ANGLE SUMS OF SCHLÄFLI ORTHOSCHEMES

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Abstract. For the simplices

$$K_n^A = \{x \in \mathbb{R}^{n+1} : x_1 \geq x_2 \geq \ldots \geq x_{n+1}, x_1 - x_{n+1} \leq 1, x_1 + \ldots + x_{n+1} = 0\}$$

and

$$K_n^B = \{x \in \mathbb{R}^n : 1 \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\},$$

called *Schläfli orthoschemes of type A and B*, respectively, we evaluate the tangent cones at their j-faces and compute explicitly the sum of the conic intrinsic volumes of these tangent cones at all j-faces of K_n^A and K_n^B, respectively. This setting contains sums of external and internal angles of K_n^A and K_n^B as special cases. The sums are evaluated in terms of Stirling numbers of both kinds. We generalize these results to finite products of Schläfli orthoschemes of type A and B and, as a probabilistic consequence, derive formulas for the expected number of j-faces of the Minkowski sums of the convex hulls of a finite number of Gaussian random walks and random bridges. Furthermore, we evaluate the analogous angle sums for the tangent cones of Weyl chambers of types A and B and finite products thereof.

1. Introduction

The Schläfli orthoscheme of type B in $\mathbb{R}^n$, denoted by K_n^B, is the simplex spanned by the $n+1$ vertices

$$(0,0,0,\ldots,0),(1,0,0,\ldots,0),(1,1,0,\ldots,0),\ldots,(1,1,1,\ldots,1)$$

or, equivalently,

$$K_n^B = \{x \in \mathbb{R}^n : 1 \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}.$$

The classical intrinsic volumes of K_n^B were computed by Gao and Vitale [8] in order to evaluate the intrinsic volumes of the so called Brownian motion body.

The Schläfli orthoscheme of type A in $\mathbb{R}^{n+1}$, denoted by K_n^A, was studied by Gao [7] in the context of Brownian bridges and is defined as the simplex spanned by the vertices $P_0, \ldots, P_{n+1}$, where $P_0 := (0,\ldots,0)$ and

$$P_i = (\underbrace{1,\ldots,1}_{i},0,\ldots,0) - \frac{i}{n+1}(1,1,\ldots,1)$$

for $i = 1,\ldots,n+1$. Equivalently, the Schläfli orthoscheme K_n^A can expressed as

$$K_n^A = \{x \in \mathbb{R}^{n+1} : x_1 \geq x_2 \geq \ldots \geq x_{n+1}, x_1 - x_{n+1} \leq 1, x_1 + \ldots + x_{n+1} = 0\}.$$

We will present further details on Schläfli orthoschemes in Section 3.1.

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In the present paper, we evaluate certain angle sums or, more generally, sums of conic intrinsic volumes, of the Schläfli orthoschemes. For a polytope P, let $\mathcal{F}_j(P)$ denote the set of all j-dimensional faces of P. The tangent cone of P at its j-dimensional face F is the convex cone $T_F(P)$ defined by

$$T_F(P) = \{x \in \mathbb{R}^n : f_0 + \varepsilon x \in P \text{ for some } \varepsilon > 0\},$$

where f_0 is any point in F not belonging to a face of smaller dimension. We explicitly compute the conic intrinsic volumes of the tangent cones of the Schläfli orthoschemes at their j-dimensional faces and, in particular, the sum of the conic intrinsic volumes over all such faces. The k-th conic intrinsic volume of a convex cone C, denoted by $\upsilon_k(C)$, is a spherical or conic analogue to the usual intrinsic volume of a convex set and will be formally introduced in Section 2.2. Among other results, we will show that

$$\sum_{F \in \mathcal{F}_j(K_n^A)} \upsilon_k(T_F(K_n^A)) = \sum_{F \in \mathcal{F}_j(K_n^B)} \upsilon_k(T_F(K_n^B)) = \frac{j!}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}, \tag{1.1}$$

where the numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are the Stirling numbers of the first and second kind, respectively.

Furthermore, we will compute the analogous angle (and conic intrinsic volume) sums for the tangent cones of *Weyl chambers* of type A and B which are convex cones in $\mathbb{R}^n$ defined by

$$A^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\}$$

and

$$B^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}.$$

The corresponding formulas are given by

$$\sum_{F \in \mathcal{F}_j(A^{(n)})} \upsilon_k(T_F(A^{(n)})) = \frac{j!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix}, \quad \sum_{F \in \mathcal{F}_j(B^{(n)})} \upsilon_k(T_F(B^{(n)})) = \frac{2^j j!}{2^n n!} B(n,k) T(k,j), \tag{1.2}$$

where the numbers $B(n,k)$ and $T(n,k)$ denote the B-analogues of the Stirling numbers of the first and second kind, respectively, which we will formally introduce in Section 2.3.

In the special cases $k = n$ and $k = j$, the equations (1.1) and (1.2) yield formulas for the sum of the internal and external angles of Schläfli orthoschemes and Weyl chambers of both types A and B.

We will generalize the above results to finite products of Schläfli orthoschemes and finite products of Weyl chambers leading to rather complicated formulas in terms of coefficients in the Taylor expansion of a certain function. The main results on angle and conic intrinsic volume sums will be stated in Section 3.2.

It turns out that the tangent cones of the Schläfli orthoschemes (and of the Weyl chambers) are essentially products of Weyl chambers of type A and B. We will then derive (1.1) and (1.2) from a more general Proposition 3.12 stated in Section 3.3. This proposition gives a formula for the sum of the conic intrinsic volumes of a product of Weyl chambers in terms of the generalized Stirling numbers of first and second kind. The main ingredients in the proof of this proposition are the known formulas for the conic intrinsic volumes of Weyl chambers; see e.g. [14, Theorem 4.2] or [9, Theorem 1.1]. We will present a different proof of Proposition 3.12 in Section 4.2, where the main ingredient is an explicit evaluation of the internal and external angles of the tangent cones and their faces.

As a probabilistic interpretation of these results, we consider convex hulls of Gaussian random walks and random bridges in Section 3.4. The expected number of j-faces of the convex hull of a single Gaussian random walk or a Gaussian bridge in $\mathbb{R}^d$ (even in a more general non-Gaussian setting) were already evaluated in [13]. Our general result on the angle sums of a product of Schläfli orthoschemes yields a formula for the expected number of j-faces of the Minkowski sum of several convex hulls of Gaussian random walks or Gaussian random bridges.

2. Preliminaries

In this section we collect notation and facts from convex geometry and combinatorics. The reader may skip this section at first reading and return to it when necessary.

2.1. **Facts from convex geometry.** For a set $M \subset \mathbb{R}^n$ denote by $\operatorname{lin} M$ (respectively, $\operatorname{aff} M$) its linear (affine) hull, that is, the minimal linear (respectively, affine) subspace containing M. Equivalently, $\operatorname{lin} M$ (respectively, $\operatorname{aff} M$) is the set of all linear (respectively, affine) combinations of elements of M. The relative interior of M, denoted by $\operatorname{relint} M$, is the set of interior points of M relative to its affine hull $\operatorname{aff} M$. Let also $\operatorname{conv} M$ denote the convex hull of M which is defined as the minimal convex set containing M, or equivalently

$$\operatorname{conv} M := \Big\{ \sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_1, \dots, x_m \in M, \lambda_1, \dots, \lambda_m \geq 0, \lambda_1 + \ldots + \lambda_m = 1 \Big\}.$$

Similarly, let $\operatorname{pos} M$ denote the positive (or conic) hull of M:

$$\operatorname{pos} M := \Big\{ \sum_{i=1}^m \lambda_i x_i : m \in \mathbb{N}, x_1, \dots, x_m \in M, \lambda_1, \dots, \lambda_m \geq 0 \Big\}.$$

A set $C \subset \mathbb{R}^n$ is called a *(convex) cone* if $\lambda_1 x_1 + \lambda_2 x_2 \in C$ for all $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$. Thus, $\operatorname{pos} M$ is the minimal cone containing M. The *dual cone* of a cone $C \subset \mathbb{R}^n$ is defined as

$$C^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \ \forall y \in C\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. We will make use of the following simple duality relation that holds for arbitrary $x_1, \dots, x_m \in \mathbb{R}^n$:

$$\operatorname{pos}\{x_1, \dots, x_m\}^\circ = \{v \in \mathbb{R}^n : \langle v, x_i \rangle \leq 0 \text{ for all } i = 1, \dots, m\}. \tag{2.1}$$

A *polyhedral set* is an intersection of finitely many closed half-spaces (whose boundaries need not pass through the origin). A bounded polyhedral set is called *polytope*. A *polyhedral cone* is an intersection of finitely many closed half-space whose boundaries contain the origin and therefore a special case of polyhedral sets. The faces of P (of arbitrary dimension) are obtained by replacing some of the half-spaces, whose intersection defines the polyhedral set, by their boundaries and taking the intersection. We denote the set of k-dimensional faces of a polyhedral set P by $\mathcal{F}_k(P)$. Furthermore, we denote the number of k-faces of P by $f_k(P) := |\mathcal{F}_k(P)|$. The *tangent cone* of P at a face $F \in \mathcal{F}_k(P)$ is defined by

$$T_F(P) = \{x \in \mathbb{R}^n : v + \varepsilon x \in P \text{ for some } \varepsilon > 0\},$$

where f_0 is any point in the relative interior of F. It is known that this definition does not depend on the choice of f_0. The *normal cone* of P at the face F is defined as the dual of the tangent cone, that is

$$N_F(P) = T_F(P)^\circ = \{w \in \mathbb{R}^d : \langle w, u \rangle \leq 0 \text{ for all } u \in T_F(P)\}.$$

It is easy to check that given a face F of a cone C, the corresponding normal cone $N_F(C)$ satisfies $N_F(C) := (\operatorname{lin} F)^\perp \cap C^\circ$, where $L^\perp$ denotes the orthogonal complement of a linear subspace L.

2.2. **Conic intrinsic volumes and angles of polyhedral sets.** Now let us introduce some geometric functionals of cones that we are going to consider. The following facts are mostly taken from [2, Section 2]; see also [16, Section 6.5]. At first, we define the conical intrinsic volumes which are the analogues of the usual intrinsic volumes in the setting of conic or spherical geometry.

Let $C \subset \mathbb{R}^n$ be a polyhedral cone, and g be an n-dimensional standard Gaussian random vector. Then, for $k \in \{0, \dots, n\}$, the k-th *conic intrinsic volume* of C is defined by

$$\upsilon_k(C) := \sum_{F \in \mathcal{F}_k(C)} \mathbb{P}(\Pi_C(g) \in \operatorname{relint} F).$$

Here, Π_C denotes the orthogonal projection on C, that is $\Pi_C(x)$ is the vector in C minimizing the Euclidean distance to $x \in \mathbb{R}^n$.

The conic intrinsic volumes of a cone C form a probability distribution on $\{0, 1, \dots, \dim C\}$, that is

$$\sum_{k=0}^{\dim C} \upsilon_k(C) = 1 \quad \text{and} \quad \upsilon_k(C) \geq 0, \quad k = 0, \dots, \dim C.$$

The Gauss-Bonnet formula [2, Corollary 4.4] for a cone C which is not a linear subspace states that

$$\sum_{k=0}^{\dim C} (-1)^k \upsilon_k(C) = 0,$$

which implies

$$\sum_{k=1,3,5,\dots} \upsilon_k(C) = \sum_{k=0,2,4,\dots} \upsilon_k(C) = \frac{1}{2}. \tag{2.2}$$

Furthermore the conic intrinsic volumes satisfy the product rule

$$\upsilon_k(C_1 \times \dots \times C_m) = \sum_{i_1 + \dots + i_m = k} \upsilon_{i_1}(C_1) \cdot \ldots \cdot \upsilon_{i_m}(C_m),$$

where $C_1 \times \dots \times C_m$ is the cartesian product of $C_1, \dots, C_m$. The product rule implies that the generating polynomial of the intrinsic volumes of C, defined by $P_C(t) := \sum_{k=0}^{\dim C} \upsilon_k(C) t^k$, satisfies

$$P_{C_1 \times \dots \times C_m}(t) = P_{C_1}(t) \cdot \ldots \cdot P_{C_m}(t). \tag{2.3}$$

For example, for an i-dimensional linear subspace L, we have $\upsilon_k(C \times L) = \upsilon_{k-i}(C)$ for $k \geq i$, since

$$\upsilon_k(L) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

holds true.

The *solid angle* (or just angle) of a cone $C \subset \mathbb{R}^n$ is defined as

$$\alpha(C) := \mathbb{P}(Z \in C),$$

where Z is uniformly distributed on the unit sphere in the linear hull $\operatorname{lin} C$. Equivalently, we can take a random vector Z having a standard Gaussian distribution on the ambient linear subspace $\operatorname{lin} C$. For a d-dimensional cone $C \subset \mathbb{R}^n$, $d \in \{1, \dots, n\}$, the d-th conical intrinsic volume coincides with the solid angle of C, that is

$$\upsilon_d(C) = \alpha(C).$$

The *internal angle* of a polyhedral set P at a face F is defined as the solid angle of its tangent cone:

$$\beta(F,P) := \alpha(T_F(P)).$$

Thus, the internal angle of a cone C at $\{0\}$ is given by $\beta(\{0\},C) = \alpha(C)$. The *external angle* of P at a face F is defined as the solid angle of the normal cone of F with respect to P, that is

$$\gamma(F,P) := \alpha(N_F(P)).$$

The conic intrinsic volumes of a cone C can be computed in terms of the internal and external angles of its faces as follows:

$$\upsilon_k(C) = \sum_{F\in\mathcal{F}_k(C)} \beta(0,F)\gamma(F,C) = \sum_{F\in\mathcal{F}_k(C)} \alpha(F)\alpha(N_F(C)) \tag{2.4}$$

for $k \in \{0,\dots,n\}$.

Let $W_{n-k} \subset \mathbb{R}^n$ be random linear subspace having the uniform distribution on the Grassmann manifold of all $(n-k)$-dimensional subspaces. Then, following Grnbaum [11], the *Grassmann angle* $\gamma_k(C)$ of a cone $C \subset \mathbb{R}^n$ is defined as

$$\gamma_k(C) := \mathbb{P}(W_{n-k} \cap C \neq \{0\}), \tag{2.5}$$

for $k \in \{0,\dots,n\}$. The Grassmann angles do not depend on the dimension of the ambient space, that is, if we embed C in $\mathbb{R}^N$ where $N \geq n$, the Grassmann angle will be the same. If C is not a linear subspace, then $\frac{1}{2}\gamma_k(C)$ is also known as the *k-th conic quermassintegral* $U_k(C)$ of C; see [12, (1)-(4)]. The conic intrinsic volumes and Grassmann angles are known to satisfy the linear relation

$$\gamma_k(C) = 2\sum_{i=1,3,5,\dots} \upsilon_{k+i}(C); \tag{2.6}$$

see e.g. [2, (2.10)], provided C is not a linear subspace.

2.3. **Stirling numbers and their generating functions.** In this section, we are going to recall the definitions of different kinds of Stirling numbers and state their generating functions. As mentioned in the introduction, these numbers appear in various results developed in this paper. The generating functions will be useful in the respective proofs.

The (signless) *Stirling numbers of the first kind* $\left[{n \atop k}\right]$ are defined as the number of permutations of the set $\{1,\dots,n\}$ having exactly k cycles. Equivalently, they can be defined as the coefficients of the polynomial

$$t(t+1)\cdot\ldots\cdot(t+n-1) = \sum_{k=0}^{n} \left[{n \atop k}\right] t^k \tag{2.7}$$

for $n \in \mathbb{N}_0$, with the convention that $\left[{n \atop k}\right] = 0$ for $n \in \mathbb{N}_0$, $k \notin \{0,\dots,n\}$ and $\left[{0 \atop 0}\right] = 1$. The Stirling numbers of the first kind can also be represented as the following sum:

$$\left[{n \atop k}\right] = \frac{n!}{k!} \sum_{\substack{m_1,\dots,m_k\in\mathbb{N}:\\ m_1+\dots+m_k=n}} \frac{1}{m_1\cdot\ldots\cdot m_k}; \tag{2.8}$$

see [15, Equations (1.9),(1.15)].

The B-analogues to the Stirling numbers of the first kind, denoted by $B(n,k)$, are defined as the coefficients of the polynomial

$$(t+1)(t+3)\cdot\ldots\cdot(t+2n-1)=\sum_{k=0}^{n}B(n,k)t^k \tag{2.9}$$

for $n\in\mathbb{N}_0$ and, by convention, $B(n,k)=0$ for $k\notin\{0,\ldots,n\}$. These numbers appear as entry A028338 in [17]. The exponential generating functions of both arrays $(\left[{n \atop k}\right])_{n,k\geq 0}$ and $(B(n,k))_{n,k\geq 0}$ are given by

$$\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\frac{t^n}{n!}=\frac{1}{k!}\big(-\log(1-t)\big)^k,\quad \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}\frac{t^n}{n!}y^k=(1-t)^{-y} \tag{2.10}$$

and

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}B(n,k)\frac{t^n}{n!}y^k=(1-2t)^{-\frac{1}{2}(y+1)}; \tag{2.11}$$

see [9, Proposition 2.3] for the proof of (2.11).

The *Stirling numbers of the second kind* $\left\{{n \atop k}\right\}$ are defined as the number of partitions of the set $\{1,\ldots,n\}$ into k non-empty subsets. Similar to (2.8), the Stirling numbers of the second kind can be represented as the following sum:

$$\begin{Bmatrix}n\\k\end{Bmatrix}=\frac{n!}{k!}\sum_{\substack{m_1,\ldots,m_k\in\mathbb{N}:\\ m_1+\ldots+m_k=n}}\frac{1}{m_1!\cdot\ldots\cdot m_k!}; \tag{2.12}$$

see [15, Equations (1.9),(1.13)].

The B-analogues to the Stirling numbers of the second kind, denoted by $T(n,k)$, are defined as

$$T(n,k)=\sum_{m=k}^{n}2^{m-k}\binom{n}{m}\begin{Bmatrix}m\\k\end{Bmatrix}.$$

They appear as Entry A039755 in [17] and were studied by Suter [20].

The exponential generating functions of the arrays $(\left\{{n \atop k}\right\})_{n,k\geq 0}$ and $(T(n,k))_{n,k\geq 0}$ are given by

$$\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}\frac{t^n}{n!}=\frac{1}{k!}(e^t-1)^k,\quad \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}\frac{t^n}{n!}y^k=e^{(e^t-1)y} \tag{2.13}$$

and

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!}y^k=e^{\frac{y}{2}(e^{2t}-1)}e^t; \tag{2.14}$$

see [20, Theorem 4] for (2.14). The numbers $T(n,k)$ and $\left\{{n \atop k}\right\}$ appear as coefficients in the formulas

$$t^n=\sum_{k=0}^{n}(-1)^{n-k}T(n,k)(t+1)(t+3)\ldots(t+2k-1),\quad t^n=\sum_{k=0}^{n}(-1)^{n-k}\begin{Bmatrix}n\\k\end{Bmatrix}t(t+1)\ldots(t+k-1);$$

see Entry A039755 in [17], which should be compared to the formulas (2.7) and (2.9) for $\left[{n \atop k}\right]$ and their B-analogues $B(n,k)$.

More generally, it is possible to define the r-Stirling numbers of the first and the second kind. For $r\in\mathbb{N}$, the (signless) *r-Stirling numbers of the first kind*, denoted by $\left[{n \atop k}\right]_r$, are defined as the number of permutations of the set $\{1,\dots,n\}$ having k cycles, such that the numbers $1,2,\dots,r$ are in distinct cycles; see [4, (1)]. The *r-Stirling numbers of the second kind*, denoted by $\left\{ {n \atop k} \right\}_r$ are defined as the number of partitions of the set $\{1,\dots,n\}$ into k non-empty disjoint subsets, such that the numbers $1,2,\dots,r$ are in distinct subsets; see [4, (2)]. Obviously, for $r \in \{0,1\}$, the r-Stirling numbers of the first and second kind coincide with the classical Stirling numbers, respectively. The r-Stirling numbers were introduced by Carlitz [5, 6] under the name weighted Stirling numbers.

The exponential generating functions in one and two variables of the r-Stirling numbers of the first kind are given by

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{t^n}{n!} = \frac{1}{k!}(1-t)^{-r}\big(-\log(1-t)\big)^k, \quad \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{t^n}{n!} y^k = \Big(\frac{1}{1-t}\Big)^{r+y}; \tag{2.15}$$

see [4, Equations (36),(37)]. For the r-Stirling numbers of the second kind they are given by

$$\sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} = \frac{1}{k!} e^{rt}(e^t-1)^k, \quad \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} y^k = e^{y(e^t-1)} e^{rt}; \tag{2.16}$$

see [4, Equations (38),(39)].

The r-Stirling numbers can equivalently be defined in terms of the regular Stirling numbers by

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \sum_{m=0}^{n-k} \binom{n-r}{m} \begin{bmatrix} n-r-m \\ k-r \end{bmatrix} r^{\overline{m}}, \tag{2.17}$$

where $r^{\overline{m}} := r(r+1)\cdot\ldots\cdot(r+m-1)$ denotes the rising factorial, $r^{\overline{0}} := 1$, and

$$\left\{ {n \atop k} \right\}_r = \sum_{m=k-r}^{n-r} \binom{n-r}{m} \left\{ {m \atop k-r} \right\} r^{n-r-m}; \tag{2.18}$$

see [4, Equations (27),(32)]. This even yields an analytic continuation of the r-Stirling numbers to non-integer (arbitrary complex) r, given by

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r = \sum_{m=0}^{n-k} \binom{n}{m} \begin{bmatrix} n-m \\ k \end{bmatrix} r^{\overline{m}} \tag{2.19}$$

and

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{m=k}^{n} \binom{n}{m} \left\{ {m \atop k} \right\} r^{n-m}. \tag{2.20}$$

Note that we obtain the following special values:

$$\begin{bmatrix} n \\ n \end{bmatrix}_r = \left\{ {n \atop n} \right\}_r = 1, \quad n \geq r \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_r = \left\{ {n \atop k} \right\}_r = 0, \quad k \notin \{r,\dots,n\}.$$

For $r = 1/2$, we observe the following relation between the r-Stirling numbers and the numbers $B(n,k)$ and $T(n,k)$:

$$\begin{bmatrix} n+1/2 \\ k+1/2 \end{bmatrix}_{1/2} = 2^{k-n} B(n,k), \quad \left\{ {n+1/2 \atop k+1/2} \right\}_{1/2} = 2^{k-n} T(n,k). \tag{2.21}$$

This can easily be verified by comparing their generating functions.

3. Main results

3.1. **Schläfli orthoschemes.** The polytopes we are interested in this paper are called *Schläfli orthoschemes*. As mentioned in the introduction, the Schläfli orthoscheme of type B in $\mathbb{R}^n$ is defined as

$$\begin{aligned} K_n^B :=& \operatorname{conv}\{(0,0,\dots,0),(1,0,\dots,0),(1,1,0,\dots,0),\dots,(1,1,\dots,1)\} \\ =&\{x\in\mathbb{R}^n : 1\geq x_1\geq x_2\geq\ldots\geq x_n\geq 0\}. \end{aligned}$$

Note that for convenience, we set $K_0^B := \{0\}$.

Similarly, the Schläfli orthoscheme of type A in $\mathbb{R}^{n+1}$ is defined as the convex hull of the $(n+1)$-dimensional vectors $P_0, P_1, \dots, P_{n+1}$, where $P_0 = (0,0,\dots,0)$ and

$$P_i = (\underbrace{1,\dots,1}_{i},0,\dots,0) - \frac{i}{n+1}(1,1,\dots,1), \quad 1\leq i\leq n+1.$$

It is not difficult to check that

$$K_n^A = \{x\in\mathbb{R}^{n+1} : x_1\geq x_2\geq\ldots\geq x_{n+1}, x_1 - x_{n+1}\leq 1, x_1+\ldots+x_{n+1}=0\}.$$

Again, we put $K_0^A = \{0\}$. The index shift from type B to type A will turn out to be convenient since $K_n^A \subset \mathbb{R}^{n+1}$ is an n-dimensional polytope. In fact, the Schläfli orthoschemes of type A and type B are simplices since they are convex hulls of $n+1$ affinely independent vectors. The Schläfli orthoscheme of type B was already considered by Gao and Vitale [8] who among other things evaluated the classical intrinsic volumes of K_n^B. Similar calculations for the Schläfli orthoscheme of type A were made by Gao [7].

The definition of the Schläfli orthoscheme can be motivated by a connection to random walks and random bridges. In fact, consider Gaussian random matrices $G_B\in\mathbb{R}^{d\times n}$ and $G_A\in\mathbb{R}^{d\times(n+1)}$, that is, the matrices have independent and standard Gaussian distributed entries. Then $G_B K_n^B$ has the same distribution as the convex hull of a d-dimensional random walk $S_0 := 0, S_1, \dots, S_n$ with Gaussian increments. Similarly, $G_A K_n^A$ has the same distribution as the convex hull of a d-dimensional Gaussian random bridge $\widetilde{S}_0 := 0, \widetilde{S}_1, \dots, \widetilde{S}_n, \widetilde{S}_{n+1} = 0$ which is essentially a Gaussian random walk conditioned on the event that it returns to 0 in the $(n+1)$-st step. We will explain these facts in Section 3.5 in more detail.

3.2. **Sums of conic intrinsic volumes in Weyl chambers and Schläfli orthoschemes.** In this section, we give the main results of this paper concerning the sums of the conic intrinsic volumes of the tangent cones of Schläfli orthoschemes of type A and B and their products. The same is done for Weyl chambers of type A and B and their products. Our first results concerning the Schläfli orthoschemes of type A and B are the following two theorems.

Theorem 3.1. *Let $j\in\{0,\dots,n\}$ and $k\in\{0,\dots,n\}$ be given. Then,*

$$\sum_{F\in\mathcal{F}_j(K_n^B)} \upsilon_k(T_F(K_n^B)) = \frac{j!}{n!}\begin{bmatrix} n+1 \\ k+1\end{bmatrix}\begin{Bmatrix} k+1 \\ j+1\end{Bmatrix}.$$

Theorem 3.2. *Let $j\in\{0,\dots,n\}$ and $k\in\{0,\dots,n\}$ be given. Then,*

$$\sum_{F\in\mathcal{F}_j(K_n^A)} \upsilon_k(T_F(K_n^A)) = \frac{j!}{n!}\begin{bmatrix} n+1 \\ k+1\end{bmatrix}\begin{Bmatrix} k+1 \\ j+1\end{Bmatrix}.$$

These theorems yield formulas for the sums of the internal and external angles of K_n^B and K_n^A at their j-faces F.

Corollary 3.3. *For $j \in \{0, \dots, n\}$ the sum of the internal angles is given by*

$$\sum_{F \in \mathcal{F}_j(K_n^B)} \alpha\big(T_F(K_n^B)\big) = \sum_{F \in \mathcal{F}_j(K_n^A)} \alpha\big(T_F(K_n^A)\big) = \frac{j!}{n!} \left\{ \begin{matrix} n+1 \\ j+1 \end{matrix} \right\}$$

and the sum of the external angles by

$$\sum_{F \in \mathcal{F}_j(K_n^B)} \alpha\big(N_F(K_n^B)\big) = \sum_{F \in \mathcal{F}_j(K_n^A)} \alpha\big(N_F(K_n^A)\big) = \frac{j!}{n!} \left[\begin{matrix} n+1 \\ j+1 \end{matrix} \right].$$

Proof. The sums of the internal angles follow from Theorems 3.1 and 3.2 with $k = n$, since K_n^B and K_n^A are both n-dimensional polytopes.

In the case of the external angles, we use that the maximal linear subspaces contained in both $T_F(K_n^B)$ and $T_F(K_n^A)$ are j-dimensional, which implies that $\upsilon_j(T_F(K_n^B)) = \upsilon_{n-j}((T_F(K_n^B))^\circ) = \upsilon_{n-j}(N_F(K_n^B)) = \alpha(N_F(K_n^B))$ and similarly for K_n^A. Using Theorems 3.1 and 3.2 with $k = j$ completes the proof. □

We obtain similar results for the tangent cones of Weyl chambers of type A and type B, where the polyhedral cone

$$B^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$$

is a *Weyl chamber of type B* (or B_n) and the polyhedral cone

$$A^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\}$$

is a *Weyl chamber of type A* (or A_{n-1}). We set $B^{(0)} = A^{(0)} = \{0\}$ for convenience.

Theorem 3.4. *Let $j \in \{0, \dots, n\}$ and $k \in \{0, \dots, n\}$ be given. Then,*

$$\sum_{F \in \mathcal{F}_j(B^{(n)})} \upsilon_k(T_F(B^{(n)})) = \frac{j!}{n!} \left[\begin{matrix} n+1/2 \\ k+1/2 \end{matrix} \right]_{1/2} \left\{ \begin{matrix} k+1/2 \\ j+1/2 \end{matrix} \right\}_{1/2} = \frac{2^j j!}{2^n n!} B(n,k) T(k,j).$$

Theorem 3.5. *Let $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, n\}$ be given. Then,*

$$\sum_{F \in \mathcal{F}_j(A^{(n)})} \upsilon_k(T_F(A^{(n)})) = \frac{j!}{n!} \left[\begin{matrix} n \\ k \end{matrix} \right] \left\{ \begin{matrix} k \\ j \end{matrix} \right\}.$$

These four theorems are special cases of Proposition 3.12 which we shall state in Section 3.3. This proposition gives a formula for sums of the intrinsic volumes of a mixed product of Weyl chambers of both types A and B.

For $k = n$ and $k = j$, Theorems 3.4 and 3.5 yield the following corollary on the sums of the internal and external angles of $B^{(n)}$ and $A^{(n)}$.

Corollary 3.6. *For $j \in \{0, \dots, n\}$ the sums of the internal and external angles of $B^{(n)}$ are given by*

$$\sum_{F \in \mathcal{F}_j(B^{(n)})} \alpha\big(T_F(B^{(n)})\big) = \frac{2^j j!}{2^n n!} T(n,j), \qquad \sum_{F \in \mathcal{F}_j(B^{(n)})} \alpha\big(N_F(B^{(n)})\big) = \frac{2^j j!}{2^n n!} B(n,j).$$

For $j \in \{1,\dots,n\}$ in the A-case, we have

$$\sum_{F\in\mathcal{F}_j(A^{(n)})} \alpha\big(T_F(A^{(n)})\big) = \frac{j!}{n!}\left\{ {n \atop j} \right\}, \qquad \sum_{F\in\mathcal{F}_j(A^{(n)})} \alpha\big(N_F(A^{(n)})\big) = \frac{j!}{n!}\left[{n \atop j} \right].$$

Finite products of Schläfli orthoschemes and Weyl chambers. The results of the above theorems may be extended to finite products of Schläfli orthoschemes and Weyl chambers leading to rather complicated formulas. Let $b \in \mathbb{N}$ and define $K^B := K^B_{n_1} \times \ldots \times K^B_{n_b}$ and $K^A := K^A_{n_1} \times \ldots \times K^A_{n_b}$ for $n_1,\dots,n_b \in \mathbb{N}_0$ with $n := n_1 + \ldots + n_b$.

Theorem 3.7. *Let $j \in \{0,\dots,n\}$ and $k \in \{0,\dots,n\}$ be given. Then,*

$$\sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k(T_F(K^B)) = R_1\big(k,j,b,(n_1,\dots,n_b)\big),$$

where for $d \in \{0,\frac{1}{2},1\}$

$$R_d\big(k,j,b,(n_1,\dots,n_b)\big) := \big[t^k\big]\big[x_1^{n_1}\cdot\ldots\cdot x_b^{n_b}\big]\big[u^j\big]\frac{(1-x_1)^{-d(t+1)}\cdot\ldots\cdot(1-x_b)^{-d(t+1)}}{\big(1-u((1-x_1)^{-t}-1)\big)\cdot\ldots\cdot\big(1-u((1-x_b)^{-t}-1)\big)}.$$

Here, $[t^N]f(t) := \frac{1}{N!}f^{(N)}(0)$ denotes the coefficient of t^N in the Taylor expansion of a function $f:\mathbb{R}\to\mathbb{R}$ around 0 and

$$\big[x_1^{N_1}\cdot\ldots\cdot x_b^{N_b}\big]g(x_1,\dots,x_b) := \frac{1}{N_1!\cdot\ldots\cdot N_b!}\frac{\partial^{N_1+\ldots+N_b}}{\partial x_1^{N_1}\ldots\partial x_b^{N_b}}g(0,\dots,0)$$

is the coefficient of $x_1^{N_1}\cdot\ldots\cdot x_b^{N_b}$ in the multidimensional Taylor expansion of a function $g:\mathbb{R}^b\to\mathbb{R}$. Note that $R_d\big(k,j,b,(n_1,\dots,n_b)\big)=0$ for $k<j$.

Theorem 3.8. *Let $j \in \{0,\dots,n\}$ and $k \in \{0,\dots,n\}$ be given. Then,*

$$\sum_{F\in\mathcal{F}_j(K^A)} \upsilon_k(T_F(K^A)) = R_1\big(k,j,b,(n_1,\dots,n_b)\big).$$

The proofs of Theorems 3.7 and 3.8 are postponed to Section 4.3. For finite products of Weyl chambers $W^B := B^{(n_1)} \times \ldots \times B^{(n_b)}$ and $W^A := A^{(n_1)} \times \ldots \times A^{(n_b)}$, we obtain the following theorems.

Theorem 3.9. *For $j \in \{0,\dots,n\}$ and $k \in \{0,\dots,n\}$, it holds that*

$$\sum_{F\in\mathcal{F}_j(W^B)} \upsilon_k(T_F(W^B)) = R_{1/2}\big(k,j,b,(n_1,\dots,n_b)\big).$$

Theorem 3.10. *For $j \in \{0,\dots,n\}$ and $k \in \{0,\dots,n\}$, it holds that*

$$\sum_{F\in\mathcal{F}_j(W^A)} \upsilon_k(T_F(W^A)) = R_0\big(k,j,b,(n_1,\dots,n_b)\big).$$

The proofs of Theorems 3.9 and 3.10 are similar to that of Theorems 3.7 and 3.8 and will be omitted. In the proofs of Theorems 3.7 and 3.8, we will observe that if we additionally sum over all possible $n_1,\dots,n_b$, the formulas in terms of Taylor coefficients simplify as follows.

Proposition 3.11. *We have*

$$\sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\ldots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k(T_F(K^B)) = \frac{j!}{n!}\binom{j+b-1}{b-1}\begin{bmatrix} n+b\\ k+b\end{bmatrix}_b \begin{Bmatrix} k+b\\ j+b\end{Bmatrix}_b$$

and

$$\sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\ldots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^A)} \upsilon_k(T_F(K^A)) = \frac{j!}{n!}\binom{j+b-1}{b-1}\begin{bmatrix} n+b\\ k+b\end{bmatrix}_b \begin{Bmatrix} k+b\\ j+b\end{Bmatrix}_b.$$

for $j\in\{0,\dots,n\}$ *and* $k\in\{0,\dots,n\}$.

The proof is postponed to Section 4.4.

3.3. **Method of proof of Theorems 3.1, 3.2, 3.4 and 3.5.** The main ingredient in proving Theorems 3.1, 3.2, 3.4 and 3.5 is the following proposition.

Proposition 3.12. *Let* $(j,b)\in\mathbb{N}_0^2\backslash\{(0,0)\}$ *and* $n\in\mathbb{N}$. *For* $l=(l_1,\dots,l_{j+b})$ *such that* $l_1,\dots,l_j\in\mathbb{N}$, $l_{j+1},\dots,l_{j+b}\in\mathbb{N}_0$ *and* $l_1+\ldots+l_{j+b}=n$ *we define*

$$T_l := A^{(l_1)}\times\ldots\times A^{(l_j)}\times B^{(l_{j+1})}\times\ldots\times B^{(l_{j+b})}.$$

Then, it follows that

$$\sum_{\substack{l_1,\dots,l_j\in\mathbb{N},l_{j+1},\dots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l) = \frac{j!}{n!}\begin{bmatrix} n+b/2\\ k+b/2\end{bmatrix}_{b/2} \begin{Bmatrix} k+b/2\\ j+b/2\end{Bmatrix}_{b/2}$$

for all $k\in\{0,\dots,n\}$.

We will give two different approaches to prove this proposition. In Section 4.1, we will prove it by computing the generating function of the intrinsic volumes. In Section 4.2, we will present a different proof in which we compute the internal and external angles of the faces of the tangent cones.

In order to see that the Theorems 3.1, 3.2, 3.4, 3.5 follow from Proposition 3.12, we describe the collections of tangent cones of the Schläfli orthoschemes and Weyl chambers of types A and B at their corresponding faces.

Schläfli orthoschemes of type B. At first, consider the B-case. The faces of K_n^B (and of any polytope in general) are obtained by replacing some of the linear inequalities in its defining conditions by equalities. Thus each j-face of K_n^B is determined by a collection of indices $0\le i_0<i_1<\ldots<i_j\le n$, $J:=\{i_0,\dots,i_j\}$, and given by

$$F_J := \{x\in\mathbb{R}^d : 1=x_1=\ldots=x_{i_0}\ge x_{i_0+1}=\ldots=x_{i_1}\ge \\ \ldots\ge x_{i_{j-1}+1}=\ldots=x_{i_j}\ge x_{i_j+1}=\ldots=x_n=0\}.$$

Note that for $i_0=0$, no x_i is required to be 1, and for $i_j=n$, no x_i is required to be 0. Take a point $x=(x_1,\dots,x_n)\in\operatorname{relint}F_J$. For this point, all inequalities in the defining condition of F_J are strict. By definition, we have

$$T_{F_J}(K_n^B) = \{v\in\mathbb{R}^n : x+\varepsilon v\in K_n^B \text{ for some } \varepsilon>0\}.$$

It follows that

$$T_{F_J}(K_n^B) = \{v\in\mathbb{R}^n : 0\ge v_1\ge\ldots\ge v_{i_0}, v_{i_0+1}\ge\ldots\ge v_{i_1},$$

$$\ldots, v_{i_{j-1}+1} \geq \ldots \geq v_{i_j}, v_{i_j+1} \geq \ldots \geq v_n \geq 0\},$$

which is isometric to the product

$$B^{(i_0)} \times A^{(i_1-i_0)} \times \ldots \times A^{(i_j-i_{j-1})} \times B^{(n-i_j)},$$

where the polyhedral cones

$$B^{(i)} := \{x \in \mathbb{R}^i : x_1 \geq \ldots \geq x_i \geq 0\}, \quad A^{(i)} := \{x \in \mathbb{R}^i : x_1 \geq \ldots \geq x_i\}, \quad i \in \mathbb{N}_0$$

are Weyl chambers of type B and A, respectively. The above results yield the following lemma.

Lemma 3.13. *The collection of tangent cones $T_F(K_n^B)$, where F runs through the set of all j-faces $\mathcal{F}_j(K_n^B)$, coincides (up to isometry) with the collection*

$$B^{(i_0)} \times A^{(i_1-i_0)} \times \ldots \times A^{(i_j-i_{j-1})} \times B^{(n-i_j)}, \quad 0 \leq i_0 < i_1 < \ldots < i_j \leq n.$$

Equivalently, it coincides (up to isometry) with the collection

$$B^{(l_0)} \times A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})}, \quad l_0 + \ldots + l_{j+1} = n,\ l_0, l_{j+1} \in \mathbb{N}_0,\ l_1, \ldots, l_j \in \mathbb{N}.$$

If the isometry type of some cone appears with some multiplicity in one collection, then it appears with the same multiplicity in the other collections.

Then, Theorem 3.1 can be deduced from Proposition 3.12 and Lemma 3.13 as follows.

Proof of Theorem 3.1 assuming Proposition 3.12. For $j \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, n\}$, we have

$$\begin{aligned}\sum_{F \in \mathcal{F}_j(K_n^B)} \upsilon_k(T_F(K_n^B)) &= \sum_{\substack{l_0, l_{j+1} \in \mathbb{N}_0, l_1, \ldots, l_j \in \mathbb{N}: \\ l_0 + \ldots + l_{j+1} = n}} \upsilon_k\big(B^{(l_0)} \times A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})}\big) \\ &= \frac{j!}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_1 \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}_1 \\ &= \frac{j!}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix},\end{aligned}$$

where we used Lemma 3.13 in the first step and Proposition 3.12 with $b = 2$ in the second step. □

Schläfli orthoschemes of type A. Now, we consider the tangent cones of Schläfli orthoschemes of type A. Recall that

$$K_n^A = \{x \in \mathbb{R}^{n+1} : x_1 \geq \ldots \geq x_{n+1}, x_1 - x_{n+1} \leq 1, x_1 + \ldots + x_{n+1} = 0\}.$$

Note that unlike the B-case, the simplex K_n^A (which has dimension n) is contained in $\mathbb{R}^{n+1}$. For us it will be easier to consider the following unbounded set

$$\widetilde{K}_n^A := \{x \in \mathbb{R}^{n+1} : x_1 \geq \ldots \geq x_{n+1}, x_1 - x_{n+1} \leq 1\}.$$

Denote by L_{n+1} the 1-dimensional linear subspace $L_{n+1} = \{x \in \mathbb{R}^{n+1} : x_1 = \ldots = x_{n+1}\}$. Then $L_{n+1}^\perp = \{x \in \mathbb{R}^{n+1} : x_1 + \ldots + x_{n+1} = 0\}$ and we have

$$\widetilde{K}_n^A = L_{n+1} \oplus K_n^A, \quad K_n^A \subset L_{n+1}^\perp.$$

where $\oplus$ denotes the orthogonal sum. Thus, there is a one-to-one correspondence between the j-faces of K_n^A and the $(j+1)$-faces of $\widetilde{K}_n^A$ given by $F \mapsto L_{n+1} \oplus F$, $\mathcal{F}_j(K_n^A) \to \mathcal{F}_{j+1}(\widetilde{K}_n^A)$. Furthermore, we have a relation between the tangent cones of K_n^A and $\widetilde{K}_n^A$ given by

$$T_{F \oplus L_{n+1}}(\widetilde{K}_n^A) = T_F(K_n^A) \oplus L_{n+1} \tag{3.1}$$

for every j-face F of K_n^A.

Now, consider the collection of tangent cones $T_F(\widetilde{K}_n^A)$, where $F \in \mathcal{F}_j(\widetilde{K}_n^A)$ for some $j \in \{1,\dots,n+1\}$, $n \in \mathbb{N}_0$ more closely. The faces of $\widetilde{K}_n^A$ are obtained by replacing some inequalities in the defining conditions of $\widetilde{K}_n^A$ by equalities. Thus, there are two types of j-faces of $\widetilde{K}_n^A$ for $j \in \{1,\dots,n+1\}$.

The j-faces of the first type are of the form

$$F_1 = \{x \in \mathbb{R}^{n+1} : x_1 = \ldots = x_{i_1} \geq x_{i_1+1} = \ldots = x_{i_2} \geq \ldots \geq x_{i_{j-1}+1} = \ldots = x_{n+1}, x_1 - x_{n+1} \leq 1\}$$

for $1 \leq i_1 < \ldots < i_{j-1} \leq n$. Note that for $j = 1$, we have the 1-face $\{x \in \mathbb{R}^{n+1} : x_1 = \ldots = x_{n+1}\}$. To determine the tangent cone at F_1, take some point in the relative interior of this face. For this point, all inequalities in the defining condition of F_1 are strict. Call this point $x = (x_1,\dots,x_{n+1}) \in \operatorname{relint} F_1$. By definition, we have

$$T_{F_1}(\widetilde{K}_n^A) = \{v \in \mathbb{R}^{n+1} : x + \varepsilon v \in \widetilde{K}_n^A \text{ for some } \varepsilon > 0\}.$$

It follows that

$$T_{F_1}(\widetilde{K}_n^A) = \{v \in \mathbb{R}^{n+1} : v_1 \geq \ldots \geq v_{i_1}, v_{i_1+1} \geq \ldots \geq v_{i_2}, \dots, v_{i_{j-1}+1} \geq \ldots \geq v_{n+1}\}.$$

Thus, $T_{F_1}(\widetilde{K}_n^A)$ is equal to

$$A^{(l_1)} \times \ldots \times A^{(l_j)},$$

where $l_1,\dots,l_j \in \mathbb{N}$ such that $l_1 + \ldots + l_j = n+1$ $(l_1 = i_1, l_2 = i_2 - i_1, \dots, l_j = n+1-i_{j-1})$.

The j-faces of $\widetilde{K}_n^A$ of the second type are of the form

$$F_2 = \{x \in \mathbb{R}^{n+1} : x_1 = \ldots = x_{i_1} \geq x_{i_1+1} = \ldots = x_{i_2} \geq \ldots \geq x_{i_j+1} = \ldots = x_{n+1}, x_1 - x_{n+1} = 1\}$$

for $1 \leq i_1 < \ldots < i_j \leq n$, whose defining condition consists of $j+1$ groups of equalities and the additional condition $x_1 - x_{n+1} = 1$. Again, take a point $x \in \operatorname{relint} F_2$. For this point all inequalities in the defining condition of F_2 are strict. Hence, the tangent cone is given by

$$\begin{aligned}
&T_{F_2}(\widetilde{K}_n^A)\\
&\quad = \{v \in \mathbb{R}^{n+1} : x + \varepsilon v \in \widetilde{K}_d^A \text{ for some } \varepsilon > 0\}\\
&\quad = \{v \in \mathbb{R}^{n+1} : v_1 \geq \ldots \geq v_{i_1}, v_{i_1+1} \geq \ldots \geq v_{i_2}, \dots, v_{i_j+1} \geq \ldots \geq v_{n+1}, v_1 \leq v_{n+1}\}\\
&\quad = \{v \in \mathbb{R}^{n+1} : v_{i_1+1} \geq \ldots \geq v_{i_2}, \dots, v_{i_{j-1}+1} \geq \ldots \geq v_{i_j}, v_{i_j+1} \geq \ldots \geq v_{n+1} \geq v_1 \geq \ldots \geq v_{i_1}\},
\end{aligned}$$

where in the last step we merged two groups of inequalities. Hence, $T_{F_2}(\widetilde{K}_n^A)$ is isometric to

$$A^{(l_1+l_{j+1})} \times A^{(l_2)} \times \ldots \times A^{(l_j)},$$

where $l_1,\dots,l_{j+1} \in \mathbb{N}$ are such that $l_1 + \ldots + l_{j+1} = n+1$, that is they form a composition of $n+1$ into $j+1$ parts.

We can combine both types of tangent cones into one type as follows. For a $(j+1)$-composition $l_1 + \ldots + l_{j+1} = n+1$, the numbers $k_1 := l_1 + l_{j+1}, k_2 := l_2, \dots, k_j := l_j$ form a j-composition of $n+1$. This association is not injective since each j-composition $k_1 + \ldots + k_j = n+1$ of $n+1$ is assigned to $k_1 - 1$ compositions of $n+1$ into $j+1$ parts. Indeed, we can represent k_1 as $1 + (k_1 - 1), 2 + (k_2 - 2), \dots, (k_1 - 1) + 1$. Thus, combining both types of tangent cones yields the following lemma.

Lemma 3.14. *The collection of tangent cones $T_F(\widetilde{K}_n^A)$, where F runs through the set of all j-faces $\mathcal{F}_j(\widetilde{K}_n^A)$, coincides (up to isometry) with the collection of cones*

$$A^{(l_1)} \times \ldots \times A^{(l_j)}, \quad l_1, \ldots, l_j \in \mathbb{N} : l_1 + \ldots + l_j = n+1,$$

where each cone of the above collection is repeated l_1 times (or taken with multiplicity l_1).

Then, Theorem 3.2 can be deduced from Proposition 3.12 and Lemma 3.14 as follows.

Proof of Theorem 3.2 assuming Proposition 3.12. Let $j \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, n\}$ be given. Using $\upsilon_k(K_n^A) = \upsilon_{k+1}(K_n^A \oplus L_{n+1})$ and (3.1), we obtain

$$\begin{aligned}
\sum_{F \in \mathcal{F}_j(K_n^A)} \upsilon_k(K_n^A) &= \sum_{F \in \mathcal{F}_j(K_n^A)} \upsilon_{k+1}(T_F(K_n^A) \oplus L_{n+1}) \\
&= \sum_{F \in \mathcal{F}_j(K_n^A)} \upsilon_{k+1}(T_{F \oplus L_{n+1}}(\widetilde{K}_n^A)) \\
&= \sum_{F \in \mathcal{F}_{j+1}(\widetilde{K}_n^A)} \upsilon_{k+1}(T_F(\widetilde{K}_n^A)).
\end{aligned}$$

Applying Lemma 3.14 $j+1$ times with multiplicity l_1 replaced by $l_1, \ldots, l_{j+1}$, we can observe that the collection of tangent cones $T_F(\widetilde{K}_n^A)$, where F runs through all $(j+1)$-faces of $\mathcal{F}_{j+1}(\widetilde{K}_n^A)$ and each cone of this collection is repeated $j+1$ times, coincides (up to isometry) with the collection of cones

$$A^{(l_1)} \times \ldots \times A^{(l_{j+1})}, \quad l_1, \ldots, l_{j+1} \in \mathbb{N} : l_1 + \ldots + l_{j+1} = n+1,$$

where each cone is taken with multiplicity $l_1 + \ldots + l_{j+1} = n+1$. Therefore, we arrive at

$$\begin{aligned}
\sum_{F \in \mathcal{F}_{j+1}(\widetilde{K}_n^A)} (j+1)\upsilon_{k+1}(T_F(\widetilde{K}_n^A)) &= \sum_{\substack{l_1, \ldots, l_{j+1} \in \mathbb{N}: \\ l_1 + \ldots + l_{j+1} = n+1}} (n+1)\upsilon_{k+1}\big(A^{(l_1)} \times \ldots \times A^{(l_{j+1})}\big) \\
&= (n+1)\frac{(j+1)!}{(n+1)!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_0 \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}_0 \\
&= \frac{(j+1)!}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix},
\end{aligned}$$

where we used Proposition 3.12. This completes the proof. □

Weyl chambers of type B. For $n \in \mathbb{N}$ recall that

$$B^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$$

and $B^{(0)} := \{0\}$ by convention. Now, let $j \in \{0, \ldots, n\}$. Each j-face of $B^{(n)}$ is determined by a collection of indices $1 \leq i_1 < \ldots < i_j \leq n$, $J := \{i_1, \ldots, i_j\}$, and given by

$$F_J := \{x \in \mathbb{R}^n : x_1 = \ldots = x_{i_1} \geq \ldots \geq x_{i_{j-1}+1} = \ldots = x_{i_j} \geq x_{i_j+1} = \ldots = x_n = 0\}.$$

Note that for $i_j = n$, no x_i's are required to be 0, and for $j = 0$, we obtain the 0-dimensional face $\{0\}$. In order to determine the tangent cone $T_{F_J}(B^{(n)})$ take a point $x = (x_1, \ldots, x_n) \in \operatorname{relint} F_J$. Again, this point satisfies the defining conditions of F_J with inequalities replaced by strict inequalities. Thus, the tangent cone is given by

$$T_{F_J}(B^{(n)}) = \{v \in \mathbb{R}^n : x + \varepsilon v \in B^{(n)} \text{ for some } \varepsilon > 0\}$$

$$= \{v \in \mathbb{R}^n : v_1 \geq \ldots \geq v_{i_1}, \ldots, v_{i_{j-1}+1} \geq \ldots \geq v_{i_j}, v_{i_j+1} \geq \ldots \geq v_n \geq 0\}$$
$$= A^{(i_1)} \times A^{(i_2-i_1)} \times \ldots \times A^{(i_j-i_{j-1})} \times B^{(n-i_j)}.$$

The above reasoning yields the following lemma.

Lemma 3.15. *The collection of tangent cones $T_F(B^{(n)})$, where F runs through the set of all j-faces $\mathcal{F}_j(B^{(n)})$, coincides with the collection of polyhedral cones*

$$A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})}, \quad l_1 + \ldots + l_{j+1} = n,\ l_1, \ldots, l_j \in \mathbb{N},\ l_{j+1} \in \mathbb{N}_0.$$

Proof of Theorem 3.4 assuming Proposition 3.12. For $j \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, n\}$, we have

$$\begin{aligned}\sum_{F \in \mathcal{F}_j(B^{(n)})} \upsilon_k\big(T_F(B^{(n)})\big) &= \sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}, l_{j+1} \in \mathbb{N}_0: \\ l_1+\ldots+l_{j+1}=n}} \upsilon_k\big(A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})}\big) \\ &= \frac{j!}{n!} \begin{bmatrix} n+1/2 \\ k+1/2 \end{bmatrix}_{1/2} \begin{Bmatrix} k+1/2 \\ j+1/2 \end{Bmatrix}_{1/2} \\ &= \frac{j!}{n!} 2^{j-n} B(n,k) T(n,k),\end{aligned}$$

where we used Lemma 3.15 in the first step, Proposition 3.12 with $b = 1$ in the last step and the formulas in (2.21) in the last step. □

Weyl chambers of type A. For $n \in \mathbb{N}$ recall that

$$A^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\}.$$

For $j \in \{1, \ldots, n\}$ every j-face of $A^{(n)}$ is determined by a collection of indices $1 \leq i_1 < \ldots < i_{j-1} \leq n-1$, $J := \{i_1, \ldots, i_{j-1}\}$, and given by

$$F_J := \{x \in \mathbb{R}^n : x_1 = \ldots = x_{i_1} \geq \ldots \geq x_{i_{j-1}+1} = \ldots = x_n\}$$

Note that for $j = 1$, we obtain the 1-face $\{x \in \mathbb{R}^n : x_1 = \ldots = x_n\}$. In order to determine the tangent cone $T_{F_J}(A^{(n)})$ consider a point $x = (x_1, \ldots, x_n) \in \operatorname{relint} F_J$. In a fashion similar to the case of a B-type Weyl chamber, we obtain the tangent cone of $A^{(n)}$ at F_J as follows:

$$\begin{aligned}T_{F_J}(A^{(n)}) &= \{v \in \mathbb{R}^n : x + \varepsilon v \in A^{(n)} \text{ for some } \varepsilon > 0\} \\ &= \{v \in \mathbb{R}^n : v_1 \geq \ldots \geq v_{i_1}, \ldots, v_{i_{j-1}+1} \geq \ldots \geq v_n\} \\ &= A^{(i_1)} \times A^{(i_2-i_1)} \times \ldots \times A^{(n-i_{j-1})}.\end{aligned}$$

This yields the following analogue of Lemma 3.15.

Lemma 3.16. *The collection of tangent cones $T_F(A^{(n)})$, where F runs through the set of all j-faces $\mathcal{F}_j(A^{(n)})$, coincides with the collection of polyhedral cones*

$$A^{(l_1)} \times \ldots \times A^{(l_j)}, \quad l_1 + \ldots + l_j = n,\ l_1, \ldots, l_j \in \mathbb{N}.$$

Proof of Theorem 3.5 assuming Proposition 3.12. For $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n\}$, we have

$$\sum_{F \in \mathcal{F}_j(A^{(n)})} \upsilon_k\big(T_F(A^{(n)})\big) = \sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}: \\ l_1+\ldots+l_j=n}} \upsilon_k\big(A^{(l_1)} \times \ldots \times A^{(l_j)}\big) = \frac{j!}{n!} \begin{bmatrix} n \\ k \end{bmatrix}_0 \begin{Bmatrix} k \\ j \end{Bmatrix}_0 = \frac{j!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix},$$

where we used Lemma 3.16 in the first step and Proposition 3.12 with $b = 0$ in the second step. □

Identities for the generalized Stirling numbers. Let us also mention that Proposition 3.12 yields the following identities for the generalized Stirling numbers.

Corollary 3.17. *For $n \in \mathbb{N}$, $j \in \{0, \dots, n\}$, $2b \in \mathbb{N}_0$ we have*

$$\sum_{k=0}^{n} \begin{bmatrix} n+b \\ k+b \end{bmatrix}_b \begin{Bmatrix} k+b \\ j+b \end{Bmatrix}_b = \frac{n!}{j!} \binom{n+2b-1}{j+2b-1} = \frac{n!}{j!} \binom{n+2b-1}{n-j} \tag{3.2}$$

and

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n+b \\ k+b \end{bmatrix}_b \begin{Bmatrix} k+b \\ j+b \end{Bmatrix}_b = 0. \tag{3.3}$$

Note that (3.3) is a special case of the orthogonality relation between the r-Stirling numbers proven by Broder [4, Theorem 25]. Relation (3.2) is also known for $b = 0, 1$, the numbers on the right-hand side being the Lah numbers.

Proof. We use the known properties

$$\sum_{k=0}^{\dim C} \upsilon_k(C) = 1 \quad \text{and} \quad \sum_{k=0}^{\dim C} (-1)^k \upsilon_k(C) = 0$$

for the conic intrinsic volumes of a cone C that is not a linear subspace. With the notation for T_l introduced in Proposition 3.12 and with b replaced by $2b$, this yields

$$\sum_{k=0}^{n} \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+2b} \in \mathbb{N}_0: \\ l_1+\dots+l_{j+2b}=n}} \upsilon_k(T_l) = \sum_{\substack{l'_1,\dots,l'_{j+2b} \in \mathbb{N}: \\ l'_1+\dots+l'_{j+2b}=n+2b}} 1 = \binom{n+2b-1}{j+2b-1} = \binom{n+2b-1}{n-j},$$

where we used the well-known fact that the number of compositions of n into k positive integers is $\binom{n-1}{k-1}$. Similarly, we obtain

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+2b} \in \mathbb{N}_0: \\ l_1+\dots+l_{j+2b}=n}} \upsilon_k(T_l) = 0.$$

On the other hand, applying Proposition 3.12 with b replaced by $2b$ yields

$$\sum_{k=0}^{n} \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+2b} \in \mathbb{N}_0: \\ l_1+\dots+l_{j+2b}=n}} \upsilon_k(T_l) = \sum_{k=0}^{n} \frac{j!}{n!} \begin{bmatrix} n+b \\ k+b \end{bmatrix}_b \begin{Bmatrix} k+b \\ j+b \end{Bmatrix}_b$$

and

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+2b} \in \mathbb{N}_0: \\ l_1+\dots+l_{j+2b}=n}} \upsilon_k(T_l) = \sum_{k=0}^{n} (-1)^k \frac{j!}{n!} \begin{bmatrix} n+b \\ k+b \end{bmatrix}_b \begin{Bmatrix} k+b \\ j+b \end{Bmatrix}_b,$$

which proves (3.2) and (3.3). □

3.4. **Expected face numbers: Convex hulls of Gaussian random walks and bridges.** The above theorems on the angle sums of the tangent cones of Schläfli orthoschemes yield results on the expected number of faces of Gaussian random walks and random bridges and their Minkowski sums. Consider independent d-dimensional standard Gaussian random vectors

$$X_1^{(1)}, X_2^{(1)}, \dots, X_{n_1}^{(1)}, X_1^{(2)}, X_2^{(2)}, \dots, X_{n_2}^{(2)}, \dots, X_1^{(b)}, X_2^{(b)}, \dots, X_{n_b}^{(b)}$$

and let $n_1 + \dots + n_b = n \geq d$. For every $i \in \{1, \dots, b\}$ we define a random walk $(S_0^{(i)}, S_1^{(i)}, \dots, S_{n_i}^{(i)})$ by

$$S_k^{(i)} = X_1^{(i)} + \dots + X_k^{(i)}, \quad k = 1, \dots, n_i$$

and $S_0^{(i)} := 0$. Consider their convex hulls

$$C_{n_i}^{(i)} := \operatorname{conv}\big\{S_0^{(i)}, S_1^{(i)}, \dots, S_{n_i}^{(i)}\big\}, \quad i = 1, \dots, b.$$

The following theorem gives a formula for the expected number of j-faces of the Minkowski sum of b such convex hulls defined by

$$C_{n_1}^{(1)} + \dots + C_{n_b}^{(b)} = \{v_1 + \dots + v_b : v_1 \in C_{n_1}^{(1)}, \dots, v_b \in C_{n_b}^{(b)}\}.$$

Theorem 3.18. *Let $0 \leq j < d \leq n$ be given and define $C_{n_1}^{(1)}, \dots, C_{n_b}^{(b)}$ as above. Then, we have*

$$\mathbb{E}\, f_j(C_{n_1}^{(1)} + \dots + C_{n_b}^{(b)}) = 2 \sum_{l \geq 1} R_1(d - 2l + 1, j, b, (n_1, \dots, n_b)),$$

where we recall that

$$\begin{aligned} &R_1(k, j, b, (n_1, \dots, n_b)) \\ &\quad := \big[t^k\big]\big[x_1^{n_1} \cdot \ldots \cdot x_b^{n_b}\big]\big[u^j\big] \frac{(1 - x_1)^{-(t+1)} \cdot \ldots \cdot (1 - x_b)^{-(t+1)}}{\big(1 - u((1 - x_1)^{-t} - 1)\big) \cdot \ldots \cdot \big(1 - u((1 - x_b)^{-t} - 1)\big)}. \end{aligned}$$

The same formula holds for the Minkowski sum of Gaussian random bridges which are essentially Gaussian random walks under the condition that they return to 0 in the last step. To state it, consider independent d-dimensional standard Gaussian random vectors

$$X_1^{(1)}, X_2^{(1)}, \dots, X_{n_1+1}^{(1)}, X_1^{(2)}, X_2^{(2)}, \dots, X_{n_2+1}^{(2)}, \dots, X_1^{(b)}, X_2^{(b)}, \dots, X_{n_b+1}^{(b)}$$

and define the random walks $(S_0^{(i)}, S_1^{(i)}, \dots, S_{n_i+1}^{(i)})$ as above. We define the Gaussian bridge $(\widetilde{S}_0^{(i)}, \widetilde{S}_1^{(i)}, \dots, \widetilde{S}_{n_i+1}^{(i)})$ as a process having the same distribution as the Gaussian random walk $(S_0^{(i)}, S_1^{(i)}, \dots, S_{n_i+1}^{(i)})$ conditioned on the event that $S_{n_i+1}^{(i)} = 0$. Equivalently, the Gaussian bridge $(\widetilde{S}_0^{(i)}, \widetilde{S}_1^{(i)}, \dots, \widetilde{S}_{n_i+1}^{(i)})$ can be constructed as

$$\widetilde{S}_k^{(i)} := S_k^{(i)} - \frac{k}{n_i + 1} S_{n_i+1}^{(i)}, \quad k = 1, \dots, n_i + 1$$

for $i = 1, \dots, b$. Note that $\widetilde{S}_{n_i+1}^{(i)} = 0$. Define the convex hulls of the Gaussian bridges by

$$\widetilde{C}_{n_i}^{(i)} := \operatorname{conv}\big\{\widetilde{S}_0^{(i)}, \widetilde{S}_1^{(i)}, \dots, \widetilde{S}_{n_i}^{(i)}\big\}, \quad i = 1, \dots, b.$$

Theorem 3.19. *Let $0 \leq j < d \leq n$ be given and $\widetilde{C}_{n_1}^{(1)}, \dots, \widetilde{C}_{n_b}^{(a)}$ as above. Then, we have*

$$\mathbb{E}\, f_j(\widetilde{C}_{n_1}^{(1)} + \dots + \widetilde{C}_{n_b}^{(b)}) = 2 \sum_{l \geq 1} R_1(d - 2l + 1, j, b, (n_1, \dots, n_b)).$$

For a single convex hull ($b = 1$), the expected number of j-faces of $C_n^{(1)}$ and $\widetilde{C}_n^{(1)}$ is already known (in a more general case), see [13, Theorems 1.2 and 5.1], and given by

$$\mathbb{E}\, f_j(C_n^{(1)}) = \mathbb{E}\, f_j(\widetilde{C}_n^{(1)}) = \frac{2 \cdot j!}{n!} \sum_{l=0}^{\infty} \begin{bmatrix} n+1 \\ d-2l \end{bmatrix} \begin{Bmatrix} d-2l \\ k+1 \end{Bmatrix}.$$

This formula is recovered as a special case of Theorems 3.18 and 3.19 with $b = 1$ and n_1 replaced by n.

3.5. **Method of proof of Theorems 3.18 and 3.19.** The main ingredient in the proof of the named theorems is the following lemma which is due to Affentranger and Schneider [1, (5)].

Lemma 3.20. *Let $P \subset \mathbb{R}^n$ be a convex polytope with non-empty interior and $G \in \mathbb{R}^{d \times n}$ be a Gaussian random matrix, that is, its entries are independent and standard normal distributed random variables. Then for all $0 \le j < d \le n$ we have*

$$\mathbb{E}\, f_j(GP) = f_j(P) - \sum_{F \in \mathcal{F}_j(P)} \gamma_d\big(T_F(P)\big),$$

where the Grassmann angles γ_d were defined in (2.5).

In fact, Affentranger and Schneider [1] proved this formula for a random orthogonal projection of P, while the fact that Gaussian matrices yield the same result follows from a result of Baryshnikov and Vitale [3].

Due to the relation between Grassmann angles and conic intrinsic volumes stated in (2.6), the lemma can be written as

$$\mathbb{E}\, f_j(GP) = f_j(P) - 2 \sum_{F \in \mathcal{F}_j(P)} \sum_{l=0}^{\infty} \upsilon_{d+2l+1}\big(T_F(P)\big).$$

Using (2.2), it follows that under the condition of Lemma 3.20 also

$$\mathbb{E}\, f_j(GP) = 2 \sum_{F \in \mathcal{F}_j(P)} \Big(\upsilon_{d-1}\big(T_F(P)\big) + \upsilon_{d-3}\big(T_F(P)\big) + \dots\Big) \tag{3.4}$$

holds true.

Now take a Gaussian matrix $G_B = (\xi_{i,j}) \in \mathbb{R}^{d \times n}$, where $\xi_{i,j}$, $i \in \{1, \dots, d\}, j \in \{1, \dots, n\}$ are independent standard Gaussian random variables. Then, we claim that $G_B K^B$ has the same distribution as the Minkowski $C_{n_1}^{(1)} + \dots + C_{n_b}^{(b)}$. And similarly, for a Gaussian matrix $G_A \in \mathbb{R}^{d \times (n+b)}$, we claim that $G_A K^A$ has the same distribution as $\widetilde{C}_{n_1}^{(1)} + \dots + \widetilde{C}_{n_b}^{(b)}$.

In order to see this, consider the case of a single Schläfli orthoscheme $K_{n_1}^B$ first. Let $G_B^{(1)} = (\xi_{i,j}) \in \mathbb{R}^{d \times n_1}$ be a Gaussian matrix. We know that the Schläfli orthoscheme $K_{n_1}^B$ is the simplex given as the convex hull of the n_1-dimensional vectors

$$(0, 0, 0, \dots, 0), (1, 0, 0 \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, 1 \dots, 1).$$

Thus, we obtain

$$\begin{aligned} G_B^{(1)} K_{n_1}^B &= \operatorname{conv}\big\{G_B^{(1)}(0, \dots, 0)^\top, G_B^{(1)}(1, 0, \dots, 0)^\top, G_B^{(1)}(1, 1, 0, \dots, 0)^\top, \dots, G_B^{(1)}(1, \dots, 1)^\top\big\} \\ &= \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_{1,1} \\ \vdots \\ \xi_{d,1} \end{pmatrix}, \begin{pmatrix} \xi_{1,1} + \xi_{1,2} \\ \vdots \\ \xi_{d,1} + \xi_{d,2} \end{pmatrix}, \dots, \begin{pmatrix} \xi_{1,1} + \dots + \xi_{1,n_1} \\ \vdots \\ \xi_{d,1} + \dots + \xi_{d,n_1} \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned}&\overset{d}{=} \operatorname{conv}\left\{S_0^{(1)}, S_1^{(1)}, \dots, S_{n_1}^{(1)}\right\}\\&= C_{n_1}^{(1)}.\end{aligned}$$

Similarly, we consider a Gaussian Matrix $G_A^{(1)} \in \mathbb{R}^{d\times(n_1+1)}$ for the Schläfli orthoscheme $K_{n_1}^A$. We know that $K_{n_1}^A$ is the convex hull of the (n_1+1)-dimensional vectors $P_0 = (0,0,\dots,0)$ and

$$P_i = (\underbrace{1,\dots,1}_{i}, 0,\dots,0) - \frac{i}{n_1+1}(1,1,\dots,1), \quad 1 \le i \le n_1.$$

Thus, in the same way we get

$$\begin{aligned}G_A^{(1)} K_{n_1}^A &= \operatorname{conv}\{G_A^{(1)}P_0, G_A^{(1)}P_1, \dots, G_A^{(1)}P_{n_1}\}\\&\overset{d}{=} \operatorname{conv}\left\{0, S_1^{(1)} - \frac{1}{n_1+1}S_{n_1+1}^{(1)}, S_2^{(1)} - \frac{2}{n_1+1}S_{n_1+1}^{(1)}, \dots, S_{n_1}^{(1)} - \frac{n_1}{n_1+1}S_{n_1+1}^{(1)}\right\}\\&\overset{d}{=} \operatorname{conv}\left\{\widetilde{S}_0^{(1)}, \widetilde{S}_1^{(1)}, \widetilde{S}_2^{(1)}, \dots, \widetilde{S}_{n_1}^{(1)}\right\}\\&= \widetilde{C}_{n_1}^{(1)}.\end{aligned}$$

The product case follows from the following observation. Let $G_B \in \mathbb{R}^{d\times n}$ be a Gaussian matrix and $n_1 + \ldots + n_b = n$. Then, we can represent G_B as the row of b independent matrices $G_B = (G_B^{(1)}, \dots, G_B^{(b)})$, where $G_B^{(i)} \in \mathbb{R}^{d\times n_i}$ is itself a Gaussian matrix for $i = 1,\dots,b$. We can represent each vector $x \in \mathbb{R}^n$ as the column of b vectors, i.e. $x = (x^{(1)}, \dots, x^{(b)})^\top$, where $x^{(i)} \in \mathbb{R}^{n_i}$ for $i = 1,\dots,b$. Then, we easily observe that

$$G_B x = (G_B^{(1)}, \dots, G_B^{(b)})(x^{(1)}, \dots, x^{(b)})^\top = G_B^{(1)}x^{(1)} + \ldots + G_B^{(b)}x^{(b)}.$$

It follows that

$$\begin{aligned}G_B K^B &= (G_B^{(1)}, \dots, G_B^{(b)})(K_{n_1}^B \times \ldots \times K_{n_b}^{(b)})\\&= G_B^{(1)}K_{n_1}^B + \ldots + G_B^{(b)}K_{n_b}^B\\&\overset{d}{=} C_{n_1}^{(1)} + \ldots + C_{n_b}^{(b)}.\end{aligned}$$

In the same way we obtain for a Gaussian matrix $G_A \in \mathbb{R}^{d\times(n+b)}$ with $n_1 + \ldots + n_b = n$ that

$$G_A K^A \overset{d}{=} \widetilde{C}_{n_1}^{(1)} + \ldots + \widetilde{C}_{n_b}^{(b)}.$$

Now, we can finally prove Theorems 3.18 and 3.19.

Proof of Theorem 3.18. Let $1 \le j < d \le n$ and $G_B \in \mathbb{R}^{d\times n}$ be a Gaussian matrix. Then, we already observed that $G_B K^B \overset{d}{=} C_{n_1}^{(1)} + \ldots + C_{n_b}^{(b)}$. Thus, (3.4) yields

$$\begin{aligned}\mathbb{E}\, f_j(C_{n_1}^{(1)} + \ldots + C_{n_b}^{(b)}) &= \mathbb{E}\, f_j(G_B K^B)\\&= 2 \sum_{F\in\mathcal{F}_j(K^B)} (\upsilon_{d-1}(T_F(K^B)) + \upsilon_{d-3}(T_F(K^B)) + \ldots)\\&= 2\sum_{l\ge1} R_1(d-2l+1, j, b, (n_1,\dots,n_b)),\end{aligned}$$

where we used Theorem 3.7 in the last step. □

Proof of Theorem 3.19. Let $1 \leq j < d \leq n$ and let $G_A \in \mathbb{R}^{d\times(n+b)}$ be a Gaussian matrix. We already saw that $G_A K^A \overset{d}{=} \widetilde{C}_{n_1}^{(1)} + \ldots + \widetilde{C}_{n_b}^{(b)}$. Although the polytope $K^A \subset \mathbb{R}^{n+b}$ is only n-dimensional, we can still apply Lemma 3.20, and therefore also (3.4), to the ambient linear subspace $\operatorname{lin} K^A$ since the Grassmann angles do not depend on the dimension of the ambient linear subspace. Combining this with Theorem 3.8, we obtain

$$\begin{aligned}\mathbb{E}\, f_j(\widetilde{C}_{n_1}^{(1)} + \ldots + \widetilde{C}_{n_b}^{(b)}) &= \mathbb{E}\, f_j(G_A K^A)\\ &= 2 \sum_{F\in\mathcal{F}_j(K^B)} (\upsilon_{d-1}(T_F(K^A)) + \upsilon_{d-3}(T_F(K^A)) + \ldots)\\ &= 2\sum_{l\geq 1} R_1(d-2l+1, j, b, (n_1,\ldots,n_b)),\end{aligned}$$

which completes the proof. □

4. Proofs: Angle sums in Weyl chambers and Schläfli orthoschemes

In this section, we present the proofs of Propositions 3.12, 3.11, and Theorems 3.7, 3.8. The most proofs rely on explicit expressions for the intrinsic volumes of the Weyl chambers. Recall that the Weyl chambers $A^{(n)}$ and $B^{(n)}$ are given by

$$A^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\}, \quad n \in \mathbb{N}$$

and

$$B^{(n)} := \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}, \quad n \in \mathbb{N},$$

and we put $B^{(0)} := \{0\}$. The intrinsic volumes of the Weyl chambers are known explicitly and given by

$$\nu_k(A^{(n)}) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!}, \quad \nu_k(B^{(n)}) = \frac{B(n,k)}{2^n n!} \tag{4.1}$$

for $k = 0,\ldots,n$; see for example [14, Theorem 4.2] or [9, Theorem 1.1]. The $B(n,k)$'s denote the B-analogues of the Stirling numbers of the first kind as defined in (2.9).

4.1. Proof of Proposition 3.12.

Let $(j,b) \in \mathbb{N}_0^2\backslash\{(0,0)\}$. Recall that for $l = (l_1,\ldots,l_{j+b})$ such that $l_1,\ldots,l_j \in \mathbb{N}$, $l_{j+1},\ldots,l_{j+b} \in \mathbb{N}_0$ and $l_1 + \ldots + l_{j+b} = n$, we define

$$T_l = A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})} \times \ldots \times B^{(l_{j+b})}.$$

Our goal is to show that

$$\sum_{\substack{l_1,\ldots,l_j\in\mathbb{N}, l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l) = \frac{j!}{n!} \begin{bmatrix} n+b/2 \\ k+b/2 \end{bmatrix}_{b/2} \begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2} \tag{4.2}$$

holds for all $k \in \{0,\ldots,n\}$.

Proof of Proposition 3.12. Let $(j,b) \in \mathbb{N}_0^2\backslash\{(0,0)\}$ and $k \in \{0,\ldots,n\}$ be given. Following the product formula for conic intrinsic volumes (2.3), the generating polynomial of the intrinsic volumes of T_l can be written as

$$P_{T_l}(t) := \sum_{k=0}^{n} \upsilon_k(T_l) t^k$$

$$= \left(\sum_{m=0}^{l_1} \upsilon_m(A^{(l_1)})t^m\right) \cdot \ldots \cdot \left(\sum_{m=0}^{l_j} \upsilon_m(A^{(l_j)})t^m\right)$$
$$\times \left(\sum_{m=0}^{l_{j+1}} \upsilon_m(B^{(l_{j+1})})t^m\right) \cdot \ldots \cdot \left(\sum_{m=0}^{l_{j+b}} \upsilon_m(B^{(l_{j+b})})t^m\right).$$

We can consider each sum on the right-hand side separately. Using the representations of the Stirling numbers of the first kind and their B-analogues from (2.7) and (2.9), and the intrinsic volumes of the Weyl chambers stated in (4.1), we obtain

$$\sum_{m=0}^{i} \upsilon_m(A^{(i)})t^m = \frac{1}{i!}\sum_{m=0}^{i} \begin{bmatrix} i \\ n \end{bmatrix} t^m = \frac{1}{i!}t(t+1)\cdot \ldots \cdot (t+i-1), \qquad i \in \mathbb{N}, \tag{4.3}$$

and

$$\sum_{m=0}^{i} \upsilon_m(B^{(i)})t^m = \frac{1}{2^i i!}\sum_{m=0}^{i} B(i,m)t^m = \frac{1}{2^i i!}(t+1)(t+3)\cdot \ldots \cdot (t+2i-1), \qquad i \in \mathbb{N}_0.$$

Note that for $i = 0$, we put $(t+1)(t+3)\cdot \ldots \cdot (t+2i-1) := 1$ by convention, to agree with $\upsilon_0(\{0\}) = 1$. This yields the following formula

$$P_{T_l}(t) = \frac{(t+1)(t+3)\cdot \ldots \cdot (t+2l_{j+1}-1)}{2^{l_{j+1}}l_{j+1}!} \cdot \ldots \cdot \frac{(t+1)(t+3)\cdot \ldots \cdot (t+2l_{j+b}-1)}{2^{l_{j+b}}l_{j+b}!}$$
$$\times \frac{t^{\overline{l_1}}}{l_1!} \cdot \ldots \cdot \frac{t^{\overline{l_j}}}{l_j!}$$

where $t^{\overline{r}} := t(t+1)\cdot \ldots \cdot (t+r-1)$ denotes the rising factorial, for $r \in \mathbb{N}$. Thus, the k-th conic intrinsic volume of T_l is the coefficient of t^k in the above polynomial $P_{T_l}(t)$. Note that this already implies $\nu_k(T_l) = 0$ for $k < j$. Thus, the left-hand side of (4.2) is 0, which coincides with the right-hand side, since

$$\begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2} = 0$$

holds for $k < j$.

Therefore, we only need to consider the case $k \geq j$. Let $P^{(n)}_{l_1,\ldots,l_{j+b}}(m)$, for $m = 0,\ldots,n$, be the coefficients of the polynomial

$$t^{\overline{l_1}} \cdot \ldots \cdot t^{\overline{l_j}}(t+1)(t+3)\cdot \ldots \cdot (t+2l_{j+1}-1)(t+1)(t+3)\cdot \ldots \cdot (t+2l_{j+b}-1) = \sum_{m=j}^{n} P^{(n)}_{l_1,\ldots,l_{j+b}}(m)t^m.$$

Using the notation just introduced, we obtain

$$\sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}, l_{j+1},\ldots,l_{j+b} \in \mathbb{N}_0: \\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l) \quad = \sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}, l_{j+1},\ldots,l_{j+b} \in \mathbb{N}_0: \\ l_1+\ldots+l_{j+b}=n}} \frac{P^{(n)}_{l_1,\ldots,l_{j+b}}(k)}{l_1! \cdot \ldots \cdot l_{j+b}! 2^{l_{j+1}+\ldots+l_{j+b}}}.$$

Now, let $[t^N]f(t) := \frac{1}{N!}f^{(N)}(0)$ be the coefficient of t^N in the Taylor expansion of a function f around 0. Define

$$C_{n,j,b}(k) := [t^k] \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \bigg(\frac{t^{\overline{l_1}}}{l_1!}\cdot\ldots\cdot\frac{t^{\overline{l_j}}}{l_j!} \times \frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l_{j+1}-1)}{2^{l_{j+1}}l_{j+1}!}\cdot\ldots\cdot\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l_{j+b}-1)}{2^{l_{j+b}}l_{j+b}!}\bigg).$$

Then, we can observe that

$$\begin{aligned}
&C_{n,j,b}(k)\\
&= \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \frac{1}{l_1!\cdot\ldots\cdot l_{j+b}!2^{l_{j+1}+\ldots+l_{j+b}}}[t^k]\Big((t+1)(t+3)\cdot\ldots\cdot(t+2l_{j+1}-1)\\
&\quad\times\ldots\cdot(t+1)(t+3)\cdot\ldots\cdot(t+2l_{j+b}-1)t^{\overline{l_1}}\cdot\ldots\cdot t^{\overline{l_j}}\Big)\\
&= \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \frac{1}{l_1!\cdot\ldots\cdot l_{j+b}!2^{l_{j+1}+\ldots+l_{j+b}}}[t^k]\sum_{m=0}^n P^{(n)}_{l_1,\ldots,l_{j+b}}(m)t^m\\
&= \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \frac{P^{(n)}_{l_1,\ldots,l_{j+b}}(k)}{l_1!\cdot\ldots\cdot l_{j+b}!2^{l_{j+1}+\ldots+l_{j+b}}}\\
&= \sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l).
\end{aligned}$$

Thus, to prove the proposition, it suffices to show that

$$C_{n,j,b}(k) = \frac{j!}{n!}\begin{bmatrix} n+b/2\\ k+b/2\end{bmatrix}_{b/2}\begin{Bmatrix} k+b/2\\ j+b/2\end{Bmatrix}_{b/2} \tag{4.4}$$

holds for all $(j,b)\in\mathbb{N}_0^2\backslash\{(0,0)\}$ and $k\in\{j,\ldots,n\}$.

In order to do this, we can introduce a new variable x and write, by expanding the product,

$$C_{n,j,b}(k) = [t^k][x^n]\Bigg(\bigg(\sum_{l=0}^\infty \frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l-1)}{2^l l!}x^l\bigg)^b \times \bigg(\sum_{l=1}^\infty\frac{t^{\bar{l}}}{l!}x^l\bigg)^j\Bigg).$$

Using (2.7) and the exponential generating function in two variables for the Stirling numbers of the first kind stated in (2.10), we obtain

$$\sum_{l=1}^\infty \frac{t^{\bar{l}}}{l!}x^l = -1 + \sum_{l=0}^\infty\sum_{m=0}^l \begin{bmatrix} l\\ m\end{bmatrix} t^m\frac{x^l}{l!} = (1-x)^{-t}-1. \tag{4.5}$$

From (2.9) and (2.11), we similarly get

$$\sum_{l=0}^\infty \frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l-1)}{2^l l!}x^l = \sum_{l=0}^\infty\sum_{m=0}^l B(l,m)t^m\Big(\frac{x}{2}\Big)^l\frac{1}{l!} = (1-x)^{-\frac{1}{2}(t+1)}. \tag{4.6}$$

Thus, we have

$$C_{n,j,b}(k) = [t^k][x^n]\Big((1-x)^{-\frac{b}{2}(t+1)}\big((1-x)^{-t}-1\big)^j\Big)$$
$$= [x^n][t^k]\big(e^{-\frac{cb}{2}(t+1)}(e^{-ct}-1)^j\big),$$

where we set $c = c(x) = \log(1-x)$. The exponential generating function of the $b/2$-Stirling numbers stated in (2.16) yields

$$e^{-\frac{cb}{2}t}(e^{-ct}-1)^j = \sum_{m=j}^{\infty} \begin{Bmatrix} m+b/2 \\ j+b/2 \end{Bmatrix}_{b/2} \frac{j!}{m!}(-ct)^m.$$

It follows that

$$[t^k]\big(e^{-\frac{cb}{2}(t+1)}(e^{-ct}-1)^j\big) = e^{-\frac{cb}{2}}[t^k]\left(\sum_{m=j}^{\infty} \begin{Bmatrix} m+b/2 \\ j+b/2 \end{Bmatrix}_{b/2} \frac{j!}{m!}(-ct)^m\right)$$
$$= e^{-\frac{cb}{2}}(-c)^k \frac{j!}{k!} \begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2}.$$

Furthermore, using (2.15) we obtain

$$[x^n]\big(e^{-\frac{cb}{2}}(-c)^k\big) = [x^n]\frac{\big(-\log(1-x)\big)^k}{(1-x)^{\frac{b}{2}}} = [x^n]\left(\sum_{m=k}^{\infty} \begin{bmatrix} m+b/2 \\ k+b/2 \end{bmatrix}_{b/2} \frac{k!}{m!}x^m\right) = \frac{k!}{n!}\begin{bmatrix} n+b/2 \\ k+b/2 \end{bmatrix}_{b/2}.$$

Taking all this into consideration, we obtain

$$C_{n,j,b}(k) = [x^n][t^k]\big(e^{-\frac{cb}{2}(t+1)}(e^{-ct}-1)^j\big)$$
$$= [x^n]\left(e^{-\frac{cb}{2}}(-c)^k \frac{j!}{k!}\begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2}\right)$$
$$= \frac{j!}{k!}\begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2}[x^n]\big(e^{-\frac{cb}{2}}(-c)^k\big)$$
$$= \frac{j!}{n!}\begin{bmatrix} n+b/2 \\ k+b/2 \end{bmatrix}_{b/2}\begin{Bmatrix} k+b/2 \\ j+b/2 \end{Bmatrix}_{b/2},$$

which coincides with (4.4) and therefore completes the proof. □

4.2. **Alternative proof of Proposition 3.12.** In this section, we present a more direct way to prove Proposition 3.12 by computing the internal and external angles of the faces of the Weyl chambers. A similar approach was already used in [8] to compute the classical intrinsic volumes of the Schläfli orthoscheme of type B and in [9] to compute the conic intrinsic volumes of the Weyl chambers.

Let $(j,b) \in \mathbb{N}_0^2 \backslash \{(0,0)\}$. Recall that for $l = (l_1, \ldots, l_{j+b})$ with $l_1, \ldots, l_j \in \mathbb{N}$, $l_{j+1}, \ldots, l_{j+b} \in \mathbb{N}_0$ and $l_1 + \ldots + l_{j+b} = n$, we have

$$T_l = A^{(l_1)} \times \ldots \times A^{(l_j)} \times B^{(l_{j+1})} \times \ldots \times B^{(l_{j+b})}.$$

Alternative proof of Proposition 3.12. Let $k \in \{0, \ldots, n\}$ be given. Using (2.4), we obtain

$$\sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}, l_{j+1},\ldots,l_{j+b} \in \mathbb{N}_0: \\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l) = \sum_{\substack{l_1,\ldots,l_j \in \mathbb{N}, l_{j+1},\ldots,l_{j+b} \in \mathbb{N}_0: \\ l_1+\ldots+l_{j+b}=n}} \sum_{F \in \mathcal{F}_k(T_l)} \alpha(F)\alpha(N_F(T_l)).$$

In order to compute this sum, we consider the k-faces of T_l. Since T_l is the product of Weyl chambers of type A and type B, its k-faces are products of faces of the individual Weyl chambers. The r-faces of the Weyl chamber $A^{(d)}$ can be represented by

$$A^{(d)}(i_1,\dots,i_{r-1}) := \{x \in \mathbb{R}^d : x_1 = \dots = x_{i_1} \geq \dots \geq x_{i_{r-1}} = \dots = x_d\} \in \mathcal{F}_r(A^{(d)})$$

for $d \in \mathbb{N}$, $1 \leq r \leq d$ and $1 \leq i_1 < \dots < i_{r-1} \leq d-1$. Note that for $r=1$, we obtain the 1-face $\{x \in \mathbb{R}^d : x_1 = \dots = x_d\}$. Thus, T_l has no k-faces for $k < j$ since the faces of $A^{(l_1)},\dots,A^{(l_j)}$ have at least dimension 1. Therefore, for $k < j$ the left-hand side of (4.2) is 0 coinciding with the right-hand side. From now on assume $k \geq j$. The s-faces of the Weyl chamber $B^{(d)}$ are given by

$$\begin{aligned} B^{(d)}(m_1,\dots,m_s) := \{x \in \mathbb{R}^d : x_1 = \dots = x_{m_1} \geq \dots \geq x_{m_{s-1}+1} = \dots = x_{m_s} \geq \\ x_{m_s+1} = \dots = x_d = 0\} \in \mathcal{F}_s(B^{(d)}) \end{aligned}$$

for $d \in \mathbb{N}$, $0 \leq s \leq d$ and $1 \leq m_1 < \dots < m_s \leq d$. Note that for $m_s = d$, no x_i's are required to be 0 and for $s=0$, we obtain the 0-dimensional face $\{0\}$.

Thus, for each k-face F of T_l, there is a collection of indices $r_1,\dots,r_j \in \mathbb{N}$, $s_1,\dots,s_b \in \mathbb{N}_0$ satisfying $r_1 + \dots + r_j + s_1 + \dots + s_b = k$, such that the face can be written as

$$\begin{aligned} F_{l,i,m} := A^{(l_1)}(i_1^{(1)},\dots,i_{r_1-1}^{(1)}) \times \dots \times A^{(l_j)}(i_1^{(j)},\dots,i_{r_j-1}^{(j)}) \\ \times B^{(l_{j+1})}(m_1^{(1)},\dots,m_{s_1}^{(1)}) \times \dots \times B^{(l_{j+b})}(m_1^{(b)},\dots,m_{s_b}^{(b)}) \end{aligned} \tag{4.7}$$

for the vectors $i = (i_1^{(1)},\dots,i_{r_1-1}^{(1)},\dots,i_1^{(j)},\dots,i_{r_j-1}^{(j)})$ and $m = (m_1^{(1)},\dots,m_{s_1}^{(1)},\dots,m_1^{(b)},\dots,m_{s_b}^{(b)})$ satisfying the conditions

$$\begin{aligned} &1 \leq i_1^{(1)} < \dots < i_{r_1-1}^{(1)} \leq l_1 - 1, \dots, 1 \leq i_1^{(j)} < \dots < i_{r_j-1}^{(j)} \leq l_j - 1, \\ &1 \leq m_1^{(1)} < \dots < m_{s_1}^{(1)} \leq l_{j+1}, \dots, 1 \leq m_1^{(b)} < \dots < m_{s_b}^{(b)} \leq l_{j+b}. \end{aligned} \tag{4.8}$$

For every $r_1,\dots,r_j,s_1,\dots,s_b$ as above and each (i,m) satisfying conditions (4.8), the cone $F_{l,i,m}$ defines a k-face of T_J.

Internal angles. Now, we want to compute the solid angle of $F_{l,i,m}$. Since $F_{l,i,m}$ is given as a product of faces, the angle $\alpha(F_{l,i,m})$ is also given as a product of angles. Thus, it is enough to evaluate the angles of $A^{(d)}(i_1,\dots,i_{r-1})$ and $B^{(d)}(m_1,\dots,m_s)$. Consider the linear hulls of $A^{(d)}(i_1,\dots,i_{r-1})$ and $B^{(d)}(m_1,\dots,m_s)$ given by

$$\operatorname{lin} A^{(d)}(i_1,\dots,i_{r-1}) = \{x \in \mathbb{R}^i : x_1 = \dots = x_{i_1},\dots,x_{i_{r-1}+1} = \dots = x_d\}$$

and

$$\operatorname{lin} B^{(d)}(m_1,\dots,m_s) = \{x \in \mathbb{R}^d : x_1 = \dots = x_{m_1},\dots,x_{m_{s-1}+1} = \dots = x_{m_s}, x_{m_s+1} = \dots = x_d = 0\}.$$

Thus, the vectors $y_1,\dots,y_r$ given by

$$\begin{aligned} y_1 = \frac{1}{\sqrt{i_1}}(\overbrace{1,\dots,1}^{1,\dots,i_1},0,\dots,0), y_2 = \frac{1}{\sqrt{i_2-i_1}}(0,\dots,0,\overbrace{1,\dots,1}^{i_1+1,\dots,i_2},0,\dots,0), \\ \dots, y_r = \frac{1}{\sqrt{d-i_{r-1}}}(0,\dots,0,\overbrace{1,\dots,1}^{i_{r-1}+1,\dots,d}) \end{aligned}$$

form an orthonormal basis of $\operatorname{lin} A^{(d)}(i_1,\dots,i_{r-1})$. Similarly, the vectors $z_1,\dots,z_s$ form an orthonormal basis of $\operatorname{lin} B^{(d)}(m_1,\dots,m_s)$, where

$$z_1=\frac{1}{\sqrt{m_1}}(\overbrace{1,\dots,1}^{1,\dots,m_1},0,\dots,0),\, z_2=\frac{1}{\sqrt{m_2-m_1}}(0,\dots,0,\overbrace{1,\dots,1}^{m_1+1,\dots,m_2},0,\dots,0),$$
$$\dots,z_s=\frac{1}{\sqrt{m_s-m_{s-1}}}(0,\dots,0,\overbrace{1,\dots,1}^{m_{s-1}+1,\dots,m_s},0,\dots,0).$$

Now let $\xi_1,\xi_2,\dots$ be independent and standard normal distributed random variables. Then, the random vector $N_1:=\xi_1y_1+\dots+\xi_ry_r$ is r-dimensional standard normal distributed on the linear subspace $\operatorname{lin} A^{(d)}(i_1,\dots,i_{r-1})$ and the random vector $N_2:=\xi_1z_1+\dots+\xi_sz_s$ is s-dimensional standard normal distributed on the linear subspace $\operatorname{lin} B^{(d)}(m_1,\dots,m_s)$. Thus, by definition, we obtain

$$\begin{aligned}\alpha\big(A^{(d)}(i_1,\dots,i_{r-1})\big)&=\mathbb{P}(N_1\in A^{(d)}(i_1,\dots,i_{r-1}))\\&=\mathbb{P}\Big(\frac{\xi_1}{\sqrt{i_1}}\geq\frac{\xi_2}{\sqrt{i_2-i_1}}\geq\dots\geq\frac{\xi_r}{\sqrt{d-i_{r-1}}}\Big)\end{aligned}$$

and similarly

$$\begin{aligned}\alpha\big(B^{(d)}(m_1,\dots,m_s)\big)&=\mathbb{P}(N_2\in B^{(d)}(m_1,\dots,m_s))\\&=\mathbb{P}\Big(\frac{\xi_1}{\sqrt{m_1}}\geq\frac{\xi_2}{\sqrt{m_2-m_1}}\geq\dots\geq\frac{\xi_s}{\sqrt{m_s-m_{s-1}}}\geq 0\Big).\end{aligned}$$

We will not evaluate these probabilities explicitly. In the course of this proof, we will divide them into groups so that each group of these probabilities adds up to 1. In summary, the solid angle of $F_{l,i,m}$ is given by

$$\begin{aligned}\alpha(F_{l,i,m})=\alpha\big(A^{(l_1)}(i_1^{(1)},\dots,i_{r_1-1}^{(1)})\big)\cdot\ldots\cdot\alpha\big(A^{(l_j)}(i_1^{(j)},\dots,i_{r_j-1}^{(j)})\big)&\\\times\alpha\big(B^{(l_{j+1})}(m_1^{(1)},\dots,m_{s_1}^{(1)})\big)\cdot\ldots\cdot\alpha\big(B^{(l_{j+b})}(m_1^{(b)},\dots,m_{s_b}^{(b)})\big).&\quad(4.9)\end{aligned}$$

External angles. Now, we need to evaluate the external angle of $F_{l,i,m}$, i.e. $\alpha(N_{F_{l,i,m}}(T_l))$. In order to do this, we consider the normal cone $N_{F_{l,i,m}}(T_l)=(\operatorname{lin}F_{l,i,m})^\perp\cap(T_l)^\circ$. We have

$$(T_l)^\circ=\big(A^{(l_1)}\big)^\circ\times\dots\times\big(A^{(l_j)}\big)^\circ\times\big(B^{(l_{j+1})}\big)^\circ\times\dots\times\big(B^{(l_{j+b})}\big)^\circ,$$

where for example

$$\begin{aligned}\big(A^{(l_1)}\big)^\circ&=\{x\in\mathbb{R}^{l_1}:x_1\geq\dots\geq x_{l_1}\}^\circ\\&=\operatorname{pos}\{e_2-e_1,e_3-e_2,\dots,e_{l_1}-e_{l_1-1}\}\\&=\{x\in\mathbb{R}^{l_1}:x_1\leq 0,x_1+x_2\leq 0,\dots,x_1+\dots+x_{l_1-1}\leq 0,x_1+\dots+x_{l_1}=0\}.\end{aligned}$$

using the duality relation (2.1) in the second line. Here, $e_1,\dots,e_{l_1}$ denotes the standard Euclidean orthonormal basis in the ambient space $\mathbb{R}^{l_1}$. Thus, the partial sums of the coordinates of the points from the dual cones of Weyl chambers of type A form "bridges" staying below zero. Similarly, we obtain

$$\begin{aligned}\big(B^{(l_{j+1})}\big)^\circ&=\operatorname{pos}\{e_2-e_1,e_3-e_2,\dots,e_{l_{j+1}}-e_{l_{j+1}-1},-e_{l_{j+1}}\}\\&=\{x\in\mathbb{R}^{l_{j+1}}:x_1\leq 0,x_1+x_2\leq 0,\dots,x_1+\dots+x_{l_{j+1}}\leq 0\},\end{aligned}$$

which we will also refer to as a "walk" staying below zero. The dual cones of the other Weyl chambers are obtained in the same way.

Recalling the definition of $F_{l,i,m}$ in (4.7), its linear hull is given as the product of linear hulls of the faces $A^{(l_1)}(i_1^{(1)},\dots,i_{r_1-1}^{(1)}),\dots,B^{(l_{j+b})}(m_1^{(b)},\dots,m_{s_b}^{(b)})$ which we already evaluated. Thus, $(\operatorname{lin} F_{l,i,m})^\perp$ is the product of the orthogonal complements of the linear hulls. For the r_1-face $A^{(l_1)}(i_1^{(1)},\dots,i_{r_1-1}^{(1)})$, we have

$$\begin{aligned}\left(\operatorname{lin} A^{(l_1)}(i_1^{(1)},\dots,l_{i_1-1}^{(1)})\right)^\perp &= \{x\in\mathbb{R}^{l_1}: x_1=\dots=x_{i_1^{(1)}},\dots,x_{i_{r_1-1}^{(1)}}=\dots=x_{l_1}\}^\perp\\ &= \{x\in\mathbb{R}^{l_1}: x_1+\dots+x_{i_1^{(1)}}=0,\dots,x_{i_{r_1-1}^{(1)}}+\dots+x_{l_1}=0\}.\end{aligned}$$

Similarly, for $B^{(l_{j+1})}(m_1^{(1)},\dots,m_{s_1}^{(1)})$, we get

$$\left(\operatorname{lin} B^{(l_{j+1})}(m_1^{(1)},\dots,m_{s_1}^{(1)})\right)^\perp = \{x\in\mathbb{R}^{l_{j+1}}: x_1+\dots+x_{m_1^{(1)}}=0,\dots,x_{m_{s_1-1}^{(1)}+1}+\dots+x_{m_{s_1}^{(1)}}=0\}.$$

Thus, $(\operatorname{lin} F_{l,i,m})^\perp$ consists of the points $x=(x^{(1)},\dots,x^{(j+b)})\in\mathbb{R}^n$ with $x^{(1)}\in\mathbb{R}^{l_1},\dots x^{(j+b)}\in\mathbb{R}^{l_{j+b}}$ whose components $x^{(q)}=(x_1^{(q)},\dots,x_{l_q}^{(q)})$ are such that the following conditions are satisfied:

$$x_1^{(q)}+\dots+x_{i_1^{(q)}}^{(q)}=0,\dots,x_{i_{r_q-1}^{(q)}}^{(q)}+\dots+x_{l_q}^{(q)}=0$$

for all $q\in\{1,\dots,j\}$ and

$$x_1^{(j+d)}+\dots+x_{m_1^{(d)}}^{(j+d)}=0,\dots,x_{m_{s_d-1}^{(d)}+1}^{(j+d)}+\dots+x_{m_{s_d}^{(d)}}^{(j+d)}=0$$

for all $d\in\{1,\dots,b\}$. In other words, each A-face corresponds to a group of l_q coordinates which is divided into smaller groups whose sums are required to be 0. Similarly, each B-face corresponds to a group of l_{j+d} coordinates which is divided into smaller subgroups whose sums are required to be 0, except for the last subgroup.

Taking both the description of the defining conditions of $(T_l)^\circ$ and $(\operatorname{lin} F_{l,i,m})^\perp$ into consideration, we obtain for the normal cone $N_{F_{l,i,m}}(T_l)=(T_l)^\circ\cap(\operatorname{lin} F_{l,i,m})^\perp$ the following conditions: $N_{F_{l,i,m}}(T_l)$ consists of the points $x=(x^{(1)},\dots,x^{(j+b)})\in\mathbb{R}^n$ with $x^{(1)}\in\mathbb{R}^{l_1},\dots x^{(j+b)}\in\mathbb{R}^{l_{j+b}}$, such that the partial sums of the groups

$$(x_1^{(q)},\dots,x_{i_1^{(q)}}^{(q)}),\dots,(x_{i_{r_q-1}^{(q)}}^{(q)},\dots,x_{l_q}^{(q)})$$

for all $q\in\{1,\dots,j\}$ and

$$(x_1^{(j+d)},\dots,x_{m_1^{(d)}}^{(j+d)}),\dots,(x_{m_{s_d-1}^{(d)}+1}^{(j+d)},\dots,x_{m_{s_d}^{(d)}}^{(j+d)}),$$

for all $d\in\{1,\dots,b\}$, form bridges staying non-positive, while the partial sums of the groups

$$(x_{m_{s_1}^{(1)}}^{(j+1)},\dots,x_{l_{j+1}}^{(j+1)}),\dots,(x_{m_b^{(b)}}^{(j+b)},\dots,x_{l_{j+b}}^{(j+b)}).$$

form walks staying non-positive. Note that each A-face corresponds to a collection of bridges, while each B-face corresponds to a collection of bridges and one walk.

The angles of the cones defined by such bridge or walk conditions can be interpreted as the probabilities that the corresponding bridges or walks stay non-positive. These probabilities are known explicitly thanks to the works of Sparre Andersen [18], [19]; see also [8], [14], [9, Section

3.1] for works relating these formulas to convex cones. The angle of a cone defined by a bridge condition is given by

$$\alpha(\{x \in \mathbb{R}^i : x_1 \le 0, \dots, x_1 + \dots + x_{i-1} \le 0, x_1 + \dots + x_i = 0\}) = 1/i.$$

Furthermore, the angle of a cone $W := \{x \in \mathbb{R}^i : x_1 \le 0, x_1 + x_2 \le 0, \dots, x_1 + \dots + x_i \le 0\}$ defined by a walk condition is given by

$$\begin{aligned}\alpha(W) &= \mathbb{P}(N \in W)\\ &= \mathbb{P}(N_1 \le 0, N_1 + N_2 \le 0, \dots, N_1 + \dots + N_i \le 0)\\ &= \binom{2i}{i}\frac{1}{2^{2i}}\end{aligned}$$

by a formula due to Sparre Andersen [18], where $N = (N_1, \dots, N_i)$ is an i-dimensional standard normal distributed vector. Taking all into consideration, we have

$$\begin{aligned}&\alpha(N_{F_{l,i,m}}(T_l))\\ &= \frac{1}{i_1^{(1)}(i_2^{(1)} - i_1^{(1)}) \cdot \ldots \cdot (l_1 - l_{r_1-1}^{(1)}) \cdots i_1^{(j)}(i_2^{(j)} - i_1^{(j)}) \cdot \ldots \cdot (l_j - i_{r_j-1}^{(j)})}\\ &\quad\times \frac{1}{m_1^{(1)}(m_2^{(1)} - m_1^{(1)}) \cdot \ldots \cdot (m_{s_1}^{(1)} - m_{s_1-1}^{(1)}) \cdots m_1^{(b)}(m_2^{(b)} - m_1^{(b)}) \cdot \ldots \cdot (m_{s_b}^{(b)} - m_{s_b-1}^{(b)})}\\ &\quad\times \frac{\binom{2(l_{j+1} - m_{s_1}^{(1)})}{l_{j+1} - m_{s_1}^{(1)}}}{2^{2(l_{j+1} - m_{s_1}^{(1)})}} \cdot \ldots \cdot \frac{\binom{2(l_{j+b} - m_{s_b}^{(b)})}{l_{j+b} - m_{s_b}^{(b)}}}{2^{2(l_{j+b} - m_{s_b}^{(b)})}}.\end{aligned} \tag{4.10}$$

Now, we can compute the sum of the k-th conic intrinsic volumes of T_l over all l:

$$\begin{aligned}&\sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+b} \in \mathbb{N}_0:\\ l_1+\dots+l_{j+b}=n}} \upsilon_k(T_l)\\ &= \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+b} \in \mathbb{N}_0:\\ l_1+\dots+l_{j+b}=n}} \sum_{F \in \mathcal{F}_k(T_l)} \alpha(F)\alpha(N_F(T_l))\\ &= \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+b} \in \mathbb{N}_0:\\ l_1+\dots+l_{j+b}=n}} \sum_{\substack{r_1,\dots,r_j \in \mathbb{N}, s_1,\dots,s_b \in \mathbb{N}_0:\\ r_1+\dots+r_j+s_1+\dots+s_b=k}} \sum_{(i,m) \text{ satisfying (4.8)}} \alpha(F_{l,i,m})\alpha(N_{F_{l,i,m}}(T_l))\\ &= \sum_{\substack{r_1,\dots,r_j \in \mathbb{N}, s_1,\dots,s_b \in \mathbb{N}_0:\\ r_1+\dots+r_j+s_1+\dots+s_b=k}} \sum_{\substack{l_1,\dots,l_j \in \mathbb{N}, l_{j+1},\dots,l_{j+b} \in \mathbb{N}_0:\\ l_1+\dots+l_{j+b}=n}} \sum_{(i,m) \text{ satisfying (4.8)}} \alpha(F_{l,i,m})\alpha(N_{F_{l,i,m}}(T_l)).\end{aligned}$$

Let us introduce the notation for the lengths of the bridges and walks. For the lenghts of the bridges corresponding to A-faces, we write

$$\begin{aligned}h_1 = i_1^{(1)}, h_2 = i_2^{(1)} - i_1^{(1)}, \dots, h_{r_1} = l_1 - i_{r_1-1}^{(1)}, \dots,&\\ h_{r_1+\dots+r_{j-1}+1} = i_1^{(j)}, h_{r_1+\dots+r_{j-1}+2} = i_2^{(j)} - i_1^{(j)}, \dots, h_{r_1+\dots+r_j} = l_j - i_{r_j-1}^{(j)}.&\end{aligned}$$

Similarly, the lengths of the bridges corresponding to B-faces are denoted by

$$h_{r_1+\ldots+r_j+1}=m_1^{(1)},h_{r_1+\ldots+r_j+2}=m_2^{(1)}-m_1^{(1)},\ldots,h_{r_1+\ldots+r_j+s_1}=m_{s_1}^{(1)}-m_{s_1-1}^{(1)},$$
$$\ldots,h_{r_1+\ldots+r_j+s_1+\ldots+s_b}=h_k=m_{s_b}^{(b)}-m_{s_b-1}^{(b)}.$$

All these numbers are in $\mathbb{N}$. Additionally, the lengths of the walks corresponding to the B-faces are denoted by

$$h_{k+1}=l_{j+1}-m_{s_1}^{(1)},h_{k+2}=l_{j+2}-m_{s_2}^{(2)},\ldots,h_{k+b}=l_{j+b}-m_{s_b}^{(b)}.$$

These numbers are in $\mathbb{N}_0$. Now, we can change the summation and arrive at

$$\begin{aligned}&\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N},s_1,\ldots,s_b\in\mathbb{N}_0\\ r_1+\ldots+r_j+s_1+\ldots+s_b=k}}\ \sum_{\substack{h_1,\ldots,h_k\in\mathbb{N},h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0\\ h_1+\ldots+h_{k+b}=n}}\frac{\binom{2h_{k+1}}{h_{k+1}}\cdot\ldots\cdot\binom{2h_{k+b}}{h_{k+b}}}{h_1h_2\cdot\ldots\cdot h_k\cdot 2^{2(h_{k+1}+\ldots+h_{k+b})}}\\
&\times\mathbb{P}\Big(\frac{\xi_1}{\sqrt{h_1}}\geq\ldots\geq\frac{\xi_{r_1}}{\sqrt{h_{r_1}}}\Big)\cdot\ldots\cdot\mathbb{P}\Big(\frac{\xi_1}{\sqrt{h_{r_1+\ldots+r_{j-1}+1}}}\geq\ldots\geq\frac{\xi_{r_j}}{\sqrt{h_{r_1+\ldots+r_j}}}\Big)\\
&\times\mathbb{P}\Big(\frac{\xi_1}{\sqrt{h_{r_1+\ldots+r_j+1}}}\geq\ldots\geq\frac{\xi_{s_1}}{\sqrt{h_{r_1+\ldots+r_j+s_1}}}\geq 0\Big)\\
&\times\ldots\times\mathbb{P}\Big(\frac{\xi_1}{\sqrt{h_{r_1+\ldots+r_j+s_1+\ldots+s_{b-1}+1}}}\geq\ldots\geq\frac{\xi_{s_b}}{\sqrt{h_{r_1+\ldots+r_j+s_1+\ldots+s_b}}}\geq 0\Big)\end{aligned}\tag{4.11}$$

using the explicit representation of the internal angles in (4.9) and external angles in (4.10).

After introducing additional (signed) permutations of the ξ_i's, we will arrange the probabilities in the above sum in groups that sum up to 1. In order to explain this, consider a permutation $\pi\in\mathrm{Sym}(r_1)$. For each tuple $(h_1,\ldots,h_{k+b})$ such that $h_1+\ldots+h_{k+b}=n$, the tuple $(h_{\pi(1)},\ldots,h_{\pi(r_1)},h_{r_1+1},\ldots,h_{k+b})$ (in which we permuted the first r_1 coordinates according to π) satisfies the same condition. Thus the above sum does not change if we replace $(h_1,\ldots,h_{r_1})$ by $(h_{\pi(1)},\ldots,h_{\pi(r_1)})$ inside the summation. Additionally, we observe that

$$\sum_{\pi\in\mathrm{Sym}(r_1)}\mathbb{P}\Big(\frac{\xi_{\pi(1)}}{\sqrt{h_{\pi(1)}}}\geq\ldots\geq\frac{\xi_{\pi(r_1)}}{\sqrt{h_{\pi(r_1)}}}\Big)=1$$

and therefore also

$$\sum_{\pi\in\mathrm{Sym}(r_1)}\mathbb{P}\Big(\frac{\xi_1}{\sqrt{h_{\pi(1)}}}\geq\ldots\geq\frac{\xi_{r_1}}{\sqrt{h_{\pi(r_1)}}}\Big)=1$$

holds true. The same argument can be applied for the first j probabilities in (4.11). For the other b probabilities, we argue in a similar way using signed permutations. For example, we have

$$\sum_{(\varepsilon,\pi)\in\{\pm1\}^{s_1}\times\mathrm{Sym}(s_1)}\mathbb{P}\Bigg(\frac{\varepsilon_1\xi_{\pi(1)}}{\sqrt{h_{r_1+\ldots+r_j+\pi(1)}}}\geq\ldots\geq\frac{\varepsilon_{s_1}\xi_{\pi(s_1)}}{\sqrt{h_{r_1+\ldots+r_j+\pi(s_1)}}}\geq 0\Bigg)=1$$

and thus, also

$$\sum_{(\varepsilon,\pi)\in\{\pm1\}^{s_1}\times\mathrm{Sym}(s_1)}\mathbb{P}\Bigg(\frac{\varepsilon_1\xi_1}{\sqrt{h_{r_1+\ldots+r_j+\pi(1)}}}\geq\ldots\geq\frac{\varepsilon_{s_1}\xi_{s_1}}{\sqrt{h_{r_1+\ldots+r_j+\pi(s_1)}}}\geq 0\Bigg)=1.$$

Thus, we can rewrite (4.11) to obtain

$$\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N},s_1,\ldots,s_b\in\mathbb{N}_0\\ r_1+\ldots+r_j+s_1+\ldots+s_b=k}} \frac{1}{r_1!\cdot\ldots\cdot r_j!s_1!\cdot\ldots\cdot s_b!2^{s_1+\ldots+s_b}} \times \sum_{\substack{h_1,\ldots,h_k\in\mathbb{N},h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0\\ h_1+\ldots+h_{k+b}=n}} \frac{\binom{2h_{k+1}}{h_{k+1}}\cdot\ldots\cdot\binom{2h_{k+b}}{h_{k+b}}}{h_1h_2\cdot\ldots\cdot h_k\cdot 2^{2(h_{k+1}+\ldots+h_{k+b})}}. \tag{4.12}$$

The sums can be evaluated separately. We start with the first sum. By summing over the possible values m of $s_1+\ldots+s_b$, we get

$$\begin{aligned}
&\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N},s_1,\ldots,s_b\in\mathbb{N}_0\\ r_1+\ldots+r_j+s_1+\ldots+s_b=k}} \frac{1}{r_1!\cdot\ldots\cdot r_j!s_1!\cdot\ldots\cdot s_b!2^{s_1+\ldots+s_b}}\\
&=\sum_{m=0}^{k-j}\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N}:\\ r_1+\ldots+r_j+m=k}} \frac{1}{r_1!\cdot\ldots\cdot r_j!}\sum_{\substack{s_1,\ldots,s_b\in\mathbb{N}_0:\\ s_1+\ldots+s_b=m}}\frac{1}{2^m s_1!\cdot\ldots\cdot s_b!}\\
&=\sum_{m=0}^{k-j}\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N}:\\ r_1+\ldots+r_j+m=k}} \frac{1}{r_1!\cdot\ldots\cdot r_j!m!2^m}\sum_{\substack{s_1,\ldots,s_b\in\mathbb{N}_0:\\ s_1+\ldots+s_b=m}}\binom{m}{s_1,\ldots,s_b}\\
&=\sum_{m=0}^{k-j}\sum_{\substack{r_1,\ldots,r_j\in\mathbb{N}:\\ r_1+\ldots+r_j=k-m}} \frac{b^m}{r_1!\cdot\ldots\cdot r_j!m!2^m},
\end{aligned} \tag{4.13}$$

where we used the multinomial theorem in the last step. Using the representation in (2.12) of the Stirling number of the second kind, we can rewrite (4.13) to obtain

$$\begin{aligned}
\sum_{m=0}^{k-j}\left(\frac{b}{2}\right)^m\frac{j!}{m!(k-m)!}\left\{{k-m \atop j}\right\} &= \frac{j!}{k!}\sum_{m=0}^{k-j}\left(\frac{b}{2}\right)^m\binom{k}{k-m}\left\{{k-m \atop j}\right\}\\
&=\frac{j!}{k!}\sum_{m=j}^{k}\binom{k}{m}\left\{{m \atop j}\right\}\left(\frac{b}{2}\right)^{k-m}\\
&=\frac{j!}{k!}\left\{{k+b/2 \atop j+b/2}\right\}_{b/2},
\end{aligned}$$

where the last step follows from the definition of the r-Stirling numbers of the second kind in (2.20).

The second sum of (4.12) can be rewritten as follows:

$$\sum_{m=0}^{n-k}\sum_{\substack{h_1,\ldots,h_k\in\mathbb{N}:\\ h_1+\ldots+h_k+m=n}}\frac{1}{h_1\cdot\ldots\cdot h_k}\sum_{\substack{h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0:\\ h_{k+1}+\ldots+h_{k+b}=m}}\frac{1}{2^{2m}}\binom{2h_{k+1}}{h_{k+1}}\cdot\ldots\cdot\binom{2h_{k+b}}{h_{k+b}}. \tag{4.14}$$

A generalization of Vandermonde's convolution [10, (3.27)] yields

$$\sum_{\substack{h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0:\\ h_{k+1}+\ldots+h_{k+b}=m}} \binom{-1/2}{h_{k+1}}\cdot\ldots\cdot\binom{-1/2}{h_{k+b}} = \binom{-b/2}{m}. \tag{4.15}$$

Using the formula

$$\binom{-1/2}{d} = \Big(-\frac{1}{4}\Big)^d\binom{2d}{d}, \quad d\in\mathbb{N}_0;$$

see [10, (5.37)], we have

$$\binom{-1/2}{h_{k+1}}\cdot\ldots\cdot\binom{-1/2}{h_{k+b}} = \frac{(-1)^m}{4^m}\binom{2h_{k+1}}{h_{k+1}}\cdot\ldots\cdot\binom{2h_{k+b}}{h_{k+b}},$$

for all $h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0$ such that $h_{k+1}+\ldots+h_{k+b}=m$. Combining this with (4.15), we obtain

$$\begin{aligned}\sum_{\substack{h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0:\\ h_{k+1}+\ldots+h_{k+b}=m}} \binom{2h_{k+1}}{h_{k+1}}\cdot\ldots\cdot\binom{2h_{k+b}}{h_{k+b}} &= (-4)^m \sum_{\substack{h_{k+1},\ldots,h_{k+b}\in\mathbb{N}_0:\\ h_{k+1}+\ldots+h_{k+b}=m}} \binom{-1/2}{h_{k+1}}\cdot\ldots\cdot\binom{-1/2}{h_{k+b}}\\ &= (-4)^m\binom{-b/2}{m}.\end{aligned}$$

Applying this to (4.14) yields

$$\begin{aligned}&\sum_{m=0}^{n-k}\sum_{\substack{h_1,\ldots,h_k\in\mathbb{N}:\\ h_1+\ldots+h_k+m=n}}\frac{1}{h_1\cdot\ldots\cdot h_k}\frac{1}{2^{2m}}(-4)^m\binom{-b/2}{m}\\ &=\sum_{m=0}^{n-k}(-1)^m\binom{-b/2}{m}\sum_{\substack{h_1,\ldots,h_k\in\mathbb{N}:\\ h_1+\ldots+h_k=n-m}}\frac{1}{h_1\cdot\ldots\cdot h_k}\\ &=\sum_{m=0}^{n-k}\frac{b}{2}\Big(\frac{b}{2}+1\Big)\cdot\ldots\cdot\Big(\frac{b}{2}+m-1\Big)\frac{k!}{m!(n-m)!}\begin{bmatrix}n-m\\ k\end{bmatrix}\\ &=\frac{k!}{n!}\sum_{m=0}^{n-k}\binom{n}{m}\begin{bmatrix}n-m\\ k\end{bmatrix}\frac{b}{2}\Big(\frac{b}{2}+1\Big)\cdot\ldots\cdot\Big(\frac{b}{2}+m-1\Big)\\ &=\frac{k!}{n!}\begin{bmatrix}n+b/2\\ k+b/2\end{bmatrix}_{b/2},\end{aligned}$$

where we used the representation in (2.8) of the Stirling numbers of the first kind and the definition of the r-Stirling numbers of the first kind in (2.19). If we insert both sums in (4.12), we finally obtain

$$\sum_{\substack{l_1,\ldots,l_j\in\mathbb{N},l_{j+1},\ldots,l_{j+b}\in\mathbb{N}_0:\\ l_1+\ldots+l_{j+b}=n}} \upsilon_k(T_l) = \frac{j!}{k!}\begin{Bmatrix}k+b/2\\ j+b/2\end{Bmatrix}_{b/2}\frac{k!}{n!}\begin{bmatrix}n+b/2\\ k+b/2\end{bmatrix}_{b/2} = \frac{j!}{n!}\begin{bmatrix}n+b/2\\ k+b/2\end{bmatrix}_{b/2}\begin{Bmatrix}k+b/2\\ j+b/2\end{Bmatrix}_{b/2},$$

which completes the proof. □

4.3. **Proofs of Theorems 3.7 and 3.8.** Let $b \in \mathbb{N}$. Recall that $K^B = K^B_{n_1} \times \ldots \times K^B_{n_b}$ for $n_1, \ldots, n_b \in \mathbb{N}_0$ such that $n := n_1 + \ldots + n_b$, where

$$K^B_d := \{x \in \mathbb{R}^d : 1 \geq x_1 \geq x_2 \geq \ldots \geq x_d \geq 0\}, \quad d \in \mathbb{N},$$

denotes the Schläfli orthoscheme of type B in $\mathbb{R}^d$. For convenience, we set $K^B_0 := \{0\}$. We want to show that

$$\sum_{F \in \mathcal{F}_j(K^B)} \upsilon_k(T_F(K^B)) = R_1\big(k, j, b, (n_1, \ldots, n_b)\big)$$

holds for all $j \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, n\}$, where for $d \in \{0, \frac{1}{2}, 1\}$ we define

$$R_d\big(k, j, b, (n_1, \ldots, n_b)\big) := \big[t^k\big]\big[x_1^{n_1} \cdot \ldots \cdot x_b^{n_b}\big]\big[u^j\big] \frac{(1-x_1)^{-d(t+1)} \cdot \ldots \cdot (1-x_b)^{-d(t+1)}}{\big(1 - u((1-x_1)^{-t} - 1)\big) \cdot \ldots \cdot \big(1 - u((1-x_b)^{-t} - 1)\big)}.$$

Recall that

$$\big[x_1^{N_1} \cdot \ldots \cdot x_b^{N_b}\big] g(x_1, \ldots, x_b) := \frac{1}{N_1! \cdot \ldots \cdot N_b!} \frac{\partial^{N_1 + \ldots + N_b}}{\partial x_1^{N_1} \ldots \partial x_b^{N_b}} g(0, \ldots, 0)$$

is the coefficient of $x_1^{N_1} \cdot \ldots \cdot x_b^{N_b}$ in the multidimensional Taylor expansion of a function $g : \mathbb{R}^b \to \mathbb{R}$.

Proof of Theorem 3.7. Let $j \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, n\}$ be given. It is easy to check that $R_d\big(k, j, b, (n_1, \ldots, n_b)\big) = 0$ for $k < j$, so that we need to consider the case $k \geq j$ only. For each j-face F of K^B, there are numbers $j_1, \ldots, j_b \in \mathbb{N}_0$ satisfying $j_1 + \ldots + j_b = j$, such that

$$F = F_1 \times \ldots \times F_b$$

for some $F_1 \in \mathcal{F}_{j_1}(K^B_{n_1}), \ldots, F_b \in \mathcal{F}_{j_b}(K^B_{n_b})$. Thus, the tangent cone of K^B at F is given by the following product formula:

$$T_F(K^B) = T_{F_1}(K^B_{n_1}) \times \ldots \times T_{F_b}(K^B_{n_b}).$$

In order to see this, observe that $\operatorname{relint} F = \operatorname{relint} F_1 \times \ldots \times \operatorname{relint} F_b$. Take $x = (x^{(1)}, \ldots, x^{(b)}) \in \operatorname{relint} F \subset \mathbb{R}^n$, where $x^{(i)} \in \operatorname{relint} F_i \subset \mathbb{R}^{n_i}$ for $i = 1, \ldots, b$. Then, we have

$$\begin{aligned}
&T_F(K^B)\\
&\quad = \{v \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ with } x + \varepsilon v \in K^B\}\\
&\quad = \{v \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_b} : \exists \varepsilon > 0 \text{ with } x + \varepsilon v \in K^B_{n_1} \times \ldots \times K^B_{n_b}\}\\
&\quad = \{v_1 \in \mathbb{R}^{n_1} : \exists \varepsilon > 0 \text{ with } x^{(1)} + \varepsilon v_1 \in K^B_{n_1}\} \times \ldots \times \{v_b \in \mathbb{R}^{n_b} : \exists \varepsilon > 0 \text{ with } x^{(b)} + \varepsilon v_b \in K^B_{n_b}\}\\
&\quad = T_{F_1}(K^B_{n_1}) \times \ldots \times T_{F_b}(K^B_{n_b}).
\end{aligned}$$

Applying Lemma 3.13 to the individual terms in the product, we observe that the collection

$$T_{F_1}(K^B_{n_1}) \times \ldots \times T_{F_b}(K^B_{n_b}),$$

where $F_1 \in \mathcal{F}_{j_1}(K^B_{n_1}), \ldots, F_b \in \mathcal{F}_{j_b}(K^B_{n_b})$, coincides (up to isometries) with the collection of cones

$$G_i := A^{(i_1^{(1)})} \times \ldots \times A^{(i_{j_1}^{(1)})} \times \cdots \times A^{(i_1^{(b)})} \times \ldots \times A^{(i_{j_b}^{(b)})} \times B^{(i_0^{(1)})} \times B^{(i_{j_1+1}^{(1)})} \times \ldots \times B^{(i_0^{(b)})} \times B^{(i_{j_b+1}^{(b)})}$$

where $i_1^{(1)},\dots,i_{j_1}^{(1)},\dots,i_1^{(b)},\dots,i_{j_b}^{(b)}\in\mathbb{N}$ and $i_0^{(1)},i_{j_1+1}^{(1)},\dots,i_0^{(b)},i_{j_b+1}^{(b)}\in\mathbb{N}_0$ such that

$$i_0^{(1)}+\ldots+i_{j_1+1}^{(1)}=n_1,\dots,i_0^{(b)}+\ldots+i_{j_b+1}^{(b)}=n_b.$$

This yields

$$\begin{aligned}\sum_{F\in\mathcal{F}_j(K^B)}\upsilon_k(T_F(K^B))&=\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\;\sum_{F_1\in\mathcal{F}_{j_1}(K^B_{n_1}),\dots,F_b\in\mathcal{F}_{j_b}(K^B_{n_b})}\upsilon_k\big(T_{F_1}(K^B_{n_1})\times\ldots\times T_{F_b}(K^B_{n_b})\big)\\ &=\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\;\sum_{\substack{i_0^{(1)}+\ldots+i_{j_1+1}^{(1)}=n_1\\ \dots\\ i_0^{(b)}+\ldots+i_{j_b+1}^{(b)}=n_b}}\upsilon_k(G_i).\end{aligned}\tag{4.16}$$

Similarly to the proof of Proposition 3.12, we observe that $\nu_k(G_i)$ is the coefficient of t^k in the polynomial

$$\begin{aligned}&\frac{t^{\overline{i_1^{(1)}}}}{i_1^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_1}^{(1)}}}}{i_{j_1}^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_1^{(b)}}}}{i_1^{(b)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_b}^{(b)}}}}{i_{j_b}^{(b)}!}\\ &\qquad\times\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_0^{(1)}-1)}{2^{i_0^{(1)}}i_0^{(1)}!}\cdot\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_{j_1+1}^{(1)}-1)}{2^{i_{j_1+1}^{(1)}}i_{j_1+1}^{(1)}!}\\ &\qquad\quad\times\ldots\times\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_0^{(b)}-1)}{2^{i_0^{(b)}}i_0^{(b)}!}\cdot\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_{j_b+1}^{(b)}-1)}{2^{i_{j_b+1}^{(b)}}i_{j_b+1}^{(b)}!}.\end{aligned}$$

Following the same arguments as in the proof of Proposition 3.12, we obtain

$$\begin{aligned}&\sum_{F\in\mathcal{F}_j(K^B)}\upsilon_k(T_F(K^B))\\ &\quad=[t^k]\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\;\sum_{\substack{i_0^{(1)}+\ldots+i_{j_1+1}^{(1)}=n_1\\ \dots\\ i_0^{(b)}+\ldots+i_{j_b+1}^{(b)}=n_b}}\Bigg(\frac{t^{\overline{i_1^{(1)}}}}{i_1^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_1}^{(1)}}}}{i_{j_1}^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_1^{(b)}}}}{i_1^{(b)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_b}^{(b)}}}}{i_{j_b}^{(b)}!}\\ &\qquad\times\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_0^{(1)}-1)}{2^{i_0^{(1)}}i_0^{(1)}!}\cdot\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_{j_1+1}^{(1)}-1)}{2^{i_{j_1+1}^{(1)}}i_{j_1+1}^{(1)}!}\\ &\qquad\times\ldots\times\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_0^{(b)}-1)}{2^{i_0^{(b)}}i_0^{(b)}!}\cdot\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2i_{j_b+1}^{(b)}-1)}{2^{i_{j_b+1}^{(b)}}i_{j_b+1}^{(b)}!}\Bigg).\end{aligned}$$

We introduce new variables $x_1,\dots,x_b$ and expand the product to obtain

$$[t^k]\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\Bigg([x_1^{n_1}]\Bigg(\sum_{l=1}^{\infty}\frac{t^{\overline{l}}}{l!}x_1^l\Bigg)^{j_1}\Bigg(\sum_{l=0}^{\infty}\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l-1)}{2^l l!}x_1^l\Bigg)^2\times\ldots$$

$$\times \left[x_b^{n_b}\right]\left(\sum_{l=1}^{\infty}\frac{t^{\bar{l}}}{l!}x_b^l\right)^{j_b}\left(\sum_{l=0}^{\infty}\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l-1)}{2^l l!}x_b^l\right)^2\Bigg).$$

By inserting the formulas (4.5) and (4.6) given by

$$\sum_{l=1}^{\infty}\frac{t^{\bar{l}}}{l!}x_i^l=(1-x_i)^{-t}-1\quad\text{and}\quad\sum_{l=0}^{\infty}\frac{(t+1)(t+3)\cdot\ldots\cdot(t+2l-1)}{2^l l!}x_i^l=(1-x_i)^{-\frac{1}{2}(t+1)},$$

for $i=1,\ldots,b$, we arrive at

$$\sum_{F\in\mathcal{F}_j(K^B)}\upsilon_k(T_F(K^B))=\left[t^k\right]\sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\Bigg(\left[x_1^{n_1}\right]\left((1-x_1)^{-(t+1)}\left((1-x_1)^{-t}-1\right)^{j_1}\right)$$

$$\times\ldots\times\left[x_b^{n_b}\right]\left((1-x_b)^{-(t+1)}\left((1-x_b)^{-t}-1\right)^{j_b}\right)\Bigg)$$

$$=\left[t^k\right]\left[x_1^{n_1}\cdot\ldots\cdot x_b^{n_b}\right]\sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\Bigg((1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}$$

$$\times\left((1-x_1)^{-t}-1\right)^{j_1}\cdot\ldots\cdot\left((1-x_b)^{-t}-1\right)^{j_b}\Bigg).$$

By introducing a new variable u and expanding the product again, we arrive at

$$\sum_{F\in\mathcal{F}_j(K^B)}\upsilon_k(T_F(K^B))=\left[t^k\right]\left[x_1^{n_1}\cdot\ldots\cdot x_b^{n_b}\right]\left[u^j\right]\left((1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}\right)$$

$$\times\left(\sum_{l=0}^{\infty}\left((1-x_1)^{-t}-1\right)^l u^l\right)\cdot\ldots\cdot\left(\sum_{l=0}^{\infty}\left((1-x_b)^{-t}-1\right)^l u^l\right)$$

$$=\left[t^k\right]\left[x_1^{n_1}\cdot\ldots\cdot x_b^{n_b}\right]\left[u^j\right]\left((1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}\right)$$

$$\times\frac{1}{\left(1-u((1-x_1)^{-t}-1)\right)\cdot\ldots\cdot\left(1-u((1-x_b)^{-t}-1)\right)}$$

$$=R_1\big(k,j,b,(n_1,\ldots,n_b)\big),$$

which completes the proof. □

Now, we want to prove Theorem 3.8. Let $b\in\mathbb{N}$. Recall that $K^A=K_{n_1}^A\times\ldots\times K_{n_b}^A$ and $\widetilde{K}^A=\widetilde{K}_{n_1}^A\times\ldots\times\widetilde{K}_{n_b}^A$ for $n_1,\ldots,n_b\in\mathbb{N}_0$ such that $n:=n_1+\ldots+n_b$, where

$$K_d^A=\{x\in\mathbb{R}^{d+1}:x_1\geq\ldots\geq x_{d+1},x_1-x_{d+1}\leq 1,x_1+\ldots+x_{d+1}=0\},\quad d\in\mathbb{N},$$

$K_0^A:=\{0\}$, denotes the Schläfli orthoscheme of type A in $\mathbb{R}^d$ and

$$\widetilde{K}_d^A=\{x\in\mathbb{R}^{d+1}:x_1\geq\ldots\geq x_{d+1},x_1-x_{d+1}\leq 1\},\quad d\in\mathbb{N}.$$

the related unbounded polyhedral set. Note that $\widetilde{K}_0^A=\mathbb{R}$. We want to show prove that

$$\sum_{F\in\mathcal{F}_j(K^A)}\upsilon_k(T_F(K^A))=R_1\big(k,j,b,(n_1,\ldots,n_b)\big),$$

holds for all $j\in\{0,\ldots,n\}$ and $k\in\{0,\ldots,n\}$.

Proof of Theorem 3.8. The proof is similar to the proof of Theorem 3.7 and we will not give each argument in full detail. We start by considering the tangent cones of $\widetilde{K}^A$. Since $\widetilde{K}^A$ is given as the product of the polyhedral sets $\widetilde{K}^A_{n_1},\dots,\widetilde{K}^A_{n_b}$, its j-faces are given as products $F_1\times\dots\times F_b$, where F_i is a j_i-face of $\widetilde{K}^A_{n_i}$ and $j_1,\dots,j_b\in\mathbb{N}$ satisfy $j_1+\dots+j_b=j$, where $j\in\{b,\dots,n+b\}$. Note that $j_1,\dots,j_b$ are different from 0 since the polyhedral sets $\widetilde{K}^A_{n_1},\dots,\widetilde{K}^A_{n_b}$ have no 0-dimensional faces. As in the proof of Theorem 3.7, we observe that the tangent cone $T_F(\widetilde{K}^A)$ is given as

$$T_F(\widetilde{K}^A)=T_{F_1}(\widetilde{K}^A_{n_1})\times\dots\times T_{F_b}(\widetilde{K}^A_{n_b}).$$

Applying Lemma 3.14 to each individual tangent cone in the product, we see that the collection of tangent cones $T_{F_1}(\widetilde{K}^A_{n_1})\times\dots\times T_{F_b}(\widetilde{K}^A_{n_b})$, where $F_1\in\mathcal{F}_{j_1}(\widetilde{K}^A_{n_1}),\dots,F_b\in\mathcal{F}_{j_b}(\widetilde{K}^A_{n_b})$, coincides (up to isometry) with the collection

$$A^{(i_1^{(1)})}\times\dots\times A^{(i_{j_1}^{(1)})}\times\dots\times A^{(i_1^{(b)})}\times\dots\times A^{(i_{j_b}^{(b)})},$$

where $i_1^{(1)},\dots,i_{j_1}^{(1)},\dots,i_1^{(b)},\dots,i_{j_b}^{(b)}\in\mathbb{N}$ such that

$$i_1^{(1)}+\dots+i_{j_1}^{(1)}=n_1+1,\dots,i_1^{(b)}+\dots+i_{j_b}^{(b)}=n_b+1$$

and each cone of the above collection is taken with multiplicity $i_1^{(1)}\cdot\ldots\cdot i_1^{(b)}$. This yields

$$\begin{aligned}\sum_{F\in\mathcal{F}_j(\widetilde{K}^A)}\upsilon_k(T_F(\widetilde{K}^A))&=\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}:\\ j_1+\dots+j_b=j}}\;\sum_{F_1\in\mathcal{F}_{j_1}(\widetilde{K}^A_{n_1}),\dots,F_b\in\mathcal{F}_{j_b}(\widetilde{K}^A_{n_b})}\upsilon_k\big(T_{F_1}(\widetilde{K}^A_{n_1})\times\dots\times T_{F_b}(\widetilde{K}^A_{n_b})\big)\\&=\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}:\\ j_1+\dots+j_b=j}}\;\sum_{\substack{i_1^{(1)}+\dots+i_{j_1}^{(1)}=n_1+1\\ \dots\\ i_1^{(b)}+\dots+i_{j_b}^{(b)}=n_b+1}} i_1^{(1)}\cdot\ldots\cdot i_1^{(b)}\upsilon_k\big(A^{(i_1^{(1)})}\times\dots\times A^{(i_{j_b}^{(b)})}\big)\qquad(4.17)\end{aligned}$$

for $k\in\{b,\dots,n+b\}$ and $j\in\{b,\dots,n+b\}$. The conic intrinsic volume on the right hand side of (4.17) is given as the coefficient of t^k in the following polynomial

$$\frac{t^{\overline{i_1^{(1)}}}}{i_1^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_1}^{(1)}}}}{i_{j_1}^{(1)}!}\times\dots\times\frac{t^{\overline{i_1^{(b)}}}}{i_1^{(b)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_b}^{(b)}}}}{i_{j_b}^{(b)}!}.$$

Thus, we obtain

$$\begin{aligned}&\sum_{F\in\mathcal{F}_j(\widetilde{K}^A)}\upsilon_k(T_F(\widetilde{K}^A))\\&=[t^k]\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}:\\ j_1+\dots+j_b=j}}\;\sum_{\substack{i_1^{(1)}+\dots+i_{j_1}^{(1)}=n_1+1\\ \dots\\ i_1^{(b)}+\dots+i_{j_b}^{(b)}=n_b+1}}\left(\frac{t^{\overline{i_1^{(1)}}}}{(i_1^{(1)}-1)!}\frac{t^{\overline{i_2^{(1)}}}}{i_2^{(1)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_1}^{(1)}}}}{i_{j_1}^{(1)}!}\times\dots\times\frac{t^{\overline{i_1^{(b)}}}}{(i_1^{(b)}-1)!}\frac{t^{\overline{i_2^{(b)}}}}{i_2^{(b)}!}\cdot\ldots\cdot\frac{t^{\overline{i_{j_b}^{(b)}}}}{i_{j_b}^{(b)}!}\right)\\&=[t^k]\sum_{\substack{j_1,\dots,j_b\in\mathbb{N}:\\ j_1+\dots+j_b=j}}\Bigg([x_1^{n_1+1}]\Big(tx_1(1-x_1)^{-(t+1)}\left((1-x_1)^{-t}-1\right)^{j_1-1}\Big)\\&\qquad\qquad\times\dots\times[x_b^{n_b+1}]\Big(tx_b(1-x_b)^{-(t+1)}\left((1-x_b)^{-t}-1\right)^{j_b-1}\Big)\Bigg)\end{aligned}$$

$$= \big[t^k\big]\big[x_1^{n_1+1}\cdot\ldots\cdot x_b^{n_b+1}\big]t^b x_1\cdot\ldots\cdot x_b(1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}$$
$$\times\sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}:\\ j_1+\ldots+j_b=j}}\Big(\big((1-x_b)^{-t}-1\big)^{j_b-1}\cdot\ldots\cdot\big((1-x_b)^{-t}-1\big)^{j_b-1}\Big),$$

where we expanded the sum and used (4.5) and

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{t^{\overline{l}}}{(l-1)!}x^l &= tx\sum_{l=0}^{\infty}\frac{(t+1)(t+2)\cdot\ldots\cdot(t+l)}{l!}x^l\\ &= tx\sum_{l=0}^{\infty}\binom{-(t+1)}{l}(-x)^l\\ &= tx(1-x)^{-(t+1)},\end{aligned}$$

which follows from the binomial series. The inner sum can be rewritten as

$$\begin{aligned}&\sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j-b}}\Big(\big((1-x_b)^{-t}-1\big)^{j_b}\cdot\ldots\cdot\big((1-x_b)^{-t}-1\big)^{j_b}\Big)\\ &= \big[u^{j-b}\big]\left(\left(\sum_{l=0}^{\infty}\big((1-x_1)^{-t}-1\big)^l u^l\right)\times\ldots\times\left(\big((1-x_b)^{-t}-1\big)^l u^l\right)\right)\\ &= \big[u^{j-b}\big]\frac{1}{\big(1-u((1-x_1)^{-t}-1)\big)\cdot\ldots\cdot\big(1-u((1-x_b)^{-t}-1)\big)}\end{aligned}$$

using the geometric series. Taking all into consideration, we have

$$\begin{aligned}&\sum_{F\in\mathcal{F}_j(\widetilde{K}^A)}\upsilon_k(T_F(\widetilde{K}^A))\\ &= \big[t^k\big]\big[x_1^{n_1+1}\cdot\ldots\cdot x_b^{n_b+1}\big]t^b x_1\cdot\ldots\cdot x_b(1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}\\ &\quad\times\big[u^{j-b}\big]\frac{1}{\big(1-u((1-x_1)^{-t}-1)\big)\cdot\ldots\cdot\big(1-u((1-x_b)^{-t}-1)\big)}\\ &= \big[t^{k-b}\big]\big[x_1^{n_1}\cdot\ldots\cdot x_b^{n_b}\big]\big[u^{j-b}\big]\frac{(1-x_1)^{-(t+1)}\cdot\ldots\cdot(1-x_b)^{-(t+1)}}{\big(1-u((1-x_1)^{-t}-1)\big)\cdot\ldots\cdot\big(1-u((1-x_b)^{-t}-1)\big)}\\ &= R_1\big(k-b,j-b,b,(n_1,\ldots,n_b)\big).\end{aligned}\tag{4.18}$$

Now, we return to K^A. Using $\upsilon_k(K_n^A)=\upsilon_{k+1}(K_n^A\oplus L_{n+1})$ and (3.1), we obtain for $j\in\{0,\ldots,n\}$ and $k\in\{0,\ldots,n\}$

$$\begin{aligned}&\sum_{F\in\mathcal{F}_j(K^A)}\upsilon_k(T_F(K^A))\\ &= \sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\ \sum_{F_1\in\mathcal{F}_{j_1}(K_{n_1}^A),\ldots,F_b\in\mathcal{F}_{j_b}(K_{n_b}^A)}\upsilon_k\big(T_{F_1}(K_{n_1}^A)\times\ldots\times T_{F_b}(K_{n_b}^A)\big)\\ &= \sum_{\substack{j_1,\ldots,j_b\in\mathbb{N}_0:\\ j_1+\ldots+j_b=j}}\ \sum_{F_1\in\mathcal{F}_{j_1+1}(\widetilde{K}_{n_1}^A),\ldots,F_b\in\mathcal{F}_{j_b+1}(\widetilde{K}_{n_b}^A)}\upsilon_{k+b}\big(T_{F_1}(\widetilde{K}_{n_1}^A)\times\ldots\times T_{F_b}(\widetilde{K}_{n_b}^A)\big)\end{aligned}$$

$$= \sum_{F\in\mathcal{F}_{j+b}(\widetilde{K}^A)} \upsilon_{k+b}(T_F(\widetilde{K}^A))$$
$$= R_1\big(k,j,b,(n_1,\dots,n_b)\big),$$

where we applied (4.18) in the last step. □

4.4. **Proof of Proposition 3.11.** Our goal is to show that

$$\sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\dots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k(T_F(K^B)) = \sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\dots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^A)} \upsilon_k(T_F(K^A))$$
$$= \frac{j!}{n!}\binom{j+b-1}{b-1}\begin{bmatrix} n+b\\ k+b\end{bmatrix}_b\begin{Bmatrix} k+b\\ j+b\end{Bmatrix}_b$$

holds for $j\in\{0,\dots,n\}$ and $k\in\{0,\dots,n\}$, where $n:=n_1+\dots+n_b$.

Proof of Proposition 3.11. At first we show the formula for K^B. In (4.16) we saw that

$$\sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k\big(T_F(K^B)\big) = \sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\dots+j_b=j}} \sum_{\substack{i_0^{(1)}+\dots+i_{j_1+1}^{(1)}=n_1\\ \dots\\ i_0^{(b)}+\dots+i_{j_b+1}^{(b)}=n_b}} \upsilon_k(G_i)$$

holds true for all $j,k\in\{0,\dots,n\}$, where

$$G_i = A^{(i_1^{(1)})}\times\dots\times A^{(i_{j_1}^{(1)})}\times\dots\times A^{(i_1^{(b)})}\times\dots\times A^{(i_{j_b}^{(b)})}\times B^{(i_0^{(1)})}\times B^{(i_{j_1+1}^{(1)})}\times\dots\times B^{(i_0^{(b)})}\times B^{(i_{j_b+1}^{(b)})}.$$

Note that in the second sum on the right hand side, the indices satisfy $i_0^{(l)}, i_{j_l+1}^{(l)}\in\mathbb{N}_0$ and $i_1^{(l)},\dots,i_{j_l}^{(l)}\in\mathbb{N}$ for all $l\in\{1,\dots,b\}$. Thus, we obtain

$$\sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\dots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k\big(T_F(K^B)\big)$$
$$= \sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\dots+j_b=j}} \sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\dots+n_b=n}} \sum_{\substack{i_0^{(1)}+\dots+i_{j_1+1}^{(1)}=n_1\\ \dots\\ i_0^{(b)}+\dots+i_{j_b+1}^{(b)}=n_b}} \upsilon_k(G_i)$$
$$= \sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\dots+j_b=j}} \sum_{\substack{l_1,\dots,l_j\in\mathbb{N},l_{j+1},\dots,l_{j+2b}\in\mathbb{N}_0:\\ l_1+\dots+l_{j+2b}=n}} \upsilon_k\big(A^{(l_1)}\times\dots\times A^{(l_j)}\times B^{(l_{j+1})}\times\dots\times B^{(l_{j+2b})}\big).$$

The last equation follows from simple renumbering. Applying Proposition 3.12 with b replaced by $2b$ yields

$$\sum_{\substack{n_1,\dots,n_b\in\mathbb{N}_0:\\ n_1+\dots+n_b=n}} \sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k\big(T_F(K^B)\big) = \sum_{\substack{j_1,\dots,j_b\in\mathbb{N}_0:\\ j_1+\dots+j_b=j}} \frac{j!}{n!}\begin{bmatrix} n+b\\ k+b\end{bmatrix}_b\begin{Bmatrix} k+b\\ j+b\end{Bmatrix}_b$$
$$= \frac{j!}{n!}\binom{j+b-1}{b-1}\begin{bmatrix} n+b\\ k+b\end{bmatrix}_b\begin{Bmatrix} k+b\\ j+b\end{Bmatrix}_b,$$

where we used the well-known fact that the number of compositions of j into b non-negative integers (which may be 0) is given by $\binom{j+b-1}{b-1}$.

The formula for K^A follows from

$$\sum_{F\in\mathcal{F}_j(K^B)} \upsilon_k(T_F(K^B)) = \sum_{F\in\mathcal{F}_j(K^A)} \upsilon_k(T_F(K^A)),$$

due to Theorems 3.7 and 3.8. This completes the proof. □

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