}_{4}, \\ (cD)^{5}f &= (cc_{1}^{4} + 11c^{2}c_{1}^{2}c_{2} + 4c^{3}c_{2}^{2} + 7c^{3}c_{1}c_{3} + c^{4}c_{4})\mathbf{f}_{1} + (15c^{2}c_{1}^{3} + 30c^{3}c_{1}c_{2} \\ &+ 5c^{4}c_{3})\mathbf{f}_{2} + (25c^{3}c_{1}^{2} + 10c^{4}c_{2})\mathbf{f}_{3} + (10c^{4}c_{1})\mathbf{f}_{4} + (c^{5})\mathbf{f}_{5}. \end{aligned}$$

TABLE 1. Expansions of $(cD)^n f$

For $n \ge 1$, we define

$$(cD)^n f = \sum_{k=1}^n A_{n,k} \mathbf{f}_k.$$
(7)

It is evident that $A_{n,k}$ is a function of c, c_1, \ldots, c_{n-k} . Thus we can write $A_{n,k}$ as follows:

$$A_{n,k} := A_{n,k}(c, c_1, c_2, \dots, c_{n-k}).$$

In particular, $A_{1,1} = c$, $A_{2,1} = cc_1$ and $A_{2,2} = c^2$. By induction, it is easy to verify that $A_{n+1,1} = cDA_{n,1}$, $A_{n,n} = c^n$ and for $2 \le k \le n$, we have

$$A_{n+1,k} = cA_{n,k-1} + cDA_{n,k}.$$
(8)

The numbers appearing in $A_{n,k}$ as coefficients can be found in [33, A139605]. We refer the reader to [4, 27, 28] for various results and examples on the expansions of $(cD)^n$.

In 1973, Comtet obtained the following result.

Proposition 2 ([12]). Let $A_{n,k}$ be defined by (7). For $1 \le k \le n$, we have

$$A_{n,k} = \frac{c}{k!} \sum (2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1})\frac{c_{k_1}}{k_1!}\cdots\frac{c_{k_{n-1}}}{k_{n-1}!},$$
(9)

where the summation is over all sequences $(k_1, k_2, \ldots, k_{n-1})$ of nonnegative integers such that $k_1 + k_2 + \cdots + k_{n-1} = n - k$ and $k_1 + \cdots + k_j \leq j$ for any $1 \leq j \leq n - 1$.

The explicit formula (9) provides a method for calculating $(cD)^n f$, see Table 1. However, to obtain the explicit coefficients in Table 1, a further step is needed. In order to state the other expansion formulas for $A_{n,k}$, we need to introduce several notations on partitions of integers.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a weakly decreasing sequence of nonnegative integers. Each λ_i is called a part of λ . The sum of the parts of a partition λ is denoted by $|\lambda|$. If $|\lambda| = n$, then we say that λ is a partition of n, also written as $\lambda \vdash n$. We denote by m_i the number of parts equal i. By using the multiplicities, we also denote λ by $(1^{m_1}2^{m_2}\cdots n^{m_n})$. The partition with all parts equal to 0 is the empty partition. The length of λ , denoted $\ell(\lambda)$, is the maximum subscript j such that $\lambda_j > 0$. The Ferrers diagram of λ is graphical representation of λ with λ_i boxes in its *i*th row and the boxes are left-justified. For a Ferrers diagram $\lambda \vdash n$ (we will often identify a partition with its Ferrers diagram), a (standard) Young tableau (SYT, for short) of shape λ is a filling of the n boxes of λ with the integers $1, 2, \ldots, n$ such that each number is used, and all rows and columns are increasing (from left to right, and from bottom to top, respectively). Given a Young tableau, we number its rows starting from the bottom and going above. Let SYT (n) be the set of standard Young tableaux of size n.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, we define

$$c_{\lambda} = \prod_{i=1}^{\ell} c_{\lambda_i}, \ c_{\emptyset} = 1.$$

Let $\begin{bmatrix} n \\ k \end{bmatrix} = \#\{\pi \in \mathfrak{S}_n : \operatorname{cyc}(\pi) = k\}$ be the Stirling numbers of the first kind. We now recall another expansion formula for $A_{n,k}$.

Proposition 3 ([1, 4]). Let $A_{n,k}$ be defined by (7). For $n \ge 1$, there exist positive integers $a(n,\lambda)$ such that

$$A_{n,k} = \sum_{\lambda \vdash n-k} a(n,\lambda) c^{n-\ell(\lambda)} c_{\lambda}, \qquad (10)$$

where λ runs over all partitions of n - k. In particular, we have

$$\sum_{\lambda \vdash n-k} a(n,\lambda) = \begin{bmatrix} n \\ k \end{bmatrix}, \ a(n,1^{n-k}) = \begin{Bmatrix} n \\ k \end{Bmatrix}, \ \sum_{\ell(\lambda) = n-k} a(n,\lambda) = \begin{Bmatrix} n \\ k \end{Bmatrix}$$

Motivated by Proposition 3, in the next section we present the other main results of this paper. More importantly, we define the g-indexes of k-Young tableau and Young tableau.

3. Inversion sequences and the g-index of Young tableau

3.1. Derivatives and inversion sequences.

An integer sequence $\mathbf{e} = (e_1, e_2, \dots, e_n)$ is an *inversion sequence* of length n if $0 \le e_i < i$ for all $1 \le i \le n$. Let \mathbf{I}_n be the set of inversion sequences of length n. There is a natural bijection ψ between \mathbf{I}_n and \mathfrak{S}_n defined by $\psi(\pi) = \mathbf{e}$, where $e_i = \#\{j \mid 1 \le j < i \text{ and } \pi(j) > \pi(i)\}$. **Definition 4.** For $\mathbf{e} \in \mathbf{I}_n$, let $|\mathbf{e}|_i = \#\{i \mid e_i = j, 1 \le i \le n\}$. Then we define

 $\phi(\mathbf{e}) = c \cdot c_{|\mathbf{e}|_1} c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot \mathbf{f}_{|\mathbf{e}|_0}.$

For example, take n = 9 and $\mathbf{e} = (0, 0, 1, 0, 4, 2, 4, 0, 1)$, then $|\mathbf{e}|_0 = 4$, $|\mathbf{e}|_1 = 2$, $|\mathbf{e}|_2 = 1$, $|\mathbf{e}|_3 = 0$, $|\mathbf{e}|_4 = 2$ and $|\mathbf{e}|_j = 0$ for $5 \le j \le 8$. So that $\phi(\mathbf{e}) = c \cdot c_2 c_1 c c_2 c c c \cdot f_4 = c^6 c_1 c_2^2 \cdot \mathbf{f}_4$.

We now present the second main result of this paper.

Theorem 5. For $n \ge 1$, we have

$$(cD)^n f = \sum_{\mathbf{e} \in \mathbf{I}_n} \phi(\mathbf{e}).$$
(11)

Proof. When n = 1, we have $I_1 = \{0\}$ and $\phi(0) = c\mathbf{f}_1$. When n = 2, we have $I_2 = \{00, 01\}$. Note that $\phi(00) = c \cdot c \cdot f_2$ and $\phi(01) = c \cdot c_1 \cdot \mathbf{f}_1$. Hence (11) is valid for n = 1, 2. Assume that (11) holds for n. Let $I_{n,k} = \{\mathbf{e} \in I_n : |\mathbf{e}|_0 = k\}$. Then for any $\mathbf{e} \in I_{n,k}$, we have

$$\phi(\mathbf{e}) = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot \mathbf{f}_k$$

Let \mathbf{e}' be obtained from $\mathbf{e} = (e_1, e_2, \dots, e_n)$ by appending e_{n+1} . We distinguish three cases:

- (*i*) If $e_{n+1} = 0$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c \cdot \mathbf{f}_{k+1}$;
- (*ii*) If $e_{n+1} = i$ and $1 \le i \le n-1$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_i+1} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c \cdot \mathbf{f}_k$;
- (*iii*) If $e_{n+1} = n$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c_1 \cdot \mathbf{f}_k$.

It is routine to check that the first case accounts for the term $cA_{n,k-1}$ and the last two cases account for the term $cDA_{n,k}$. Then $\sum_{\mathbf{e}\in I_{n+1,k}}\phi(\mathbf{e}) = (cA_{n,k-1} + cDA_{n,k})\mathbf{f}_k = A_{n+1,k}\mathbf{f}_k$, which follows from (8). This completes the proof.

Example 6. When n = 3, the correspondence between $\mathbf{e} \in I_3$ and $\phi(\mathbf{e})$ is illustrated as follows:

So that

$$\sum_{\mathbf{e}\in\mathbf{I}_3}\phi(\mathbf{e}) = (cc_1^2 + c^2c_2)\mathbf{f}_1 + (3c^2c_1)\mathbf{f}_2 + c^3\mathbf{f}_3.$$

Example 7. When c = x, we have $c_0 = x$, $c_1 = Dx = 1$ and $c_i = 0$ for $i \ge 2$. Then $\phi(\mathbf{e}) \neq 0$ unless $|\mathbf{e}|_i = 0$ or $|\mathbf{e}|_i = 1$ for all $i \ge 1$. In this case, let $k = |\mathbf{e}|_0$, then

 $n-k = \#\{j : |\mathbf{e}|_j = 1, 1 \le j \le n-1\}, \ (n-1) - (n-k) = \#\{j : |\mathbf{e}|_j = 0, 1 \le j \le n-1\}.$

Thus $\phi(\mathbf{e}) = cc_{|\mathbf{e}|_1} \cdots c_{|\mathbf{e}|_{n-1}} \mathbf{f}_{|\mathbf{e}|_0} = x^k \mathbf{f}_k$. It follows from (6) that $(xD)^n f = \sum_{k=0}^n {n \choose k} x^k \mathbf{f}_k$. Hence

$$\binom{n}{k} = \#\{\mathbf{e} \in \mathbf{I}_n : |\mathbf{e}|_0 = k, |\mathbf{e}|_j = 0 \text{ or } 1 \text{ for any } 1 \le j \le n-1\}.$$

We can derive Comtet's formula (9) by using Theorem 5. For $\mathbf{e} \in \mathbf{I}_n$, let $k = |\mathbf{e}|_0$ and $k_i = |\mathbf{e}|_{n-i}$ for $1 \le i \le n-1$. Note that

$$k_1 + k_2 + \dots + k_{n-1} = n - k$$

and $k_1 + \cdots + k_j \leq j$ for each j. Therefore, the number of such **e** is equal to

$$\binom{1}{k_1}\binom{2-k_1}{k_2}\binom{3-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{n-1}}{k} = \frac{(2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1})}{k!k_1!k_2!\cdots k_{n-1}!}.$$

3.2. Derivatives and k-Young tableaux.

Since the $c_{k_1}, c_{k_2}, \ldots, c_{k_{n-1}}$ are commutative, we have to group the terms in (9) which produce the same product $c_{k_1}c_{k_2}\cdots c_{k_{n-1}}$. We say that the *type* of *n* is a pair (k,μ) , denoted by $(k,\mu) \vdash n$, where $k \in [n]$ and $\mu = (\mu_1, \ldots, \mu_{n-1})$ is a partition of n - k, i.e., μ is written up to n - 1 terms by appending 0's at the end. Let (k,μ) be a type of *n*. We define

Set
$$(\mu) = \{\mu_j \mid 1 \le j \le n-1\}, \ |\mu|_j = \#\{i \mid \mu_i = j, 1 \le i \le n-1\}.$$

Let $(|\mathbf{e}|_0, \mu(\mathbf{e}))$ be the *type* of $\mathbf{e} \in I_n$, where $\mu(\mathbf{e})$ is the decreasing order of $|\mathbf{e}|_1, \ldots, |\mathbf{e}|_{n-1}$. For each type (k, μ) of n, let $p_{k,\mu}$ be the number of inversion sequences of type (k, μ) . It follows from Theorem 5 that

$$(cD)^{n}f = \sum_{(k,\mu)\vdash n} p_{k,\mu}cc_{\mu_{1}}c_{\mu_{2}}\cdots c_{\mu_{n-1}}\mathbf{f}_{k},$$
 (12)

where the summation is taken over all types (k, μ) of n.

Example 8. For $1 \le n = k + |\mu| \le 3$, the numbers $p_{k,\mu}$ are $p_{1,(0)} = 1$, $p_{2,(0)} = 1$, $p_{1,(1)} = 1$, $p_{3,(0,0)} = 1$, $p_{2,(1,0)} = 3$, $p_{1,(2,0)} = 1$ and $p_{1,(1,1)} = 1$.

Lemma 9. By convention, set $p_{0,\mu} = 0$. If $(k,\mu) = (1, (1, 1, ..., 1))$, then let $p_{k,\mu} = 1$. For other type (k,μ) of n, we have

$$p_{k,\mu} = \sum_{j \in \text{Set}\,(\mu) \setminus \{0\}} (|\mu|_{j-1} + 1) p_{k,\mu^{(j)}} + p_{k-1,\mu^{(0)}},\tag{13}$$

where $\mu^{(j)}$ is obtained from μ by replacing the last occurrences of the part j by j-1 and by deleting the last 0 and $\mu^{(0)}$ is obtained from μ by deleting the last 0. Thus $(k, \mu^{(j)}) \vdash (n-1)$ and $(k-1, \mu^{(0)}) \vdash (n-1)$.

Proof. Take an inversion sequence $\mathbf{e} \in \mathbf{I}_n$ of type (k, μ) . Let $\mathbf{e}' = (e_1, e_2, \dots, e_{n-1}) \in \mathbf{I}_{n-1}$ be obtained from \mathbf{e} by deleting the last e_n . If $e_n = 0$, then, the type of \mathbf{e}' is $(k - 1, \mu^{(0)})$. This operation is reversible. If $e_n = i$ $(1 \le i \le n-1)$ and $|\mathbf{e}|_i = j \in \text{Set}(\mu) \setminus \{0\}$, then the type of \mathbf{e}' is $(k, \mu^{(j)})$. In this case, the operation is not reversible. We have exactly $(|\mu|_{j-1} + 1)$ ways to do the inverses. In fact we can append $e_n = i' \ne i$ at the end of \mathbf{e}' with the condition of $|\mathbf{e}|_i - 1 = j - 1 = |\mathbf{e}|_{i'}$ to obtain an inversion sequence in \mathbf{I}_n of type (k, μ) .

As an illustration of (13), in order to get inversion sequences of type $(k, \mu) = (3, (2, 1, 1, 0, 0, 0))$, we distinguish three cases:

- (i) For each $\mathbf{e} \in I_6$ that counted by $p_{2,(2,1,1,0,0)}$, we can get exactly one inversion sequence of type (k, μ) by appending $e_7 = 0$ at the end of \mathbf{e} ;
- (ii) Let $\mathbf{e} \in \mathbf{I}_6$ be an inversion sequence counted by $p_{3,(1,1,1,0,0)}$. If $|\mathbf{e}|_i = 1$ then we can append $e_7 = i$ at the end of \mathbf{e} . As we have three choices for i, we get the term $3p_{3,(1,1,1,0,0)}$;

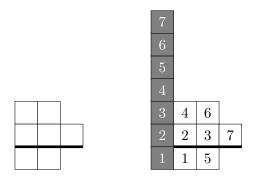


FIGURE 1. $(k = 2, \mu = (3, 2, 0, 0, 0, 0))$ -diagram and k-Young tableau of shape (k, μ)

(iii) Let $\mathbf{e} \in \mathbf{I}_6$ be an inversion sequence counted by $p_{3,(2,1,0,0,0)}$. If $|\mathbf{e}|_i = 0$ or i = 6 then we can append $e_7 = i$ at the end of \mathbf{e} . As we have four choices for i, we get the term $4p_{3,(2,1,0,0,0)}$.

Repeatedly, it is routine to verify that

$$p_{3,(2,1,1,0,0,0)} = 4p_{3,(2,1,0,0,0)} + 3p_{3,(1,1,1,0,0)} + p_{2,(2,1,1,0,0)} = 4 \times 120 + 3 \times 90 + 146 = 896.$$

Each type (k, μ) of n can be represented by a picture which contains k boxes in the bottom row, and the Young diagram of the partition μ in the top. Such picture is called a (k, μ) -diagram. See Figure 1 (left diagram).

Definition 10. Let (k, μ) be a type of n. A k-Young tableau Z of shape (k, μ) is a filling of the n boxes of the (k, μ) -diagram by the integers 1, 2, ..., n such that (i) each number is used, (ii) all rows and columns in the top Young diagram are increasing (from left to right, and from bottom to top, respectively), (iii) the bottom row becomes an increasing sequence of lenght k, starting with 1.

The filling of the top Young diagram of the partition μ is called the *top Young tableau* of the k-Young tableau. Unlike the ordinary Young tableau, there is no condition between the bottom row and the top Young tableau. We always put a special column of n boxes at the left of k-Young tableaux, and labelled by the integers $1, 2, \ldots, n$ from bottom to top. See Figure 1 (right diagram) for an example.

Definition 11. Let Z be a k-Young tableau of shape (k, μ) , where $k + |\mu| = n$. For each $v \in [n]$, suppose that v is in the box (i, j) of the top Young diagram, we define the g-index of v, denoted by $g_Z(v)$, to be the number of boxes (i - 1, j') such that $j' \ge j$ and the letter in this box is less than or equal to v (see Figure 3, right diagram). If v is in the bottom row, then we define $g_Z(v) = 1$. The g-index of Z is given by $G_Z = g_Z(1)g_Z(2)\cdots g_Z(n)$.

For the k-Young tableau given in Figure 1 (right diagram), we have

$$g_Z(1) = 1, \ g_Z(2) = 1, \ g_Z(3) = 1, \ g_Z(4) = 2, \ g_Z(5) = 1, \ g_Z(6) = 1, \ g_Z(7) = 2.$$

The third main result of this paper is given as follows.

Theorem 12. If $(k, \mu) \vdash n$, then we have

$$p_{k,\mu} = \sum_{Z} G_Z \tag{14}$$

where the summation is taken over all k-Young tableaux of shape (k, μ) .

Proof. Identity (14) is obtained from Lemma 9 by induction on n. The maximum letter n in the k-Young tableaux Z can be at the end of the bottom row, or a corner in the top Young tableau of Z. In the first case, $g_Z(n) = 1$, and removing the letter n yields a (k - 1)-Young tableau of shape $(k - 1, \mu)$. In the second case, $g_Z(n) = |\mu|_{j-1} + 1$, and removing the letter n yields a k-Young tableaux of shape $(k, \mu^{(j)})$, where j is the length of the row contained n. We recover all terms in (13).

The Stirling numbers of the first kind ${n\brack k}$ can be defined as follows:

$$\sum_{k=1}^{n} {n \brack k} x^{k} = x(x+1)\cdots(x+n-1).$$

According to [2, Proposition A. 2], we have $(e^x D)^n f = e^{nx} \sum_{k=1}^n {n \choose k} \mathbf{f}_k$. We replace $c = e^x$ and $c_j = e^x$ in (12). By Theorem 12, we obtain

$$(e^{x}D)^{n}f = e^{nx}\sum_{(k,\mu)\vdash n}\sum_{Z}G_{Z}\mathbf{f}_{k},$$

Hence

$$\sum_{(k,\mu)\vdash n} \sum_{Z} G_Z x^k = x(x+1)(x+2)\cdots(x+n-1),$$
(15)

where the first summation is taken over all type (k, μ) of n, and the second summation is taken over all k-Young tableaux of shape (k, μ) . For example, when n = 4, the k-Young tableaux with their *q*-indexes are listed in Figure 2.

As an application of Theorem 12, we give the following result.

Proposition 13. Let $\binom{n}{k}$ be the Stirling numbers of the second kind. Then we have

$$\binom{n}{k} = \sum_{Z} G_{Z},$$

where the summation is taken over all k-Young tableaux of shape $(k, (1^{n-k}0^{k-1}))$.

Proof. Let c = x and f = 1/(1-x). Then $c_1 = 1$ and $c_j = 0$ for $j \ge 2$, and $\mathbf{f}_k = k!/(1-x)^{k+1}$. It follows from (12) that

$$(xD)^{n} \frac{1}{1-x} = \sum_{(k,\mu)\vdash n} p_{k,\mu} cc_{\mu_{1}} c_{\mu_{2}} \cdots c_{\mu_{n-1}} \mathbf{f}_{k}$$
$$= \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot \frac{k! x^{k}}{(1-x)^{k+1}}$$
$$= \frac{1}{(1-x)^{n+1}} \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot k! x^{k} (1-x)^{n-k}.$$

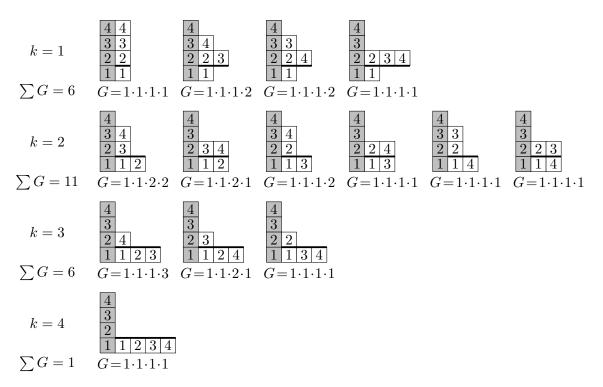


FIGURE 2. All k-Young tableaux of size 4 and their g-indexes

By Theorem 12, we have

$$A_n(x) = \sum_{k=0}^n p_{k,(1^{n-k}0^{k-1})} \cdot k! x^k (1-x)^{n-k}$$
$$= \sum_{k=0}^n \sum_Z G_Z \cdot k! x^k (1-x)^{n-k},$$
(16)

where the second summation is taken over all k-Young tableaux of shape $(k, (1^{n-k}0^{k-1}))$. Recall that the Frobenius formula for Eulerian polynomials is given as follows (see [10] for instance):

$$A_n(x) = \sum_{k=0}^n k! {n \\ k} x^k (1-x)^{n-k}$$

By comparing with (16), we get the desired result.

3.3. The *g*-index of Young tableaux.

Let T be a standard Young tableau of shape λ . We always put a special column of n boxes at the left of T, and labelled by $1, 2, 3, \ldots, n$ from bottom to top. For each $v \in [n]$, suppose that v is in the box (i, j), we define the *g*-index of v, denoted by $g_T(v)$, to be the number of boxes (i-1, j') such that $j' \geq j$ and the letter in this box is less than or equal to v (see Figure 3, right diagram). The *g*-index of T is defined by

$$G_T = g_T(1)g_T(2)\cdots g_T(n).$$

For the Young tableau given in Figure 3 (left diagram), we have

$$g_T(1) = 1, \ g_T(2) = 1, \ g_T(3) = 2, \ g_T(4) = 1, \ g_T(5) = 1, \ g_T(6) = 4, \ g_T(7) = 1.$$

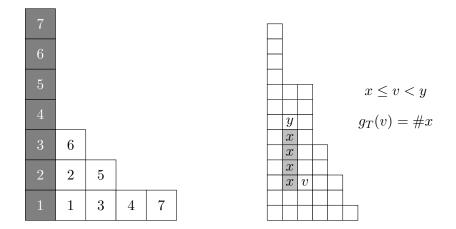


FIGURE 3. Young tableaux and *g*-index

Let $\lambda(T)$ be the corresponding partition of the Young tableau T. If $\lambda(T) = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, then let $\lambda(T)! = \lambda_1! \lambda_2! \cdots \lambda_\ell!$.

We now present the fourth main result of this paper.

Theorem 14. Let $C_n(x)$ be the second-order Eulerian polynomials. Then we have

$$C_n(x) = \sum_{T \in \text{SYT}(n)} G_T \ \lambda(T)! \ x^{n+1-\ell(\lambda(T))}.$$
(17)

Take x = 1 in (17), we obtain the following corollary.

Corollary 15. We have

$$(2n-1)!! = \sum_{T \in \text{SYT}(n)} G_T \ \lambda(T)!.$$

Example 16. For n = 4, the 10 standard Young tableaux and their g-indexes are listed in Figure 4. We verify that $C_4(x) = 24x^4 + 58x^3 + 22x^2 + x$.

We now present the fifth main result of this paper.

Theorem 17. Let $A_n(x)$ be the Eulerian polynomials. Then we have

$$A_n(x) = \sum_{T \in \text{SYT}(n)} G_T \ x^{n+1-\ell(\lambda(T))}.$$
(18)

So the following corollary is immediate.

Corollary 18. We have

$$n! = \sum_{T \in \text{SYT}(n)} G_T.$$

Example 19. For n = 4, the 10 standard Young tableaux and their g-indexes are listed in Figure 4. We verify that $A_4(x) = x^4 + 11x^3 + 11x^2 + x$.

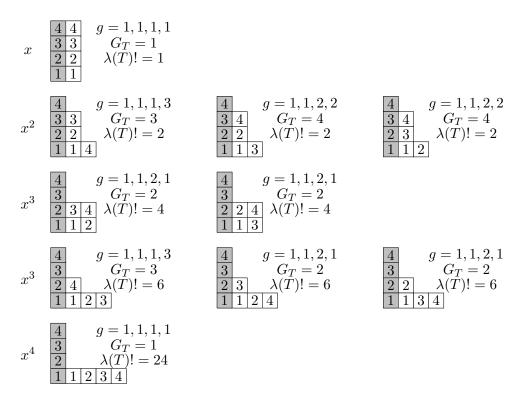


FIGURE 4. SYT (4) and *g*-indexes

Let $\pi \in \mathfrak{S}_n$. We say that π has no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. The permutation π is called *simsun* if for each $k \in [n]$, the subword of π restricted to [k] (in the order they appear in π) contains no double descents. Simsun permutations are useful in describing the action of the symmetric group on the maximal chains of the partition lattice (see [35, 36]). There has been much recent work devoted to simsun permutation and its variations, see [11, 24] and references therein.

Denote by \mathcal{RS}_n the set of simsun permutations in \mathfrak{S}_n . Let

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\operatorname{des}(\pi)} = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} S(n,i) x^i$$

be the descent polynomial of simsum permutations. It follows from [11, Theorem 1] that the polynomials $S_n(x)$ satisfy the recurrence relation

$$S_n(x) = (n+1)xS_{n-1}(x) + x(1-2x)S'_{n-1}(x)$$

for $n \ge 2$, with the initial conditions $S_1(x) = x$, $S_2(x) = x + x^2$ and $S_3(x) = x + 4x^2$.

An increasing tree on [n] is a rooted tree with vertex set $\{0, 1, 2, ..., n\}$ in which the labels of the vertices are increasing along any path from the root 0. The *degree* of a vertex in a rooted tree is the number of its children. A 0-1-2 increasing tree is an increasing tree in which the degree of any vertex is at most two. It should be noted that the number S(n, i) counts 0-1-2 increasing trees on [n] with *i* leaves (see [33, A094503]), and the polynomial $S_n(x)$ is also known as the André polynomial (see [9, 17]). Now we present the sixth main result of this paper.

Theorem 20. Let $S_n(x)$ be the André polynomials. For $n \ge 1$, we have

$$S_n(x) = \sum_T G_T \ x^{n+1-\ell(\lambda(T))},$$
(19)

where the summation is taken over all Young tableaux in SYT(n) with at most two columns.

We say that $\pi \in \mathfrak{S}_n$ is alternating if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. In other words, $\pi(i) < \pi(i+1)$ if *i* is even and $\pi(i) > \pi(i+1)$ if *i* is odd. The Euler number E_n is the number of alternating permutations in \mathfrak{S}_n (see [34]). A remarkable property of simsun permutations is that $\#\mathcal{RS}_n = E_{n+1}$ (see [35, p. 267]). So we get the following corollary.

Corollary 21. Let E_n be the nth Euler number. Then we have

$$E_{n+1} = \sum_T G_T,$$

where the summation is taken over all Young tableaux in SYT(n) with at most two columns.

An index $i \in [n]$ is a peak (resp. exterior double descent) of π if $\pi(i-1) < \pi(i) > \pi(i+1)$ (resp. $\pi(i-1) > \pi(i) > \pi(i+1)$), where $\pi(0) = \pi(n+1) = 0$. Let a(n,i) be the number of permutations in \mathfrak{S}_n with *i* peaks and without exterior double descents. The following gamma expansion of Eulerian polynomials was first given by Foata and Schützenberger [16]:

$$A_n(x) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} a(n,i) x^i (1+x)^{n+1-2i},$$

which implies that Eulerian polynomials are symmetric and unimodal. In recent years there has been much interest in studying gamma expansions of combinatorial polynomials, see [22, 26] and the references therein. Combining [3, Corollary 3.2] and [24, Proposition 1], we get another gamma expansion of Eulerian polynomials:

$$A_{n+1}(x) = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} 2^{i-1} S(n,i) x^i (1+x)^{n+2-2i}.$$

Let T be a Young tableau in SYT (n) with at most two columns. If $\lambda(T) = (1^{n-2i+2}2^{i-1})$, then $n + 1 - \ell(\lambda(T)) = i$, where $1 \le i \le \lfloor (n+2)/2 \rfloor$. Then by using (19), we immediately get the following result.

Theorem 22. Let S(n,i) be the number of 0-1-2 increasing trees on [n] with i leaves. Then

$$\sum_{i=1}^{\lfloor (n+2)/2 \rfloor} 2^{i-1} S(n,i) x^i = \sum_T G_T \lambda(T)! \ x^{n+1-\ell(\lambda(T))},$$

where the summation is taken over all Young tableaux in SYT(n) with at most two columns.

4. Proof of Theorem 14

Setting c = x/(1-x) and f = 1/(1-x), then we have

$$c_j = \frac{j!}{(1-x)^{j+1}}$$
 $(j \ge 1);$ $\mathbf{f}_k = \frac{k!}{(1-x)^{k+1}}$ $(k \ge 0).$

By using (12), we obtain

$$\left(\frac{x}{1-x}D\right)^{n} \frac{1}{1-x} = \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot cc_{\mu_{1}}c_{\mu_{2}} \cdots c_{\mu_{n-1}}\mathbf{f}_{k}$$
$$= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot \frac{x^{|\mu|_{0}+1}}{1-x} \frac{\mu_{1}!}{(1-x)^{\mu_{1}+1}} \cdots \frac{\mu_{n-1}!}{(1-x)^{\mu_{n-1}+1}} \frac{k!}{(1-x)^{k+1}}$$
$$= \frac{1}{(1-x)^{2n+1}} \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k! \mu_{1}! \cdots \mu_{n-1}! x^{|\mu|_{0}+1},$$

where the summation is taken over all types (k, μ) of n. Combining (5) and Theorem 12, we have

$$C_{n}(x) = \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k! \mu_{1}! \cdots \mu_{n-1}! x^{|\mu|_{0}+1}$$
$$= \sum_{(k,\mu)\vdash n} \sum_{Z} G_{Z} \cdot k! \mu_{1}! \cdots \mu_{n-1}! x^{|\mu|_{0}+1}.$$
(20)

In view of (17) and (20), we need to establish some relations between k-Young tableaux and standard Young tableaux. Let Z be a k-Young tableau of shape (k, μ) . We define $T = \rho(Z)$ to be the unique standard Young tableau such that the sets of the letters in the j-th column in Z and T are the same for all j. Let us list some basic facts of this map $Z \mapsto T = \rho(Z)$:

- (i) We can obtain T from Z by ordering the letters in each column in increasing order. One can check that if T is obtained in this way, then T is a standard Young tableau;
- (*ii*) The partition $\lambda(T)$ is the decreasing ordering of the sequence $(k, \mu_1, \ldots, \mu_{n-1})$, removing the 0's at the end. Hence, $\lambda(T)! = k! \mu_1! \mu_2! \cdots \mu_{n-1}!$;
- (*iii*) We have $n \ell(\lambda(T)) = |\mu|_0$;
- (*iv*) In general $G_Z \neq G_T$.

For example, take the k-Young tableau given in Figure 1, we obtain the standard Young tableau given in Figure 5. However the map ρ is not bijective. Let

$$\rho^{-1}(T) = \{ (k, \mu, Z) \mid \rho(Z) = T \}.$$

By the above properties of ρ and (20), we have

$$C_{n}(x) = \sum_{T \in \text{SYT}(n)} \sum_{(k,\mu,Z) \in \rho^{-1}(T)} G_{Z} \cdot k! \mu_{1}! \cdots \mu_{n-1}! x^{|\mu|_{0}+1}$$
$$= \sum_{T \in \text{SYT}(n)} \lambda(T)! x^{n+1-\ell(\lambda(T))} \sum_{(k,\mu,Z) \in \rho^{-1}(T)} G_{Z}.$$
(21)

The following lemma is fundamental.

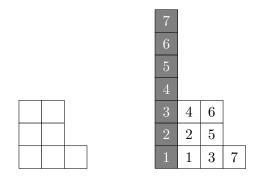


FIGURE 5. $T = \rho(Z)$ for Z given in Figure 1

Lemma 23. For each standard Young tableau T, we have

$$\sum_{Z \in \rho^{-1}(T)} G_Z = G_T,\tag{22}$$

where we write $Z \in \rho^{-1}(T)$ instead of $(k, \mu, Z) \in \rho^{-1}(T)$ since we can recover (k, μ) from Z.

Proof. We will proof (22) by induction on the size of T. Suppose that (22) is true for all standard Young tableau T of size n-1. Given a $T \in SYT(n)$. Let T' is a standard Young tableau of size n-1 obtained from T by removing the letter n. This operation is reversible if $\lambda(T)$ is known. By the hypothesis of induction, we have

$$\sum_{Z'\in\rho^{-1}(T')}G_{Z'}=G_{T'},$$
(23)

It should be noted that

 $G_T = G_{T'} \times g_T(n).$

On the other hand, for a k-Young tableau $Z \in \rho^{-1}(T)$ of size n, if we remove the letter n, we obtain a k'-Young tableau $Z' \in \rho^{-1}(T')$ of sie n-1. However, unlike Young tableau, this operation is not always reversible. Let us analyse in detail. Let β be the length of the row containing the letter n in k-Young tableau $Z \in \rho^{-1}(T)$ with shape (k, μ) if n is in the top Young tableau of Z. The set $\rho^{-1}(T)$ can be divided into four subsets: $\rho^{-1}(T) = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are respectively defined as follows:

$$\Gamma_1 = \{ Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k = \beta - 1 \},$$

$$\Gamma_2 = \{ Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k - 1 \in \mu \},$$

$$\Gamma_3 = \{ Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k \neq \beta - 1 \},$$

$$\Gamma_4 = \{ Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k - 1 \notin \mu \}.$$

See Figure 6 for two examples. It should be noted that some of the Γ_i may be empty according to T. We claim that the set Γ_1 and Γ_2 have the same carnality. Moreover, for each $Z_1 \in \Gamma_1$, there exists $Z_2 \in \Gamma_2$ in a unique manner, such that $Z'_1 = Z'_2 \in \rho^{-1}(T')$, See Figure 7. Moreover, we have the relations for the g-indexes (see Figure 7): $g_{Z_1}(n) = g_T(n) - 1$ and $g_{Z_2}(n) = 1$. For

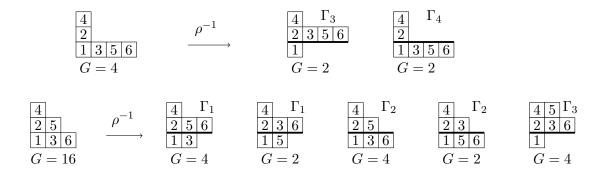


FIGURE 6. Decomposition of $\rho^{-1}(T)$ into $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$

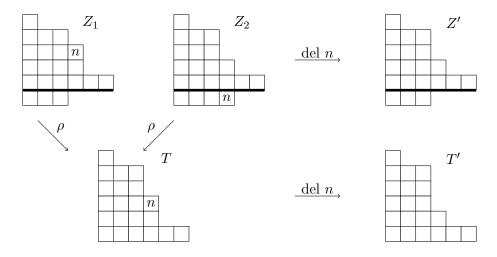


FIGURE 7. $Z_1, Z_2 \in \rho^{-1}(T)$ are mapped to the same $Z' \in \rho^{-1}(T')$ by removing the letter n

 $Z_3 \in \Gamma_3$ and $Z_4 \in \Gamma_4$ we have $g_{Z_3}(n) = g_T(n)$ and $g_{Z_4}(n) = g_T(n)$. By all these observations, we have

$$\begin{split} \sum_{Z \in \rho^{-1}(T)} G_Z &= \sum_{Z_1 \in \Gamma_1, Z_2 \in \Gamma_2} (G_{Z_1} + G_{Z_2}) + \sum_{Z_3 \in \Gamma_3} G_{Z_3} + \sum_{Z_4 \in \Gamma_4} G_{Z_4} \\ &= \sum_{Z_1 \in \Gamma_1, Z_2 \in \Gamma_2} (g_{Z_1}(n)G_{Z'} + g_{Z_2}(n)G_{Z'}) + \sum_{Z_3 \in \Gamma_3} g_T(n)G_{Z'_3} + \sum_{Z_4 \in \Gamma_4} g_T(n)G_{Z'_4} \\ &= g_T(n) \sum_{Z' \in \rho^{-1}(T')} G_{Z'} \\ &= g_T(n)G_{T'} \\ &= G_T. \end{split}$$

Hence (22) holds. This completes the proof.

Proof of Theorem 14. Combining (21) and Lemma 23, we get that

$$C_n(x) = \sum_{T \in \text{SYT}(n)} \lambda(T)! x^{n+1-\ell(\lambda(T))} \sum_{\substack{(k,\mu,Z) \in \rho^{-1}(T)}} G_Z$$
$$= \sum_{T \in \text{SYT}(n)} G_T \lambda(T)! x^{n+1-\ell(\lambda(T))},$$

as desired. This completes the proof.

5. Proof of Theorem 17

We shall prove Theorem 17 by using context-free grammars. For an alphabet V, let $\mathbb{Q}[[V]]$ be the ring of the rational commutative ring of formal power series in monomials formed from letters in V. Following Chen [8], a context-free grammar over V is a function $G: V \to \mathbb{Q}[[V]]$ that replaces each letter in V with an element of $\mathbb{Q}[[V]]$. The formal derivative D_G is a linear operator defined with respect to the grammar G. In other words, D_G is the unique derivation satisfying $D_G(x) = G(x)$ for $x \in V$. For example, if $V = \{x, y\}$ and $G = \{x \to xy, y \to y\}$, then $D_G(x) = xy, D_G^2(x) = D_G(xy) = xy^2 + xy$. For two formal functions u and v, we have $D_G(u + v) = D_G(u) + D_G(v)$ and $D_G(uv) = D_G(u)v + uD_G(v)$. For a constant c, we have $D_G(c) = 0$. It follows from Leibniz's rule that

$$D_{G}^{n}(uv) = \sum_{k=0}^{n} \binom{n}{k} D_{G}^{k}(u) D_{G}^{n-k}(v).$$

We refer the reader to [9, 26] for the recent progress on context-free grammars.

Setting $u_i = D_G^i(u)$, it follows from (12) and (14) that

$$(uD_G)^n = \sum_{(k,\mu)\vdash n} \sum_Z G_Z u u_{\mu_1} u_{\mu_2} \cdots u_{\mu_{n-1}} D_G^k,$$
(24)

where the first summation is taken over all types (k, μ) of n and the second summation is taken over all k-Young tableaux of shape (k, μ) . It is well-known that Eulerian polynomials are symmetric, i.e., $A_0(x) = 1$ and

$$A_n(x) = \sum_{i=1}^n {\binom{n}{i}} x^i = \sum_{i=1}^n {\binom{n}{i}} x^{n+1-i} \text{ for } n \ge 1.$$

There is a grammatical interpretation of Eulerian numbers due to Dumont [13], which can be restated as follows.

Proposition 24. If $G = \{x \rightarrow y, y \rightarrow y\}$, then we have

$$(xD_G)^n(y) = \sum_{i=1}^n \left\langle {n \atop i} \right\rangle x^{n+1-i} y^i \text{ for } n \ge 1.$$

Proof of Theorem 17. Let $G = \{x \to y, y \to y\}$. From (24), we have

$$(xD_G)^n(y) = \sum_{(k,\mu) \vdash n} \sum_Z G_Z x x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{n-1}} D_G^k(y),$$

where $x_0 = x$ and $x_i = D_G^i(x) = y$ for $i \ge 1$ and $D_G^k(y) = y$ for $k \ge 0$. Hence

$$(xD_G)^n(y) = \sum_{(k,\mu)\vdash n} \sum_Z G_Z y^{n-|\mu|_0} x^{|\mu|_0+1}$$

Comparing this with Proposition 24, we get

$$A_n(x) = \sum_{i=1}^n {\binom{n}{i}} x^{n+1-i} = (xD_G)^n(y)|_{y=1} = \sum_{(k,\mu)\vdash n} \sum_Z G_Z x^{|\mu|_0+1},$$
(25)

where the first summation is taken over all types (k, μ) of n and the second summation is taken over all k-Young tableaux of shape (k, μ) . In the same way as the proof of Theorem 14, by using Lemma 23, we get (18).

6. Proof of Theorem 20

We now recall a grammatical interpretation of $S_n(x)$.

Proposition 25 ([9, 13]). Let $G_1 = \{x \to xy, y \to x\}$. For $n \ge 1$, we have

$$D_{G_1}^n(x) = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} S(n,i) x^i y^{n+2-2i}.$$

Thus $S_n(x) = D_{G_1}^n(x)|_{y=1}$.

It is routine to verify that Proposition 25 can be restated as follows.

Proposition 26. Let $G_2 = \{x \rightarrow y, y \rightarrow 1\}$. For $n \ge 1$, we have

$$(xD_{G_2})^n(x) = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} S(n,i) x^i y^{n+2-2i}.$$

Proof of Theorem 20. Let $G_2 = \{x \to y, y \to 1\}$. From (24), we have

$$(xD_{G_2})^n(x) = \sum_{(k,\mu)\vdash n} \sum_Z G_Z x x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{n-1}} D_{G_2}^k(x).$$
(26)

Note that

$$x_0 = D_{G_2}^0(x) = x, \ x_1 = D_{G_2}(x) = y, \ x_2 = D_{G_2}^2(x) = 1, \ x_i = D_{G_2}^i(x) = 0 \text{ for } i \ge 3.$$

Recall that for $(k,\mu) \vdash n$, we have $k \in [n]$. Then $x_{\mu_1}x_{\mu_2}\cdots x_{\mu_{n-1}}D_{G_2}^k(x) \neq 0$ if and only if $0 \leq \mu_j \leq 2$ for all $j \in [n-1]$ and $1 \leq k \leq 2$. Thus

$$\mu = (1^{m_1} 2^{m_2} 0^{n-1-m_1-m_2}), \tag{27}$$

where m_1 and m_2 are nonnegative integers. Let Z be a k-Young tableau of shape (k, μ) , where μ is given by (27). As in the proof of Theorem 14, we define $T = \rho(Z)$ to be the unique standard Young tableau such that the sets of the letters in the *j*-th column in Z and T are the same for all *j*. Then Z has at most two columns. Therefore, by using Proposition 26, we get

$$S_n(x) = (xD_{G_2})^n(x)|_{y=1} = \sum_{(k,\mu)\vdash n} \sum_Z G_Z x^{|\mu|_0+1},$$
(28)

where the first summation is taken over all types (k, μ) of n, the second summation is taken over all k-Young tableaux of shape (k, μ) and the partitions μ have the form (27). In the same way as the proof of Theorem 14, by using Lemma 23, we get (19).

7. Concluding Remarks

In this paper, we present combinatorial expansions of $(c(x)D)^n$ in terms of inversion sequences as well as k-Young tableaux. By introducing the g-index of Young tableau, we find that Eulerian polynomials, second-order Eulerian polynomials, André polynomials and the generating polynomials of gamma coefficients of Eulerian polynomials can be expressed in terms of standard Young tableaux, which imply a deep connection among these polynomials.

References

- G. Benkart, S.A. Lopes, M. Ondrus, A parametric family of subalgebras of the Weyl algebra II. Irreducible modules Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory, *Contemp. Math.*, vol. 602, Amer. Math. Soc., Providence, RI (2013), pp. 73–98.
- [2] P. Blasiak, P. Flajolet, Combinatorial models of creation-annihilation, Sém. Lothar. Combin., 65 (2010/12), Art. B65c, 78 pp.
- [3] P. Brändén, Actions on permutations and unimodality of descent polynomials, *European J. Combin.*, 29 (2008), 514–531.
- [4] E. Briand, S. Lopes, M. Rosas, Normally ordered forms of powers of differential operators and their combinatorics, J. Pure Appl. Algebra, 224(8) (2020), 106312.
- [5] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, European J. Combin., 15 (1994), 417-441.
- [6] F. Brenti, A class of q-symmetric functions arising from plethysm, J. Combin. Theory Ser. A, 91 (2000), 137–170.
- [7] Ch.A. Charalambides, J. Singh, A review of the Stirling numbers, their generalizations and statistical applications, *Comm. Statist. Theory Methods*, 17 (8) (1988), 2533–2595.
- [8] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.*, 117 (1993), 113–129.
- [9] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. in Appl. Math., 82 (2017), 58–82.
- [10] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. in Appl. Math., 41 (2008), 133–157.
- [11] C-O. Chow, W. C. Shiu, Counting simsun permutations by descents, Ann. Comb., 15 (2011), 625–635.
- [12] L. Comtet, Une formule explicite pour les puissances successives de l'operateur de dérivations de Lie, C. R. Hebd. Seances Acad. Sci., 276 (1973), 165–168.
- [13] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sém. Lothar. Combin., 37, Art. B37a (1996), 1–21.
- [14] A. Dzhumadil'daev, D. Yeliussizov, Stirling permutations on multisets, Europ. J. Combin., 36 (2014), 377– 392.
- [15] S.-P. Eu, T.-S. Fu, Y.-C. Liang, T.-L. Wong, On *xD*-generalizations of Stirling numbers and Lah numbers via graphs and rooks, *Electron. J. Combin.*, 24 (2017), P2.9.
- [16] D. Foata, M. P. Schützenberger, Théorie géometrique des polynômes eulériens, Lecture Notes in Math., vol. 138, Springer, Berlin, 1970.
- [17] D. Foata, G.-N. Han, Arbres minimax et polynômes d'André, Adv. in Appl. Math., 27 (2001), 367–389.
- [18] D. Foata, G.-N. Han, New permutation coding and equidistribution of set-valued statistics, *Theoret. Comput. Sci.*, 410 (2009), 3743–3750.

- [19] I. Gessel, R.P. Stanley, Stirling polynomials, J. Combin. Theory Ser. A, 24 (1978), 25–33.
- [20] J. Haglund, M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, *European J. Combin.*, 33 (2012), 477–487.
- [21] H.-K. Hwang, H.-H. Chern, G.-H. Duh, Guan-Huei, An asymptotic distribution theory for Eulerian recurrences with applications, Adv. in Appl. Math., 112 (2020), 101960.
- [22] Z. Lin, J. Zeng, The γ-positivity of basic Eulerian polynomials via group actions, J. Combin. Theory Ser. A, 135 (2015), 112–129.
- [23] S.-M. Ma, T. Mansour, The 1/k-Eulerian polynomials and k-Stirling permutations, Discrete Math., 338 (2015), 1468-1472.
- [24] S.-M. Ma, Y.-N. Yeh, The peak statistics on simsun permutations, *Electron. J. Combin.*, 23(2) (2016), #P2.14.
- [25] S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, Stirling permutations of the second kind and perfect matchings, *Electron. J. Combin.*, 24(4) (2017), #P4.27.
- [26] S.-M. Ma, J. Ma, Y.-N. Yeh, γ-positivity and partial γ-positivity of descent-type polynomials, J. Combin. Theory Ser. A, 167 (2019), 257–293.
- [27] T. Mansour, M. Schork, Commutation relations, normal ordering, and Stirling numbers, in: Discrete Mathematics and Its Applications, Boca Raton, CRC Press, Boca Raton, FL, 2016.
- [28] Ž. Mijajlović, Z. Marković, Some recurrence formulas related to the differential operator θD, Facta Univ., (NIŠ) 13 (1998), 7–17.
- [29] T.K. Petersen, Eulerian Numbers. Birkhäuser/Springer, New York, 2015.
- [30] G. Rządkowski, M. Urlińska, Some applications of the generalized Eulerian numbers, J. Combin. Theory Ser. A, 163 (2019), 85–97.
- [31] C.D. Savage, G. Viswanathan, The 1/k-Eulerian polynomials, Electron J. Combin., 19 (2012), #P9.
- [32] H. Scherk, De evolvenda functione $(yd \cdot yd \cdot yd \cdot ydx)/dx^n$ disquisitiones nonnullae analyticae (Ph.D. Thesis), University of Berlin, 1823.
- [33] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.
- [34] R.P. Stanley, A survey of alternating permutations, Contemp. Math., 531 (2010), 165–196.
- [35] S. Sundaram, The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice, Adv. Math., 104(2) (1994), 225–296.
- [36] S. Sundaram, The homology of partitions with an even number of blocks, J. Algebraic Combin., 4 (1995), 69–92.
- [37] B.-X. Zhu, A generalized Eulerian triangle from staircase tableaux and tree-like tableaux, J. Comb. Theory, Ser. A, 172 (2020), 105206.

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