# Additive transversality of fractal sets in the reals and the integers

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**Abstract**

By juxtaposing ideas from fractal geometry and dynamical systems, Hillel Furstenberg proposed a series of conjectures in the late 1960's that explore the relationship between digit expansions with respect to multiplicatively independent bases. In this work, we introduce analogues of some of the notions and results surrounding Furstenberg's work in the discrete setting of the integers. In particular, we define a new class of fractal sets of integers that parallels the notion of $\times r$-invariant sets on the 1-torus, and we investigate the additive independence between these fractal sets when they are structured with respect to different bases. We obtain

- an integer analogue of a result of Furstenberg regarding the classification of all sets that are simultaneously $\times 2$ and $\times 3$ invariant (see Theorem B);
- an integer analogue of a result of Lindenstrauss-Meiri-Peres on iterated sumsets of $\times r$-invariant sets (see Theorem C);
- an integer analogue of Hochman and Shmerkin's solution to Furstenberg's sumset conjecture regarding the dimension of the sumset $X + Y$ of a $\times r$-invariant set $X$ and a $\times s$-invariant set $Y$ (see Theorem D).

To obtain the latter, we provide a quantitative strengthening of a theorem of Hochman and Shmerkin which provides a lower bound on the dimension of $\lambda X + \eta Y$ uniformly in the scaling-parameters $\lambda$ and $\eta$ at every finite scale (see Theorem A). Our methods yield a new combinatorial proof of the theorem of Hochman and Shmerkin that avoids the machinery of local entropy averages and CP-processes.

## Contents

# 1. Introduction

The purpose of this paper is to investigate the additive independence between fractal sets that are structured with respect to multiplicatively independent bases. We explore this topic in two different regimes: the unit interval $[0,1]$ and the non-negative integers $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$. We will begin by explaining the precedent for this inquiry at the intersection of combinatorial number theory, fractal geometry, and ergodic theory.

## 1.1. History and context

Number theorists in the first half of the 20th century were among the first to consider the degree to which base 2 and base 3 representations of real numbers are independent: an open conjecture attributed to Mahler [MF] postulates, for example, that if $(a_n)_{n=1}^\infty \subseteq \{0,1\}$ is not eventually periodic, then at least one of the numbers $\sum_{n=1}^\infty a_n 2^{-n}$ and $\sum_{n=1}^\infty a_n 3^{-n}$ is transcendental. Cassels [Cas], answering a question of Steinhaus about Cantor's middle thirds set $C$, proved that almost every number in $C/2$ (with respect to the $\log 2/\log 3$-dimensional Hausdorff measure) is normal to every base which is not a power of 3.

In the language of fractal geometry and dynamical systems, Furstenberg [Fur1, Fur2] established a number of conjectures and results that explore the relationship between multiplicative structures with respect to different bases. The notion of structure particularly relevant to this work is that of multiplicative invariance: a set $X \subseteq [0,1]$ is *$\times r$-invariant* if it is closed and $T_r X \subseteq X$, where $T_r \colon [0,1] \to [0,1]$ denotes the map

$$T_r \colon x \mapsto rx \bmod 1.$$

One of Furstenberg's first and most well-known results concerning multiplicatively invariant sets is the following theorem, the measure-theoretic analogue of which is the infamous $\times 2$, $\times 3$ conjecture, a central open problem in ergodic theory.

**Theorem 1.1** ([Fur1, Theorem 4.2])**.** *If $X \subseteq [0,1]$ is simultaneously $\times 2$- and $\times 3$-invariant then either $X$ is finite or $X = [0,1]$.*

The numbers 2 and 3 in Theorem 1.1 can be replaced by any pair of *multiplicatively* *independent* positive integers, i.e., integers $r, s \in \mathbb{N}$ for which $\log r/\log s \notin \mathbb{Q}$. Furstenberg conjectured that if $r$ and $s$ are multiplicatively independent and $X, Y \subseteq [0,1]$ are $\times r$- and $\times s$-invariant, respectively, then $X$ and $Y$ are *transverse* in a sense made precise below. While some of Furstenberg's "transversality conjectures" remain open, two of them were resolved recently by Hochman and Shmerkin [HS], Shmerkin [Shm], and Wu [Wu]. The transversality conjecture resolved by Hochman and Shmerkin is of particular relevance to this work, so we will expound on it further now.

In Euclidean geometry, linear subspaces $U, V \subseteq \mathbb{R}^d$ are said to be *in general position* (or *transverse*) if

$$\dim(U+V) \;=\; \min\big(\dim U + \dim V,\; d\big). \tag{1.1}$$

In analogy, Furstenberg conjectured that if $r$ and $s$ are multiplicatively independent and $X$ and $Y$ are $\times r$- and $\times s$-invariant subsets of $[0,1]$, then

$$\dim_{\mathrm{H}}(X+Y) \;=\; \min\big(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y,\; 1\big), \tag{1.2}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. Besides the obvious geometric analogy between (1.1) and (1.2), this expression gives meaning to the statement that the sets $X$ and $Y$ are additively combinatorially independent. Indeed, the size of the sumset $X+Y$ is a rudimentary measure of the additive structure shared between $X$ and $Y$. If $X$ and $Y$ are finite sets of real numbers, then it is easy to check that

$$|X| + |Y| - 1 \leqslant |X+Y| \leqslant |X||Y|. \tag{1.3}$$

Equality holds on the left if and only if $X$ and $Y$ are arithmetic progressions of the same step size. When $|X+Y|$ is near this lower bound, inverse theorems in combinatorial number theory provide additive structural information on the sets $X$ and $Y$. At the other end of the spectrum, equality holds on the right in (1.3) if and only if none of the sums $x+y$, with $x \in X$ and $y \in Y$, coincide. In this case, the sets $X$ and $Y$ lie in general position from an additive combinatorial point of view.

When $X$ and $Y$ are (infinite) subsets of $[0,1]$, their sizes and the size of the sumset $X+Y$ can be measured by the Hausdorff dimension. If $X$ and $Y$ are such that $\dim_{\mathrm{H}}(X \times Y) = \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y$, the analogues of the inequalities in (1.3) are

$$\max\big(\dim_{\mathrm{H}} X,\; \dim_{\mathrm{H}} Y\big) \leqslant \dim_{\mathrm{H}}\big(X+Y\big) \leqslant \min\big(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y,\; 1\big).$$

Equality in the lower bound happens when there are significantly many coincidences amongst the sums $x+y$, an indication that $X$ and $Y$ share mutual additive structures. Equality in the upper bound happens when the sums $x+y$ are mostly as unique as they can be, an indication that $X$ and $Y$ are additive combinatorially transverse. It is precisely this equality that was conjectured by Furstenberg to hold in (1.2) in the case that $X$ and $Y$ are multiplicatively structured with respect to multiplicatively independent bases.

With no structural assumptions on the sets $X, Y \subseteq [0,1]$, it is not difficult to find examples for which the equality in (1.2) does not hold. Nevertheless, it is a consequence of Marstrand's projection theorem that for all Borel sets $X$ and $Y$, the typical dilated sets $\lambda X$ and $\eta Y$ are additive combinatorially transverse in the sense of (1.2).

**Theorem 1.2** ([Mar, Theorem II][1]). *Let $X$ and $Y$ be Borel subsets of $[0,1]$. For Lebesgue-a.e. $(\lambda,\eta)\in\mathbb{R}^2$,*

$$\dim_H\big(\lambda X+\eta Y\big)=\min\big(\dim_H(X\times Y),\ 1\big). \tag{1.4}$$

In this context, Furstenberg's conjecture in (1.2) says that the multiplicative structure of the sets $X$ and $Y$ can be leveraged to change the result in Marstrand's theorem from concerning the typical sumset $\lambda X+\eta Y$ to concerning the specific one $X+Y$. In fact, Furstenberg conjectured that for $\times r$- and $\times s$-invariant sets $X$ and $Y$, the equality in (1.4) holds for *all* non-zero $\lambda$ and $\eta$. Hochman and Shmerkin resolved this conjecture by proving a stronger result for multiplicatively invariant measures; the following theorem is a corollary of their main result, [HS, Theorem 1.3].

**Theorem 1.3** ([HS]). *Let $r$ and $s$ be multiplicatively independent positive integers, and let $X,Y\subseteq[0,1]$ be $\times r$- and $\times s$-invariant sets, respectively. For all $\lambda,\eta\in\mathbb{R}\backslash\{0\}$,*

$$\dim_H\big(\lambda X+\eta Y\big)=\min\big(\dim_H X+\dim_H Y,\ 1\big). \tag{1.5}$$

A number of partial results preceded Theorem 1.3 both for multiplicatively invariant sets and for attractors of iterated function systems (IFSs). Carlos Moreira [Mor] considered sumsets of attractors of IFSs with certain irrationality and non-linearity conditions. Peres and Shmerkin [PS] proved (1.5) for attractors of IFSs with rationally independent contraction ratios; this resolved Theorem 1.3 in the special case that $X$ and $Y$ are restricted digit Cantor sets with respect to multiplicatively independent bases. This work of Peres and Shmerkin is particularly relevant to the arguments in this paper, as we will explain further in the next section and in Section 3. Hochman and Shmerkin [HS] developed Furstenberg's CP processes [Fur2] and introduced local entropy averages to prove (1.5) both for invariant sets and measures and for attractors of IFSs satisfying some general minimality conditions.

In an effort to better understand the role that the multiplicative independence between the bases plays in Theorem 1.3, it is natural to ask about the sum of sets that are all structured with respect to the same base $r$. Taking $X\subseteq[0,1]$ to be those numbers that can be written in decimal with only the digits 0, 1, and 2, we see that the equality in (1.2) need not hold:

$$\frac{\log 5}{\log 10}=\dim_{\mathrm{H}}(X+X)<2\dim_{\mathrm{H}}X=\frac{2\log 3}{\log 10}.$$

Nevertheless, it is a consequence of the following theorem of Lindenstrauss, Meiri, and Peres that the dimension of the iterated sumset $X+\cdots+X$ approaches 1 as the number of summands increases.

**Theorem 1.4** ([LMP, Corollary 1.2]). *Let $(X_i)_{i=1}^\infty$ be a sequence of $\times r$-invariant subsets of $[0,1]$. If $\sum_{i=1}^\infty \dim_H X_i/|\log\dim_H X_i|$ diverges, then*

$$\lim_{n\to\infty}\dim_H\big(X_1+\cdots+X_n\big)=1.$$

This theorem demonstrates that the multiplicative structure captured by multiplicative invariance sits transversely to the additive structure captured by additive closure: because

[1] Marstrand's projection theorem originally concerns orthogonal projections of subsets of the plane. Images of the Cartesian product $X\times Y$ under orthogonal projections are, up to affine transformations which preserve dimension, sumsets of the form $\lambda X+\eta Y$. Also note that for sufficiently regular sets $X$ and $Y$, $\dim_{\mathrm{H}}(X\times Y)=\dim_{\mathrm{H}}X+\dim_{\mathrm{H}}Y$; see, for example, [Mat, Corollary 8.11].

the sumset $X_1 + \cdots + X_n$ fills out the entire space (with respect to the Hausdorff dimension), the sets $X_i$ are not contained in an additively closed set of dimension less than 1. A stronger conclusion is reached under the assumption of multiplicative independence of the bases with respect to which $X_1$ and $X_2$ are structured: Theorem 1.3 gives that relatively few of the sums $x_1 + x_2$, with $x_i \in X_i$, coincide.

While there is a strong historical precedent for the study of $\times r$-invariant subsets of the unit interval, less seems to be known in the integer and $r$-adic settings, despite the fact that many of the same objects and questions can be naturally formulated there. Furstenberg [Fur2], assuming a positive answer to one of his yet-unresolved transversality conjectures in the reals, drew a connection between the real regime and the integers by showing that given any finite collection of finite strings from the symbols $\{0, \ldots, 9\}$, the number $2^n$, written in decimal, contains all of those strings provided that $n$ is sufficiently large. In the same work, he also proved an analogue of Theorem 1.1 in the $rs$-adics showing that no non-trivial set is structured with respect to multiplicatively independent bases.

Integer restricted digit Cantor sets, to which we now turn our focus, are sets of integers which exhibit a special case of the structure we consider in this paper and which have received some attention in the literature. An *integer base-$r$ restricted digit Cantor set* is a set of non-negative integers whose base-$r$ expansion includes only digits from a fixed set $\mathcal{D} \subseteq \{0, 1, \ldots, r-1\}$, i.e.,

$$\left\{ \sum_{i=0}^{n} a_i r^i \;\middle|\; n \in \mathbb{N}_0,\ a_0, \ldots, a_n \in \mathcal{D} \right\}. \tag{1.6}$$

Given the direct analogy between these sets and their real counterparts, integer restricted digit Cantor sets are natural first candidates for analogues of the transversality results mentioned above. While a number of arithmetic properties of restricted digit Cantor sets in the positive integers are well studied – divisibility [BS], distribution in arithmetic progressions [EMS, Kon], number of prime factors [KMS], character sums [BCS] – much less appears to be known about the relationship between integer restricted digit Cantor sets with respect to different bases.

An unresolved conjecture of Erdős [Erd] posits that for all but finitely many positive integers $n$, the number $2^n$ requires the digit 1 in order to be expressed in base 3; see [DW, Lag] for some recent progress. In the same vein, it is a folklore conjecture in number theory [Inc] that $\{0, 1, 82000\}$ is equal to the set $A$ of non-negative integers that can be written in bases 2, 3, 4, and 5 using only the digits 0 and 1; Burrell and Yu [BY] proved that for all $\varepsilon > 0$, $\left|A \cap [0, N]\right| \leqslant C_\varepsilon N^\varepsilon$ (that is, the set has zero upper mass dimension, as defined in the next section). These statements are all profitably understood in terms of intersections of restricted digit Cantor sets; as such, they strongly resemble the intersection transversality conjectures of Furstenberg in [Fur2]. The recent resolution of Furstenberg's intersection conjecture in the reals by Shmerkin [Shm] and Wu [Wu] perhaps lends some evidence in favor of analogous intersection transversality statements in other regimes.

Much less appears to be known regarding additive transversality in the integers. Yu [Yu] achieves some results on the number of solutions to the equation $x + y = z$ in which the variables come from different integer restricted digit Cantor sets. Our main results, to which we turn next, concern solutions to the equation $x_1 + y_1 = x_2 + y_2$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, for much more generally structured sets $X$ and $Y$ in the real and the integer settings.

### 1.2. Main results

The main results in this article are best categorized by two settings: the unit interval and the non-negative integers. For subsets of the unit interval, we enhance Theorem 1.3 by demonstrating uniformity in the quantifiers on $\lambda$ and $\eta$. We provide a new, combinatorial proof of Theorem 1.3 that avoids the machinery of local entropy averages and CP-processes. For subsets of the non-negative integers, we introduce a natural class of $\times r$-invariant sets. We prove a result on iterated sumsets of $\times r$-invariant sets, and we demonstrate how invariance with respect to multiplicatively independent bases leads to analogues of transversality results of Furstenberg and Hochman-Shmerkin.

#### 1.2.1. Sums of multiplicatively invariant subsets of the reals

In order to capture the way in which the conclusion of Theorem 1.3 is uniform in the parameters $\lambda$ and $\eta$, and in order to define a discrete Hausdorff dimension for subsets of the non-negative integers, we make use of the *discrete Hausdorff content*, defined for $X \subseteq \mathbb{R}^d$ to be

$$\mathcal{H}^{\gamma}_{\geqslant\rho}(X) = \inf \left\{ \sum_{i\in I} \delta_i^{\gamma} \;\middle|\; X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i \geqslant \rho \right\}.$$

The discrete Hausdorff content is discussed at more length in Section 2.1 (see Definition 2.3). For the current discussion, it is helpful to know that for compact sets $X$,

$$\dim_{\mathrm{H}} X = \sup \left\{ \gamma \geqslant 0 \;\middle|\; \lim_{\rho\to 0^+} \mathcal{H}^{\gamma}_{\geqslant\rho}(X) > 0 \right\}; \tag{1.7}$$

see Lemma 2.4. Thus, bounding the discrete Hausdorff content from below at all scales uniformly across a family of sets allows us to quantify a "uniform lower bound on the Hausdorff dimension" across the family.

**Theorem A.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $X, Y \subseteq [0,1]$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min\big(\dim_H X + \dim_H Y, 1\big)$. For all compact $I \subseteq \mathbb{R}\backslash\{0\}$ and all $\gamma < \overline{\gamma}$,*

$$\lim_{\rho\to 0^+} \inf_{\lambda,\eta\in I} \mathcal{H}^{\gamma}_{\geqslant\rho}\big(\lambda X + \eta Y\big) > 0. \tag{1.8}$$

We use Theorem A in a critical way in our main sumset result for the integers, Theorem D. Beyond that, utilizing the fact in (1.7), Theorem 1.3 follows as an immediate corollary to Theorem A. Thus, we provide a new proof of [HS, Theorem 1.3] for sets that avoids use of the main tools in [HS], namely CP-processes and local entropy averages.

The core of our argument in the proof of Theorem A can be traced back to Peres and Shmerkin [PS], who showed, among other things, that Theorem 1.3 holds for restricted digit Cantor sets. We are able to modify and strengthen their argument to achieve uniformity in the parameters $\lambda$ and $\eta$ and extend the result to sets which are only assumed to be multiplicatively invariant. The latter is achieved by combining a flexible combinatorial discrete Marstrand theorem in Section 3.1 with a combinatorial result on trees in Section 3.2.

### 1.2.2. Multiplicatively invariant subsets of the non-negative integers

A primary goal of this article is to introduce the study of multiplicatively invariant subsets of the non-negative integers and demonstrate some transversality results analogous to those on the real line. To that end, we begin by introducing an analogue of a $\times r$-invariant set for the integers. Let $r \in \mathbb{N}$, $r \geqslant 2$. Define $\mathfrak{R}_r \colon \mathbb{N}_0 \to \mathbb{N}_0$ and $\mathfrak{L}_r \colon \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\mathfrak{R}_r \colon n \mapsto \lfloor n/r \rfloor \qquad \text{and} \qquad \mathfrak{L}_r \colon n \mapsto n - r^k \lfloor n/r^k \rfloor,$$

where $k = \lfloor \log n / \log r \rfloor$ when $n \geqslant 1$ and $\lfloor \cdot \rfloor$ denotes the floor function. The maps $\mathfrak{R}_r$ and $\mathfrak{L}_r$ are best understood using the base-$r$ representations of non-negative integers: if $n = a_k r^k + \cdots + a_1 r + a_0$ is the base-$r$ representation of $n$, then

$$\mathfrak{R}_r(n) \;=\; a_k r^{k-1} + \cdots + a_2 r + a_1 \qquad \text{and} \qquad \mathfrak{L}_r(n) \;=\; a_{k-1} r^{k-1} + \cdots + a_1 r + a_0.$$

In other words, the map $\mathfrak{R}_r$ "forgets" the least significant digit (the right-most digit, hence the letter $\mathfrak{R}$) while the map $\mathfrak{L}_r$ "forgets" the most significant digit (the left-most digit, hence the letter $\mathfrak{L}$) in base $r$. For example, in base $r = 10$ we have that $\mathfrak{R}_{10}(71393) = 7139$ and $\mathfrak{L}_{10}(71393) = 1393$.

**Definition 1.5.** A set $A \subseteq \mathbb{N}_0$ is $\times r$*-invariant* if $\mathfrak{R}_r(A) \subseteq A$ and $\mathfrak{L}_r(A) \subseteq A$.

It may be helpful to note that a $\times r$-invariant set $A$ need not satisfy $rA \subseteq A$ and that there are examples showing that the condition $rA \subseteq A$ does not yield a natural integer analogue of the notion of $\times r$-invariance on the unit interval; see Section 4.6.

There are many natural examples of $\times r$-invariant subsets of $\mathbb{N}_0$. Integer base-$r$ restricted digit Cantor sets, defined in (1.6), are clearly $\times r$-invariant. More general examples arise from symbolic subshifts of $\{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$. For any closed and left-shift-invariant set $\Sigma \subseteq \{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$, the corresponding *language set* is defined by

$$\mathcal{L}(\Sigma) = \big\{ (w_0, w_1, \ldots, w_k) \;\big|\; (w_0, w_1, \ldots) \in \Sigma,\ k \in \mathbb{N}_0 \big\}.$$

Any language set naturally embeds into the non-negative integers as

$$\big\{ w_k r^k + \cdots + w_1 r + w_0 \;\big|\; (w_0, w_1, \ldots, w_k) \in \mathcal{L}(\Sigma) \big\},$$

yielding a set that is $\times r$-invariant. For more details, see Definition 4.1 and Proposition 4.3, and for more such examples, see Examples 4.2. As yet another source of $\times r$-invariant subsets of the non-negative integers, we note that if $X$ is a $\times r$-invariant subset of $[0,1]$, then the set

$$\bigcup_{k \in \mathbb{N}_0} \big\{ \lfloor r^k x \rfloor \;\big|\; x \in X \big\}$$

can be shown to be $\times r$-invariant; see Section 4.2 for more details.

Our first result in the integer setting is a natural analogue of Theorem 1.1 that demonstrates that there are no non-trivial examples of sets which exhibit structure simultaneously with respect to multiplicatively independent bases.

**Theorem B.** *Let $r$ and $s$ be multiplicatively independent positive integers. If $A \subseteq \mathbb{N}_0$ is simultaneously $\times r$- and $\times s$-invariant then either $A$ is finite or $A = \mathbb{N}_0$.*

To measure the size of multiplicatively invariant subsets of $\mathbb{N}_0$ and their sumsets, we make use of two notions of dimension in the integers that parallel the classical Minkowski and Hausdorff dimensions from geometric measure theory. The discrete analogue of the lower and upper Minkowski dimension are the *lower and upper mass dimension*, defined for $A \subseteq \mathbb{N}_0$ as

$$\begin{aligned} \underline{\dim}_{\mathrm{M}} A &= \liminf_{N\to\infty} \frac{\log|A\cap[0,N)|}{\log N} = \sup\left\{\gamma \geqslant 0 \;\middle|\; \liminf_{N\to\infty} \frac{|A\cap[0,N)|}{N^\gamma} > 0\right\}, \\ \overline{\dim}_{\mathrm{M}} A &= \limsup_{N\to\infty} \frac{\log|A\cap[0,N)|}{\log N} = \sup\left\{\gamma \geqslant 0 \;\middle|\; \limsup_{N\to\infty} \frac{|A\cap[0,N)|}{N^\gamma} > 0\right\}. \end{aligned}$$

Whenever $\underline{\dim}_{\mathrm{M}} A = \overline{\dim}_{\mathrm{M}} A$, we say that the mass dimension of $A$ exists and denote it by $\dim_{\mathrm{M}} A$. In analogy to (1.7), the *lower and upper discrete Hausdorff dimension* of $A$ are defined to be

$$\begin{aligned} \underline{\dim}_{\mathrm{H}} A &= \sup\left\{\gamma \geqslant 0 \;\middle|\; \liminf_{N\to\infty} \frac{\mathcal{H}^\gamma_{\geqslant 1}(A\cap[0,N))}{N^\gamma} > 0\right\}, \\ \overline{\dim}_{\mathrm{H}} A &= \sup\left\{\gamma \geqslant 0 \;\middle|\; \limsup_{N\to\infty} \frac{\mathcal{H}^\gamma_{\geqslant 1}(A\cap[0,N))}{N^\gamma} > 0\right\}, \end{aligned}$$

and if these two quantities agree then we say that the *discrete Hausdorff dimension* of $A$, $\dim_{\mathrm{H}} A$, exists and is equal to this quantity.

The mass dimension and the upper discrete Hausdorff dimension are systematically studied along with a host of other discrete dimensions in [BT2]. We discuss these notions of dimension and the interplay between them at greater length in Section 2.3. For the current discussion, it is helpful to know that

$$\underline{\dim}_{\mathrm{H}} \leqslant \underline{\dim}_{\mathrm{M}} \leqslant \overline{\dim}_{\mathrm{M}} \qquad \text{and} \qquad \underline{\dim}_{\mathrm{H}} \leqslant \overline{\dim}_{\mathrm{H}} \leqslant \overline{\dim}_{\mathrm{M}}, \tag{1.9}$$

and that for any $\times r$-invariant set $A \subseteq \mathbb{N}_0$, both the mass dimension $\dim_{\mathrm{M}} A$ and the discrete Hausdorff dimension $\dim_{\mathrm{H}} A$ exist and coincide; see Lemma 2.18 and Remark 4.7.

Our second main result in the integer setting is an analogue of Theorem 1.4 concerning the dimension of iterated sumsets of $\times r$-invariant sets.

**Theorem C.** *Let $(A_i)_{i=1}^\infty$ be a sequence of $\times r$-invariant subsets of $\mathbb{N}_0$. If $\sum_{i=1}^\infty \dim_H A_i / |\log \dim_H A_i|$ diverges, then*

$$\lim_{n\to\infty} \underline{\dim}_H\big(A_1 + \cdots + A_n\big) = 1.$$

In the same way as in the continuous regime, this theorem demonstrates that the structure captured by $\times r$-invariance in $\mathbb{N}_0$ sits transversely to the additive structure captured by additive closure. It also demonstrates the connection between $\times r$-invariant subsets of the integers and $\times r$-invariant subsets of $[0,1]$, and it will serve to emphasize the role multiplicative independence plays in the other results in this section.

Our final results in the integer setting concern the dimension of sumsets of $\times r$- and $\times s$-invariant sets. To better frame these results, consider the following discrete analogue of Marstrand's theorem from [Gla, Theorem 1.4]: *for all $A, B \subseteq \mathbb{Z}$ satisfying a necessary* *dimension condition*[2] *and for Lebesgue-a.e.* $(\lambda, \eta) \in \mathbb{R}^2$,

$$\overline{\dim}_{\mathrm{M}} \big( \lfloor \lambda A + \eta B \rfloor \big) = \min \big( \overline{\dim}_{\mathrm{M}} (A \times B),\ 1 \big), \tag{1.10}$$

*where* $\lfloor \lambda A + \eta B \rfloor := \big\{ \lfloor \lambda a + \eta b \rfloor \ \big| \ a \in A,\, b \in B \big\}$. When $A$ and $B$ are sufficiently regular – for example, when $A$ and $B$ are $\times r$- and $\times s$-invariant, respectively – $\overline{\dim}_{\mathrm{M}} (A \times B) = \overline{\dim}_{\mathrm{M}} A + \overline{\dim}_{\mathrm{M}} B$, and thus the typical sumset has dimension as large as possible. In the same way that Theorem 1.3 improves Marstrand's theorem under the assumption of multiplicative invariance, the following theorem improves this discrete version of Marstrand's theorem under the analogous structural assumptions.

**Theorem D.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $A, B \subseteq \mathbb{N}_0$ be $\times r$- and $\times s$-invariant sets, respectively. For all $\lambda, \eta > 0$, the mass and discrete Hausdorff dimensions of $\lfloor \lambda A + \eta B \rfloor$ exist and*

$$\dim_H \big( \lfloor \lambda A + \eta B \rfloor \big) \ = \ \dim_M \big( \lfloor \lambda A + \eta B \rfloor \big) \ = \ \min \big( \dim_H A + \dim_H B,\ 1 \big).$$

In particular, Theorem D tells us that for multiplicatively independent bases $r$ and $s$, any $\times r$-invariant set $A$ and $\times s$-invariant set $B$ are additively combinatorially transverse in the sense that

$$\dim_{\mathrm{H}} \big( A + B \big) \ = \ \min \big( \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B,\ 1 \big), \tag{1.11}$$

thus realizing a natural analogue to Furstenberg's sumset conjecture in the integers.

Bounding $\dim_{\mathrm{H}}(A+B)$ from above is accomplished by a straight-forward combination of the trivial upper sumset estimate in (1.3), the dimension bounds in (1.9), and the fact that the mass and discrete Hausdorff dimensions coincide for $\times r$- and $\times s$-invariant sets. Much more work goes into bounding $\dim_{\mathrm{H}}(A+B)$ from below. The following theorem gives the lower bound required for (1.11) by demonstrating more: the same sort of uniformity in the real regime described in Theorem A is present in the integer regime.

**Theorem E.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $A, B \subseteq \mathbb{N}_0$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min \big( \dim_H A + \dim_H B, 1 \big)$. For all compact $I \subseteq (0, \infty)$ and all $\gamma < \overline{\gamma}$,*

$$\liminf_{N \to \infty} \ \inf_{\lambda, \eta \in I} \ \frac{\mathcal{H}^{\gamma}_{\geqslant 1} \big( \lfloor \lambda A + \eta B \rfloor \cap [0, N) \big)}{N^{\gamma}} > 0.$$

Following the outline in the previous paragraph, Theorem D follows from Theorem E; a complete proof is given in Section 4.5. Since Theorem E is, in turn, derived from Theorem A, one can view Theorem D as a consequence of Theorem A. It is natural to ask whether Theorem D can be derived from Theorem 1.3 directly, but it seems to the authors that this is not possible and that one needs the full strength of Theorem A to obtain Theorem D.

[2] The condition is that the upper mass dimension of $A \times B$ is equal to the upper counting dimension of $A \times B$. The upper mass dimension of $A \times B$ is $\overline{\dim}_{\mathrm{M}} (A \times B) := \limsup_{N \to \infty} \log \big| (A \times B) \cap \{-N, \ldots, N\}^2 \big| / \log N$, while the upper counting dimension of $A \times B$ is equal to $\limsup_{N \to \infty} \max_{z \in \mathbb{Z}^2} \log \big| (A \times B) \cap (z + \{-N, \ldots, N\}^2) \big| / \log N$.

### 1.3. Overview of the paper

In Section 2, we present several preliminary results that are used later, including some basic facts from discrete and continuous fractal geometry, some properties of $\times r$-invariant subsets of $[0,1]$, and notions of dimension for subsets of $\mathbb{N}_0$. Section 3 contains a proof of Theorem A, adapting the argument from [PS] and combining it with two new crucial ingredients: a discrete version of Marstrand's projection theorem which includes topological information on the exceptional set and handles projections of "large subsets"; and a combinatorial theorem about existence of regular subtrees. In Section 4, we study $\times r$-invariant sets of integers and their dimensions. In particular, we prove Theorems B, C, D and E. Theorem B is proved with a self contained elementary argument, and Theorem C is derived from Theorem 1.4. Theorem E, from which Theorem D follows as a corollary, is proved using Theorem A and a correspondence between $\times r$-invariant sets of $[0,1]$ and $\times r$-invariant sets of $\mathbb{N}_0$. Finally, in Section 5, we collect a number of open questions concerning $\times r$-invariant subsets of $\mathbb{N}_0$.

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## 2. Preliminary definitions and results

The positive and non-negative integers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. For $x \in \mathbb{R}$, denote the fractional part by $\{x\}$ and the integer part (or floor) by $\lfloor x \rfloor$. The Lebesgue measure on the real line is denoted by Leb. Throughout the paper, $\mathbb{R}^d$ is equipped with the Euclidean norm which we denote by $|\cdot|$. Given two positive-valued functions $f$ and $g$, we write $f \ll_{a_1,\ldots,a_k} g$ or $g \gg_{a_1,\ldots,a_k} f$ if there exists a constant $K > 0$ depending only on the quantities $a_1, \ldots, a_k$ for which $f(x) \leqslant K g(x)$ for all $x$ in the domain common to both $f$ and $g$. We write $f \asymp_{a_1,\ldots,a_k} g$ if both $f \ll_{a_1,\ldots,a_k} g$ and $f \gg_{a_1,\ldots,a_k} g$.

### 2.1. Continuous and discrete fractal geometry

In this section, we lay out the notation, tools, and results we need from continuous and discrete fractal geometry. A good general reference for the standard material in this section is [Mat, Ch. 4]. In the definitions that follow, $\rho, \gamma, c > 0$, $d \in \mathbb{N}$, and $X \subseteq \mathbb{R}^d$ is non-empty.

**Definition 2.1.**

- The set $X$ is *$\rho$-separated* if for all distinct $x_1, x_2 \in X$, $|x_1 - x_2| \geqslant \rho$.
- The *metric entropy of $X$ at scale $\rho$* is

$$\mathcal{N}(X, \rho) = \sup \big\{ |X_0| \;\big|\; X_0 \subseteq X \text{ is } \rho\text{-separated} \big\}.$$

- The *lower Minkowski dimension* of $X$ is

$$\underline{\dim}_{\mathrm{M}} X = \liminf_{\delta \to 0^+} \frac{\log \mathcal{N}(X, \delta)}{\log \delta^{-1}}. \tag{2.1}$$

The *upper Minkowski dimension*, $\overline{\dim}_{\mathrm{M}} X$, is defined analogously with a limit supremum in place of the limit infimum. If $\underline{\dim}_{\mathrm{M}} X = \overline{\dim}_{\mathrm{M}} X$, then this value is the *Minkowski* *dimension* of $X$, $\dim_{\mathrm{M}} X$.

It is a well-known fact that we will use without further mention that if $\rho < 1$, then $\underline{\dim}_{\mathrm{M}} X = \liminf_{N\to\infty} \log \mathcal{N}(X, \rho^{-N}) / \log \rho^N$ and $\overline{\dim}_{\mathrm{M}} X = \limsup_{N\to\infty} \log \mathcal{N}(X, \rho^{-N}) / \log \rho^N$.

**Definition 2.2.**

- The *unlimited Hausdorff content at dimension* $\gamma$ of $X$ is

$$\mathcal{H}^{\gamma}_{\infty}(X) = \inf \left\{ \sum_{i\in I} \delta_i^{\gamma} \;\middle|\; X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i \right\}.$$

  Note that when $X$ is compact, the index set $I$ may be taken to be finite.
- The *Hausdorff dimension* of $X$ is

$$\begin{aligned}\dim_{\mathrm{H}} X &= \sup\{\gamma \in \mathbb{R} \mid \mathcal{H}^{\gamma}_{\infty}(X) > 0\} \\ &= \inf\{\gamma \in \mathbb{R} \mid \mathcal{H}^{\gamma}_{\infty}(X) = 0\}.\end{aligned}$$

In the following definition, we introduce two notions meant to capture the dimensionality of discrete sets.

**Definition 2.3.**

- (cf. [KT, Definition 1.2]) The set $X$ is a $(\rho,\gamma)_c$*-set* if it is $\rho$-separated and for all $\delta \geqslant \rho$ and all open balls $B$ of diameter $\delta$,

$$\left|X \cap B\right| \leqslant c \left(\frac{\delta}{\rho}\right)^{\gamma}. \tag{2.2}$$

- The *discrete Hausdorff content of* $X$ *at scale* $\rho$ *and dimension* $\gamma$ is

$$\mathcal{H}^{\gamma}_{\geqslant\rho}(X) = \inf \left\{ \sum_{i\in I} \delta_i^{\gamma} \;\middle|\; X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i \geqslant \rho \right\}.$$

  Note that when $X$ is compact, the index set $I$ may be taken to be finite.

In the definition of a $(\rho,\gamma)_c$-set, we think of $\rho$ as being positive and close to 0, $\gamma \in [0, d]$ as the "dimension" of the set, and $c > 0$ as an uninteresting parameter that exists only to make our arguments explicit. The inequality in (2.2) guarantees that the points of a $(\rho,\gamma)_c$-set cannot be too concentrated in any ball. It follows from that inequality that the maximum cardinality of a $(\rho,\gamma)_c$ set in $[0,1]^d$ is on the order of $\rho^{-\gamma}$. A $(\rho,\gamma)_c$-set with cardinality $\gg \rho^{-\gamma}$ can be thought of as a discrete approximation to a set with Hausdorff dimension $\gamma$; this is made more precise in Remark 2.5 below and is realized in Lemma 2.13.

The discrete Hausdorff content at scale $\rho$ is a "$\rho$-resolution" analogue of the unlimited Hausdorff content. The discrete Hausdorff contents of two sets that look the same at scale $\rho$ are approximately equal; the more formal statement can be found in Lemma 2.7.

The following lemma provides a connection between the discrete and the continuous regimes and proves the equality in (1.7) from the introduction.

**Lemma 2.4.** *Let* $X \subseteq \mathbb{R}^d$ *be compact. For all* $\gamma \geqslant 0$,

$$\lim_{\rho\to 0^+} \mathcal{H}^{\gamma}_{\geqslant\rho}(X) = \mathcal{H}^{\gamma}_{\infty}(X). \tag{2.3}$$

*Consequently, if* $\lim_{\rho\to 0}\mathcal{H}^{\gamma}_{\geqslant\rho}(X) > 0$, *then* $\dim_H X \geqslant \gamma$.

*Proof.* Let $\gamma \geqslant 0$. The limit in (2.3) exists because the function $\rho \mapsto \mathcal{H}^{\gamma}_{\geqslant\rho}(X)$ is non-increasing as $\rho$ tends to $0^+$ and is bounded from below by $\mathcal{H}^{\gamma}_{\infty}(X)$. Equality in the limit follows from the fact that $X$ is compact, allowing for the index set in the definition of $\mathcal{H}^{\gamma}_{\infty}(X)$ to be taken to be finite. If $\lim_{\rho\to 0}\mathcal{H}^{\gamma}_{\geqslant\rho}(X) > 0$, then $\mathcal{H}^{\gamma}_{\infty}(X) > 0$, and it follows from the definition of the Hausdorff dimension that $\dim_{\mathrm{H}} X \geqslant \gamma$. □

**Remark 2.5.** It would be natural to define the *metric entropy at scale* $\rho$ *and dimension* $\gamma$ of the set $X$ as

$$\mathcal{N}(X,(\rho,\gamma)_c) = \sup\big\{|X_0| \;\big|\; X_0 \subseteq X \text{ is a } (\rho,\gamma)_c\text{-set}\big\}.$$

Using a max flow, min cut argument similar to the one in [BP, Ch. 3], it can be shown that for $X$ compact,

$$\frac{\mathcal{N}\big(X,(\rho,\gamma)_c\big)}{\rho^{-\gamma}} \asymp_{c,d} \mathcal{H}^{\gamma}_{\geqslant\rho}(X). \tag{2.4}$$

Thus, $(\rho,\gamma)_c$-sets of cardinality $\gg \rho^{-\gamma}$ can be thought of as discrete fractal sets of dimension $\gamma$. We will not need (2.4), so we omit the details.

The following is a discrete version of the well-known mass distribution principle, cf. [BP, Lemma 1.2.8].

**Lemma 2.6.** *Let* $\mu$ *be a Borel probability measure on* $\mathbb{R}^d$, *and let* $\rho,\kappa > 0$. *If for all balls* $B$ *of diameter* $\delta \geqslant \rho$, $\mu(B) \leqslant \kappa\delta^{\gamma}$, *then the support* $\operatorname{supp}\mu$ *of* $\mu$ *satisfies* $\mathcal{H}^{\gamma}_{\geqslant\rho}(\operatorname{supp}\mu) \geqslant \kappa^{-1}$.

*Proof.* Let $\varepsilon > 0$, and let $\{B_i\}_{i\in I}$ be a cover of $\operatorname{supp}\mu$ with ball $B_i$ of diameter $\delta_i \geqslant \rho$ and with $\sum_{i\in I}\delta_i^{\gamma} \leqslant \mathcal{H}^{\gamma}_{\geqslant\rho}(\operatorname{supp}\mu) + \varepsilon$. Then the conclusion follows because $\varepsilon > 0$ was arbitrary. □

Denote by $[X]_\delta$ the closed $\delta$-neighborhood of $X$:

$$[X]_\delta := \big\{z \in [0,1] \;\big|\; \exists x \in X \text{ with } |z-x| \leqslant \varepsilon\big\}.$$

The *Hausdorff distance* between two non-empty, compact sets $X, Y \subseteq \mathbb{R}$ is

$$d_H(X,Y) := \inf\big\{\varepsilon > 0 \;\big|\; X \subseteq [Y]_\varepsilon \text{ and } Y \subseteq [X]_\varepsilon\big\}.$$

By the Blaschke selection theorem, the set of all non-empty, compact subsets of $\mathbb{R}$ equipped with the Hausdorff distance is a complete metric space.

**Lemma 2.7.** *Let* $a \geqslant 1$ *and* $\rho > 0$. *If* $X, Y \subseteq \mathbb{R}$ *are compact and* $X \subseteq [Y]_{a\rho}$, *then*

$$\mathcal{N}\big(X,\rho\big) \ll_a 1 + \mathcal{N}\big(Y,\rho\big)$$

*and*

$$\mathcal{H}^{\gamma}_{\geqslant\rho}(X) \ll_a \mathcal{H}^{\gamma}_{\geqslant\rho}(Y).$$

*Proof.* If $n = \mathcal{N}(X,\rho)$ then, by the definition of metric entropy, there exist $u_1,\ldots,u_n \in X$ such that $u_{i+1} - u_i \geqslant \rho$ for all $i \in \{1,\ldots,n-1\}$. Since $X \subseteq [Y]_{a\rho}$, we can find for every $i \in \{1,\ldots,n\}$ a point $v_i \in Y$ such that $|u_i - v_i| \leqslant a\rho$. Then, by the triangle inequality, we have $v_{i+2a+1} - v_i \geqslant \rho$. Therefore the set $Y' := \{v_{(2a+1)i} \mid 1 \leqslant i \leqslant \lfloor n/(2a+1) \rfloor\}$ is a $\rho$-separated subset of $Y$. This implies

$$\mathcal{N}(Y,\rho) \geqslant |Y'| \geqslant \left\lfloor \frac{n}{2a+1} \right\rfloor \geqslant \frac{\mathcal{N}(X,\delta)}{2a+1} - 1.$$

For the proof of $\mathcal{H}^\gamma_{\geqslant\rho}(X) \ll_a \mathcal{H}^\gamma_{\geqslant\rho}(Y)$, let $\{B_i\}_{i\in I}$ be a collection of open balls that covers $Y$ and where $B_i$ has diameter $r_i \geqslant \rho$ and $\sum_{i\in I} r_i^\gamma < 2\mathcal{H}^\gamma_{\geqslant\rho}(Y)$. It follows that $X \subseteq \bigcup_{i\in I}[B_i]_{a\rho}$ and $[B_i]_{a\rho}$ is a ball of diameter $r_i+2a\rho \leqslant (2a+1)r_i$. Therefore $\mathcal{H}^\gamma_{\geqslant\rho}(X) \leqslant \sum_{i\in I}((2a+1)r_i)^\gamma \leqslant 2(2a+1)\mathcal{H}^\gamma_{\geqslant\rho}(Y)$. □

## 2.2. Multiplicatively invariant subsets of the reals and their finite approximations

Multiplicatively invariant subsets of $[0,1]$ and $\mathbb{N}_0$ are the main objects of study in this paper. In this section, we record some basic facts about multiplicatively invariant subsets of $[0,1]$ and their discrete approximations.

**Definition 2.8.** Let $r \in \mathbb{N}$ and $X \subseteq [0,1]$.

- The map $T_r : [0,1] \to [0,1]$ is defined by $T_r x = \{rx\}$, the fractional part of the real number $rx$.
- The set $X$ is *$\times r$-invariant* if it is closed and $T_r X \subseteq X$.

The Hausdorff and Minkowski dimensions of a multiplicatively invariant set coincide. As a consequence of this regularity, the Hausdorff dimension of products of such sets is also well-behaved. We record these facts here for later use.

**Theorem 2.9** ([Fur1, Proposition III.1])**.** *If $X \subseteq [0,1]$ is $\times r$-invariant, then* $\dim_H X = \dim_M X$.

**Lemma 2.10.** *If $X, Y \subseteq [0,1]$ are $\times r, \times s$-invariant, respectively, then* $\dim_H(X \times Y) = \dim_H X + \dim_H Y$.

*Proof.* This follows immediately from [Mat, Corollary 8.11] and the fact that $\dim_{\mathrm{H}} X = \overline{\dim}_{\mathrm{M}} X$. □

Since we will work almost exclusively with finite approximations to multiplicatively invariant sets, we establish some useful notation.

**Definition 2.11.** Let $X \subseteq [0,1]$ be $\times r$-invariant. For $n \in \mathbb{N}_0$, the set $X_n$ denotes the set $X$ rounded down to the lattice $r^{-n}\mathbb{Z}$. That is, the point $i/r^n$ is an element of $X_n$ if and only if $X \cap [i/r^n, (i+1)/r^n)$ is non-empty.

The next results show that finite approximations to a multiplicatively invariant set are multiplicatively invariant and are discrete models of fractal sets as captured by Definition 2.3.

**Lemma 2.12.** *Let $X \subseteq [0,1]$ be $\times r$-invariant. For all $n \in \mathbb{N}_0$, $T_r X_n \subseteq X_{n-1}$.*

*Proof.* Let $n \in \mathbb{N}_0$, and let $i/r^n \in X_n$ with $i \in \{0,\dots,r^n-1\}$. Write $i = i_0 + d_{n-1}r^{n-1}$ with $i_0 \in \{0,\dots,r^{n-1}-1\}$ and $d_{n-1} \in \{0,\dots,r-1\}$. Note that $T_r(i/r^n) = i_0/r^{n-1}$ and $T_r((i+1)/r^n) = (i_0+1)/r^{n-1}$. We must show that $i_0/r^{n-1} \in X_{n-1}$.

Since $i/r^n \in X_n$, there exists $x \in X \cap [i/r^n, (i+1)/r^n)$. Since $T_r x \in X$, $T_r x \in X \cap [i_0/r^{n-1}, (i_0+1)/r^{n-1})$. It follows by the definition of $X_{n-1}$ that $i_0/r^{n-1} \in X_{n-1}$, as was to be shown. □

**Lemma 2.13.** *Let $r \geqslant 2$, and let $X \subseteq [0,1]$ be $\times r$-invariant with $\dim_H X > 0$. For all $0 < \gamma_4 < \dim_H X < \gamma_5$, there exists $c > 0$ such that for all sufficiently large $N \in \mathbb{N}$, the set $X_N$ is a $(r^{-N}, \gamma_5)_c$-set satisfying $r^{N\gamma_4} \leqslant |X_N| \leqslant r^{N\gamma_5}$.*

*Proof.* Put $\gamma = \dim_{\mathrm{H}} X$. It follows from Theorem 2.9 that there exists $c_0 > 0$ such that for all $N \in \mathbb{N}$,

$$|X_N| \leqslant c_0 r^{N\gamma_5}, \tag{2.5}$$

and for all sufficiently large $N \in \mathbb{N}$,

$$r^{N\gamma_4} \leqslant |X_N| \leqslant r^{N\gamma_5}.$$

Using the fact that $X$ is $\times r$-invariant and the bound in (2.5), for all $0 \leqslant n \leqslant N$ and for all $i \in \{0, \ldots, r^n - 1\}$,

$$\left| X_N \cap \left[ \frac{i}{r^n}, \frac{i+1}{r^n} \right] \right| \leqslant |X_{N-n}| \leqslant c_0 r^{(N-n)\gamma_5}.$$

Put $c = 2r^{\gamma_5} c_0$. To show that $X_N$ is a $(r^{-N}, \gamma_5)_c$-set, let $B \subseteq \mathbb{R}$ be a ball of diameter $\delta \geqslant r^{-N}$. Put $n = \lfloor -\log_r \delta \rfloor$ so that $r^{-(n+1)} < \delta \leqslant r^{-n}$, and note that a union of two intervals of length $r^n$ of the form above suffice to cover $B$. Therefore,

$$|X_N \cap B| \leqslant 2c_0 r^{(N-n)\gamma_5} \leqslant c \left( \frac{\delta}{r^{-N}} \right)^{\gamma_5},$$

as was to be shown. □

The following notation, borrowed from [PS], allows us to easily compare powers of $r$ and powers of $s$. This is useful when considering the finite approximations to the Cartesian product of a $\times r$- and a $\times s$-invariant set.

**Definition 2.14.** For $n \in \mathbb{N}_0$, we set $n' = \lfloor n \log r / \log s \rfloor$ to be the greatest integer so that $s^{n'} \leqslant r^n$. (The bases $r$ and $s$ do not appear in this notation but should always be clear from context.)

Recall from Definition 2.11 that $X_N$ is the set $X$ rounded to the lattice $r^{-N}\mathbb{Z}$. Extending this notation to $Y$, the set $Y_N$ is the set $Y$ rounded to the lattice $s^{-N}\mathbb{Z}$. Henceforth, this discrete approximation will always appear as $Y_{N'}$, which is the set $Y$ rounded to the lattice $s^{-N'}\mathbb{Z}$.

**Corollary 2.15.** *Let $2 \leqslant r < s$, and let $X, Y \subseteq [0,1]$ be $\times r$- and $\times s$-invariant sets with $\dim_H X, \dim_H Y > 0$. For all $0 < \gamma_4 < \dim_H(X \times Y) < \gamma_5$, there exist $c_1, c_2 > 0$ such that for all sufficiently large $N \in \mathbb{N}$, the sets $X_N \times Y_{N'}$ and $X_N \times Y_{N'+1}$ are $(c_1 r^{-N}, \gamma_5)_{c_2}$-sets satisfying $r^{N\gamma_4} \leqslant |X_N \times Y_{N'}| \leqslant r^{N\gamma_5}$ and $r^{N\gamma_4} \leqslant |X_N \times Y_{N'+1}| \leqslant r^{N\gamma_5}$.*

*Proof.* Let $0 < g_4 < \dim_{\mathrm{H}} X < g_5$ and $0 < h_4 < \dim_{\mathrm{H}} Y < h_5$ be such that

$$\gamma_4 < g_4 + h_4 < \dim_{\mathrm{H}}(X \times Y) < g_5 + h_5 < \gamma_5.$$

Applying Lemma 2.13, there exist $c, d > 0$ such that for sufficiently large $N \in \mathbb{N}$, the set $X_N$ is a $(r^{-N}, g_5)_c$-set satisfying $r^{Ng_4} \leqslant |X_N| \leqslant r^{Ng_5}$ and $Y_{N'}$ is a $(s^{-N'}, h_5)_d$-set satisfying $s^{N'h_4} \leqslant |Y_{N'}| \leqslant s^{N'h_5}$. It follows that for sufficiently large $N \in \mathbb{N}$, $r^{N\gamma_4} \leqslant |X_N \times Y_{N'}| \leqslant r^{N\gamma_5}$ and $r^{N\gamma_4} \leqslant |X_N \times Y_{N'+1}| \leqslant r^{N\gamma_5}$.

Set $c_1 = s^{-1}$ and $c_2 = s^{g_5}cd$. Since $s^{N'} < r^N < s^{N'+1}$, the sets $X_N \times Y_{N'}$ and $X_N \times Y_{N'+1}$ are $c_1 r^{-N}$-separated. Since $X_N$ is a $(r^{-N}, g_5)_c$-set, it is a $(c_1 r^{-N}, g_5)_{s^{g_5}c}$-set.[3] Let $B \subseteq \mathbb{R}^2$ be a ball of diameter $\delta \geqslant c_1 r^{-N}$. Note that

$$\begin{aligned}\big|(X_N \times Y_{N'}) \cap B\big| &\leqslant s^{g_5} c \left(\frac{\delta}{c_1 r^{-N}}\right)^{g_5} d \left(\frac{\delta}{s^{-N'}}\right)^{h_5} \\ &\leqslant s^{g_5} c \left(\frac{\delta}{c_1 r^{-N}}\right)^{g_5} d c_1^{h_5} \left(\frac{\delta}{c_1 r^{-N}}\right)^{h_5} \leqslant c_2 \left(\frac{\delta}{c_1 r^{-N}}\right)^{\gamma_5},\end{aligned}$$

which shows that the set $X_N \times Y_{N'}$ is a $(c_1 r^{-N}, \gamma_5)_{c_2}$-set. By a similar calculation,

$$\begin{aligned}\big|(X_N \times Y_{N'+1}) \cap B\big| &\leqslant s^{g_5} c \left(\frac{\delta}{c_1 r^{-N}}\right)^{g_5} d \left(\frac{\delta}{s^{-(N'+1)}}\right)^{h_5} \\ &\leqslant s^{g_5} c \left(\frac{\delta}{c_1 r^{-N}}\right)^{g_5} d \left(\frac{\delta}{c_1 r^{-N}}\right)^{h_5} \leqslant c_2 \left(\frac{\delta}{c_1 r^{-N}}\right)^{\gamma_5},\end{aligned}$$

which shows that the set $X_N \times Y_{N'+1}$ is a $(c_1 r^{-N}, \gamma_5)_{c_2}$-set. □

## 2.3. Dimension of subsets of integers

To measure the size of subsets of $\mathbb{N}_0$, we will make use of the (upper and lower) mass dimension and the (upper and lower) discrete Hausdorff dimension, defined below. The upper and lower mass dimensions and the upper Hausdorff dimension are treated systematically in [BT2]; we will state the properties we require from these quantities with the aim of making this presentation self-contained. These dimensions join a bevy of other natural notions of dimension for subsets of the integers, integer lattices, and more general discrete sets; see [Nau, BT1, IRUT, LM2].

**Definition 2.16.** Let $A \subseteq \mathbb{N}_0$ be non-empty.

- The *lower mass dimension* of $A$ is
$$\underline{\dim}_{\mathrm{M}} A \;=\; \liminf_{N \to \infty} \frac{\log \big|A \cap [0, N)\big|}{\log N}.$$
The *upper mass dimension*, $\overline{\dim}_{\mathrm{M}} A$, is defined analogously with a limit supremum in place of the limit infimum. If $\underline{\dim}_{\mathrm{M}} A = \overline{\dim}_{\mathrm{M}} A$, then this value is the *mass dimension* of $A$, $\dim_{\mathrm{M}} A$.
- The *lower Hausdorff dimension* of $A$ is
$$\underline{\dim}_{\mathrm{H}} A = \sup \left\{ \gamma \geqslant 0 \;\middle|\; \liminf_{N \to \infty} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0, N)\big)}{N^{\gamma}} > 0 \right\}.$$
The *upper Hausdorff dimension*, $\overline{\dim}_{\mathrm{H}} A$, is defined analogously with a limit supremum

[3] More generally, if $0 < c_1 < 1$, then every $(\delta, \gamma)_c$-set is a $(c_1\delta, \gamma)_{c_1^{-\gamma} c}$-set. This is a quick exercise left to the reader.

in place of the limit infimum. If $\underline{\dim}_{\mathrm{H}} A = \overline{\dim}_{\mathrm{H}} A$, then this value is the *discrete Hausdorff dimension* of $A$, $\dim_{\mathrm{H}} A$.

As the notation suggests, the mass and discrete Hausdorff dimensions are defined in analogy to the Minkowski and Hausdorff dimensions, respectively. The analogies becomes clearer on noting that

$$\big|A \cap [0,N)\big| = \mathcal{N}\left(\frac{A \cap [0,N)}{N},\ N^{-1}\right), \tag{2.6}$$

$$\frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,N)\big)}{N^{\gamma}} = \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\left(\frac{A \cap [0,N)}{N}\right), \tag{2.7}$$

so that the mass and discrete Hausdorff dimensions are capturing, in some sense, the Minkowski and Hausdorff dimensions of the sequence of sets $N \mapsto A/N$ in the unit interval. It should always be clear from context which dimension is understood: we will never consider Minkowski or Hausdorff dimension of subsets of $\mathbb{N}_0$.

As a word of caution, note that our terminology does not match exactly with the terminology used in [BT2]. What we call the upper discrete Hausdorff dimension is called $\dim_L$ in [BT2] (see Lemma 2.3 in that paper), while the discrete Hausdorff dimension defined in that work does not appear in our work.

In the following lemmas, we collect the required properties of the mass and discrete Hausdorff dimensions.

**Lemma 2.17.** *Let $A, B \subseteq \mathbb{N}_0$, $\lambda > 0$, and $\eta \in \mathbb{R}$.*

*(I) For all* $\dim \in \{\underline{\dim}_M, \overline{\dim}_M, \underline{\dim}_H, \overline{\dim}_H\}$, $\dim A \in [0,1]$.

*(II) For all* $\dim \in \{\underline{\dim}_M, \overline{\dim}_M, \underline{\dim}_H, \overline{\dim}_H\}$, $\dim A = \dim\big(\lfloor \lambda A + \eta \rfloor\big)$.

*(III) For all* $\dim \in \{\overline{\dim}_M, \overline{\dim}_H\}$, $\dim(A \cup B) = \max\big(\dim A, \dim B\big)$.

*(IV) For all $r \in \mathbb{N}$, $r \geqslant 2$,*

$$\underline{\dim}_H A = \sup\left\{\gamma \geqslant 0 \;\middle|\; \liminf_{N\to\infty} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,r^N)\big)}{r^{N\gamma}} > 0\right\},$$

*and the analogous statement with $\overline{\dim}_H$ in place of $\underline{\dim}_H$ and limit supremum in place of limit infimum holds.*

*(V)* $\overline{\dim}_M(A+B) \leqslant \overline{\dim}_M A + \overline{\dim}_M B$.

*Proof.* The statements in (I), (II), and (III) follow from straightforward calculations which are left to the reader. The sets in Examples 2.19 (II) below show that (III) does not hold for the lower mass and discrete Hausdorff dimensions.

Both of the statements in (IV) follow from the fact that for all $\gamma > 0$ and all $r^K \leqslant N \leqslant r^{K+1}$,

$$\frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,r^K)\big)}{r^{K\gamma}} \leqslant r^{\gamma} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,N)\big)}{N^{\gamma}} \leqslant r^{2\gamma} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,r^{K+1})\big)}{r^{(K+1)\gamma}}.$$

Indeed, this shows that the limit infimum (resp. limit supremum) of the sequence $N \mapsto \mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,r^N)\big)/r^{N\gamma}$ is non-zero if and only if the limit infimum (resp. limit supremum) of the sequence $N \mapsto \mathcal{H}^{\gamma}_{\geqslant 1}\big(A \cap [0,N)\big)/N^{\gamma}$ is non-zero.

To show the statement in (V), note that for finite sets $F, G \subseteq \mathbb{N}_0$, $|F+G| \leqslant |F||G|$.

Applying this to a finite segment of the set $A+B$, we see

$$\big|(A+B)\cap[0,N)\big| \leqslant \big|\big(A\cap[0,N)\big)+\big(B\cap[0,N)\big)\big| \leqslant \big|A\cap[0,N)\big|\big|B\cap[0,N)\big|.$$

The statement in (V) follows by taking logarithms and a limit supremum as $N$ tends to infinity. The sets in Examples 2.19 (II) and (IV) below show that the statement in (V) does not hold for the lower mass dimension or the upper or lower discrete Hausdorff dimensions. □

**Lemma 2.18.** *For all* $A\subseteq\mathbb{N}_0$,

$$\begin{aligned}\underline{\dim}_H A \leqslant \underline{\dim}_M A \leqslant \overline{\dim}_M A,\\ \underline{\dim}_H A \leqslant \overline{\dim}_H A \leqslant \overline{\dim}_M A,\end{aligned}$$

*and no other comparisons are possible in general.*

*Proof.* It is immediate from the definitions that $\underline{\dim}_{\mathrm{M}} A \leqslant \overline{\dim}_{\mathrm{M}} A$ and $\underline{\dim}_{\mathrm{H}} A \leqslant \overline{\dim}_{\mathrm{H}} A$, and the set in Examples 2.19 (I) below shows that neither of these inequalities are, in general, equalities.

To see that $\underline{\dim}_{\mathrm{H}} A \leqslant \underline{\dim}_{\mathrm{M}} A$ and that $\overline{\dim}_{\mathrm{H}} A \leqslant \overline{\dim}_{\mathrm{M}} A$, note that it follows by covering $A\cap[0,N)$ by $|A\cap[0,N)|$ many balls of diameter 1 that

$$\frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(A\cap[0,N)\big)}{N^{\gamma}} \leqslant \frac{|A\cap[0,N)|}{N^{\gamma}}.$$

If $\gamma > \underline{\dim}_{\mathrm{M}} A$ (resp. $\gamma > \overline{\dim}_{\mathrm{M}} A$), then the limit infimum (resp. limit supremum) of the right hand side is zero, implying that $\gamma \geqslant \underline{\dim}_{\mathrm{H}} A$ (resp. $\gamma \geqslant \overline{\dim}_{\mathrm{H}} A$). It follows that $\underline{\dim}_{\mathrm{H}} A \leqslant \underline{\dim}_{\mathrm{M}} A$ and $\overline{\dim}_{\mathrm{H}} A \leqslant \overline{\dim}_{\mathrm{M}} A$. The set in Examples 2.19 (III) below shows that neither of these inequalities are, in general, equalities.

To see that no other comparisons are possible, it suffices to show that there can in general be no comparison between $\overline{\dim}_{\mathrm{H}}$ and $\underline{\dim}_{\mathrm{M}}$. This is demonstrated by the sets in Examples 2.19 (I) and (III) below. □

We conclude this section with some examples meant to illustrate the extent to which the mass and discrete Hausdorff dimensions relate. These examples do not feature the type of structures that we are concerned with in this work, so we leave some of the details to the reader.

**Examples 2.19.**

(I) Let $(x_n)_{n=0}^{\infty}\subseteq\mathbb{N}_0$ be any sequence which satisfies $\lim_{n\to\infty}\log(x_{n+1}-x_n)/\log x_{n+1}=1$, and define

$$A := \{0\}\cup\bigcup_{n=0}^{\infty}\{x_{2n}, x_{2n}+1,\ldots,x_{2n+1}\}.$$

It is easy to check that $\underline{\dim}_{\mathrm{M}} A = \underline{\dim}_{\mathrm{H}} A = 0$ and that $\overline{\dim}_{\mathrm{M}} A = \overline{\dim}_{\mathrm{H}} A = 1$.

(II) Let $A$ be the set from (I). Put $B=\{0\}\cup\big(\mathbb{N}_0\backslash A\big)$. Then $\underline{\dim}_{\mathrm{M}} B = \underline{\dim}_{\mathrm{H}} B = 0$ while $\overline{\dim}_{\mathrm{M}} B = \overline{\dim}_{\mathrm{H}} B = 1$, and $A+B=A\cup B=\mathbb{N}_0$.

(III) Define

$$A = \{0,\ldots,16\}\cup\bigcup_{n=2}^{\infty}\big\{2^n,\ldots,2^n+\lfloor 2^{n-n/\log n}\rfloor\big\}.$$

It is quick to check that the mass dimension of $A$ exists and $\dim_{\mathrm{M}} A = 1$. On the other hand, by covering $A$ with the intervals in its definition, it can be shown that the discrete Hausdorff dimension of $A$ exists and $\dim_{\mathrm{H}} A = 0$.

(IV) Let $A$ be the set from (III). Define $B \subseteq \mathbb{N}_0$ to contain $\{0, \ldots, 16\}$ and to be such that for all $n \geqslant 2$, $B$ on the interval $[2^n, 2^{n+1})$ is comprised of $2\lfloor 2^{n/\log n} \rfloor$ many integers, as equally spaced as possible. It is quick to check that $\dim_{\mathrm{M}} B = 0$. By the definitions of $A$ and $B$ on the interval $[2^n, 2^{n+1})$, it is easy to see that $[2^{n+1}, 2^{n+2})$ is contained in $A + B$. Therefore, $A + B = \mathbb{N}_0$.

## 3. Sums of multiplicatively invariant subsets of the reals

In this section, we prove Theorem A, the main theorem in the first half of this work. Several auxiliary results go into the proof: a discrete version of Marstrand's projection theorem in Section 3.1, a regularity result for finite trees in Section 3.2, and a quantitative equidistribution result in Section 3.3. We outline the proof of Theorem A in Section 3.4 before presenting the full details in Sections 3.5 and 3.6.

Theorem A has a geometric formulation in terms of orthogonal projections; while we will not make any particular use of the theorem in this form, it is worth formulating for its historical connection to the topic. Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line that contains the origin and forms an angle $\theta$ with the positive $x$-axis.

**Theorem 3.1.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $X, Y \subseteq [0,1]$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min\big(\dim_H X + \dim_H Y, 1\big)$. For all compact $I \subseteq (0,\pi)\backslash\{\pi/2\}$ and all $\gamma < \overline{\gamma}$,*

$$\lim_{\rho \to 0^+} \inf_{\theta \in I} \mathcal{H}^{\gamma}_{\geqslant \rho}\big(\pi_\theta(X \times Y)\big) > 0.$$

The proof of the equivalence between Theorem A and Theorem 3.1 is standard and not needed in this work, so it is omitted.

### 3.1. A discrete Marstrand projection theorem

In this section, we prove a discrete analogue of Marstrand's projection theorem from geometric measure theory. The theorem – stated for sumsets in the introduction as Theorem 1.2 – says that for every Borel set $A \subseteq [0,1]^2$, for Lebesgue-a.e. $\theta \in [0,\pi)$, $\dim_{\mathrm{H}} \pi_\theta A = \min(1, \dim_{\mathrm{H}} A)$, where $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto $\ell_\theta$, the line that contains the origin and forms an angle $\theta$ with the positive $x$-axis. Marstrand's theorem and its relatives have enjoyed much recent attention: we refer the interested reader to the survey [FFJ] and to the end of this section where we put Theorem 3.3 into more context.

The key idea behind Marstrand's theorem is that of "geometric transversality" and is captured in the following lemma. An immediate consequence of the lemma is that there are not many projections which map two distant points close together. The proof follows from a simple geometric argument and is left to the reader.

**Lemma 3.2.** *For all nonzero $x \in \mathbb{R}^2$ and all $\rho > 0$, the set of angles $\theta \in [0,\pi)$ for which $|\pi_\theta x| \leqslant \rho$ is contained in at most two balls of diameter $\ll \rho |x|^{-1}$.*

Our discrete analogue of Marstrand's theorem, Theorem 3.3, reaches a conclusion similar to that of Marstrand's by quantifying the size of the set $E$ of exceptional directions, those directions in which the image of the set $A$ is small. On a first reading, it is safe to think of $\gamma < 1$, $n \approx \rho^{-\gamma}$, $\delta = 1$, and $m \approx \rho^{-(\gamma-\varepsilon)}$. In this case, the set $A$ is a discrete analogue of a set of Hausdorff dimension $\gamma$ and the set $E$ is the set of exceptional directions in which the set $A$ loses at least a proportion $\rho^{\varepsilon}$ of its points.

**Theorem 3.3.** *Let $\gamma, \rho, c > 0$. Put $\overline{\gamma} = \min(\gamma, 1)$. If $A \subseteq [0,1]^2$ is a $(\rho,\gamma)_c$-set with $n := |A| > -\log c$, then for all $\delta > 0$ and all $0 \leqslant m \leqslant \delta^2 n/4$, the set*

$$E = \big\{\theta \in [0,\pi) \;\big|\; \exists A' \subseteq A,\ |A'| \geqslant \delta n,\ \mathcal{N}(\pi_\theta A', \rho) \leqslant m\big\} \tag{3.1}$$

*satisfies*

$$\mathcal{N}(E,\rho) \ll_{\gamma,c} \rho^{-1}\frac{m}{\delta^2 n}\begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1\end{cases}.$$

*Proof.* Let $A \subseteq [0,1]^2$ be a $(\rho,\gamma)_c$-set of cardinality $n > -\log c$. Let $\delta > 0$, and let $0 \leqslant m \leqslant \delta^2 n/4$.

Define $S(\theta) = \big\{(a_1,a_2) \in A^2 \;\big|\; |\pi_\theta(a_1 - a_2)| < \rho\big\}$. Let $E'$ be a maximal $\rho$-separated subset of $E$; thus, $|E'| = \mathcal{N}(E,\rho)$. The goal is to bound $\sum_{\theta\in E'} \big|S(\theta)\big|$ from above and below to get the desired bound on $|E'|$.

Let $\theta \in E'$ and $A'$ be the subset of $A$ corresponding to $\theta$. Since the set $\pi_\theta A'$ lies on a line and $\mathcal{N}(\pi_\theta A', \rho) \leqslant m$, there exists a collection $\{B\}_{B\in\mathcal{B}}$ of no more than $2m$ closed balls $B$ of diameter $\rho$ whose union covers $\pi_\theta A'$. By Cauchy-Schwarz,

$$\begin{aligned}(\delta n)^2 \leqslant |A'|^2 &\leqslant \left(\sum_{B\in\mathcal{B}} \big|\{a_0 \in A' \mid \pi_\theta a_0 \in B\}\big|\right)^2 \\ &\leqslant \big|\mathcal{B}\big| \sum_{B\in\mathcal{B}} \big|\{a_0 \in A' \mid \pi_\theta a_0 \in B\}\big|^2 \\ &\leqslant 2m \sum_{B\in\mathcal{B}} \big|\{a \in A \mid \pi_\theta a \in B\}\big|^2 \\ &= 2m \sum_{B\in\mathcal{B}} \big|\{(a_1,a_2) \in A^2 \mid \pi_\theta a_1, \pi_\theta a_2 \in B\}\big| \\ &\leqslant 2m\big|S(\theta)\big|.\end{aligned}$$

It follows that

$$\frac{\delta^2 n^2}{2m}|E'| \leqslant \sum_{\theta\in E'} \big|S(\theta)\big|. \tag{3.2}$$

Now we use Lemma 3.2 to bound the right hand side of (3.2) from above: for $a_1, a_2 \in [0,1]^2$, the set

$$\Theta(a_1,a_2) = \big\{\theta \in [0,\pi) \;\big|\; \big|\pi_\theta(a_1 - a_2)\big| < \rho\big\}$$

is contained in at most two balls of diameter $\ll \rho/|a_1 - a_2|$. Therefore, $\mathcal{N}(\Theta(a_1,a_2),\rho) \ll 1/|a_1 - a_2|$, and using the fact that $E'$ is $\rho$-separated, we see that

$$\sum_{\theta \in E'} 1_{S(\theta)}(a_1, a_2) = \sum_{\theta \in E'} 1_{\Theta(a_1,a_2)}(\theta) \leqslant K \frac{1}{|a_1 - a_2|}$$

for some constant $K$ depending on the result in Lemma 3.2. It follows that

$$\begin{aligned}
\sum_{\theta \in E'} \left|S(\theta)\right| &= \sum_{\theta \in E'} \sum_{a_1, a_2 \in A} 1_{S(\theta)}(a_1, a_2) \\
&= n|E'| + \sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} \sum_{\theta \in E'} 1_{\Theta(a_1,a_2)}(\theta) \\
&\leqslant n|E'| + K \sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} |a_1 - a_2|^{-1},
\end{aligned}$$

and so we are left to bound the second term from above.

For $\ell \in \mathbb{N}_0$, let $H_\ell = \{x \in \mathbb{R}^2 \mid |x| \in [\rho e^\ell, \rho e^{\ell+1})\}$. Breaking up the sum $\sum |a_1 - a_2|^{-1}$ by fixing $a_1$ and partitioning the $a_2$'s by shells, and using the fact that $A$ is $\rho$-separated, we see

$$\begin{aligned}
\sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} |a_1 - a_2|^{-1} &= \sum_{a_1 \in A} \sum_{\ell=0}^{\infty} \sum_{a_2 \in A \cap (a_1 + H_\ell)} |a_1 - a_2|^{-1} \\
&\leqslant \rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^{\infty} e^{-\ell} \left|A \cap (a_1 + H_\ell)\right|.
\end{aligned}$$

Since $A$ is a $(\rho, \gamma)_c$-set, for all $\ell \geqslant 0$, $\left|A \cap (a_1 + H_\ell)\right| \leqslant c\left(2\rho e^{\ell+1}/\rho\right)^\gamma$. On the other hand, $\sum_{\ell=0}^{\infty} \left|A \cap (a_1 + H_\ell)\right| = |A| - 1$. It follows then from the fact that $\ell \mapsto e^{-\ell}$ is decreasing that $\sum_{\ell=0}^{\infty} e^{-\ell}\left|A \cap (a_1 + H_\ell)\right| \leqslant \sum_{\ell=0}^{\ell_0} 2^\gamma c e^{\ell(\gamma-1)+\gamma}$, where $\ell_0 = \lceil \log((n/c)^{1/\gamma}) \rceil$ is the smallest value such that the set $A$ could be contained in a ball of diameter $\rho e^{\ell_0}$ about $a_1$. Therefore,

$$\begin{aligned}
\rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^{\infty} e^{-\ell}\left|A \cap (a_1 + H_\ell)\right| &\ll_{\gamma,c} \rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^{\ell_0} \left(e^{\gamma-1}\right)^\ell \\
&\ll_{\gamma,c} \rho^{-1} n \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases}.
\end{aligned}$$

Combining the upper and lower bounds on $\sum_{\theta \in E'} \left|S(\theta)\right|$, we see that there exists a constant $K$ depending on the result in Lemma 3.2, $\gamma$, and $c$ such that

$$\frac{\delta^2 n^2}{2m}|E'| \leqslant n|E'| + K\rho^{-1} n \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases}.$$

Dividing both sides by $n$ and using the fact that $m \leqslant \delta^2 n/4$, we see that

$$\frac{\delta^2 n}{4m}|E'| \leqslant \left(\frac{\delta^2 n}{2m} - 1\right)|E'| \leqslant K\rho^{-1} \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases},$$

which rearranges to the desired conclusion. □

The proof of Theorem A will feature oblique projections instead of orthogonal ones. The following corollary concerns oblique projections and is stated in a way that will make it immediately applicable in the proof of Theorem A.

Denote by $\Pi_t : \mathbb{R}^2 \to \mathbb{R}$ the oblique projection $\Pi_t(x, y) = x + ty$. Let $\varphi : (0, \pi/2) \to \mathbb{R}$ be the diffeomorphism $\varphi(\theta) = \log\tan\theta$. Note that $\Pi_{e^{\varphi(\theta)}}$ is the oblique projection that is the "continuation" of the orthogonal projection $\pi_\theta$, meaning that the points $(x, y)$, $(\Pi_{e^{\varphi(\theta)}}(x, y), 0)$, and $\pi_\theta(x, y)$ are collinear.

**Corollary 3.4.** *Let $0 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_5$ be such that $\gamma_2 < 1$ and*

$$2(\gamma_5 - \gamma_3) < \gamma_4 - \gamma_2. \tag{3.3}$$

*For all compact $I \subseteq \mathbb{R}$, all $\varepsilon, c_1, c_2, c_3 > 0$, all sufficiently small $\rho > 0$, and all $(c_1\rho, \gamma_5)_{c_2}$-sets $A \subseteq [0, 1]^2$ with $|A| \geqslant \rho^{-\gamma_4}$, there exists $T \subseteq I$ with the following properties:*

*(I) the set $I \backslash T$ is covered by a disjoint union of finitely many half-open intervals of total Lebesgue measure less than $\varepsilon$;*

*(II) for all $t \in T$ and all $A' \subseteq A$ with $|A'| \geqslant \rho^{-\gamma_3}$, there exists a subset $A'_t \subseteq A'$ with $|A'_t| \geqslant \rho^{-\gamma_2}$ such that the points of $\Pi_{e^t} A'_t$ are distinct and $c_3\rho$-separated.*

*Proof.* Let $I \subseteq \mathbb{R}$ be compact and $\varepsilon, c_1, c_2, c_3 > 0$. Let $\sigma \in \big(\gamma_5 - \gamma_3, (\gamma_4 - \gamma_2)/2\big)$. Let $\rho > 0$ be sufficiently small (to be specified later). Let $A \subseteq [0, 1]^2$ be a $(c_1\rho, \gamma_5)_{c_2}$-set with $|A| \geqslant \rho^{-\gamma_4}$. Put $\overline{\gamma_5} = \min(1, \gamma_5)$, $n = |A|$, $\delta = \rho^\sigma$, and $m = 2c_3\rho^{-\gamma_2}$. Note that since $A$ is a $(c_1\rho, \gamma_5)_{c_2}$-set contained in a ball of diameter $\sqrt{2}$, $n \leqslant 2c_2(c_1\rho)^{-\gamma_5}$.

We want to apply Theorem 3.3 with $\gamma_5$ as $\gamma$, $c_1\rho$ as $\rho$, $c_2$ as $c$, and with $A$, $n$, $\delta$, and $m$ as they are. We see that the inequality $n > -\log c_2$ holds for $\rho$ sufficiently small, as does $m \leqslant \delta^2 n/4$ since $\sigma < (\gamma_4 - \gamma_2)/2$. Since the conditions of Theorem 3.3 hold, the set $E \subseteq [0, \pi)$ defined in (3.1) satisfies

$$\begin{aligned} \mathcal{N}(E, \rho) &\ll_{\gamma_5, c_2} \rho^{-1} \frac{m}{\delta^2 n} n^{1 - \overline{\gamma_5}/\gamma_5} \log n \\ &\ll_{\gamma_5, c_1, c_2, c_3} \rho^{-1} \frac{\rho^{-\gamma_2}}{\rho^{2\sigma} \rho^{-\gamma_4 \overline{\gamma_5}/\gamma_5}} \log\left(\rho^{-\gamma_5}\right). \end{aligned} \tag{3.4}$$

Let $J = \varphi^{-1}(I)$, and put $T = I \backslash \varphi|_J(E)$. Since the map $\varphi|_J$ is bi-Lipschitz,

$$\mathcal{N}(\varphi|_J(E), \rho) \asymp_I \mathcal{N}(E, \rho).$$

Combining this with (3.4) and the fact that $\sigma < (\gamma_4 - \gamma_2)/2$, we have that for sufficiently small $\rho$, $\mathcal{N}(I \backslash T, \rho) \leqslant \varepsilon\rho^{-1}/6$. It follows that the set $I \backslash T$ can be covered by a disjoint union of not more than $\varepsilon\rho^{-1}/2$-many half-open intervals of length $\rho$, a cover of total measure less than $\varepsilon$. This establishes (I).

To prove (II), let $t \in T$, and let $A' \subseteq A$ with $|A'| \geqslant \rho^{-\gamma_3}$. Since $n \leqslant 2c_2(c_1\rho)^{-\gamma_5}$ and $\sigma > \gamma_5 - \gamma_3$, for sufficiently small $\rho$, $\rho^{-\gamma_3} \geqslant \delta n$. It follows that $|A'| \geqslant \delta n$. Because $\theta := \varphi^{-1}(t) \notin E$, $\mathcal{N}(\pi_\theta A', \rho) \geqslant m$. It follows that $\mathcal{N}(\pi_\theta A', c_3\rho) \geqslant \rho^{-\gamma_2}$. By choosing points in $A'$ in each fiber of a maximally $\rho$-separated set of the projection, we see that there exists a subset $A'_t \subseteq A'$ of cardinality at least $\rho^{-\gamma_2}$ such that the orthogonal projection of the points in $A'_\theta$ onto $\ell_\theta$ are disjoint and $c_3\rho$-separated. Since the oblique projection $\Pi_{e^t}$ increases distances between points that lie on $\ell_\theta$, the images of points of $A'_t$ under $\Pi_{e^t}$ are $c_3\rho$-separated. ∎

The results in this section add to a number of other discrete Marstrand-type theorems in the recent literature: [LM2, Lemma 5.2], [LM1, Prop. 3.2], [Gla, Lemma 3.8], [PS, Prop. 7], to name a few. Let us highlight some distinguishing features of Lemma 3.2 and Theorem 3.3 that play an important role in this work. Analogues of Lemma 3.2 more commonly found in the literature, such as the one in [Mat, Lemma 3.11], bound the *measure* of the set of projections which map $x$ close to 0. The result in Lemma 3.2 uses coverings to capture topological information on the set of projections. This information is carried into Theorem 3.3 and is important in the application to Theorem A. Another useful feature of Theorem 3.3 is the allowance of a subset $A'$ in (3.1); this will allow us to treat sets in Theorem A that exhibit multiplicative invariance without necessarily being self-similar.

### 3.2. Trees and a regularity result

Trees are combinatorial objects that are convenient for describing fractal sets. We will be concerned solely with finite trees throughout this work. After giving the main definitions, we motivate their importance by explaining how they will be used in the proof of Theorem A. We move then to prove the main result in this section.

The following definitions describe the familiar notion of a rooted tree, a graph with no cycles whose vertices can be arranged on levels and whose edges only connect vertices on adjacent levels.

**Definition 3.5.**

- A *tree of height* $N \in \mathbb{N}_0$ is a finite set of *nodes* $\Gamma$ together with a partition $\Gamma = \Gamma_0 \cup \cdots \cup \Gamma_N$ with $|\Gamma_0| = 1$ and a *parent* function $P : \Gamma \backslash \Gamma_0 \to \Gamma \backslash \Gamma_N$ such that for every $n \in \{1, \ldots, N\}$, $P(\Gamma_n) = \Gamma_{n-1}$.
- The nodes in $\Gamma_n$ have *height* $n$. The single node with height 0 is the *root* and the nodes with height $N$ are called *leaves*.
- The node $Q$ is the *parent* of each of its *children*, nodes in the set $C_\Gamma(Q) := P^{-1}(Q)$.
- If $Q$ is a node of height $n$, the *induced tree based at* $Q$ is the tree $\Gamma_Q := \cup_{i=0}^{N-n} C_\Gamma^{n+i}(Q)$ of height $N - n$ with root $Q$ and the same parent function as $\Gamma$, restricted to the set $\Gamma_Q$.
- A *subtree* of $\Gamma$ is a tree $\Gamma' \subseteq \Gamma$ of the same height as $\Gamma$ with parent function $P|_{\Gamma' \backslash \Gamma'_0}$. (A subtree is uniquely determined by its non-empty set of leaves $\Gamma'_N \subseteq \Gamma_N$.)

Continuing with terminology inspired by genealogy trees, the ancestors of a node $Q$ are those nodes that lie between $Q$ and the root. For the reasons described below in Remark 3.8, it will be important to count the number of ancestors of $Q$ that have many children. To this end, we introduce the following terminology and notation.

**Definition 3.6.** Let $\Gamma$ be a tree, $c > 0$, and $\omega \in [0, 1]$.

- The *ancestry* of $Q \in \Gamma_n$ is the set
$$\mathcal{A}_\Gamma(Q) := \{P^k(Q) \mid 1 \leqslant k \leqslant n\}.$$
Note that $|\mathcal{A}_\Gamma(Q)|$ is equal to the height of $Q$.
- The node $Q$ is *c-fertile* if $|C_\Gamma(Q)| \geqslant c$. The set of $c$-fertile ancestors of $Q$ is denoted
$$\mathcal{F}_{\Gamma,c}(Q) := \{A \in \mathcal{A}_\Gamma(Q) \mid A \text{ is } c\text{-fertile}\}.$$
A node $Q$ has *$(c, \omega)$-fertile ancestry* if $|\mathcal{F}_{\Gamma,c}(Q)| \geqslant \omega |\mathcal{A}_\Gamma(Q)|$.

The following definitions allow us to capture the dimension of a finite tree by giving costs to the nodes and measuring the cost of the least expensive cut.

**Definition 3.7.** Let $\Gamma$ be a tree, $r \in \mathbb{N}$, $r \geqslant 2$, and $\gamma > 0$.

- A *cut* of $\Gamma$ is a subset $\mathcal{C} \subseteq \Gamma$ such that for every leaf $L$ of $\Gamma$, $\big(\{L\} \cup \mathcal{A}_\Gamma(L)\big) \cap \mathcal{C} \neq \emptyset$.
- The *Hausdorff content of* $\Gamma$ *with base* $r$ *at dimension* $\gamma$ is

$$\mathcal{H}_r^\gamma(\Gamma) := \min \left\{ \sum_{Q \in \mathcal{C}} r^{-\mathrm{height}(Q)\gamma} \;\middle|\; \mathcal{C} \text{ is a cut of } \Gamma \right\}.$$

The main result in this section, Theorem 3.11, says, roughly speaking, that any tall enough tree with Hausdorff content bounded from below and with a uniform upper bound on the number of children of any node has a subtree in which most nodes have fertile ancestry. Before making this statement precise and beginning with the details of the proof, let us make two observations about the concept of fertile ancestry that will help explain why it will be useful later on in the proof of Theorem A.

**Remark 3.8.**

(I) The property of having fertile ancestry is preserved under a type of tree thinning process that we will employ in the proof of Theorem A. More specifically, suppose that $\Gamma$ is a tree in which every node has either one child or at least $c$ many children and in which every node has $(c, \omega)$-fertile ancestry. Suppose further that for every node $Q$, there exists a subset $\tilde{C}(Q) \subseteq C_\Gamma(Q)$ of the children of $Q$ with $|\tilde{C}(Q)| \geqslant \min\big(\tilde{c}, |C_\Gamma(Q)|\big)$. These subsets naturally give rise to a subtree $\tilde{\Gamma}$ obtained by thinning the tree $\Gamma$: the subtree $\tilde{\Gamma}$ is uniquely defined by the property that if $Q$ is a node of $\tilde{\Gamma}$, then $C_{\tilde{\Gamma}}(Q) = \tilde{C}(Q)$. It is not hard to see that every node in $\tilde{\Gamma}$ has $(\tilde{c}, \omega)$-fertile ancestry, regardless of how the subsets of children $\tilde{C}(Q)$ were chosen.

(II) A tree in which every node has fertile ancestry necessarily has large Hausdorff content. This is a simple consequence of the mass distribution principle (or the max flow-min cut theorem) for trees, the real analogue of which is stated in Lemma 2.6. More specifically, let $\Gamma$ be a tree, and consider a "flow" through $\Gamma$ of magnitude 1 starting at the root that splits equally amongst children. The value of the flow at any node $Q$ with fertile ancestry can be bounded from above using the fact that many times, much of the flow is split amongst a large set of children before reaching $Q$. If all nodes of $\Gamma$ have fertile ancestry, then the flow is not concentrated too highly at any node. According to the mass distribution principle, the Hausdorff content of a tree that supports such a flow is high.

We now proceed with the main results in this subsection. In the next two results, fix $r \geqslant 2$ and $0 < \gamma_3 < \gamma_4 < \gamma_5$ such that setting

$$A := \gamma_5 - \gamma_4 + \log_r 2,$$
$$B := \gamma_4 - \gamma_3 - \log_r 2,$$

ensures the quantity $B$ is positive. The following lemma describes the fundamental dichotomy behind Theorem 3.11.

**Lemma 3.9.** *If* $\Gamma$ *is a tree with the property that*

$$\text{every node in the tree has at most } r^{\gamma_5} \text{ many children,} \tag{3.5}$$

*then at least one of the following holds:*

*(I) there are at least $r^{\gamma_3}$ many children $Q$ of the root, each of which satisfies*

$$\mathcal{H}_r^{\gamma_4}(\Gamma_Q) \geqslant \mathcal{H}_r^{\gamma_4}(\Gamma) r^{-A};$$

*(II) there is at least one child $Q$ of the root satisfying*

$$\mathcal{H}_r^{\gamma_4}(\Gamma_Q) \geqslant \mathcal{H}_r^{\gamma_4}(\Gamma) r^{B}.$$

*Proof.* Let $\Gamma$ be a tree satisfying (3.5). Let $Q_1, Q_2, \ldots, Q_I$ be the children of the root of $\Gamma$, ordered so that $\mathcal{H}_r^{\gamma_4}(\Gamma_{Q_i}) \geqslant \mathcal{H}_r^{\gamma_4}(\Gamma_{Q_{i+1}})$. If neither (I) nor (II) holds, then $\mathcal{H}_r^{\gamma_4}(\Gamma_{Q_1}) < \mathcal{H}_r^{\gamma_4}(\Gamma) r^B$ and $\mathcal{H}_r^{\gamma_4}(\Gamma_{Q_{\lceil r^{\gamma_3}\rceil}}) < \mathcal{H}_r^{\gamma_4}(\Gamma) r^{-A}$. It follows by the ordering of the $Q_i$'s and the definition of the Hausdorff content and induced trees that

$$\begin{aligned} \mathcal{H}_r^{\gamma_4}(\Gamma) &\leqslant r^{-\gamma_4} \sum_{i=1}^{I} \mathcal{H}_r^{\gamma_4}(\Gamma_{Q_i}) \\ &= \sum_{i=1}^{\lfloor r^{\gamma_3}\rfloor} r^{-\gamma_4} \mathcal{H}_r^{\gamma_4}(\Gamma_{Q_i}) + \sum_{i=\lceil r^{\gamma_3}\rceil}^{I} r^{-\gamma_4} \mathcal{H}_r^{\gamma_4}(\Gamma_{Q_i}) \\ &< r^{\gamma_3} r^{-\gamma_4} \mathcal{H}_r^{\gamma_4}(\Gamma) r^B + r^{\gamma_5} r^{-\gamma_4} \mathcal{H}_r^{\gamma_4}(\Gamma) r^{-A} = \mathcal{H}_r^{\gamma_4}(\Gamma), \end{aligned}$$

a contradiction. □

**Lemma 3.10.** *Every finite tree $\Gamma$ that satisfies (3.5) has a subtree $\Gamma'$ with the property that for all nodes $Q$ in $\Gamma'$,*

$$|\mathcal{F}_{\Gamma', r^{\gamma_3}}(Q)| \geqslant \frac{|\mathcal{A}_{\Gamma'}(Q)| B + \log_r \mathcal{H}_r^{\gamma_4}(\Gamma)}{A+B}. \tag{3.6}$$

*Proof.* We will prove the lemma by induction on the height $N$ of the tree $\Gamma$. To verify the base case, let $\Gamma$ be the tree of height $N = 0$: a single node with no children. Taking $\Gamma' = \Gamma$, the inequality (3.6) for this single node follows from the fact that $\log_r \mathcal{H}_r^{\gamma_4}(\Gamma) = 0$.

Suppose that $N \in \mathbb{N}$ is such that the theorem holds for all trees of height $N-1$. Let $\Gamma$ be a tree of height $N$ that satisfies (3.5). By Lemma 3.9, at least one of Case (I) or Case (II) holds.

Suppose Case (I) of Lemma 3.9 holds. Let $Q$ be any one of the $r^{\gamma_3}$-many children guaranteed by Case (I). By the induction hypothesis, there exists a subtree $\Gamma'_Q$ of $\Gamma_Q$ in which every node satisfies (3.6) with $\Gamma_Q$ in place of $\Gamma$ and $\Gamma'_Q$ in place of $\Gamma'$. Define the subtree $\Gamma'$ of $\Gamma$ to be the root node of $\Gamma$ with the collection of at least $r^{\gamma_3}$ many children $Q$, each of those children followed by its subtree $\Gamma'_Q$.

We will now verify that (3.6) holds for all nodes of $\Gamma'$. Let $Q$ be any node of $\Gamma'$. If $Q$ is the root node of $\Gamma'$, then (3.6) holds because $\log_r \mathcal{H}_r^{\gamma_4}(\Gamma) \leqslant 0$. If $Q$ is a non-root node of $\Gamma'$, then it belongs to one of the subtrees $\Gamma'_S$ for some child $S$ of the root of $\Gamma'$. By property (3.6) for the subsubtree $\Gamma'_S$, we see

$$\begin{aligned} |\mathcal{F}_{\Gamma', r^{\gamma_3}}(Q)| - 1 &= |\mathcal{F}_{\Gamma'_S, r^{\gamma_3}}(Q)| \\ &\geqslant \frac{|\mathcal{A}_{\Gamma'_Q}(Q)| B + \log_r \mathcal{H}_r^{\gamma_4}(\Gamma_S)}{A+B} \\ &\geqslant \frac{(|\mathcal{A}_{\Gamma'}(Q)| - 1) B + \log_r \mathcal{H}_r^{\gamma_4}(\Gamma) - A}{A+B}. \end{aligned}$$

This simplifies to the inequality in (3.6), verifying the inductive step if Case (I) of Lemma 3.9 holds.

Suppose Case (II) of Lemma 3.9 holds. Let $Q$ be the child guaranteed by Case (II). By the induction hypothesis, there exists a subtree $\Gamma'_Q$ of $\Gamma_Q$ in which every node satisfies (3.6) with $\Gamma_Q$ in place of $\Gamma$ and $\Gamma'_Q$ in place of $\Gamma'$. Define the subtree $\Gamma'$ of $\Gamma$ to be the root of $\Gamma$ with only the child $Q$ followed by its subtree $\Gamma'_Q$.

We will now verify that (3.6) holds for all nodes of $\Gamma'$. Let $Q$ be any node of $\Gamma'$. If $Q$ is the root node of $\Gamma'$, then (3.6) holds because $\log_r \mathcal{H}_r^{\gamma_4}(\Gamma) \leqslant 0$. If $Q$ is a non-root node of $\Gamma'$, then by property (3.6) for the subtree containing $Q$, we see

$$|\mathcal{F}_{\Gamma',r^{\gamma_3}}(Q)| \geqslant \frac{(|\mathcal{A}_{\Gamma'}(Q)|-1)B + \mathcal{H}_r^{\gamma_4}(\Gamma) + B}{A+B}.$$

This simplifies to the inequality in (3.6), verifying the inductive step if Case (II) of Lemma 3.9 holds. The proof of the inductive step is complete, and the theorem follows. ☐

**Theorem 3.11.** *For all* $0 < \varepsilon < 1$, *for all* $0 < \gamma_3 < \gamma_4 < \gamma_5 < \gamma_4 + \varepsilon(\gamma_4 - \gamma_3)$, *for all sufficiently large* $r \in \mathbb{N}$, *and for all* $V > 0$, *there exists* $N_0 \in \mathbb{N}$ *for which the following holds. For all* $N \geqslant N_0$ *and for all trees* $\Gamma$ *of height* $N$ *with* $\mathcal{H}_r^{\gamma_4}(\Gamma) \geqslant V$ *that satisfy (3.5), there exists a subtree* $\Gamma'$ *of* $\Gamma$ *such that all nodes* $Q \in \Gamma'$ *with height at least* $N_0$ *have* $(r^{\gamma_3}, 1-\varepsilon)$*-fertile ancestry in* $\Gamma'$.

*Proof.* Let $0 < \varepsilon < 1$ and $0 < \gamma_3 < \gamma_4 < \gamma_5 < \gamma_4 + \varepsilon(\gamma_4 - \gamma_3)$. Let $r \in \mathbb{N}$ be sufficiently large so that $\gamma_4 - \gamma_3 - \log_r 2 > (1-\varepsilon)(\gamma_5 - \gamma_3)$. Define $A = \gamma_5 - \gamma_4 + \log_r 2$ and $B = \gamma_4 - \gamma_3 - \log_r 2$, and note by the inequality in the previous sentence, $B/(A+B) > (1-\varepsilon)$. Let $V > 0$. Choose $N_0 \in \mathbb{N}$ such that

$$\frac{N_0 B + \log_r V}{N_0(A+B)} > 1 - \varepsilon, \tag{3.7}$$

and note that for all $N \geqslant N_0$, the inequality in (3.7) holds with $N_0$ replaced by $N$.

Let $N \geqslant N_0$, and let $\Gamma$ be a tree of height $N$ with $\mathcal{H}_r^{\gamma_4}(\Gamma) \geqslant V$ that satisfies (3.5). By Lemma 3.10, there exists a subtree $\Gamma'$ of $\Gamma$ such that for all nodes $Q$ of $\Gamma'$, the inequality in (3.6) holds.

Let $Q$ be a node of $\Gamma'$ with height at least $N_0$. By (3.6) and (3.7), we see that

$$\frac{|\mathcal{F}_{\Gamma',r^{\gamma_3}}(Q)|}{|\mathcal{A}_{\Gamma'}(Q)|} \geqslant \frac{|\mathcal{A}_{\Gamma'}(Q)|B + \log_r V}{|\mathcal{A}_{\Gamma'}(Q)|(A+B)} > 1 - \varepsilon.$$

It follows that $Q$ has $(r^{\gamma_3}, 1-\varepsilon)$-fertile ancestry in $\Gamma'$, as was to be shown. ☐

### 3.3. A quantitative equidistribution lemma

The main result in this short section, Lemma 3.13, gives a lower bound on the number of visits of an equidistributed sequence to a set as a function only of the measure and topological complexity of the set's complement. This result is certainly not new; we state it explicitly here for convenience in a way that highlights the uniformity in the quantifiers.

For $U \in \mathbb{N}$, denote by $\mathcal{I}_U$ the collection of those subsets of $[0,1)$ that are a union of no more than $U$ disjoint intervals of the form $[a,b)$.

**Lemma 3.12.** *For any uniformly distributed sequence* $(x_n)_{n\in\mathbb{N}_0} \subseteq [0,1)$, $U \in \mathbb{N}$, *and* $\varepsilon > 0$, *there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \geqslant N_0$ *and all* $B \in \mathcal{I}_U$,

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n \in B\}\big| \leqslant \mathrm{Leb}(B) + \varepsilon.$$

*Proof.* Let $(x_n)_{n\in\mathbb{N}_0} \subseteq [0,1)$ be uniformly distributed, $U \in \mathbb{N}$, and $\varepsilon > 0$. The discrepancy of $(x_n)_{n=0}^{N-1}$ (cf. [KN, Def. 1.3]) is

$$D_N = \sup_I \left| \frac{\{0 \leqslant n \leqslant N-1 \mid x_n \in I\}}{N} - \mathrm{Leb}(I) \right|,$$

where the supremum is taken over all half-open intervals $I$ in $[0,1)$. Because $(x_n)_n$ is uniformly distributed, $D_N \to 0$ as $N \to \infty$. By the definition of discrepancy, for any half-open interval $I \subseteq [0,1)$,

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n \in I\}\big| \leqslant \mathrm{Leb}(I) + D_N.$$

It follows that for every $B \in \mathcal{I}_U$,

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n \in B\}\big| \leqslant \mathrm{Leb}(B) + UD_N.$$

Let $N_0 \in \mathbb{N}$ be large enough so that for all $N \geqslant N_0$, $UD_N \leqslant \varepsilon$. The conclusion follows. □

**Lemma 3.13.** *Let* $\beta > 0$. *For any uniformly distributed sequence* $(x_n)_{n\in\mathbb{N}_0} \subseteq [0,\beta)$ *with respect to the Lebesgue measure,* $U \in \mathbb{N}$, *and* $\varepsilon > 0$, *there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \geqslant N_0$ *and all* $J \subseteq [0,\beta)$ *whose complement is covered by a union of no more than* $U$ *many disjoint, half-open intervals of total Lebesgue measure less than* $\varepsilon\beta/2$,

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n \in J\}\big| \geqslant 1 - \varepsilon.$$

*Proof.* Let $(x_n)_{n\in\mathbb{N}_0} \subseteq [0,\beta)$ be uniformly distributed, $U \in \mathbb{N}$, and $\varepsilon > 0$. Let $N_0$ be from Lemma 3.12 with $(x_n/\beta)_{n\in\mathbb{N}_0}$, $U$, and $\varepsilon/2$.

Let $N \geqslant N_0$ and $J \subseteq [0,\beta)$. Put $B = [0,\beta)\backslash J$, and note that by assumption, $B/\beta \in \mathcal{I}_U$ and $\mathrm{Leb}(B/\beta) < \varepsilon/2$. It follows from Lemma 3.12 that

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n/\beta \in B/\beta\}\big| < \varepsilon.$$

Therefore,

$$\frac{1}{N}\big|\{0 \leqslant n \leqslant N-1 \mid x_n \in J\}\big| \geqslant 1 - \varepsilon,$$

as was to be shown. □

### 3.4. Outline of the proof of Theorem A

Before beginning with the details of the proof of Theorem A, we explain the main ideas behind it. To understand the argument, it helps to begin by assuming that the set $X \times Y$ is self-similar in the sense that for every $n \in \mathbb{N}_0$, it is a union of approximately $r^{n(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y)}$ many translates of the set $r^{-n}X \times s^{-n'}Y$. (Recall that $n' = \lfloor n \log r / \log s \rfloor$ so that $s^{-n'} \approx r^{-n}$.) This is the case, for example, if $X$ and $Y$ are both restricted digit Cantor sets. In this case, Peres and Shmerkin [PS] proved that for all $\lambda, \eta \in \mathbb{R}\backslash\{0\}$, $\dim_{\mathrm{H}}(\lambda X + \mu Y) = \overline{\gamma}$. Our argument follows along the same lines as theirs.

Recall that $\Pi_t : \mathbb{R}^2 \to \mathbb{R}$ is the oblique projection $\Pi_t(x, y) = x + ty$. A quick calculation shows that

$$\Pi_{e^t}(r^{-n}X \times s^{-n'}Y) = r^{-n}\Pi_{e^t r^n/s^{n'}}(X \times Y),$$

which implies that the images of the translates of $r^{-n}X \times s^{-n'}Y$ under the map $\Pi_{e^t}$ are affinely equivalent to the image of the full set $X \times Y$ under the map $\Pi_{e^t r^n/s^{n'}}$. It follows that the set $\Pi_{e^t}(X \times Y)$ contains affine images of the sets $\Pi_{e^t r^n/s^{n'}}(X \times Y)$ and hence that

$$\dim_{\mathrm{H}} \Pi_{e^t}(X \times Y) \geqslant \sup_{n \in \mathbb{N}_0} \dim_{\mathrm{H}} \Pi_{e^t r^n/s^{n'}}(X \times Y).$$

Thus, to bound $\dim_{\mathrm{H}} \Pi_{e^t}(X \times Y)$ from below, it suffices to show that there is some $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a "good angle" for $X \times Y$, in the sense that $\dim_{\mathrm{H}} \Pi_{e^t r^n/s^{n'}}(X \times Y) > \overline{\gamma} - \varepsilon$. It follows from Marstrand's theorem that the set of such "good angles" for $X \times Y$ (indeed, for any set) has full measure in $\mathbb{R}$, and it will be shown that the sequence $n \mapsto \log(e^t r^n/s^{n'})$ has image in $[t, t + \log s)$ and is the orbit of $t$ under the irrational $x \mapsto x + \log r \pmod{\log s}$ translated by $t$. When combined, these facts fall just short of allowing us to conclude the existence of $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a good angle: it is possible that the image of an equidistributed sequence misses a set of full measure.

To make use of the above outline, one needs to gain some topological information on the set of good angles from Marstrand's theorem. This can be accomplished by moving the argument to a discrete setting. Discretizing introduces a number of technical nuisances, but the core of the argument remains the same. Recall that $X_n$ and $Y_{n'}$ are the sets $X$ and $Y$ rounded to the lattices $r^{-n}\mathbb{Z}$ and $s^{-n'}\mathbb{Z}$, respectively. The discrete analogue of Marstrand's theorem in Theorem 3.3 tells us that the complement of the set of "good angles" for a finite set such as $X_n \times Y_{n'}$ can be covered by a disjoint union of few half-open intervals. This topological information combines with the equidistribution of the irrational rotation described above to allow us to find many $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a good angle for $X_n \times Y_{n'}$.

The argument described thus far is essentially due to Peres and Shmerkin in [PS] and allows them to conclude that for all $t \in \mathbb{R}\backslash\{0\}$, $\dim_{\mathrm{H}} \Pi_{e^t}(X \times Y) = \overline{\gamma}$. We will now describe the two primary modifications we make to this argument in the course of the proof of Theorem A.

The first modification allows us to show that the discrete Hausdorff content of $\Pi_{e^t}(X \times Y)$ at all small scales is uniform in $t$. Ultimately, this uniformity stems from the fact that the irrational rotation described above is uniquely ergodic: changing $t$ in the argument above changes only the point whose orbit we consider. Exposing the uniformity in the argument after this is then mainly a matter of taking care with the quantifiers in the auxiliary results.

The second modification allows us to handle sets $X$ and $Y$ which are only assumed to be $\times r$- and $\times s$-invariant. Such sets need not be self-similar, but they do exhibit some "near self similarity" in the following sense. Consider the discrete set $X_m$ for some large $m \in \mathbb{N}$. Because $X$ is $\times r$-invariant, the set $X_{(n+1)m} \cap \big[i/r^{nm}, (i+1)/r^{nm}\big)$, when dilated by $r^{mn}$ and considered modulo 1, is a subset of $X_m$. While this set is generally not equal to $X_m$, it is, by an averaging argument, very often of cardinality greater than $r^{-\varepsilon}|X_m|$. This is profitably re-interpreted in the language of trees: in the tree with levels $X_{nm} \times Y_{(nm)'}$, $n \in \mathbb{N}_0$, many nodes have nearly the maximum allowed number of children. The tree thinning result in

Theorem 3.11 exploits this abundance by finding a sufficiently "regular" subtree on which we focus our attention. Then, we invoke our discrete analogue of Marstrand's theorem – which provides information on the set of angles that are good not only for the original set $X_m \times Y_{m'}$, but also for large subsets of it – to further thin the subtree. Following the reasoning given in Remark 3.8, the resulting subtree has fertile ancestry and hence has large Hausdorff content. By the construction of the subtree, its image under $\Pi_{e^t}$ is large, and this yields the lower bound on the Hausdorff dimension in the conclusion of the theorem.

### 3.5. Proof of Theorem A

In this section and the next, let $r$, $s$, $X$, $Y$, and $\overline{\gamma}$ be as given as in the statement of Theorem A. The proof of Theorem A begins with a number of reductions, the last of which in Claim 3.14 is a statement about the existence of measures on the images of the discrete product sets under oblique projections. We prove Claim 3.14 in the next subsection.

By Lemma 2.10, $\dim_{\mathrm{H}}(X \times Y) = \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y$. Note that if $\dim_{\mathrm{H}} X = 0$, then the conclusion is clear by considering, for any $x \in X$, images of the set $\{x\} \times Y$. The same is true if $\dim_{\mathrm{H}} Y = 0$. Thus, we will proceed under the assumption that $\dim_{\mathrm{H}} X, \dim_{\mathrm{H}} Y > 0$. Note that the set $1 - X$ is $\times r$-invariant and that $-\lambda X + \eta Y$ is a translate of the set $\lambda(1-X) + \eta Y$. The analogous statement holds for $Y$. Combining these facts, it is easy to see that it suffices to prove Theorem A in the case that $I \subseteq (0, \infty)$.

The next step is to formulate a statement sufficient to prove Theorem A in terms of oblique projections of discrete sets. Recall that $n' = \lfloor n \log r / \log s \rfloor$ and that $X_n$, $Y_{n'}$ are the sets $X$ and $Y$ rounded to the lattices $r^{-n}\mathbb{Z}$ and $s^{-n'}\mathbb{Z}$, respectively. For $n \in \mathbb{N}_0$, define

$$\mathcal{Q}_n = X_n \times Y_{n'} \qquad \text{and} \qquad \widetilde{\mathcal{Q}}_n = X_n \times Y_{n'+1}. \tag{3.8}$$

**Claim 3.14.** *For all compact $I \subseteq \mathbb{R}$ and all $0 < \gamma < \overline{\gamma}$, there exists $m, N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$ and all $t \in I$, there exists a probability measure $\mu$ supported on the finite set $\Pi_{e^t} \mathcal{Q}_{Nm}$ with the property that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geqslant r^{-Nm}$, $\mu(B) \leqslant r^{N_0 m} \delta^{\gamma}$.*

To deduce Theorem A from Claim 3.14, let $I \subseteq (0, \infty)$ be compact and $0 < \gamma < \overline{\gamma}$. Apply Claim 3.14 with $\tilde{I} := \{ \log(\eta/\lambda) \mid \eta, \lambda \in I \}$ as $I$ and $\gamma$ as it is. Let $m, \mathbb{N}_0 \in \mathbb{N}$ be as guaranteed by Claim 3.14.

Note that the limit in (1.8) is guaranteed to exist because the function $\rho \mapsto \inf_{\lambda, \eta \in I} \mathcal{H}^{\gamma}_{\geqslant \rho} \big(\lambda X + \eta Y\big)$ is non-increasing (as $\rho$ decreases) and is bounded from below by zero. Therefore, to show that (1.8) holds, it suffices to prove that

$$\lim_{N \to \infty} \inf_{\lambda, \eta \in I} \mathcal{H}^{\gamma}_{\geqslant r^{-Nm}} \big(\lambda X + \eta Y\big) > 0. \tag{3.9}$$

It follows from the fact that

$$d_H(\lambda X_{Nm} + \eta Y_{(Nm)'}, \lambda X + \eta Y) \ll_{I,r,s} r^{-Nm}$$

and Lemma 2.7 that for all $\lambda, \eta \in I$,

$$\begin{aligned} \mathcal{H}^{\gamma}_{\geqslant r^{-Nm}} \big(\lambda X + \eta Y\big) &\asymp_{I,r,s} \mathcal{H}^{\gamma}_{\geqslant r^{-Nm}} \big(\lambda X_{Nm} + \eta Y_{(Nm)'}\big) \\ &\asymp_{I,r,s} \mathcal{H}^{\gamma}_{\geqslant r^{-Nm}} \big(X_{Nm} + e^{\log(\eta/\lambda)} Y_{(Nm)'}\big). \end{aligned} \tag{3.10}$$

Therefore, to show (3.9), it suffices to prove that

$$\lim_{N\to\infty} \inf_{t\in\tilde{I}} \mathcal{H}^{\gamma}_{\geqslant r^{-Nm}}\big(\Pi_{e^t}\mathcal{Q}_{Nm}\big) > 0. \tag{3.11}$$

Combining the conclusion of Claim 3.14 with Lemma 2.6, we see that for all $N \geqslant N_0$ and $t \in \tilde{I}$, $\mathcal{H}^{\gamma}_{\geqslant r^{-Nm}}\big(\Pi_{e^t}\mathcal{Q}_{Nm}\big) \geqslant r^{-N_0 m}$. This shows that the limit in (3.11) is positive and completes the deduction of Theorem A from Claim 3.14.

## 3.6. Proof of Claim 3.14

**Choosing the parameter $m$ and scale $\rho$.** Recall that $r$, $s$, $X$, $Y$, and $\overline{\gamma}$ are as given as in the statement of Theorem A. Without loss of generality, we can assume that $r < s$. Put $\beta = \log s$, let $0 < \gamma < \overline{\gamma}$, and define $\varepsilon := \overline{\gamma} - \gamma$ and $\gamma_1 := \gamma$.

We claim that there exist $\gamma_2$, $\gamma_3$, $\gamma_4$, and $\gamma_5$ such that

(I) $0 < \gamma_1/(1-\varepsilon/2) < \gamma_2 < \gamma_3 < \gamma_4 < \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y < \gamma_5$;

(II) $\gamma_5 < \gamma_4 + \varepsilon(\gamma_4 - \gamma_3)/6$;

(III) $\gamma_2 < 1$;

(IV) $2(\gamma_5 - \gamma_3) < \gamma_4 - \gamma_2$ (this is the inequality in (3.3)).

To see why, note that if we put $\gamma_2 = \gamma_1/(1-\varepsilon/2)$, $\gamma_4 = \gamma_5 = \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y$, and $\gamma_3 = \gamma_2/3 + 2\gamma_4/3$, then the inequalities in (I) holds with "$<$" replaced by "$\leqslant$", while the inequalities in (II), (III), and (IV) hold as written. It follows that $\gamma_2$ and $\gamma_5$ can be increased and $\gamma_4$ can be decreased (with the corresponding change $\gamma_3 = \gamma_2/3 + 2\gamma_4/3$) so that all of the inequalities hold.

Let $c_1$ and $c_2$ be the constants guaranteed in the statement of Corollary 2.15. Let $I \subseteq (0,\infty)$ be compact, and define $I_\beta = I + [0,\beta]$. Let $P > 0$ be a Lipschitz constant for all of the maps $\Pi_{e^t}$, $t \in I_\beta$, and let $c_3 = 4Ps^{-1} + 1$. Choose $m \in \mathbb{N}$ large enough so that we can apply

- Theorem 3.11 with $\varepsilon/6$ as $\varepsilon$ and $r^m$ as $r$;
- Corollary 3.4 with $I_\beta$ as $I$, $\varepsilon\beta/12$ as $\varepsilon$ and $r^{-m}$ as $\rho$;
- Corollary 2.15 with $m$ as $N$.

Put $\rho = r^{-m}$.

**A uniformly distributed sequence.** Let $\alpha = \log\big(r^m/s^{m'}\big)$ and let $R : [0,\beta) \to [0,\beta)$ be the transformation $R : x \mapsto x + \alpha \pmod{\beta}$. As $\beta = \log s$ and $m' = \lfloor m \log r/\log s\rfloor$, we have

$$\alpha/\beta = m\log r/\log s - m' = \big\{m \log r/\log s\big\}. \tag{3.12}$$

Since $\log r/\log s$ is irrational, we conclude that $\alpha/\beta$ is irrational, whereby the sequence $(R^n(0))_{n\in\mathbb{N}_0}$ is uniformly distributed on $[0,\beta)$.

**Claim 3.15.** *For all $n \in \mathbb{N}_0$,*

(V) $R^n(0) + (nm)'\log s = nm\log r$;

(VI)
$$\big((n+1)m\big)' = \begin{cases} (nm)' + m' & \text{if } R^n(0) + \alpha < \beta \\ (nm)' + m' + 1 & \text{if } R^n(0) + \alpha > \beta \end{cases}.$$

*Proof.* Since for all $n \in \mathbb{N}$, $R^n(0) = n\alpha \pmod{\beta}$, using (3.12), we can write $R^n(0)/\beta = \big\{n\alpha/\beta\big\} = \big\{n\{m\log r/\beta\}\big\} = \big\{nm\log r/\beta\big\}$. Recalling that $(nm)' = \lfloor nm\log r/\beta\rfloor$, this establishes (V).

Next, note that for any real numbers $x, y$,

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if } \{x\}+\{y\} < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } \{x\}+\{y\} \geqslant 1 \end{cases}.$$

The equality in (VI) follows from this by substituting $x = nm \log r/\beta$ and $y = m \log r/\beta$ and using $R^n(0)/\beta = \{nm \log r/\beta\}$ and (3.12). □

**Choosing the parameter $N_0$.** From Corollary 2.15, the sets $\mathcal{Q}_m$ and $\widetilde{\mathcal{Q}}_m$ (defined in (3.8)) are $(c_1\rho, \gamma_5)_{c_2}$-sets and satisfy

$$\rho^{-\gamma_4} \leqslant |\mathcal{Q}_m|, |\widetilde{\mathcal{Q}}_m| \leqslant \rho^{-\gamma_5}. \tag{3.13}$$

Let $T_1$ (resp. $T_2$) be the subset of $I_\beta$ obtained from applying Corollary 3.4 with $I_\beta$ as $I$, $\varepsilon\beta/12$ as $\varepsilon$ and $\mathcal{Q}_m$ (resp. $\widetilde{\mathcal{Q}}_m$) as $A$. Put $T = T_1 \cap T_2$. It follows from Corollary 3.4 that $I_\beta \backslash T$ is covered by a disjoint union of finitely many half-open intervals of Lebesgue measure less than $\varepsilon\beta/6$. Let $U \in \mathbb{N}$ be the number of half-open intervals which suffice to cover the set $I_\beta \backslash T$.

Let $N_0 \in \mathbb{N}$ be the larger of

- the $N_0$ from Theorem 3.11 with $\varepsilon/6$ as $\varepsilon$, $r^m$ as $r$, and $2^{-\gamma_4}\mathcal{H}^{\gamma_4}_\infty(X \times Y)$ as $V$;
- the $N_0$ from Lemma 3.13 with $(R^n(0))_{n\in\mathbb{N}_0}$ as $(x_n)_{n\in\mathbb{N}_0}$ and $\varepsilon/3$ as $\varepsilon$.

**Fixing the parameters $N$ and $t$.** To prove Claim 3.14, we will show that for all $N \geqslant N_0$ and all $t \in I$ there exists a probability measure $\mu$ supported on the set $\Pi_{e^t}\mathcal{Q}_{Nm}$ with the property that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geqslant \rho^N$, $\mu(B) \leqslant \rho^{-N_0}\delta^\gamma$. Let $N \geqslant N_0$ and $t \in I$. From this point on, all new quantities and objects can depend on $N$ and $t$.

**Constructing the tree $\Gamma$.** Let $\Gamma$ be the tree (see Definition 3.5) of height $N$ with node set at height $n \in \{0, 1, \ldots, N\}$ equal to $\mathcal{Q}_{nm}$. Associating the point $(i/r^{mn}, j/s^{(mn)'}) \in \mathcal{Q}_{nm}$ with the rectangle

$$\left[\frac{i}{r^{mn}}, \frac{i+1}{r^{mn}}\right) \times \left[\frac{j}{s^{(mn)'}}, \frac{j+1}{s^{(mn)'}}\right),$$

parentage in the tree $\Gamma$ is determined by containment amongst associated rectangles.

Denote by $C_\Gamma(Q)$ the children of the node $Q$ in $\Gamma$. Denote by $\odot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the binary operation of pointwise multiplication.

**Claim 3.16.** *Let $n < N$ and $Q \in \mathcal{Q}_{nm}$.*

(VII) *If $R^n(0) + \alpha < \beta$, then $C_\Gamma(Q) \subseteq Q + (r^{-nm}, s^{-(nm)'}) \odot \mathcal{Q}_m$.*

(VIII) *If $R^n(0) + \alpha > \beta$, then $C_\Gamma(Q) \subseteq Q + (r^{-nm}, s^{-(nm)'}) \odot \widetilde{\mathcal{Q}}_m$.*

(IX) *$\mathcal{H}^{\gamma_4}_{r^m}(\Gamma) \geqslant 2^{-\gamma_4}\mathcal{H}^{\gamma_4}_\infty(X \times Y)$.*

*Proof.* We first prove parts (VII) and (VIII). By Lemma 2.12, $rX_n \subseteq X_{n-1}$ (mod 1) and $sY_{n'} \subseteq Y_{n'-1}$ (mod 1). By (VI), if $R^n(0) + \alpha < \beta$, then $\big((n+1)m\big)' = (nm)' + m'$, and hence $(r^{nm}, s^{(nm)'}) \odot \mathcal{Q}_{(n+1)m} \subseteq \mathcal{Q}_m$ (mod 1), and in particular $(r^{nm}, s^{(nm)'}) \odot C_\Gamma(Q) \subseteq \mathcal{Q}_m$ (mod 1). If $R^n(0) + \alpha > \beta$, then $\big((n+1)m\big)' = (nm)' + m' + 1$, and hence $(r^{nm}, s^{(nm)'}) \odot \mathcal{Q}_{(n+1)m} \subseteq \widetilde{\mathcal{Q}}_m$ (mod 1), and in particular $(r^{nm}, s^{(nm)'}) \odot C_\Gamma(Q) \subseteq \widetilde{\mathcal{Q}}_m$ (mod 1).

Write $Q = (i/r^{nm}, j/s^{(nm)'})$ and let $Q' \in C_\Gamma(Q)$. Because $Q'$ is a child of $Q$, we can write $Q' = Q + (i_0/r^{(n+1)m}, j_0/s^{((n+1)m)'})$ where $0 \leqslant i_0 < r^m$ and $0 \leqslant j_0 < s^{m'}$. It follows that $(r^{nm}, s^{(nm)'}) \odot (C_\Gamma(Q) - Q) \subseteq \mathcal{Q}_m$ (in the first case $R^n(0) + \alpha < \beta$) or $(r^{nm}, s^{(nm)'}) \odot (C_\Gamma(Q) - Q) \subseteq \widetilde{\mathcal{Q}}_m$ (in the second case $R^n(0) + \alpha > \beta$), where the containment now is understood without reducing modulo 1.

To prove (IX), take a cut $\{Q_1, \dots, Q_\ell\} \subseteq \Gamma$ of $\Gamma$ with node $Q_i$ at height $n_i$. Then, by construction of $\Gamma$, there exists a cover $X \times Y \subseteq \cup_{i=1}^{\ell} B_i$ where ball $B_i$ has diameter at most $2\rho^{n_i}$. Since the cut was arbitrary, it follows that $\mathcal{H}^{\gamma_4}_{r^m}(\Gamma) \geqslant 2^{-\gamma_4}\mathcal{H}^{\gamma_4}_\infty(X \times Y)$. □

**Constructing the tree $\Gamma'$.** Combining (3.13) with (VII) and (VIII), it follows that $|C_\Gamma(Q)| \leqslant r^{m\gamma_5}$ for every non-leaf node $Q$ of $\Gamma$. The tree $\Gamma$ has now been shown to satisfy all the hypothesis of Theorem 3.11 (with $\varepsilon/6$ as $\varepsilon$, $r^m$ as $r$, and $2^{-\gamma_4}\mathcal{H}^{\gamma_4}_\infty(X \times Y)$ as $V$), thus there exists a subtree $\Gamma'$ of $\Gamma$ with the property that every node with height at least $N_0$ has $(r^{m\gamma_3}, 1 - \varepsilon/6)$-fertile ancestry in $\Gamma'$.

**Constructing the tree $\Gamma''$.** Now we will use Corollary 3.4, the corollary to the discrete version of Marstrand's theorem, to further thin out the tree $\Gamma'$; an outline for this step was described in Remark 3.8 (I). For each non-leaf node $Q \in \Gamma'$, we will define a subset $C^m_{\Gamma'}(Q)$ of $C_{\Gamma'}(Q)$. Define $J = (T - t) \cap [0, \beta)$. Since $I_\beta \backslash T$ is covered by at most $U$ many half-open intervals of measure less than $\varepsilon\beta/6$, the same is true for the set $[0, \beta)\backslash J$. Define $\mathcal{J} = \{0 \leqslant n \leqslant N - 1 \mid R^n(0) \in J\}$. Note that for all $n \geqslant N_0$, by Lemma 3.13, $|\mathcal{J} \cap \{0, \dots, n-1\}| \geqslant (1 - \varepsilon/3)n$.

Let $Q$ be a non-leaf node of $\Gamma'$, and let $n \in \{0, \dots, N-1\}$ be the height of $Q$. Consider the following cases:

(X) $n \notin \mathcal{J}$ or $|C_{\Gamma'}(Q)| < \rho^{-\gamma_3}$. Select a single child $Q'$ of $Q$ and put $C^m_{\Gamma'}(Q) = \{Q'\}$.

(XI) $n \in \mathcal{J}$, $|C_{\Gamma'}(Q)| \geqslant \rho^{-\gamma_3}$, and $R^n(0) + \alpha < \beta$. By Theorem 3.11 and (VII), the set $A' := (r^{nm}, s^{(nm)'}) \odot (C_{\Gamma'}(Q) - Q)$ is a subset of $\mathcal{Q}_m$ of cardinality at least $\rho^{-\gamma_3}$. Since $n \in \mathcal{J}$, we have that $t + R^n(0) \in T$. Applying Corollary 3.4 (II) with $t + R^n(0)$ in the role of $t$, there exists a subset $A'_t \subseteq A'$ with $|A'_t| \geqslant \rho^{-\gamma_2}$ and such that the points of $\Pi_{e^{t+R^n(0)}} A'_t$ are distinct and $c_3\rho$-separated. Define $C^m_{\Gamma'}(Q) = Q + (r^{-nm}, s^{-(nm)'}) \odot A'_t$ so that $(r^{nm}, s^{(nm)'}) \odot (C^m_{\Gamma'}(Q) - Q) = A'_t$.

(XII) $n \in \mathcal{J}$, $|C_{\Gamma'}(Q)| \geqslant \rho^{-\gamma_3}$, and $R^n(0) + \alpha > \beta$. We do exactly as in (XI) with $\mathcal{Q}_m$ replaced by $\widetilde{\mathcal{Q}}_m$ and using (VIII) to get the set $C^m_{\Gamma'}(Q)$.

Let $\Gamma''$ be the subtree of $\Gamma'$ with the property that if $Q$ is a non-leaf node of $\Gamma''$, then $C_{\Gamma''}(Q) = C^m_{\Gamma'}(Q)$. We claim that

$$\text{every node of } \Gamma'' \text{ with height at least } N_0 \text{ has } (r^{m\gamma_2}, 1 - \varepsilon/2)\text{-fertile ancestry.} \tag{3.14}$$

Indeed, let $Q$ be a node of $\Gamma''$ with height $n \geqslant N_0$. The ancestry of $Q$ in $\Gamma'$ is $(r^{m\gamma_3}, 1 - \varepsilon/6)$-fertile. Each $r^{m\gamma_3}$-fertile ancestor of $Q$ in $\Gamma'$ with height in the set $\mathcal{J}$ is an $r^{m\gamma_2}$-fertile ancestor of $Q$ in $\Gamma''$. Since $\left|\mathcal{J} \cap \{0, \dots, n-1\}\right| \geqslant (1 - \varepsilon/3)n$, there are at least $(1 - \varepsilon/2)n$ many $r^{m\gamma_3}$-fertile ancestor of $Q$ in $\Gamma'$ with height in the set $\mathcal{J}$. It follows that $Q$ has $(r^{m\gamma_2}, 1 - \varepsilon/2)$-fertile ancestry in $\Gamma''$.

**Claim 3.17.** *If $L_1$ and $L_2$ are two distinct leaves of $\Gamma''$ and $n$ is maximal such that $L_1$ and $L_2$ have a common ancestor at height $n$, then $|\Pi_{e^t} L_1 - \Pi_{e^t} L_2| \geqslant \rho^{n+1}$.*

*Proof.* Let $Q$ be the common ancestor of $L_1$ and $L_2$ in $\Gamma''$ of height $n$. Note that by the definition of $\Gamma''$ and maximality of $n$, it must be that $Q$ has more than one child and hence that $n \in \mathcal{J}$. Let $Q_1$ and $Q_2$ be the children of $Q$ in $\Gamma''$ that are ancestors of $L_1$ and $L_2$, respectively. Note that $Q_1 \neq Q_2$ but that $Q_i$ may be equal to $L_i$.

We will show first that $\Pi_{e^t} Q_1$ and $\Pi_{e^t} Q_2$ are $c_3\rho^{n+1}$-separated. Write $Q = (p, q)$ and $Q_i = (p_i, q_i)$. Suppose that $R^n(0) + \alpha < \beta$. It follows from (V) that

$$\begin{aligned}\Pi_{e^t} Q_i &= r^{-nm}\left(r^{nm}\Pi_{e^t}(Q_i - Q)\right) + \Pi_{e^t} Q \\ &= \rho^n \left(r^{nm}(p_i - p) + e^{t+R^n(0)} s^{(nm)'}(q_i - q)\right) + \Pi_{e^t} Q \\ &= \rho^n \left(\Pi_{e^{t+R^n(0)}}\left((r^{nm}, s^{(nm)'}) \odot (Q_i - Q)\right)\right) + \Pi_{e^t} Q.\end{aligned} \tag{3.15}$$

By (XI), the points of $\Pi_{e^{t+R^n(0)}}\left((r^{nm}, s^{(nm)'}) \odot (Q_i - Q)\right)$, $i = 1, 2$, are $c_3\rho$-separated. It follows then from (3.15) that the points of $\Pi_{e^t} Q_i$, $i = 1, 2$, are $c_3\rho^{n+1}$-separated. A similar argument works to reach the same conclusion if $R^n(0) + \alpha > \beta$ using (XII).

By the definition of the $\mathcal{Q}_{nm}$ sets, $|Q_i - L_i| \leqslant 2s^{-1}\rho^{n+1}$. By the triangle inequality and the fact that $c_3 = 4Ps^{-1} + 1$,

$$\begin{aligned}\left|\Pi_{e^t} L_1 - \Pi_{e^t} L_2\right| &\geqslant \left|\Pi_{e^t} Q_1 - \Pi_{e^t} Q_2\right| - \left|\Pi_{e^t}(Q_1 - L_1)\right| - \left|\Pi_{e^t}(Q_1 - L_1)\right| \\ &\geqslant (4Ps^{-1} + 1)\rho^{n+1} - 4Ps^{-1}\rho^{n+1} \geqslant \rho^{n+1}.\end{aligned}$$

It follows that $\left|\Pi_{e^t} L_1 - \Pi_{e^t} L_2\right| \geqslant \rho^{n+1}$, as was to be shown. □

**Constructing the measure $\mu$.** The proof of Claim 3.14 will be concluded by demonstrating that 1) the fertile ancestry property of $\Gamma''$ in (3.14) guarantees that $\Gamma''$ supports a "measure" which is not too concentrated on any node (an outline for this step was described in Remark 3.8 (II)); and 2) by Claim 3.17, the projection of this measure is not too concentrated on any ball.

Let $\nu : \Gamma'' \to [0, 1]$ be the unique function that takes 1 on the root of $\Gamma''$ and has the properties that for all non-leaf nodes $Q$ of $\Gamma''$, $\nu$ is constant on $\mathcal{C}_{\Gamma''}(Q)$ and $\nu(Q) = \sum_{C \in \mathcal{C}_{\Gamma''}(Q)} \nu(C)$. (Colloquially, a mass of 1 begins at the root of $\Gamma''$ and spreads down the tree by splitting equally amongst the children of each node.) Let $\nu_N$ be the function $\nu$ restricted to $\Gamma''_N$, the set of leaves of $\Gamma''$. By the defining properties of $\nu$, the function $\nu_N$ is a probability measure on $\Gamma''_N$.

Since $\Gamma''_N \subseteq \mathcal{Q}_{Nm}$, the measure $\mu = \Pi_{e^t}\nu_N$, the push-forward of $\nu_N$ through the map $\Pi_{e^t}$, is a probability measure supported on the set $\Pi_{e^t}\mathcal{Q}_{Nm}$. We will conclude the proof of Claim 3.14 by verifying that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geqslant \rho^N$, $\mu(B) \leqslant \rho^{-N_0}\delta^{\gamma_1}$. (Recall that $\gamma_1 = \gamma$.)

Let $B \subseteq \mathbb{R}$ be an interval of length $\delta \geqslant \rho^N$. Put $n = \lfloor \log_\rho \delta \rfloor + 1$ and note that $\rho^n < \delta \leqslant \rho^{n-1}$. It follows from Claim 3.17 that there exists a node $Q$ of $\Gamma''$ with height at least $n$ with the property that if $L$ is a leaf of $\Gamma''$ with $\Pi_{e^t} L \in B$, then $Q$ is an ancestor of $L$. This implies that $\mu(B) \leqslant \nu(Q)$, and so it suffices to show that

$$\nu(Q) \leqslant \rho^{-N_0}\delta^{\gamma_1}. \tag{3.16}$$

If $n \leqslant N_0$, then $\rho^{-N_0}\delta^{\gamma_1} > 1$ and (3.16) holds trivially. If $n > N_0$, then by the definition of $\nu$ and the fact that $Q$ has $(r^{m\gamma_2}, 1 - \varepsilon/2)$-fertile ancestry (cf. (3.14)),

$$\nu(Q) \leqslant \frac{1}{r^{m\gamma_2(1-\varepsilon/2)n}} = \rho^{\gamma_2(1-\varepsilon/2)n} \leqslant \rho^{-N_0}\delta^{\gamma_1},$$

since $(1 - \varepsilon/2)\gamma_2 > \gamma_1$. This verifies (3.16), completing the proof of Claim 3.14 and hence of

Theorem A.

# 4. Multiplicatively invariant subsets of the non-negative integers

This section is devoted to the study of $\times r$-invariant sets in the non-negative integers (see Definition 1.5) and contains the proofs of Theorems B, C, D, and E.

As was already mentioned in Section 1.2.2, $\times r$-invariant subsets of $\mathbb{N}_0$ are closely related to symbolic subshifts in $r$ symbols. This connection is explored in more detail in Section 4.1, where we establish numerous fundamental properties about $\times r$-invariant sets, including the existence of their mass dimension (see Proposition 4.3). Thereafter, in Section 4.2, we show that the study of $\times r$-invariant sets in the integers also connects to fractal geometry in the unit interval $[0,1]$. Among other things, we describe a natural way of identifying $\times r$-invariant subsets of $\mathbb{N}_0$ with $\times r$-invariant subsets of $[0,1]$ which respects the Hausdorff content at every finite scale (see Proposition 4.6). This plays a crucial role in the derivations of Theorems C and E from their continuous counterparts in the later subsections. In Section 4.3 we give an elementary and self-contained proof of Theorem B. Finally, Theorem C is proven in Section 4.4, and Theorems D and E are proven in Section 4.5.

## 4.1. Connections to symbolic dynamics

Throughout this subsection, we use $\sigma$ to denote the left-shift on $\{0,1,\ldots,r-1\}^{\mathbb{N}_0}$, which is defined by

$$\sigma : (w_n)_{n\in\mathbb{N}_0} \mapsto (w_{n+1})_{n\in\mathbb{N}_0}.$$

In the language of symbolic dynamics, any closed subset of $\{0,1,\ldots,r-1\}^{\mathbb{N}_0}$ satisfying $\sigma(\Sigma) \subseteq \Sigma$ is called a *subshift*. The *language set* associated to a subshift $\Sigma$ is the set of all the finite words appearing in the elements of $\Sigma$, i.e.,

$$\mathcal{L}(\Sigma) = \big\{(w_0, w_1, \ldots, w_k) \;\big|\; (w_0, w_1, w_2, \ldots) \in \Sigma,\ k \in \mathbb{N}_0\big\}.$$

The language set of any subshift can be naturally embedded into the integers by identifying finite words in the alphabet $\{0,1,\ldots,r-1\}$ with the base-$r$ expansion in the integers. This gives rise to the following definition.

**Definition 4.1.** The *$r$-language set* associated to a subshift $\Sigma \subseteq \{0,1,\ldots,r-1\}^{\mathbb{N}_0}$ is the set $A_\Sigma \subseteq \mathbb{N}_0$ defined by

$$A_\Sigma \;=\; \{0\} \cup \big\{w_0 + w_1 r + \ldots + w_{k-1} r^{k-1} \;\big|\; (w_0, w_1, \ldots, w_{k-1}) \in \mathcal{L}(\Sigma)\big\}.$$

Note that the set $A_\Sigma$ is a $\times r$-invariant subset of $\mathbb{N}_0$ (recall Definition 1.5). Indeed, the invariance of $A_\Sigma$ under the map $\mathfrak{R}_r$ follows directly from the shift-invariance of $\Sigma$, whereas the invariance of $A_\Sigma$ under $\mathfrak{L}_r$ follows from the simple observation that for any word $(w_0, w_1, \ldots, w_k) \in \mathcal{L}(\Sigma)$, all of the prefixes $(w_0, w_1, \ldots, w_i)$, $i \leqslant k$, belong to $\mathcal{L}(\Sigma)$. Therefore, the notion of $r$-language sets provides us with a natural way of constructing from any subshift of the full symbolic shiftspace in $r$ letters a $\times r$-invariant subsets of the non-negative integers.

**Examples 4.2.**

- Let $\mathcal{F}$ be an arbitrary collection of finite words from the alphabet $\{0,1,\ldots,r-1\}$.

The *shift of finite type* with forbidden words $\mathcal{F}$ is the subshift of $\{0,1,\ldots,r-1\}^{\mathbb{N}_0}$ consisting of all infinite words that do not contain a word from $\mathcal{F}$ as a sub-word. The $r$-language sets corresponding to shifts of finite type form a natural class of $\times r$-invariant subsets of $\mathbb{N}_0$ which contains the class of restricted digit Cantor sets defined in (1.6) as a rather special subclass. Integer sets corresponding to subshifts of finite type were also considered by Lima and Moreira in [LM2].

- The classical *golden mean shift* is the subshift of $\{0,1\}^{\mathbb{N}_0}$ consisting of all binary sequences with no two consecutive 1's. This leads to a natural example of a $\times 2$-invariant set $A_{\text{golden}} \subseteq \mathbb{N}_0$ consisting of all integers whose binary digit expansion does not contain two consecutive 1's. Since the topological entropy of the golden mean shift is known the equal $\log((1+\sqrt{5})/2)$ (cf. [LM3, Example 4.1.4]), it follows from Proposition 4.3 below that the mass dimension of $A_{\text{golden}}$ equals $\log((1+\sqrt{5})/2)/\log 2$.
- The *even shift* is the subshift of $\{0,1\}^{\mathbb{N}_0}$ consisting of all binary sequences so that between any two 1's there are an even number of 0's. The corresponding $\times 2$-invariant set $A_{\text{even}} \subseteq \mathbb{N}_0$ consists of all integers whose binary digit expansion has an even number of 0's between any two 1's. Since the topological entropy of the golden mean shift coincides with the topological entropy of the even shift (cf. [LM3, Example 4.1.6]), we conclude that $A_{\text{even}}$ and $A_{\text{golden}}$ have the same mass dimension.
- The *prime gap shift* is the subshift of $\{0,1\}^{\mathbb{N}_0}$ consisting of all binary sequences such that there is a prime number of 0's between any two 1's. This corresponds to the $\times 2$-invariant set $A_{\text{prime}} \subseteq \mathbb{N}_0$ of all those numbers written in binary in which there is a prime number of 0's between any two 1's. For example, the first 17 elements of $A_{\text{prime}}$ are: $0, 1, 2, 4, 8, 9, 16, 17, 18, 32, 34, 36, 64, 65, 68, 72, 73$. The entropy of the prime gap shift is approximately 0.30293, (cf. [LM3, Exercise 4.3.7]) which implies that the dimension of $A_{\text{prime}}$ is approximately 0.437.

As we have observed, every $r$-language set is a $\times r$-invariant set. The converse is also true: Every $\times r$-invariant subset $A \subseteq \mathbb{N}_0$ coincides with the $r$-language set associated to some subshift $\Sigma \subseteq \{0,1,\ldots,r-1\}^{\mathbb{N}_0}$. Additionally, the dimension of $A$ coincides with the normalized topological entropy of $(\Sigma, \sigma)$.

**Proposition 4.3.** *For any $\times r$-invariant set $A \subseteq \mathbb{N}_0$, there exists a closed and shift-invariant set $\Sigma \subseteq \{0,1,\ldots,r-1\}^{\mathbb{N}_0}$ such that $A$ coincides with $A_\Sigma$, the $r$-language set associated to $\Sigma$. Moreover, the mass dimension of $A$ exists and equals the normalized topological entropy of the symbolic subshift $(\Sigma, \sigma)$, i.e.,*

$$\dim_M A = \frac{h_{\text{top}}(\Sigma, \sigma)}{\log r}.$$

We remark that Proposition 4.3 is a generalization of some of the results in [LM2, Section 3], where subsets of integers arising from shifts of finite type are defined and studied. Also note that Proposition 4.3 only gives the existence of the mass dimension, not the discrete Hausdorff dimension, for $\times r$-invariant sets. The discrete Hausdorff dimension of such sets also always exists, but this is only proved in the next subsection using different methods (see Proposition 4.6 below). Finally, we remark that the identification of $\times r$-invariant subsets of $\mathbb{N}_0$ and subshifts of $\{0,1,\ldots,r-1\}^{\mathbb{N}_0}$ given by Proposition 4.3 is not bijective. In fact, for every $\times r$-invariant set $A \subseteq \mathbb{N}_0$, there exists an infinite family of subshifts $\Sigma$ such that $A = A_\Sigma$.

*Proof of Proposition 4.3.* Let $\Sigma^{(k)}$ denote the set of all infinite words $w = (w_0, w_1, \ldots) \in \{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$ for which $w_0 + w_1 r + \ldots + w_k r^k$ is an element of $A$, and define

$$\Sigma \coloneqq \bigcap_{k \in \mathbb{N}_0} \Sigma^{(k)}.$$

Being an intersection of closed sets, $\Sigma$ is closed. From $\mathfrak{R}_r(A) \subseteq A$, it follows that $\sigma(\Sigma) \subseteq \Sigma$, which proves that $(\Sigma, \sigma)$ is a subshift. From the construction it is clear that the language set $A_\Sigma$ of $\Sigma$ is contained in $A$. On the other hand, if $a = w_0 + \cdots + w_k r^k \in A$ then, using $\mathfrak{L}_r(A) \subseteq A$, it follows that the word $(w_0, \ldots, w_k, 0, 0, \ldots)$ belongs to $\Sigma$. It follows that $a \in A_\Sigma$, showing that $A = A_\Sigma$.

It remains to verify that $\dim_{\mathrm{M}} A = h_{\mathrm{top}}(\Sigma, T)/\log r$. Let $\mathcal{L}_N(\Sigma)$ denote the set of words of length $N$ appearing in the language set $\mathcal{L}(\Sigma)$, i.e.,

$$\mathcal{L}_N(\Sigma) \;=\; \big\{(w_0, w_1, \ldots, w_{N-1}) \;\big|\; (w_0, w_1, w_2, \ldots) \in \Sigma\big\}.$$

It is well known (see, for instance, [Wal, Theorem 7.13 (i)]) that the topological entropy of $(\Sigma, \sigma)$ is given by

$$h_{\mathrm{top}}(\Sigma, \sigma) \;=\; \lim_{N \to \infty} \frac{1}{N} \log |\mathcal{L}_N(\Sigma)|,$$

where the limit as $N \to \infty$ on the right hand side is known to exist. Since $|\mathcal{L}_N(\Sigma)| = |A \cap [0, r^N)|$, we have

$$\frac{\log |\mathcal{L}_N(\Sigma)|}{N} \;=\; \frac{\log |A \cap [0, r^N)|}{N}. \tag{4.1}$$

It follows that

$$h_{\mathrm{top}}(\Sigma, \sigma) \;=\; \lim_{N \to \infty} \frac{\log |A \cap [0, r^N)|}{N} \;=\; (\log r)(\dim_{\mathrm{M}} A),$$

which implies $\dim_{\mathrm{M}} A = h_{\mathrm{top}}(\Sigma, \sigma)/\log r$. □

As a corollary of Proposition 4.3 we obtain the following result, which plays an important role in our proof of Theorem D.

**Corollary 4.4.** *For any $\times r$-invariant $A \subseteq \mathbb{N}_0$, the set*

$$A' \coloneqq \bigcap_{k \in \mathbb{N}_0} \bigcap_{\ell \in \mathbb{N}_0} \mathfrak{R}_r^k \mathfrak{L}_r^\ell (A)$$

*satisfies $\mathfrak{R}_r(A') = \mathfrak{L}_r(A') = A'$ (in particular, $A'$ is $\times r$-invariant) and* $\dim_M A' = \dim_M A$.

*Proof.* Note that $A'$ is the largest subset of $A$ satisfying $\mathfrak{R}_r(A') = \mathfrak{L}_r(A') = A'$; in particular, it is $\times r$-invariant. Therefore, to prove $\dim_{\mathrm{M}} A' = \dim_{\mathrm{M}} A$, it suffices to find any subset $A'' \subseteq A$ satisfying $\mathfrak{R}_r(A'') = \mathfrak{L}_r(A'') = A''$ and $\dim_{\mathrm{M}} A'' = \dim_{\mathrm{M}} A$. If $\dim_{\mathrm{M}} A = 0$, then there is nothing to show, so let us proceed under the assumption that $\dim_{\mathrm{M}} A > 0$.

According to Proposition 4.3, we can find a closed and shift-invariant set $\Sigma \subseteq \{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$ such that $A$ coincides with the $r$-language set $A_\Sigma$ associated to $\Sigma$. Let $\mu$ be an ergodic $\sigma$-invariant Borel probability measure on $\Sigma$ of maximal entropy (the existence of such a measure follows from, eg. [Wal, Theorem 8.2 + Theorem 8.7 (v)]). Let $\Sigma''$ denote the support of $\mu$ and observe that $(\Sigma'', \sigma)$ is a subshift of $(\Sigma, \sigma)$ with $h_{\mathrm{top}}(\Sigma, \sigma) = h_{\mathrm{top}}(\Sigma'', \sigma)$. Moreover, since

$\mu$ is ergodic, almost every point in $\Sigma''$ has a dense orbit (by Birkhoff's ergodic theorem) and almost every point is recurrent (by Poincaré's recurrence theorem). Therefore there exists a point $x \in \Sigma''$ which visits every non-empty open set in $\Sigma''$ infinitely often.

Let $A'' \subseteq \mathbb{N}_0$ be the $r$-language set associated to $\Sigma''$. Since $\Sigma'' \subseteq \Sigma$, we have $A'' \subseteq A$. Also, $\dim_{\mathrm{M}} A = h_{\mathrm{top}}(\Sigma, \sigma) \log r$, $\dim_{\mathrm{M}} A'' = h_{\mathrm{top}}(\Sigma'', \sigma) \log r$, and $h_{\mathrm{top}}(\Sigma, \sigma) = h_{\mathrm{top}}(\Sigma'', \sigma)$, which implies $\dim_{\mathrm{M}} A = \dim_{\mathrm{M}} A''$. All that remains to be shown is that $\mathfrak{R}_r(A'') = \mathfrak{L}_r(A'') = A''$.

Since $A''$ is a $r$-language set, we already have the inclusions

$$\mathfrak{R}_r(A'') \subseteq A'' \qquad \text{and} \qquad \mathfrak{L}_r(A'') \subseteq A''.$$

To prove the inclusion $A'' \subseteq \mathfrak{R}_r(A'')$, let $n \in A''$ be arbitrary. We can write this number as $n = w_0 + w_1 r + \ldots + w_k r^k$ where $(w_0, w_1, \ldots, w_k)$ is a word in the language $\mathcal{L}(\Sigma'')$. Since the point $x$ visits every open set of $\Sigma''$ infinitely often, the word $(w_0, \ldots, w_k)$ appears in $x$ infinitely often. Therefore, there exists a letter $u \in \{0, 1, \ldots, r-1\}$ such that the word $(u, w_0, w_1, \ldots, w_k)$ appears in $x$ and hence belongs to $\mathcal{L}(\Sigma'')$. Take $n_1 \coloneqq u + w_0 r + w_1 r^2 + \ldots + w_k r^{k+1}$ and note that $n_1 \in A''$ and that $\mathfrak{R}_r(n_1) = n$, which proves $n \in \mathfrak{R}_r(A'')$.

Finally, to prove the inclusion $A'' \subseteq \mathfrak{L}_r(A'')$, let $n = w_0 + w_1 r + \ldots + w_k r^k \in A''$ be arbitrary. Since $h_{\mathrm{top}}(\Sigma'', \sigma) > 0$, some non-zero letter must appear in $\mathcal{L}(\Sigma'')$. Invoking again the fact that $x$ visits every open set infinitely often, a word of the form $(w_0, w_1, \ldots, w_k, v_1, v_2, \ldots, v_\ell)$ must appear in $x$ (and hence belong to $\mathcal{L}(\Sigma'')$), where $v_1$, $v_2$, $\ldots$, $v_\ell \in \{0, 1, \ldots, r-1\}$ for some $\ell \in \mathbb{N}$, with $v_\ell \neq 0$ but $v_i = 0$ for all $i < \ell$. Defining $n_2 \coloneqq w_0 + w_1 r + \ldots + w_k r^k + v_1 r^{k+1} + \ldots + v_\ell r^{k+\ell} \in A''$, we see that $\mathfrak{L}_r(n_2) = n$ and hence $n \in \mathfrak{L}_r(A'')$. □

Here is another corollary of Proposition 4.3, which may be of independent interest.

**Corollary 4.5.** *If $A \subseteq \mathbb{N}_0$ is $\times r$-invariant and $\dim_M A = 1$, then $A = \mathbb{N}_0$.*

*Proof.* Suppose $A$ is $\times r$-invariant with $\dim_{\mathrm{M}} A = 1$. There exists a closed and shift-invariant set $\Sigma \subseteq \{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$ such that $A$ coincides with the $r$-language set coming from $\Sigma$ and $h_{\mathrm{top}}(\Sigma, \sigma) = \log r$. However, the only subshift of $\{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$ with full entropy is the full shift. Hence $\Sigma = \{0, 1, \ldots, r-1\}^{\mathbb{N}_0}$, which implies $A = \mathbb{N}_0$. □

## 4.2. Connections to fractal subsets of the reals

The purpose of this subsection is to establish a connection between $\times r$ invariant subsets of the non-negative integers and $\times r$-invariant subsets of $[0, 1]$. Recall that $X \subseteq [0, 1]$ is called *$\times r$-invariant* if it is closed and $T_r X \subseteq X$, where $T_r \colon x \mapsto rx \bmod 1$.

First, we remark that every $\times r$-invariant subset of $[0, 1]$ can be "lifted" to a $\times r$-invariant subset of $\mathbb{N}_0$. Indeed, if $X \subseteq [0, 1]$ is $\times r$-invariant, then one can show that the set

$$\left\{ \lfloor r^k x \rfloor \mid x \in X,\ k \in \mathbb{N}_0 \right\}$$

is $\times r$-invariant. We will not make use of this fact, so we leave the details to the interested reader. Of more importance to us is the converse direction, stated in the following proposition. Recall from Section 2.1 the definition of Hausdorff distance.

**Proposition 4.6.** *For any $\times r$-invariant set $A \subseteq \mathbb{N}_0$, the sequence $X_k \coloneqq (A \cap [0, r^k))/r^k$ converges with respect to the Hausdorff metric $d_H$ as $k \to \infty$ to a $\times r$-invariant set $X \subseteq [0, 1]$*

*satisfying* $\dim_H X = \dim_M A = \dim_H A$. *In particular, the discrete Hausdorff dimension of* $A$ *exists and equals the discrete mass dimension of* $A$.

**Remark 4.7.** It follows from Proposition 4.6 that for a $\times r$-invariant set $A \subseteq \mathbb{N}_0$, $\dim_{\mathrm{M}} A = \dim_{\mathrm{H}} A$. Therefore Proposition 4.3, Corollary 4.4, and Corollary 4.5 remain true when the mass dimension $\dim_{\mathrm{M}}$ is replaced by the discrete Hausdorff dimension $\dim_{\mathrm{H}}$.

For the proof of Proposition 4.6 we will need two technical lemmas.

**Lemma 4.8.** *Let* $A \subseteq \mathbb{N}_0$, *and define* $X_k := (A \cap [0, r^k))/r^k$.
*(I) If* $\mathfrak{R}_r(A) \subseteq A$, *then for any* $k, l \in \mathbb{N}$ *with* $l \geqslant k$, *we have* $X_l \subseteq [X_k]_{r^{-k}}$.
*(II) If* $\mathfrak{R}_r(A) \supseteq A$, *then for any* $k, l \in \mathbb{N}$ *with* $l \geqslant k$, *we have* $X_k \subseteq [X_l]_{r^{-k}}$.
*In particular, if* $\mathfrak{R}_r(A) = A$ *then for all* $l \geqslant k$, *we have* $d_H(X_l, X_k) \leqslant r^{-k}$.

*Proof.* It is helpful to note first that for all $n, l, k \in \mathbb{N}$ with $l \geqslant k$,

$$\left| \frac{n}{r^l} - \frac{\mathfrak{R}_r^{l-k}(n)}{r^k} \right| \leqslant \frac{1}{r^k}. \tag{4.2}$$

This inequality follows easily from the fact that $\mathfrak{R}_r^{l-k}(n) = \lfloor n/r^{l-k} \rfloor$. For the proof of part (I), let $y \in X_l$ and write $y = m/r^l$ for some $m \in A$. Note that $\tilde{m} := \mathfrak{R}_r^{l-k}(m)$ belongs to $A \cap [0, r^k)$ because $\mathfrak{R}_r(A) \subseteq A$. Then, setting $\tilde{y} := \tilde{m}/r^k$, we see that $\tilde{y} \in X_k$ and, by (4.2), $d(y, \tilde{y}) \leqslant r^{-k}$. This proves $X_l \subseteq [X_k]_{r^{-k}}$.

Next, we prove part (II). For any $x \in X_k$ we can find $n \in A \cap [0, r^k)$ such that $x = n/r^k$. Since $A \subseteq \mathfrak{R}_r^{l-k}(A)$, there exists $\tilde{n} \in A \cap [0, r^l)$ such that

$$\mathfrak{R}_r^{l-k}(\tilde{n}) = n.$$

Now $\tilde{x} := \tilde{n}/r^l$ belongs to $X_l$ and it follows from (4.2) that $d(x, \tilde{x}) \leqslant r^{-k}$. This proves $X_k \subseteq [X_l]_{r^{-k}}$. ∎

**Lemma 4.9.** *Suppose* $A \subseteq \mathbb{N}_0$ *satisfies* $\mathfrak{R}_r(A) \subseteq A$, *and define* $A' := \bigcap_{k \in \mathbb{N}} \mathfrak{R}_r^k(A)$. *Also, set* $X_k := (A \cap [0, r^k))/r^k$ *and* $X'_k := (A' \cap [0, r^k))/r^k$. *Then* $\lim_{k\to\infty} d_H(X_k, X'_k) = 0$.

*Proof.* Let $\varepsilon > 0$, and let $m \in \mathbb{N}$ such that $2r^{-m} < \varepsilon$. Since $\mathfrak{R}_r(A) \subseteq A$, we have

$$A \cap [0, r^m) \supseteq \mathfrak{R}_r(A) \cap [0, r^m) \supseteq \mathfrak{R}_r^2(A) \cap [0, r^m) \supseteq \mathfrak{R}_r^3(A) \cap [0, r^m) \supseteq \ldots.$$

The sequence $k \mapsto \mathfrak{R}_r^k(A) \cap [0, r^m)$ eventually stabilizes. This happens exactly when $\mathfrak{R}_r^k(A) \cap [0, r^m) = A' \cap [0, r^m)$, or equivalently, when $\mathfrak{R}_r^k(A) \cap [0, r^m) = \mathfrak{R}_r^k(A') \cap [0, r^m)$, because $\mathfrak{R}_r^k(A') = A'$. It follows from (4.2) that the Hausdorff distance between $X_k$ and $(\mathfrak{R}_r^{k-m}(A) \cap [0, r^m))/r^m$ is bounded from above by $r^{-m}$. The same holds for $X'_k$ and $(\mathfrak{R}_r^{k-m}(A') \cap [0, r^m))/r^m$. Therefore, for all $k \geqslant m$ for which $\mathfrak{R}_r^{k-m}(A) \cap [0, r^m) = \mathfrak{R}_r^{k-m}(A') \cap [0, r^m)$, we have $d_H(X_k, X'_k) \leqslant 2r^{-m}$ by the triangle inequality. Since there exists a cofinite set of $k$ for which $\mathfrak{R}_r^{k-m}(A) \cap [0, r^m) = \mathfrak{R}_r^{k-m}(A') \cap [0, r^m)$, we conclude that $\limsup_{k\to\infty} d_H(X_k, X'_k) < \varepsilon$. The conclusion follows since $\varepsilon > 0$ was arbitrary. ∎

*Proof of Proposition 4.6.* Define $A' := \bigcap_{k \in \mathbb{N}_0} \mathfrak{R}_r^k(A)$ and $X'_k := (A' \cap [0, r^k))/r^k$. In view of Lemma 4.9, the sequence $k \mapsto X_k$ converges with respect to the Hausdorff metric if and only if the sequence $k \mapsto X'_k$ converges. Since $A' = \mathfrak{R}_r(A')$, it follows from Lemma 4.8 that

$$d_H(X'_k, X'_l) \leqslant r^{-k}, \qquad \text{for all } k, l \in \mathbb{N} \text{ with } l \geqslant k.$$

This implies that $k \mapsto X'_k$ is a Cauchy sequence, and hence it is convergent (recall that by the Blaschke selection theorem, the set of all non-empty, compact subsets of $[0,1]$ equipped with the Hausdorff distance is a complete metric space).

Next, let us show that $X$ is $\times r$-invariant. Since $\mathfrak{L}_r(A) \subseteq A$, a simple computation shows $T_r(X_k) \subseteq X_{k-1}$. Therefore, using $X = \lim_{k\to\infty} X_k$, we get that $T_r(X) \subseteq X$.

Finally, we have to show $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} A = \dim_{\mathrm{M}} A$. It follows form Theorem 2.9 that the Minkowski and Hausdorff dimensions of $\times r$-invariant subsets of $[0,1]$ coincide. In other words, we have

$$\dim_{\mathrm{M}} X \;=\; \dim_{\mathrm{H}} X. \tag{4.3}$$

It therefore suffices to show that

$$\dim_{\mathrm{M}} X \;=\; \dim_{\mathrm{M}} A, \tag{4.4}$$

$$\dim_{\mathrm{H}} X \;=\; \dim_{\mathrm{H}} A. \tag{4.5}$$

We begin with (4.4). As guaranteed by Corollary 4.4, $\dim_{\mathrm{M}} A = \dim_{\mathrm{M}} A'$. By combining part (I) of Lemma 4.8 with Lemma 2.7, we see that

$$\begin{aligned} 0 &\leqslant \liminf_{k\to\infty} \left( \frac{\log \mathcal{N}\big(X_k, r^{-k}\big)}{k \log r} - \frac{\log \mathcal{N}\big(X, r^{-k}\big)}{k \log r} \right) \\ &= \dim_{\mathrm{M}} A - \limsup_{k\to\infty} \frac{\log \mathcal{N}\big(X, r^{-k}\big)}{k \log r}, \end{aligned} \tag{4.6}$$

where the equality follows from the fact that $\dim_{\mathrm{M}} A = \lim_{k\to\infty} \frac{1}{k \log r} \log \mathcal{N}(X_k, r^{-k})$ (cf. equation (2.6)). On the other hand, using part (II) of Lemma 4.8, Lemma 2.7, and the fact that $\dim_{\mathrm{M}} A' = \lim_{k\to\infty} \frac{1}{k \log r} \log \mathcal{N}(X'_k, r^{-k})$, we see

$$\begin{aligned} 0 &\leqslant \liminf_{k\to\infty} \left( \frac{\log \mathcal{N}\big(X', r^{-k}\big)}{k \log r} - \frac{\log \mathcal{N}\big(X'_k, r^{-k}\big)}{k \log r} \right) \\ &= \liminf_{k\to\infty} \frac{\log \mathcal{N}\big(X', r^{-k}\big)}{k \log r} - \dim_{\mathrm{M}} A'. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) with the fact that $X' \subseteq X$, we see

$$\dim_{\mathrm{M}} A' \leqslant \liminf_{k\to\infty} \frac{\log \mathcal{N}\big(X', r^{-k}\big)}{k \log r} \leqslant \limsup_{k\to\infty} \frac{\log \mathcal{N}\big(X, r^{-k}\big)}{k \log r} \leqslant \dim_{\mathrm{M}} A.$$

Since $\dim_{\mathrm{M}} A = \dim_{\mathrm{M}} A'$ and $X' \subseteq X$, we conclude that $\dim_{\mathrm{M}} X$ exists and is equal to $\dim_{\mathrm{M}} A$.

Next, let us turn to the proof of (4.5). In view of (4.3), (4.4), and Lemma 2.18, instead of (4.5) it suffices to show

$$\dim_{\mathrm{H}} X \;\leqslant\; \underline{\dim}_{\mathrm{H}} A. \tag{4.8}$$

Note, however, that $\underline{\dim}_{\mathrm{H}} A' \leqslant \underline{\dim}_{\mathrm{H}} A$ because $A' \subseteq A$, and that $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} X'$ because $\dim_{\mathrm{M}} A = \dim_{\mathrm{M}} A'$. Also, we have $\dim_{\mathrm{H}} X = \dim_{\mathrm{M}} X = \dim_{\mathrm{M}} A$ and $\dim_{\mathrm{H}} X' = \dim_{\mathrm{M}} X' = \dim_{\mathrm{M}} A'$, where $\dim_{\mathrm{H}} X' = \dim_{\mathrm{M}} X'$ follows from the fact that $X'$ is also $\times r$-invariant and $\dim_{\mathrm{M}} X' = \dim_{\mathrm{M}} A'$ follows from the work above with $A'$ in place of $A$ and $X'$ in place of $X$.

This means that (4.8) will follow from

$$\dim_{\mathrm{H}} X' \leqslant \underline{\dim}_{\mathrm{H}} A'. \tag{4.9}$$

Next, recall from equation (1.7) that

$$\dim_{\mathrm{H}} X' = \sup\left\{\gamma \geqslant 0 \;\middle|\; \lim_{\rho\to 0^+} \mathcal{H}^{\gamma}_{\geqslant\rho}(X') > 0\right\}. \tag{4.10}$$

Also, by definition,

$$\underline{\dim}_{\mathrm{H}} A' = \sup\left\{\gamma \geqslant 0 \;\middle|\; \liminf_{N\to\infty} \mathcal{H}^{\gamma}_{\geqslant 1}\big(A' \cap [0,N)\big)/N^{\gamma} > 0\right\},$$

which in view of part (IV) of Lemma 2.17 and equation (2.7) can be rewritten as

$$\underline{\dim}_{\mathrm{H}} A' = \sup\left\{\gamma \geqslant 0 \;\middle|\; \liminf_{k\to\infty} \mathcal{H}^{\gamma}_{\geqslant r^{-k}}\big(X'_k\big) > 0\right\}. \tag{4.11}$$

By combining part (I) of Lemma 4.8 with Lemma 2.7 we see that

$$\mathcal{H}^{\gamma}_{\geqslant r^{-k}}\big(X'\big) \ll \mathcal{H}^{\gamma}_{\geqslant r^{-k}}\big(X'_k\big),$$

which togehter with (4.10) and (4.11) implies (4.9). □

### 4.3. Proof of Theorem B

In this subsection we give a proof of Theorem B. For $w = (w_0, \ldots, w_{\ell-1}) \in \{0, \ldots, r-1\}^{\ell}$, define

$$(w)_r := w_0 r^{\ell-1} + w_1 r^{\ell-2} + \cdots + w_{\ell-2} r^1 + w_{\ell-1}.$$

We will say that a non-negative integer $n$ *begins with $w$ in base $r$* if there exists $d \in \mathbb{N}_0$ and $n_0 \in [0, r^d)$ such that

$$n = (w)_r r^d + n_0. \tag{4.12}$$

If $w_0 \neq 0$, this means that the $n$ most significant digits in the base-$r$ expansion of $n$ are $w_0$, $w_1$, …, $w_{\ell-1}$, in order.

**Lemma 4.10.** *For all $w \in \{0, \ldots, r-1\}^{\ell}$, there is an arc $I_w \subseteq [0,1)$ modulo 1 (meaning that $I$ is an interval when 0 and 1 are identified) with the property that for all $x \geqslant (w)_r$, if $\{\log x/\log r\} \in I_w$, then $\lfloor x \rfloor$ begins with $w$ in base $r$.*

*Proof.* Let $w \in \{0, \ldots, r-1\}^{\ell+1}$. It follows from (4.12) that a positive integer $n$ begins with $w$ in base $r$ if and only if there exists $d \in \mathbb{N}_0$ such that

$$(w)_r r^d \leqslant n < \big((w)_r + 1\big) r^d.$$

Therefore, a positive real number $x$ has the property that $\lfloor x \rfloor$ begins with $w$ in base $r$ if and only if

$$(w)_r r^d \leqslant x < \big((w)_r + 1\big) r^d.$$

The previous inequality is equivalent to

$$\frac{\log(w)_r}{\log r} + d \leqslant \frac{\log x}{\log r} < \frac{\log\big((w)_r + 1\big)}{\log r} + d. \tag{4.13}$$

Let $I_w$ be the modulo 1 arc from the fractional part of $\log(w)_r/\log r$ to the fractional part of $\log\big((w)_r+1\big)/\log r$ in the positive direction. We see that if $x \geqslant (w)_r$ and $\big\{\log x/\log r\big\} \in I_w$, then (4.13) holds, and $\lfloor x \rfloor$ begins with $w$ in base $r$. □

**Lemma 4.11.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $A \subseteq \mathbb{N}_0$ be $\times s$ invariant and infinite. For all $w \in \{0, \ldots, r-1\}^\ell$, there exists an element of $A$ that begins with $w$ in base $r$.*

*Proof.* Let $w \in \{0, \ldots, r-1\}^\ell$, and let $\delta$ be half the length of the interval $I_w$ from Lemma 4.10. Define $K_0 = \lceil \log(w)_r/\log s \rceil$ and $\alpha = \log s/\log r$. Since $\alpha$ is irrational, there exists $K \in \mathbb{N}$ such that the set $\big\{\{i\alpha\} \mid i \in \{K_0, \ldots, K\}\big\}$ is $\delta$-dense in $[0, 1)$.

Since $A$ is infinite, there exists $n \in A$ such that $k := \lfloor \log n/\log s \rfloor \geqslant K$. Since $A$ is $\mathfrak{R}_s$-invariant, $n$, $\lfloor n/s \rfloor$, $\ldots$, $\lfloor n/s^k \rfloor$ are all elements of $A$. Define $x = n/s^k$, and note that $\lfloor x \rfloor$, $\lfloor sx \rfloor$, $\ldots$, $\lfloor s^k x \rfloor$ is the same list of integers which are, therefore, all elements of $A$.

We will show that there exists $0 \leqslant i \leqslant k$ for which $s^i x \geqslant (w)_r$ and $\{\log(s^i x)/\log r\} \in I_w$. It will follow by Lemma 4.10 that $\lfloor s^i x \rfloor$ is an element of $A$ that begins with $w$ in base $r$.

Note that $\log(s^i x)/\log r = i\alpha + \log x/\log r$. Since $\big\{\{i\alpha\} \mid i \in \{K_0, \ldots, K\}\big\}$ is $\delta$-dense in $[0, 1)$ and $I_w$ is an interval of length equal to $2\delta$, there exists $i \in \{K_0, \ldots, K\}$ such that $\{\log(s^i x)/\log r\} \in I_w$. Since $i \geqslant K_0$, we have that $s^i x \geqslant (w)_r$. We have found $0 \leqslant i \leqslant k$ for which $s^i x \geqslant (w)_r$ and $\{\log(s^i x)/\log r\} \in I_w$, as was to be shown. □

*Proof of Theorem B.* Let $A \subseteq \mathbb{N}_0$ be simultaneously $\times r$- and $\times s$-invariant, and suppose $A$ is infinite. To show that $A = \mathbb{N}_0$, we will show that for all $w \in \{0, \ldots, r-1\}^\ell$, $(w)_r \in A$. Let $w \in \{0, \ldots, r-1\}^\ell$. By Lemma 4.11, there exists $n \in A$ that begins with $w$ in base $r$. Writing $n$ as in (4.12), we see that $\mathfrak{R}_r^d(n) = (w)_r$. Since $A$ is $\mathfrak{R}_r$-invariant, $(w)_r \in A$, as was to be shown. □

## 4.4. Proof of Theorem C

In this section, we will prove Theorem C. The strategy is to use tools from Section 4.2 to derive Theorem C from the theorem of Lindenstrauss-Meiri-Peres, Theorem 1.4. Throughout this section, $r \geqslant 2$ is fixed and all of the asymptotic notation may implicitly depend on it.

**Remark 4.12.** There are some useful remarks to make before the proof. Let $X_1$, $X_2$, $\ldots$, $X_n \subseteq [0, 1]$ be $\times r$-invariant sets. The sumset $X_1 + \cdots + X_n$ may be interpreted in $\mathbb{R}/\mathbb{Z}$ or in $\mathbb{R}$. Denote temporarily by $W_n$ the set $X_1 + \cdots + X_n$ interpreted modulo 1 as a subset of $[0, 1]$ and by $Y_n$ the set $X_1 + \cdots + X_n$ interpreted in $\mathbb{R}$ as a subset of $[0, n]$. Two facts of particular relevance to us are: 1) set $W_n$ is $\times r$-invariant, and 2) $\dim_{\mathrm{H}} W_n = \dim_{\mathrm{H}} Y_n$. The first fact follows easily from the fact that multiplication by $r$ is a group endomorphism of $(\mathbb{R}/\mathbb{Z}, +)$. (In contrast, note that the sumset of $\times r$-invariant subsets of $\mathbb{N}_0$ is not necessarily $\times r$-invariant: if $A$ is the base-10 restricted digit Cantor set with allowed digits 0 and 5, then $A + A$ contains 10 but does not contain $\mathfrak{R}_{10}(10) = 1$, for example). The second fact follows immediately by writing $W_n = \cup_{i=0}^{n-1}\big((Y_n \cap [i, i+1]) - i\big)$ and using the translation-invariance and finite (countable) stability under unions of the Hausdorff dimension.

*Proof of Theorem C.* Recall that $(A_i)_{i=1}^{\infty}$ is a sequence of $\times r$-invariant subsets of $\mathbb{N}_0$. For each $i \in \mathbb{N}$, let $A_i'$ be the set described in Corollary 4.4, and define $X_i \subseteq [0,1]$ to be the Hausdorff limit of the sequence $(A_i' \cap [0, r^N)/r^N)_{N=1}^{\infty}$ as in Proposition 4.6. Since $\dim_{\mathrm{H}} X_i = \dim_{\mathrm{H}} A_i' = \dim_{\mathrm{H}} A_i$ and $\sum_{i=1}^{\infty} \dim_{\mathrm{H}} A_i / |\log \dim_{\mathrm{H}} A_i|$ diverges, we have that $\sum_{i=1}^{\infty} \dim_{\mathrm{H}} X_i / |\log \dim_{\mathrm{H}} X_i|$ diverges. It follows by Theorem 1.4 that

$$\lim_{n\to\infty} \dim_{\mathrm{H}} \big(X_1 + \cdots + X_n\big) = 1. \tag{4.14}$$

According to Remark 4.12, we can and will interpret the sum $X_1 + \cdots + X_n$ to be in $\mathbb{R}$.

We claim now that for all $n \in \mathbb{N}$, the discrete Hausdorff dimension of the set $A_1' + \cdots + A_n'$ exists and

$$\dim_{\mathrm{H}} \big(A_1' + \cdots + A_n'\big) = \dim_{\mathrm{H}} \big(X_1 + \cdots + X_n\big). \tag{4.15}$$

Combined with (4.14), this suffices to conclude the proof of Theorem C since $A_i' \subseteq A_i$ implies that $\dim_{\mathrm{H}} \big(A_1' + \cdots + A_n'\big) \leqslant \underline{\dim}_{\mathrm{H}} \big(A_1 + \cdots + A_n\big)$.

To show (4.15), let $n \in \mathbb{N}$, and define $k = \lfloor \log n / \log r \rfloor + 1$. Define $B_n = A_1' + \cdots + A_n'$ and $Y_n = X_1 + \cdots + X_n$, where the sum defining $Y_n$ is understood to be in $\mathbb{R}$. Note that for all $N \geqslant k$,

$$\sum_{i=1}^{n} \frac{A_i' \cap [0, r^{N-k})}{r^N} \subseteq \frac{B_n \cap [0, r^N)}{r^N} \subseteq \sum_{i=1}^{n} \frac{A_i' \cap [0, r^N)}{r^N}, \tag{4.16}$$

where the sums indicate sumsets. The goal now is to compare the discrete Hausdorff contents of each of these sets at scale $r^{-N}$.

By the definition of the set $X_i$, it follows from Lemma 4.8 that

$$d_H \left( \frac{A_i' \cap [0, r^N)}{r^N}, X_i \right) \ll r^{-N}, \tag{4.17}$$

which implies by Lemma 2.7 that for all $\gamma \in [0,1]$,

$$\mathcal{H}^{\gamma}_{\geqslant r^{-N}} \left( \sum_{i=1}^{n} \frac{A_i' \cap [0, r^N)}{r^N} \right) \asymp_n \mathcal{H}^{\gamma}_{\geqslant r^{-N}} \big(Y_n\big). \tag{4.18}$$

It also follows from (4.17) that

$$d_H \left( \frac{A_i' \cap [0, r^{N-k})}{r^N}, \frac{X_i}{r^k} \right) \ll_n r^{-N},$$

which implies by Lemma 2.7 that

$$\mathcal{H}^{\gamma}_{\geqslant r^{-N}} \left( \sum_{i=1}^{n} \frac{A_i' \cap [0, r^{N-k})}{r^N} \right) \asymp_n \mathcal{H}^{\gamma}_{\geqslant r^{-N}} \left( \frac{Y_n}{r^k} \right). \tag{4.19}$$

Combining (4.16) with (4.18) and (4.19), we see that

$$\mathcal{H}^{\gamma}_{\geqslant r^{-N}} \left( \frac{Y_n}{r^k} \right) \ll_n \mathcal{H}^{\gamma}_{\geqslant r^{-N}} \left( \frac{B_n \cap [0, r^N)}{r^N} \right) = \frac{\mathcal{H}^{\gamma}_{\geqslant 1} \left( B_n \cap [0, r^N) \right)}{r^{N\gamma}} \ll_n \mathcal{H}^{\gamma}_{\geqslant r^{-N}} \big(Y_n\big).$$

Letting $N$ tend to $\infty$ and noting that $n$, and hence $k$, are fixed, these inequalities combine with Lemma 2.4, Lemma 2.17 (IV), (4.14), and the fact that $\dim_{\mathrm{H}}(Y_n/r^k) = \dim_{\mathrm{H}} Y_n$ to prove the equality in (4.15). □

### 4.5. Proofs of Theorems D and E

*Proof of Theorem D assuming Theorem E.* Suppose $A, B \subseteq \mathbb{N}_0$ are $\times r$- and $\times s$-invariant, where $r$ and $s$ are multiplicatively independent, and let $\lambda, \eta > 0$ be fixed. Define $\overline{\gamma} = \min(\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B, 1)$, which according to Proposition 4.6 is the same as $\min(\dim_{\mathrm{M}} A + \dim_{\mathrm{M}} B, 1)$. By Lemma 2.17, parts (II) and (V), we have the upper bound

$$\overline{\dim}_{\mathrm{M}}\big(\lfloor \lambda A + \eta B \rfloor\big) \leqslant \overline{\gamma}.$$

On the other hand, from Theorem E we obtain for all $\gamma < \overline{\gamma}$ the lower bound

$$\underline{\dim}_{\mathrm{H}}\big(\lfloor \lambda A + \eta B \rfloor\big) \geqslant \gamma.$$

In view of Lemma 2.18, this proves that the discrete mass dimension and the discrete Hausdorff dimension of $\lfloor \lambda A + \eta B \rfloor$ both exist and equal $\overline{\gamma}$. □

*Proof that Theorem A implies Theorem E.* Suppose $A, B \subseteq \mathbb{N}_0$ are $\times r$- and $\times s$-invariant, where $r$ and $s$ are multiplicatively independent, and $I \subseteq (0, \infty)$ is compact. Assuming Theorem A, we want to show that

$$\liminf_{N\to\infty} \inf_{\lambda,\eta\in I} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(\lfloor \lambda A + \eta B \rfloor \cap [0,N)\big)}{N^{\gamma}} > 0, \tag{4.20}$$

for all $\gamma < \overline{\gamma} := \min(\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B, 1)$.

First, let us make the observation that

$$\mathcal{H}^{\gamma}_{\geqslant 1}\big(\lfloor \lambda A + \eta B \rfloor \cap [0,N)\big) \geqslant \frac{1}{2}\mathcal{H}^{\gamma}_{\geqslant 1}\big((\lambda A + \eta B) \cap [0,N)\big).$$

Next, note that if $M \in \mathbb{N}$ is chosen sufficiently large depending on $I$, then for every $\lambda, \eta \in I$, the set $\lambda(A \cap [0, N/M)) + \eta(B \cap [0, N/M))$ is a subset of $(\lambda A + \eta B) \cap [0, N)$. This implies

$$\begin{aligned}\frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big((\lambda A + \eta B) \cap [0,N)\big)}{N^{\gamma}} &\geqslant \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(\lambda(A\cap[0,N/M)) + \eta(B\cap[0,N/M))\big)}{N^{\gamma}} \\ &\geqslant M^{-\gamma}\left(\frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(\lambda(A\cap[0,N/M)) + \eta(B\cap[0,N/M))\big)}{(N/M)^{\gamma}}\right).\end{aligned}$$

Therefore, (4.20) is implied by

$$\liminf_{N\to\infty} \inf_{\lambda,\eta\in I} \frac{\mathcal{H}^{\gamma}_{\geqslant 1}\big(\lambda(A\cap[0,N)) + \eta(B\cap[0,N))\big)}{N^{\gamma}} > 0,$$

which according to equation (2.7) is the same as

$$\liminf_{N\to\infty} \inf_{\lambda,\eta\in I} \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\left(\lambda\Big(\frac{A\cap[0,N)}{N}\Big) + \eta\Big(\frac{B\cap[0,N)}{N}\Big)\right) > 0. \tag{4.21}$$

Define for every $k, \ell \in \mathbb{N}$ the sets $X_k := (A \cap [0, r^k))/r^k$ and $Y_\ell := (B \cap [0, s^\ell))/s^\ell$. Define $k_N := \lfloor \log N/\log r \rfloor$ and $\ell_N := \lfloor \log N/\log s \rfloor$, and note that $N = r^{k_N} r^{\{\log N/\log r\}} =$

$s^{\ell_N} s^{\{\log N/\log s\}}$. Since $A\cap[0,N)\supseteq A\cap[0,r^{k_N})$ and $B\cap[0,N)\supseteq B\cap[0,s^{\ell_N})$, we have

$$\begin{aligned}\mathcal{H}^{\gamma}_{\geqslant N^{-1}}\bigg(\lambda\Big(\frac{A\cap[0,N)}{N}\Big)+\eta\Big(\frac{B\cap[0,N)}{N}\Big)\bigg)\\ &\geqslant \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\bigg(\lambda\Big(\frac{A\cap[0,r^{k_N})}{r^{k_N} r^{\{\log N/\log r\}}}\Big)+\eta\Big(\frac{B\cap[0,s^{\ell_N})}{s^{\ell_N} s^{\{\log N/\log s\}}}\Big)\bigg)\\ &\geqslant \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\Big(\lambda r^{-\{\log N/\log r\}}X_{k_N}+\eta s^{-\{\log N/\log s\}}Y_{\ell_N}\Big).\end{aligned}$$

Since $I\subseteq(0,\infty)$ is compact, we can choose $t>0$ such that $I\subseteq[t^{-1},t]$. So if $\lambda$ and $\eta$ belong to $I$ then $\lambda r^{-\{\log N/\log r\}}$ and $\eta s^{-\{\log N/\log s\}}$ belong to the interval $J:=[\min(r^{-1},s^{-1})\cdot t^{-1},t]$. We conclude that

$$\inf_{\lambda,\eta\in I}\ \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\bigg(\lambda\Big(\frac{A\cap[0,N)}{N}\Big)+\eta\Big(\frac{B\cap[0,N)}{N}\Big)\bigg)\geqslant\inf_{\lambda,\eta\in J}\ \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\Big(\lambda X_{k_N}+\eta Y_{\ell_N}\Big).\quad(4.22)$$

Next, let $X=\lim_{k\to\infty}X_k$ and $Y=\lim_{\ell\to\infty}Y_\ell$ in the Hausdorff metric. The existence of these limits is guaranteed by Proposition 4.6, which also gives that $\dim_{\mathrm{H}}X=\dim_{\mathrm{H}}A$ and $\dim_{\mathrm{H}}Y=\dim_{\mathrm{H}}B$. Since $r^{k_N}\leqslant N<r^{k_N+1}$ and $s^{\ell_N}\leqslant N<s^{\ell_N+1}$, it follows from part (I) of Lemma 4.8 that

$$X\subseteq\big[X_{k_N}\big]_{rN^{-1}}\qquad\text{and}\qquad Y\subseteq\big[Y_{\ell_N}\big]_{sN^{-1}}.$$

Therefore there exists $a\geqslant 1$ such that

$$\lambda X+\eta Y\ \subseteq\ \Big[\lambda X_{k_N}+\eta Y_{\ell_N}\Big]_{aN^{-1}}$$

uniformly over all $\lambda,\eta\in J$. In view of Lemma 2.7 this implies

$$\inf_{\lambda,\eta\in J}\ \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\Big(\lambda X_{k_N}+\eta Y_{\ell_N}\Big)\gg_a\inf_{\lambda,\eta\in J}\ \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\Big(\lambda X+\eta Y\Big).\quad(4.23)$$

Since $\gamma<\overline{\gamma}$, Theorem A gives

$$\lim_{N\to\infty}\inf_{\lambda,\eta\in J}\ \mathcal{H}^{\gamma}_{\geqslant N^{-1}}\Big(\lambda X+\eta Y\Big)>0.\quad(4.24)$$

The claim in (4.21) now follows from (4.22), (4.23), and (4.24). □

### 4.6. An example that shows $\mathfrak{R}$-invariance does not suffice

In this subsection, we exhibit sets $A,B\subseteq\mathbb{N}_0$ that demonstrate that $\mathfrak{R}_r$- and $\mathfrak{R}_s$-invariance alone does not suffice to reach the conclusions in Theorem D. This is in contrast to Theorem B: the conclusion holds under the weaker assumption that $A$ is simultaneously $\mathfrak{R}_r$- and $\mathfrak{R}_s$-invariant. We do not know whether $\mathfrak{L}$-invariance alone suffices in either Theorem B or Theorem D, but invariance under multiplication by $r$ and $s$ alone does not suffice to reach the conclusions in either theorem: the set of squares is invariant under multiplication by both 4 and 9 simultaneously, but has dimension equal to 1/2, while the sets in the example below demonstrate that Theorem D does not hold under the assumption of invariance under multiplication.

Fix $2\leqslant r<s$. We will construct two sets $A,B\subseteq\mathbb{N}_0$ which satisfy the following properties:

(I) the mass dimensions of $A$ and $B$ exist and $\dim_{\mathrm{M}} A = \dim_{\mathrm{M}} B = 1/2$;
(II) $rA \subseteq A$ and $sB \subseteq B$;
(III) $\mathfrak{R}_r(A) = A$ and $\mathfrak{R}_s(B) = B$; and
(IV) $\overline{\dim}_{\mathrm{M}} (A+B) \leqslant 4/5$.

This shows that neither $\mathfrak{R}$-invariance alone nor multiplication-invariance alone suffice to obtain the result in Theorem D.

In what follows, the interval notation $[a,b]$ is understood to mean $[a,b] \cap \mathbb{N}_0$. For $i,j \in \mathbb{N}_0$, let

$$I_i = [r^i, r^i + r^{(i+1)/2}], \qquad J_j = [s^j, s^j + s^{(j+1)/2}],$$

and then define

$$A = \{0\} \cup \bigcup_{i,\ell \geqslant 0} r^\ell I_i, \qquad B = \{0\} \cup \bigcup_{j,m \geqslant 0} s^m J_j.$$

First we will verify (I) by showing that the mass dimension of $A$ exists and is equal to $1/2$; the argument for $B$ is the same. It is easy to see that for all $N \geqslant 1$,

$$I_{N-1} \subseteq A \cap [1, r^N) \subseteq \bigcup_{\substack{i,\ell \geqslant 0 \\ i+\ell \leqslant N}} r^\ell I_i,$$

from which it follows that

$$r^{N/2} \leqslant \left| A \cap [0, r^N) \right| \leqslant (N+1)^2 (r^{(N+1)/2} + 1).$$

This shows that $\underline{\dim}_{\mathrm{M}} A = \overline{\dim}_{\mathrm{M}} A = \dim_{\mathrm{M}} A = 1/2$.

It is clear from the definition of the sets $A$ and $B$ that (II) holds.

Next we will verify (III) by showing that $\mathfrak{R}_r(A) = A$; the same argument works to show that $\mathfrak{R}_s(B) = B$. Since $rA \subseteq A$, we have that

$$A = \mathfrak{R}_r(rA) \subseteq \mathfrak{R}_r(A) = \{0\} \cup \bigcup_{i,\ell \geqslant 0} \mathfrak{R}_r(r^\ell I_i).$$

Since $0 \in A$, we need only to verify that for all $i, \ell \geqslant 0$, $\mathfrak{R}_r(r^\ell I_i) \subseteq A$. If $\ell \geqslant 1$, then $\mathfrak{R}_r(r^\ell I_i) = r^{\ell-1} I_i \subseteq A$. If $\ell = 0$ and $i = 0$, then we see $\mathfrak{R}_r(I_0) = \{0\} \subseteq A$. If $\ell = 0$ and $i \geqslant 1$, then we see $\mathfrak{R}_r(I_i) = [r^{i-1}, r^{i-1} + r^{(i-1)/2}] \subseteq I_{i-1} \subseteq A$. Thus, $\mathfrak{R}_r(A) = A$.

Finally we will verify (IV) by showing that for all $N$ sufficiently large,

$$\left| (A+B) \cap [0, r^N) \right| \leqslant 4N^4 r^{4N/5}. \tag{4.25}$$

Let $\sigma = \log s / \log r$. Because

$$B \cap [1, r^N) \subseteq \bigcup_{\substack{i,\ell \geqslant 0 \\ \sigma(j+m) \leqslant N}} s^m J_j,$$

we have that

$$\left| (A+B) \cap [0, r^N) \right| \leqslant 1 + \sum_{i,j,\ell,m} |r^\ell I_i + s^m J_j|, \tag{4.26}$$

where the sum is over all $i, j, \ell, m \geqslant 0$ for which $i + \ell \leqslant N$ and $\sigma(j+m) \leqslant N$. We will estimate this sum from above by splitting the sum indices into two sets depending on the "type" of the pair $(i,j)$, which we now define.

A pair $(i,j)$ is of Type I if

$$\frac{i+1}{2}+\sigma\frac{j+1}{2}\leqslant\frac{4N}{5}. \tag{4.27}$$

Using the trivial bound $|C+D|\leqslant|C||D|$ for finite sets $C,D\subseteq\mathbb{N}_0$, we see that if $i,j,\ell$, and $m$ are such that $(i,j)$ is of Type I, then

$$|r^\ell I_i+s^m J_j|\leqslant|I_i||J_j|=r^{(i+1)/2}s^{(j+1)/2}\leqslant r^{4N/5}. \tag{4.28}$$

A pair $(i,j)$ is of Type II if it is not of Type I, that is, if

$$\frac{i+1}{2}+\sigma\frac{j+1}{2}>\frac{4N}{5}. \tag{4.29}$$

Using the fact that $\sigma j\leqslant N$ and that $N$ is sufficiently large, we see from (4.29) that $(i-1)/2>N/4$. It follows then from the fact that $i+\ell\leqslant N$ that

$$\ell+\frac{i+1}{2}<\frac{4N}{5}. \tag{4.30}$$

Similarly, using that $i\leqslant N$ and the fact that $N$ is sufficiently large, we see from (4.29) that $\sigma(j-1)/2>N/4$. It follows from the fact that $\sigma(j+m)\leqslant N$ that

$$\sigma\left(m+\frac{j+1}{2}\right)<\frac{4N}{5}. \tag{4.31}$$

Now we are in a position to use the following fact: if $C,D\subseteq\mathbb{N}_0$ are contained in intervals of length $L$, $M$, respectively, then $C+D$ is contained in an interval of length $L+M$ and hence $|C+D|\leqslant L+M+1$. If $i,j,\ell$, and $m$ are such that $(i,j)$ is of Type II, then

$$|r^\ell I_i+s^m J_j|\leqslant r^{\ell+(i+1)/2}+s^{m+(j+1)/2}+1.$$

Using (4.30) and (4.31), we have that

$$|r^\ell I_i+s^m J_j|\leqslant 3r^{4N/5}. \tag{4.32}$$

Finally, by splitting up the sum in (4.26) into tuples for which the pairs $(i,j)$ are of Type I or Type II, we see by combining (4.28) and (4.32) that the desired inequality in (4.25) holds.

# 5. Open directions

We collect in this section a number of interesting and potentially fruitful open questions concerning multiplicatively invariant subsets of the non-negative integers. Though these questions and conjectures are stated for arbitrary $\times r$-invariant subsets of $\mathbb{N}_0$, many are already open and interesting for the special case of base-$r$ restricted digit Cantor sets.

## 5.1. Positive density for sumsets of full dimension

In [Hoc, Problem 4.10], Hochman asks whether the sumset $X+Y$ of a $\times r$-invariant set $X\subseteq[0,1)$ and a $\times s$-invariant set $Y\subseteq[0,1)$ satisfying $\dim_{\mathrm{H}}X+\dim_{\mathrm{H}}Y>1$ has positive Lebesgue measure. We remark that a projection theorem of Marstrand [Mar, Theorem I] implies that $\lambda X+\eta Y$ has positive Lebesgue measure for a.e. $(\lambda,\eta)\in\mathbb{R}^2$, suggesting a possible affirmative answer. In [Gla, Theorem 1.4], a version of Marstrand's projection theorem for subsets of the integers was obtained, with Lebesgue measure replaced by the notion of upper natural density.[4] It therefore makes sense to consider the following integer analogue of the the question.

**Question 5.1.** *Let $r,s\in\mathbb{N}$ be multiplicatively independent, and let $A,B\subseteq\mathbb{N}_0$ be $\times r$- and $\times s$-invariant, respectively. If $\dim_M A+\dim_M B>1$, then does the sumset $A+B$ have positive upper natural density?*

## 5.2. Difference sets

For closed subsets $X,Y\subseteq[0,1]$, working with the difference set $X-Y$ is no harder than working with the sumset $X+Y$. In particular, proving that

$$\dim_{\mathrm{M}}\big(X-Y\big)=\min\big(\dim_{\mathrm{M}}X+\dim_{\mathrm{M}}Y,\ 1\big)$$

in Theorem 1.3 requires no additional work. The story changes in the setting of the non-negative integers, where difference sets are much more cumbersome to handle, ultimately because the fibers of the map $(a,b)\mapsto a-b$ are not compact. This is why our main results in the integer setting only deal with sumsets $\lambda A+\eta B$ with $\lambda$ and $\eta$ both positive. This naturally leads us to the following question.

**Question 5.2.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $A,B\subseteq\mathbb{N}_0$ be $\times r$- and $\times s$-invariant, respectively. Is it true that*

$$\dim_M(A-B)\ =\ \min\big(\dim_M A+\dim_M B,\ 1\big)?$$

The methods used in Section 4 allow us to establish the lower bound $\dim_{\mathrm{M}}(A-B)\geqslant\min(\dim_{\mathrm{M}}A+\dim_{\mathrm{M}}B,\ 1)$. However, the upper bound $\dim_{\mathrm{M}}(A-B)\leqslant\min(\dim_{\mathrm{M}}A+\dim_{\mathrm{M}}B,\ 1)$, which is straightforward for sums, remains open for differences.

## 5.3. Analogous results for the counting dimension

The *upper Banach dimension* (or *upper counting dimension*, cf. [LM2] and [Gla]) of a set $A\subseteq\mathbb{N}_0$ is

$$\dim^* A:=\limsup_{N-M\to\infty}\frac{\log\big|A\cap[M,N]\big|}{\log(N-M)}.$$

In general, we only have the inequality $\dim^* A\geqslant\overline{\dim}_{\mathrm{M}}A$, but if $A\subseteq\mathbb{N}_0$ is $\times r$-invariant, then it can be shown that $\dim_{\mathrm{M}}A=\dim_{\mathrm{H}}A=\dim^* A$.

**Question 5.3.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $A,B\subseteq\mathbb{N}_0$ be $\times r$- and $\times s$-invariant, respectively. Is it true that*

$$\dim^*(A+B)\ =\ \min\big(\dim^* A+\dim^* B,\ 1\big)?$$

Note that the lower bound $\dim^*(A+B)\geqslant\min\big(\dim^* A+\dim^* B,\ 1\big)$ follows from Theorem D using the fact that $\dim^*\geqslant\overline{\dim}_{\mathrm{M}}$.

[4]Given a set $E\subseteq\mathbb{Z}$, its *upper natural density* is defined by $\bar{d}(E):=\limsup_{N\to\infty}|E\cap\{-N,\dots,N\}|/(2N+1)$.

## 5.4. Polynomial functions of multiplicatively invariant sets

Our main results for integer sets, Theorems D and E, concern the dimension of the sumset $A+B$. It is natural to ask about different functions of $A$ and $B$. The following conjecture is a (special case of a) natural polynomial extension of Theorem D.

**Conjecture 5.4.** *Fix $n, m \in \mathbb{N}$ and let $\Delta(A,B) := \{a^n + b^m \mid a \in A, b \in B\}$ for $A, B \subseteq \mathbb{N}_0$. Let $r, s \in \mathbb{N}$ be multiplicatively independent, and let $A, B \subseteq \mathbb{N}_0$ be a $\times r$- and a $\times s$-invariant subsets, respectively. Then*

$$\dim_M \Delta(A,B) = \min\left(\frac{1}{n}\dim_M A + \frac{1}{m}\dim_M B\,,\, 1\right). \tag{5.1}$$

It is easy to see that for any $A \subseteq \mathbb{N}_0$, the set $A^n := \{a^n \mid a \in A\}$ has dimension $\dim_{\mathrm{M}} A^n = \frac{1}{n}\dim_{\mathrm{M}} A$, however it is not true in general that $A^n$ is $\times r$-invariant when $A$ is.

Setting $\Delta(A,B) = \Delta_{\lambda,\eta}(A,B) := \{\lambda a^n + \eta b^m \mid a \in A, b \in B\}$, it follows from (1.10) that the equality in (5.1) holds for Lebesgue almost every $(\lambda, \eta) \in \mathbb{R}^2$ when $A$ and $B$ satisfy a natural dimension condition (see footnote 2).

## 5.5. Geometric transversality in the integers

In [Fur2], Furstenberg conjectured that given $\times r$- and a $\times s$-invariant subsets $X$ and $Y$ of $[0,1]$, where $r, s \in \mathbb{N}$ are multiplicatively invariant, the intersection satisfies

$$\dim_{\mathrm{H}}\big(X \cap Y\big) \leqslant \max\big(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y - 1, 0\big).$$

This conjecture was recently proved, independently, by Shmerkin [Shm] and Wu [Wu]. It is natural to formulate an analogous conjecture for subsets of $\mathbb{N}_0$.

**Conjecture 5.5.** *Let $r, s \in \mathbb{N}$ be multiplicatively independent, and let $A, B \subseteq \mathbb{N}_0$ be $\times r$- and $\times s$-invariant, sets respectively. Then*

$$\overline{\dim}_M\big(A \cap B\big) \leqslant \max\big(\dim_M A + \dim_M B - 1, 0\big).$$

This conjecture and some related problems will be addressed in a forthcoming paper.

## 5.6. Multiplicatively invariant sets in relation to other arithmetic sets in the integers

In this paper, we are concerned with transversality between $\times r$- and $\times s$-invariant sets whenever $r$ and $s$ are multiplicatively independent. In principle, it makes sense to inquire about transversality (or *independence*) between any two sets which are structured in different ways. To keep the discussion short, we restrict to infinite arithmetic progressions (or congruence classes), the set of perfect squares, and the set of primes.

**Question 5.6.** *Let $A \subseteq \mathbb{N}_0$ be a $\times r$-invariant set, and let $P$ be an infinite arithmetic progression. Is is true that $\dim_M(A \cap P)$ is either $0$ or $\dim_M(A)$?*

The answer is yes for restricted digit Cantor sets. In fact, it is proved in [EMS] that such sets satisfy "good equidistribution properties" in residue classes.

More generally, one could ask about the sum or the intersection of a $\times r$-invariant set and the image of an arbitrary polynomial with integer coefficients, for instance the set of perfect squares, $S = \{n^2 \mid n \in \mathbb{N}_0\}$. Note that $\dim_{\mathrm{M}} S = 1/2$.

**Question 5.7.** *Let $A \subseteq \mathbb{N}_0$ be a $\times r$-invariant set. Is is true that*

$$\dim_M \big(A + S\big) \ = \ \min\big(\dim_M A + 1/2,\ 1\big)$$

*and/or*

$$\overline{\dim}_M \big(A \cap S\big) \ \leqslant \ \max\big(\dim_M A - 1/2,\ 0\big)?$$

In a similar vein, one can ask about intersections with the set of prime numbers, $\mathbb{P}$. Note that $\dim_{\mathrm{M}} \mathbb{P} = 1$.

**Question 5.8.** *Let $A \subseteq \mathbb{N}_0$ be a $\times r$-invariant set. Is is true that $\dim_M(A \cap \mathbb{P})$ is either $0$ or $\dim_M(A)$?*

Maynard showed in [May] that the answer to Question 5.8 is positive when $A$ is a restricted digit Cantor set where the number of restricted digits is small enough with respect to the base. In fact, he obtains a Prime Number Theorem in such sets, which is stronger than simply $\dim_{\mathrm{M}}(A \cap \mathbb{P}) = \dim_{\mathrm{M}} A$. Question 5.8 is open for general restricted digit Cantor sets, and may be very difficult in general. The methods in this paper do not appear to shed new light on this line of inquiry.

### 5.7. Transversality of multiplicatively invariant sets in the $rs$-adics

The $rs$-adics is a non-Archimedean regime in which it is easy to ask questions analogous to those asked in this work. Following Furstenberg [Fur2], note that the maps $\mathfrak{R}_r$ and $\mathfrak{R}_s$, with domains extended to $\mathbb{Z}$, are uniformly continuous with respect to the $rs$-adic metric on $\mathbb{Z}$, and therefore extend to continuous transformations of the set of $rs$-adic integers, $\mathbb{Z}_{rs}$. As a compact metric space, there is a natural Hausdorff dimension to measure the size of subsets of $\mathbb{Z}_{rs}$. Let us call a set $X \subseteq \mathbb{Z}_{rs}$ *$\times r$-invariant* if it is closed and $\mathfrak{R}_r X \subseteq X$.

**Question 5.9.** *Let $r$ and $s$ be multiplicatively independent positive integers, and let $X, Y \subseteq \mathbb{Z}_{rs}$ be $\times r$- and $\times s$-invariant sets, respectively. Is it true that $\dim_H \big(X + Y\big) = \min\big(\dim_H X + \dim_H Y,\ \dim_H \mathbb{Z}_{rs}\big)$?*

**Conjecture 5.10** ([Fur2, Conjecture 3]). *Let $r$ and $s$ be multiplicatively independent positive integers, and let $X, Y \subseteq \mathbb{Z}_{rs}$ be $\times r$- and $\times s$-invariant sets, respectively. One has*

$$\dim_H \big(X \cap Y\big) \leqslant \max\big(\dim_H X + \dim_H Y - \dim_H \mathbb{Z}_{rs}, 0\big).$$

Furstenberg [Fur2, Theorem 3] proved an analogue of Theorem 1.1 in the $rs$-adics; positive answers to the previous question and conjecture would combine with that result to bring transversality results in the $rs$-adics in line with those in the real and integer settings.

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