INTERVAL POSETS FOR PERMUTATIONS

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ABSTRACT. The interval poset of a permutation catalogues the intervals that appear in its one-line notation, according to set inclusion. We study this poset, describing its structural, characterizing, and enumerative properties.

1. INTRODUCTION

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, \ldots, n\}$. For the purposes of this work, we write permutations as words in one-line notation

$$w = w(1)w(2)\cdots w(n) \in \mathfrak{S}_n.$$

Definition 1.1. An *interval* in a permutation w is an interval of values that appear in consecutive positions of w. That is, $[h, h+j]$ is an interval of w if $\{w(t) : t \in [i, i+j]\} = [h, h+j]$, for some i. An interval of $w \in \mathfrak{S}_n$ is proper if it has between 2 and $n-1$ elements.

Clearly $[1, n]$ is an interval of every $w \in \mathfrak{S}_n$, as is $\{i\}$ for each $i \in [n]$.

Example 1.2. The proper intervals of 43187562 are $[3, 4]$, $[5, 6]$, $[5, 7]$, $[5, 8]$, and $[7, 8]$.

A permutation need not have any proper intervals, as is the case for $2413 \in \mathfrak{S}_4$ and $35142 \in \mathfrak{S}_5$. Such permutations are *simple permutations*, and appear throughout the literature of permutations and permutation patterns (see, for example, [1, 4]). Let

$$\mathsf{simp}(n)$$

denote the number of simple permutations in $\mathfrak{S}_n$. These are enumerated in [5, A111111]. Note that $\mathsf{simp}(3) = 0$, whereas $\mathsf{simp}(n) > 0$ for all other positive integers n.

As shown in Example 1.2, the intervals of a permutation need not be disjoint. The relationships among the intervals of a permutation have a natural poset structure, and this is what we study in this work. The reader is referred to [7] for basic terminology.

Definition 1.3. The *interval poset* of $w \in \mathfrak{S}_n$ is the poset $\mathcal{P}(w)$ whose elements are the intervals of w and whose order relations are defined by set inclusion. The *closed interval poset* $\overline{\mathcal{P}}(w)$ is obtained from $\mathcal{P}(w)$ by adjoining a minimum element $\widehat{0}$, which we think of as representing the empty interval.

Put another way, $\overline{\mathcal{P}}(w)$ is the induced subposet of the boolean algebra on $[n]$ formed by the (possibly empty) intervals of w. The interval poset for the permutation 43187562 is shown in Figure 1.2.

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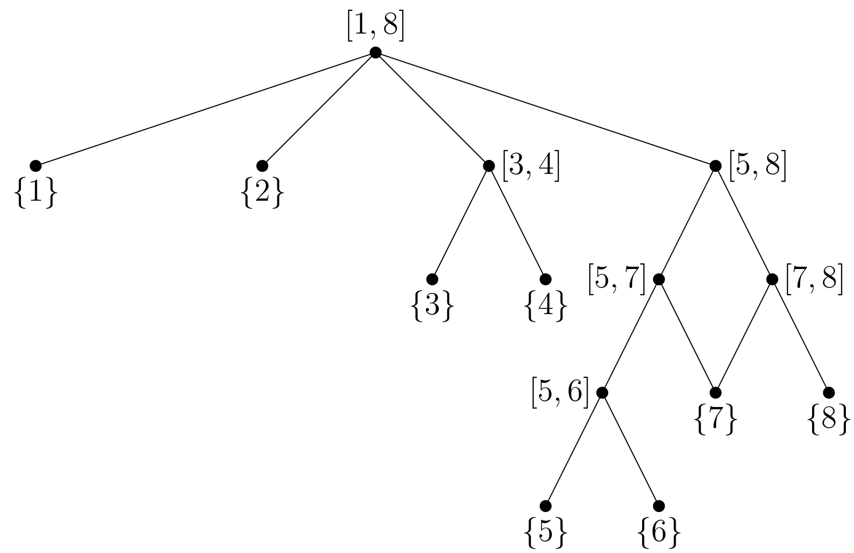


FIGURE 1. The interval poset $\mathcal{P}(43187562)$.

The goal of this paper is to study the interval poset and closed interval poset of a permutation. We lay out terminology and basic facts about interval posets in Section 2. Many of these follow immediately from the definition of the poset. In Section 3, we answer deeper questions about the structure of these posets (Theorems 3.1, 3.2, 3.5, and 3.11). Section 4 turns to questions of characterization; namely, which posets are of the form $\mathcal{P}(w)$ for some w (Theorems 4.8 and 4.9). In Section 5, we enumerate the permutations w satisfying $\mathcal{P}(w) = \mathcal{P}$ for a given $\mathcal{P}$ (Theorem 5.1). Special families of interval posets are studied in Section 6, with characterizations in terms of pattern avoidance (Theorems 6.2 and 6.3 and Corollary 6.5) and enumerations (Corollaries 6.4 and 6.6). We close the paper with proposed directions for further research.

2. PRELIMINARIES

Fix a permutation $w \in \mathfrak{S}_n$. The permutation $1 \in \mathfrak{S}_1$ is of limited intrigue, and so unless specified otherwise, we will always assume that permutations have at least two letters and interval posets have at least two minimal elements. Both $\mathcal{P}(w)$ and $\overline{\mathcal{P}}(w)$ have a unique maximal element: the interval $[1, n]$. Both posets are finite, with $|\mathcal{P}(w)| = |\overline{\mathcal{P}}(w) - 1| \le 2^n - 1$. The posets themselves need not be graded, and the rank of $\mathcal{P}(w)$ can be anything between 1 and $n-1$ (with the exception that the rank is always 2 when $n = 3$). The rank of $\overline{\mathcal{P}}(w)$ is one more than the rank of $\mathcal{P}(w)$. It follows from the definitions that each element in $\mathcal{P}(w)$ represents an interval of consecutive integers, and no element of $\mathcal{P}(w)$ covers exactly one element. The number of minimal elements in $\mathcal{P}(w)$ is the number of letters permuted by w, and an arbitrary element in $\mathcal{P}(w)$ represents the interval formed by the union of all minimal elements in its principal order ideal.

It will occasionally be useful to refer to an arbitrary word on distinct letters as a "permutation," and this should be understood in the obvious way.

The next result is straightforward, but useful enough to warrant a reference.

Lemma 2.1. The principal order ideal of an interval $I \in \mathcal{P}(w)$ is isomorphic to the interval poset for the permutation formed by restricting the word w to the letters of I; similarly for principal order ideals in the closed interval poset.

For example, the principal order ideal of $[5, 8] \in \mathcal{P}(43187562)$, depicted in Figure 1, is isomorphic to the interval poset of the permutation (that is, the subword) 8756; equivalently, to the interval poset of its order isomorphic permutation 4312.

Another feature of interval posets that follows directly from definitions will also be critical to some of our later arguments.

Lemma 2.2. An interval $H \in \mathcal{P}(w)$ is covered by exactly one interval I if and only if for every interval $J \supset H$, we also have $J \supseteq I$.

A familiar involution on permutations is relevant to this work.

Definition 2.3. The *reverse* of the permutation $w = w(1) \cdots w(n)$ is the permutation

$$w^R := w(n) \cdots w(1).$$

Because intervals depend on being consecutive, with no directional preference, a permutation and its reverse have the same intervals.

Lemma 2.4. $\mathcal{P}(w) = \mathcal{P}(w^R)$ for all w.

These pairs of permutations will come in handy later, and so we name them here.

Definition 2.5. Consider a permutation $w \in \mathfrak{S}_n$, and its reverse w^R. The element of $\{w, w^R\}$ in which 1 appears to the left of n is the *rising twin*. The other permutation, in which n appears to the left of 1, is the *falling twin*.

It is tempting to hope that the map $w \mapsto \mathcal{P}(w)$ might be 2-to-1, mapping only w and w^R to the same poset, but that is false. The easiest counterexample comes from the six simple permutations in $\mathfrak{S}_5$, all of which have the interval poset depicted in Figure 2.

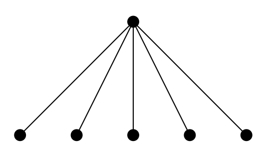


FIGURE 2. The interval poset for $24153, 35142, 25314, 41352, 31524, 42513 \in \mathfrak{S}_5$.

Throughout this work, we will prefer to draw interval posets in a particular way.

Definition 2.6. If $I_1, I_2, \ldots, I_r$ are non-nesting intervals so that $\min(I_j) < \min(I_{j+1})$ for all j (equivalently, $\max(I_j) < \max(I_{j+1})$), then the sequence $I_1, \ldots, I_r$ is in *increasing order*.

By definition of the poset $\mathcal{P}(w)$, no two intervals in an antichain are nesting; that is, neither interval is a subset of the other.

Definition 2.7. The *canonical* Hasse diagram of $\mathcal{P}(w)$ is drawn so that the (antichain of) elements of a fixed depth from the maximal element $[1, n]$ appear at the same height, and in increasing order from left to right across the poset. The canonical Hasse diagram of $\overline{\mathcal{P}}(w)$ is obtained by adjoining a minimum element to that diagram of $\mathcal{P}(w)$.

Figure 1 depicts the canonical Hasse diagram of $\mathcal{P}(43187562)$.

We will always draw the canonical Hasse diagrams of $\mathcal{P}(w)$ and $\overline{\mathcal{P}}(w)$. Noting that the minimal elements of $\mathcal{P}(w)$ (atoms of $\overline{\mathcal{P}}(w)$) are the 1-element intervals $\{i\}$ for $i \in [n]$, we can omit the element labels in a canonical Hasse diagram. It will sometimes be useful to think of Hasse diagrams as directed graphs, with the maximal element as the root, and we will use this language (e.g., "tree," "child") as appropriate.

We close this section by recalling a standard operation on permutations.

Definition 2.8. Fix $w \in \mathfrak{S}_n$ and $p_i \in \mathfrak{S}_{m_i}$ for $i \in [1, n]$. The *inflation* of w by $p_1, \ldots, p_n$ is the permutation $w[p_1, \ldots, p_n] \in \mathfrak{S}_{\sum m_i}$ defined by replacing each $w(i)$ by an interval that is order isomorphic to p_i, so that the letters in the segment replacing $w(i)$ are all larger than the letters in the segment replacing $w(j)$ if and only if $w(i) > w(j)$. When there is a j such that $m_i = 1$ for all $i \neq j$, we refer to this as *inflating the j in w by p_j*.

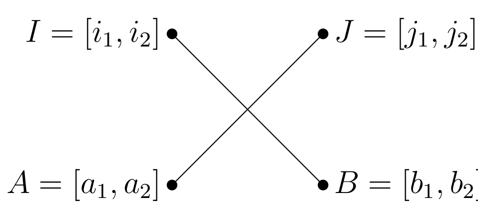
Example 2.9. $3142[21, 1, 4312, 1] = 43187562$. Inflating the 3 in 3176452 by 21 produces that same permutation: 43187562.

3. Structural properties of interval posets

In this section, we describe the structural properties of the interval poset $\mathcal{P}(w)$ of a permutation, and of $\overline{\mathcal{P}}(w)$ where relevant. We begin with a notion of planarity, such as was formalized by Lakser and presented by Quackenbush in [6]: a poset is *planar* if it has a planar Hasse diagram. The interval poset depicted in Figure 1 is planar, and indeed this is no coincidence. Unlike the other structural properties that we explore below, planarity depends on finding a specific embedding of the poset in the plane. In fact, we can explicitly describe an embedding of the poset $\mathcal{P}(w)$ that produces a planar diagram.

Theorem 3.1. For any permutation w, the posets $\mathcal{P}(w)$ and $\overline{\mathcal{P}}(w)$ are planar.

Proof. It is enough to prove the result for $\mathcal{P}(w)$. Consider the canonical Hasse diagram of $\mathcal{P}(w)$, and suppose that it is not planar. Then there must be a configuration of covering relations such as this.



Because $A \subset J$ and $I \subset B$, we have $j_1 \leq a_1 \leq a_2 \leq j_2$ and $i_1 \leq b_1 \leq b_2 \leq i_2$. The set $\{A, B\}$ is an antichain, so $a_1 < b_1$ and $a_2 < b_2$. Similarly, $i_1 < j_1$ and $i_2 < j_2$. Putting this information together, we have

$$i_1 < j_1 \leq a_1 < b_1 \leq b_2 \leq i_2 < j_2,$$

with $a_2 \in [a_1, b_2)$. The set $I \cap J = [j_1, i_2]$ must be an interval in w because I and J were intervals. More precisely, suppose that it is not. That is, that the elements of $I \cap J$ do not appear in consecutive positions of w. Because the elements of I do appear consecutively, it must be that the elements of $I \cap J$ are interrupted by some $x \in I \setminus (I \cap J)$. But then the elements of J would not appear consecutively in w, contradicting the fact that J is an interval in w. Thus $I \cap J$ is an interval in w, and so it is an element of $\overline{\mathcal{P}}(w)$.

But then $B \subset I$ is not a covering relation, because $B \subsetneq [j_1, i_2] \subsetneq I$, which is a contradiction. □

We now examine whether $\overline{\mathcal{P}}(w)$ belongs to several well-known poset classes. Once again, the reader is referred to [7] for definitions and further details.

We start with lattices. In fact, this result follows from Theorem 3.1 and a theorem of Lakser (see [6]), but we present a proof specific to the interval poset here, to highlight how the lattice structure works in this context.

Theorem 3.2. For any permutation w, the poset $\overline{\mathcal{P}}(w)$ is a lattice.

Proof. Consider two intervals I and J in w. We will construct their unique greatest lower bound (meet, denoted $\wedge$) and unique least upper bound (join, denoted $\vee$).

Consider $I \cap J$. Anything less than both I and J in $\overline{\mathcal{P}}(w)$ must also be less than $I \cap J$ in $\overline{\mathcal{P}}(w)$. Because I and J are intervals, their intersection must also be an interval. If this set is empty, then $I \wedge J = \widehat{0}$. Suppose, now, that $I \cap J$ is not empty. As argued in the proof of Theorem 3.1, this $I \cap J$ is an interval in w. Therefore, $I \cap J \in \mathcal{P}(w)$ is the meet of I and J.

Determining the join of two elements is less straightforward. We cannot simply consider the set $I \cup J$, because this might not be an interval in w. For example, $[3, 4] \cup [5, 6] = [3, 6]$ is not an interval in 43187562. Set $a_0 := \min(I \cup J)$ and $b_0 := \max(I \cup J)$. Because elements of $\overline{\mathcal{P}}(w)$ are intervals, anything greater than both I and J in $\overline{\mathcal{P}}(w)$ must also be greater than $[a_0, b_0]$ in $\overline{\mathcal{P}}(w)$. If $[a_0, b_0]$ is an interval in w, then this is $I \vee J$ and we are done. If not, then some collection of values C_0 appears among the elements of $[a_0, b_0]$ in the one-line notation for w. Set $a_1 := \min([a_0, b_0] \cup C_0)$ and $b_1 := \max([a_0, b_0] \cup C_0)$. If $[a_1, b_1]$ is an interval in w, then this is $I \vee J$ and we are done. If not, then define C_1 to be the values appearing among the elements of $[a_1, b_1]$ in w and proceed as above. Eventually, this process terminates with some $[a_k, b_k]$ that is an interval in w. It must terminate because each interval $[a_i, b_i]$ is larger than the previous interval, and none can be larger than $[1, n] \in \overline{\mathcal{P}}(w)$. Moreover, the choices of a_i, b_i, and C_i are determined at each step. This final $[a_k, b_k]$ is the unique smallest interval in w that contains both I and J, and thus it is the join $I \vee J$. □

We demonstrate Theorem 3.2's construction of the join by continuing Example 1.2.

Example 3.3. Let $w = 43187562$, with $I = [3, 4]$ and $J = [5, 6]$. Then $a_0 = 3$ and $b_0 = 6$. Because the elements of $[3, 6]$ do not appear consecutively in w, we must look at $C_0 = \{1, 7, 8\}$. Then $a_1 = 1$ and $b_1 = 8$. The elements of $[1, 8]$ appear consecutively in w (obviously), and so we find that $[3, 4] \vee [5, 6] = [1, 8]$. This can also be seen in Figure 1.

The lattice $\overline{\mathcal{P}}(w)$ is clearly atomic, by construction, but it is not always complemented as we see in the following example.

Example 3.4. Let $w = 123$. There is no element $p \in \overline{\mathcal{P}}(w)$ such that $p \vee \{2\} = [1, 3]$ and $p \wedge \{2\} = \widehat{0}$. The poset $\overline{\mathcal{P}}(w)$ is shown in Figure 3, with $\{2\}$ circled in that figure.

We now turn to classifying modular interval posets for permutations. In fact, these are quite a small class.

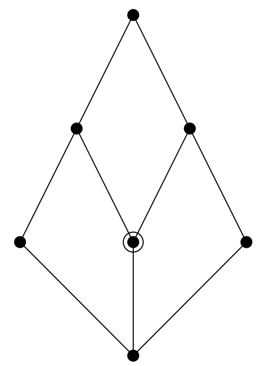


FIGURE 3. The lattice $\overline{\mathcal{P}}(123)$, with canonical Hasse diagram, which is not complemented. The element $\{2\}$ has been circled.

Theorem 3.5. An interval poset $\overline{\mathcal{P}}(w)$ is modular if and only if w is a simple permutation.

Proof. First consider $w \in \mathfrak{S}_n$ for which $\overline{\mathcal{P}}(w)$ is modular. Suppose that w has a proper interval I containing 1. Because I is proper, $n \notin I$. Then the poset $\{\widehat{0}, \{1\}, \{n\}, I, I \vee \{n\} = [1, n]\}$ is isomorphic to the pentagon lattice N_5, meaning that $\overline{\mathcal{P}}(w)$ would not be modular. Therefore there can be no such interval I. A similar argument shows that w has no proper intervals containing n. Now suppose that w has a proper interval I containing j for some $j \in [2, n-1]$ (and necessarily $1 \notin I$). Then the poset $\{\widehat{0}, \{1\}, \{j\}, I, \{1\} \vee I = [1, n]\}$ is isomorphic N_5, and so $\overline{\mathcal{P}}(w)$ would not be modular. Thus there are no proper intervals in w, so w is simple.

Now suppose that $w \in \mathfrak{S}_n$ is simple. The poset $\overline{\mathcal{P}}(w)$ has rank 2: a minimal element, a maximal element, and n atoms/coatoms. This certainly contains no sublattice isomorphic to N_5, and therefore it is modular. □

Before classifying distributive interval posets for permutations, we need some terminology.

Definition 3.6. Fix a permutation w and consider its interval poset $\mathcal{P}(w)$. An element $I \in \overline{\mathcal{P}}(w)$ is *fruitful* if I covers more than 2 elements.

Fruitful elements highlight a particular phenomenon among the intervals of a permutation.

Lemma 3.7. Fix a fruitful element $I \in \mathcal{P}(w)$, covering $\{H_1, \ldots, H_k\}$ for some $k > 2$. The interval I, as it appears in the one-line notation for w, is order isomorphic to the inflation of a simple permutation $v \in \mathfrak{S}_k$. Moreover, the intervals $\{H_i\}$ are pairwise disjoint.

Proof. The maximal proper intervals in I are $H_1, \ldots, H_k$, which we can assume to be listed in increasing order. If H_j and H_{j+1} intersect nontrivially, then $H_j \cup H_{j+1}$ would also be an interval in I, contradicting either maximality of the intervals or the value of k. Thus all elements of H_j are less than all elements of H_{j+1}. Consider $w|_I$, the word obtained by restricting the one-line notation of w to the letters in I. For each j, replace the entire interval H_j in $w|_I$ by the letter j, and let the resulting permutation of $[k]$ be called v. Then $w|_I$ is the inflation $v[H_1, \ldots, H_k]$. The permutation v is simple because it has no proper intervals. □

This result gives an immediate restriction on the possible structure of $\mathcal{P}(w)$, because $\mathfrak{S}_3$ has no simple permutations. Thus, in fact, we could revise Definition 3.6 to say that an element is fruitful if it covers more than three elements. We will use that bound henceforth.

Corollary 3.8. No element of $\mathcal{P}(w)$ covers exactly three elements.

Because simple permutations have no proper intervals, the intervals H_i and H_{i+1} cannot appear consecutively in w, yielding the following handy result.

Lemma 3.9. Fix a permutation w. Suppose that $I \in \mathcal{P}(w)$ is a fruitful element, covering $\{H_1, \ldots, H_k\}$ for $k > 3$. Then I is the only element to cover each H_i.

Proof. Without loss of generality, assume that the sequence $H_1, \ldots, H_k$ is in increasing order. By Theorem 3.1, the only intervals from this list that could be covered by something in addition to I are H_1 and H_k. By Lemma 3.7, the letters of I as they appear in w are isomorphic to the inflation of a simple permutation in $\mathfrak{S}_k$ for $k > 3$, and neither 1 nor k can appear at either end of such a permutation. Thus no element can cover H_1 without also covering some other H_i, which will violate Theorem 3.1. That the intervals $\{H_i\}$ are pairwise disjoint follows as argued in the proof of Lemma 3.7. □

Proposition 3.10. If $\overline{\mathcal{P}}(w)$ has a fruitful element, then it is not a distributive lattice.

Proof. Let $I \in \overline{\mathcal{P}}(w)$ be a fruitful element, and H_1, H_2, H_3 be three of the elements that it covers. By Lemma 3.7, the intervals H_1, H_2, and H_3 are pairwise disjoint. Thus, as discussed in Theorem 3.2, their pairwise meets are $\widehat{0}$. Then $\{I, H_1, H_2, H_3, \widehat{0}\}$ is isomorphic to the diamond lattice M_3, and so $\overline{\mathcal{P}}(w)$ is not distributive. □

The configuration referenced in the proof of Proposition 3.10 is depicted in Figure 4.

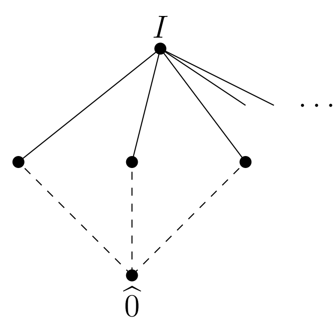


FIGURE 4. The diamond lattice constructed in the proof of Proposition 3.10.

We can now combine Theorem 3.5 and Proposition 3.10 to (quite strictly!) characterize distributive interval posets for permutations.

Theorem 3.11. An interval poset $\overline{\mathcal{P}}(w)$ is distributive if and only if $w \in \mathfrak{S}_1 \cup \mathfrak{S}_2$.

Proof. Suppose that $w \in \mathfrak{S}_n$ and $\overline{\mathcal{P}}(w)$ is distributive. Distributive lattices are modular, so w must be simple, by Theorem 3.5. To avoid a sublattice isomorphic to the diamond lattice M_3, we need $n < 3$.

It is straightforward to check that $\overline{\mathcal{P}}(w)$ is distributive for all $w \in \mathfrak{S}_1 \cup \mathfrak{S}_2$. □

4. Characterizing interval posets

The goal of this section is, in a sense, to learn how to invert the map $w \mapsto \mathcal{P}(w)$. To that end, we define the following (possibly empty) set.

Definition 4.1. Fix a poset P. The permutations whose interval poset is P are the *interval generators* of P, and the set of interval generators of P is denoted

$$\mathcal{I}(P) := \{w : \mathcal{P}(w) = P\}.$$

The results of the last section gave some restrictions on the types of posets that can appear as interval posets. Here, motivated by Lemma 3.7, we analyze that issue more thoroughly.

Definition 4.2. A *k-fruitful poset* has a unique maximal element, and $k > 3$ minimal elements that are all covered by that maximal element. An *argyle poset* is the interval poset of $12 \cdots k$ for some $k > 1$. A *binary tree poset* is a poset whose Hasse diagram is a full binary tree.

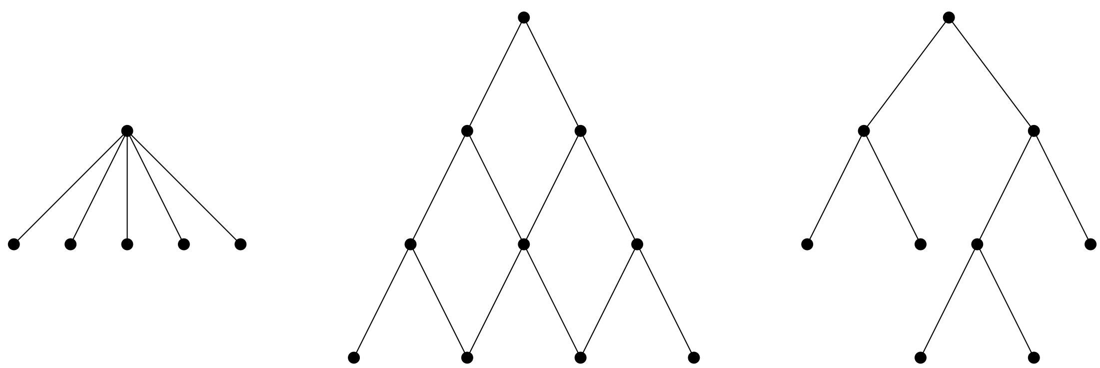


Figure 5. From left to right: a 5-fruitful poset, an argyle poset, and a binary tree poset.

These three classes of posets, illustrated in Figure 5, are, in fact, interval posets. Moreover, we can describe the generating set $\mathcal{I}(P)$ in each case.

Proposition 4.3. For any $k > 3$, the k-fruitful poset is the interval poset for exactly $\mathsf{simp}(k)$ permutations: the simple permutations in $\mathfrak{S}_k$.

Proof. The k-fruitful poset is the interval poset of a permutation $w \in \mathfrak{S}_k$ if and only if there are no proper intervals in w. This is the definition of being a simple permutation. $\square$

Showing that argyle posets are interval posets is similarly easy.

Proposition 4.4. The argyle poset with $k > 1$ minimal elements is the interval poset for exactly two permutations: $12 \cdots k$ and $(12 \cdots k)^R = k \cdots 21$.

Proof. An argyle poset with k minimal elements is the interval poset of a permutation $w \in \mathfrak{S}_k$ if and only if each j and $j+1$ appear consecutively. The only permutations with this property are the identity and its reverse. $\square$

Binary tree posets are more subtle. For one thing, the embedding matters, which was not an issue with k-fruitful posets or argyle posets.

Definition 4.5. Fix a binary tree poset P and its embedding, with $k > 1$ minimal nodes labeled $1, \ldots, k$ according to a depth-first search that always chooses left edges before right edges. An *alternating depth-first search* of P is a depth-first search that choose left edges before right edges at all choices that are of even (respectively, odd) distance from the maximal element, and right edges before left edges at all choices that are odd (resp., even) distance from the maximal element. The *ADFS* words of P are the two words formed by minimal element labels in the order seen when performing alternating depth-first searches of P.

Example 4.6. The ADFS words of the binary tree poset in Figure 5 are 21534 and 43512.

The ADFS words of a binary tree poset are reverses of each other. Thus one is a rising twin and the other is a falling twin.

Proposition 4.7. A binary tree posets with $k > 1$ minimal elements is the canonical Hasse diagrams of interval posets for exactly two permutations: the ADFS words of the poset.

Proof. We prove the result by induction on k. If $k = 2$, then the poset is the interval poset of both 12 and 21, and of nothing else.

Now suppose that we have a binary tree poset P with $k > 2$ minimal elements, and that the result holds when there are $k - 1$ minimal elements. Because P is a full binary tree, two of those minimal elements are covered by the same element x, which covers nothing else. Say that these are the jth and $(j + 1)$st minimal elements in P, when read from left to right in depth-first order.

Let Q be the binary tree poset obtained from P by deleting these two minimal elements. Thus x is minimal in Q (and the jth minimal element in the poset, when read from left to right in depth-first order), and Q has $k - 1$ minimal elements. By the inductive hypothesis, we have $\mathcal{I}(Q) = \{v, v^R\}$, where v and v^R are the ADFS words of Q.

Then P is the interval poset of a permutation $w \in \mathfrak{S}_k$ if and only if w is obtained from v by inflating j by 12 or 21, perhaps according to some rule. Without loss of generality, suppose that the element x was the right child of its covering element y, and that the left child of y represents the interval I in Q. Because Q is the canonical Hasse diagram of $\mathcal{P}(v)$, all letters of I are less than j, and $I \cup \{j\}$ is an interval; that is, $I = [i, j - 1]$. Then, in the ADFS word that prefers the left child of y, producing

$$\cdots \boxed{\text{elements of } I}\, j \, \cdots ,$$

it will be necessary, by Lemma 2.2, to prefer the right child of x. That is, we must inflate the j in this word by 21 to obtain the permutation

$$\cdots \boxed{\text{elements of } I}\, (j+1)j \, \cdots .$$

Similarly, in the ADFS word that prefers the right child of y, it will be necessary by Lemma 2.2 to prefer the left child of x. In other words, the permutations whose interval posets are P are exactly the ADFS words of P. □

Lemma 2.1 suggests the recursion that we will use to describe exactly which posets occur as $\mathcal{P}(w)$ for some w. We describe this recursion as a type of poset inflation. Consider the following procedure.

Procedure INT-POSET

Step 0. Define P_0 to be the 1-element poset. Set $i := 0$.

Step 1. Replace a minimal element of P_i by Q_i, which is either a k-fruitful poset, an argyle poset, or a binary tree poset. Set P_{i+1} to be the result after this replacement.
Step 2. Set $i := i + 1$.
Step 3. Either output P_i or return to Step 1.

Note that the option to use binary tree posets in INT-POSET is redundant because we can build any binary tree poset by iteratively replacing minimal elements by argyle posets having 2 minimal elements. We do not fuss over that redundancy in the description of INT-POSET because the ability to build an entire binary poset in one step can be both handy and illuminating (as in, for example, Proposition 4.7).

Theorem 4.8. A poset constructed by INT-POSET is the interval poset of a permutation.

Proof. Let $P = P_t$ be a poset constructed by INT-POSET. We will prove the result by induction on t. If $t = 1$, then P is the interval poset of the permutations described by Propositions 4.3, 4.4, or 4.7.

Now suppose that $t > 1$ and that the result holds for P_{t-1} constructed by INT-POSET; that is, there exists $v \in \mathcal{I}(P_{t-1})$. To get from $P_{t-1} = \mathcal{P}(v)$ to P_t, we replace a minimal element of P_{t-1} by a poset $Q := Q_{t-1}$ having one of the three types described in Step 1. Note the implications of Lemma 2.2, since the coatoms of Q are covered by nothing else in P_t. Let $\{j\}$ be the label of the minimal element that gets replaced in $P_{t-1} = \mathcal{P}(v)$. Say that Q has k minimal elements. We know from Propositions 4.3, 4.4, and 4.7 that $\mathcal{I}(Q) \subseteq \mathfrak{S}_k$ has at least two elements.

We would like to inflate j by some $\pi \in \mathcal{I}(Q)$. By Lemma 2.2, this must occur in such a way as to guarantee that any proper interval containing $x \in [j, j + k - 1]$ also contains all of the inflated π. If this can be done to satisfy the lemma, then P is the interval poset of the permutation resulting from that replacement.

If Q is a fruitful poset or if, in P_{t-1}, the element labeled $\{j\}$ was covered by a fruitful element, then, by Lemmas 3.7 and 3.9, we will not violate Lemma 2.2. Thus, in such a setting, inflating j in v by any $\pi \in \mathcal{I}(Q)$ will produce a permutation with interval poset P.

Suppose that Q is not a fruitful poset and that, in P_{t-1}, the element labeled $\{j\}$ was covered by an element that was not fruitful; that is, there is an element covering exactly two elements: $\{j\}$ and an interval I. This I must contain either $j - 1$ or $j + 1$. Without loss of generality, say $j - 1 \in I$. In v, if $j - 1$ appears to the left (respectively, right) of j, then inflating by any rising (resp., falling) twin from $\mathcal{I}(Q)$ would produce an interval containing $j-1$ and j, and not containing $j+k-1$. This would violate Lemma 2.2. On the other hand, inflating by any falling (resp., rising) twin from $\mathcal{I}(Q)$, of which there is at least one, does not violate Lemma 2.2 and hence produces a permutation whose interval poset is P. In this last case, if $\{j\}$ shares a cover both with $I \ni j - 1$ and with $I' \ni j + 1$, then, by Proposition 4.4, $j - 1$ and $j + 1$ must appear on opposite sides of j in v. Therefore the choice of whether to use rising or falling twins dictated by $j - 1$ will agree with the choice dictated by $j + 1$. In other words, interval generators of P are exactly those formed from any such $v \in \mathcal{I}(P_{t-1})$ and exactly half of the elements $\mathcal{I}(Q)$. $\square$

Theorem 4.9. Any interval poset can be constructed by INT-POSET.

Proof. Consider an interval poset $\mathcal{P}(w)$, for $w \in \mathfrak{S}_n$. We prove the theorem by induction on the number of proper intervals in $w \in \mathfrak{S}_n$. If w has no proper intervals, then w is a simple permutation, and so $\mathcal{P}(w)$ is an n-fruitful poset if $n > 3$ and $\mathcal{P}(w)$ is a binary tree poset (or argyle poset) with 2 leaves if $n = 2$. Each of these options can be constructed by INT-POSET.

Now fix $m > 0$ and suppose that the result holds for any permutation with $m-1$ proper intervals. Suppose that the permutation w has m proper intervals. In particular, consider the coatoms of $\mathcal{P}(w)$; that is, the inclusion-wise maximal proper intervals in w: $M_1, \ldots, M_a$. If $a > 2$ (in fact, $a > 3$), then the intervals $\{M_i\}$ are mutually disjoint, by Lemma 3.7. Then we can build the poset $\mathcal{P}(w)$ by the procedure above, letting P_1 be the a-fruitful poset, and then inflating the minimal elements of this poset inductively, as appropriate, using Lemma 2.1.

Suppose, instead, that $a = 2$. That is, there are two inclusion-wise maximal proper intervals in w. Consider the largest argyle poset that can used for P_1 in INT-POSET. It has $b \geq 2$ minimal elements: $I_1, \ldots, I_b$. Because b is maximal, these intervals $\{I_j\}$ must be disjoint: otherwise $\{I_j \setminus (I_j \cap (I_{j-1} \cup I_{j+1}))\} \cup \{I_j\}$ would have formed a larger argyle poset here. Then we can build the poset $\mathcal{P}(w)$ by the procedure above, letting P_1 be the argyle poset with b minimal elements, and inflating those minimal elements inductively, as appropriate, using Lemma 2.1. □

We demonstrate Theorems 4.8 and 4.9 using the permutation 43187562 whose interval poset was depicted in Figure 1.

Example 4.10. $\mathcal{P}(43187562)$ can be built by INT-POSET, as shown in Figure 6.

Now consider the poset constructed as shown in Figure 6, and let us recover the permutation 43187562. As described in the proof of Theorem 4.8, we see first that $\mathcal{I}(P_1) = \{2413, 3142\}$. Set $v_1 := 3142$. To find a permutation in $\mathcal{I}(P_2)$, we will inflate the 4 in v_1 by an element of $\mathcal{I}(Q_1) = \{123, 321\}$. Because P_1 was a 4-fruitful poset, either choice will work. Let us select 321, and so we get $v_2 := 316542$. To find a permutation in $\mathcal{I}(P_3)$, we will inflate the 3 in v_2 by an element of $\mathcal{I}(Q_2) = \{12, 21\}$. Again, because P_1 was a 4-fruitful poset either option will work. We select 21, which yields $v_3 := 4317652$. Finally, to find an element of $\mathcal{I}(P_4)$, we inflate the 5 in v_3 by an element of $\mathcal{I}(Q_3) = \{12, 21\}$. Because 6 appears to the left of 5 in v_3, we need to use the rising twin 12, producing $v_4 := 43187562$, as desired.

5. Enumerative properties of interval posets

Lemma 2.4 showed that a permutation and its reverse have the same interval poset. Therefore, there are at most $n!/2$ interval posets with n minimal elements. In fact, that is an overcount when $n > 3$, because simple permutations of a given size have the same poset. The enumeration below comes from infusing INT-POSET with Propositions 4.3, 4.4, and 4.7, and Lemma 2.2.

Theorem 5.1. Fix an interval poset P. Using the notation from INT-POSET,

$$|\mathcal{I}(P)| = \prod_{i=0} |\mathcal{I}(Q_i)|^{\varepsilon_i},$$

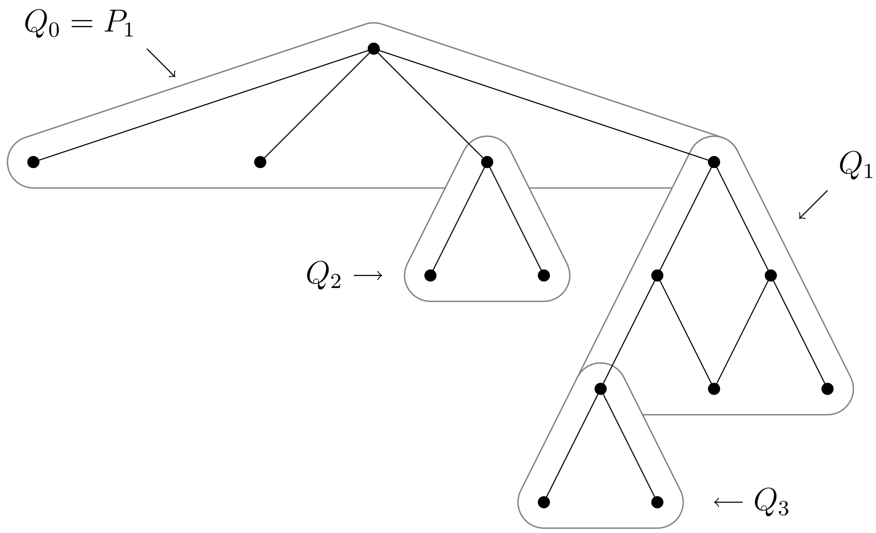


FIGURE 6. Using the notation from INT-POSET, the poset $\mathcal{P}(43187562)$ is constructed in four steps.

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i = 0, \\ 1 & \text{if } Q_i \text{ is a fruitful tree,} \\ 1 & \text{if } Q_i \text{ replaces the child of a fruitful element, and} \\ 0 & \text{otherwise.} \end{cases}$$

Each Q_i in the statement of Theorem 5.1 has three possibilities, and $|\mathcal{I}(Q_i)|$ was computed for each of these in Propositions 4.3, 4.4, and 4.7:

$$|\mathcal{I}(Q_i)| = \begin{cases} \mathsf{simp}(k) & \text{if } Q_i \text{ is a } k\text{-fruitful tree, and} \\ 2 & \text{otherwise.} \end{cases}$$

Theorems 4.8 and 4.9 show the centrality of INT-POSET to the nature of interval posets. Although there may be multiple ways to use INT-POSET to construct a given poset, this does not affect the enumeration of the interval generators: one cannot alter the position or size of any fruitful trees that appear.

Example 5.2. Let P be the interval poset depicted in Figure 1 and constructed as in Figure 6:

$$|\mathcal{I}(P)| = 2 \cdot 2 \cdot 2 \cdot 1 = 8.$$

Indeed, as outlined in Example 4.10, we can choose interval generators of $Q_0 = P_1$, of Q_1, and of Q_2, but we have no choice when it comes to interval generators of Q_3. The 8 interval generators of P are constructed from the components Q_i as follows.

$$2413 \rightsquigarrow 245613 \rightsquigarrow 2567134 \rightsquigarrow 26578134$$
$$2413 \rightsquigarrow 245613 \rightsquigarrow 2567143 \rightsquigarrow 26578143$$
$$2413 \rightsquigarrow 265413 \rightsquigarrow 2765134 \rightsquigarrow 28756134$$
$$2413 \rightsquigarrow 265413 \rightsquigarrow 2765143 \rightsquigarrow 28756143$$
$$3142 \rightsquigarrow 314562 \rightsquigarrow 3415672 \rightsquigarrow 34165782$$
$$3142 \rightsquigarrow 314562 \rightsquigarrow 4315672 \rightsquigarrow 43165782$$
$$3142 \rightsquigarrow 316542 \rightsquigarrow 3417652 \rightsquigarrow 34187562$$
$$3142 \rightsquigarrow 316542 \rightsquigarrow 4317652 \rightsquigarrow 43187562$$

The definition of ε_i in Theorem 5.1 has immediate implications for a certain family of interval posets (to be studied in more depth in the next section).

Corollary 5.3. An interval poset with no fruitful elements has exactly two interval generators. Moreover, the only interval posets with exactly two interval generators are the 4-fruitful poset and those interval posets with no fruitful elements.

6. Special families of interval posets

The poset types used in the procedure INT-POSET motivate us to analyze certain types of interval posets in further detail. There are, of course, many options for what to study, and we focus on three. An interval poset is:

- a *tree* if its Hasse diagram is a tree,
- *binary* if it has no fruitful elements, and
- a *binary tree* if its Hasse diagram is a binary tree.

We begin by classifying tree interval posets. Theorems 4.8 and 4.9 suggest that argyle posets are what need to be avoided, and Proposition 4.4 hints at how that can be done. We describe this in terms of bivincular pattern containment, introduced in [3]. We need only a very specific case of this here, which we define below.

Definition 6.1. Let $p \in \mathfrak{S}_k$. A permutation $w \in \mathfrak{S}_n$ contains the *bivincular* pattern $\underline{\overline{p}}$ if w has a p-pattern in consecutive positions, using consecutive values; that is, if there exist i and j such that $w(i+h) = p(h) + j$ for all $h \in [1, k]$.

Theorem 6.2. $\mathcal{P}(w)$ is a tree interval poset if and only if w does contain an inflation of the bivincular patterns $\underline{\overline{123}}$ or $\underline{\overline{321}}$.

Proof. $\mathcal{P}(w)$ is a tree if and only if $\mathcal{P}(w)$ can be constructed by INT-POSET without using argyle posets. Because argyle posets with two minimal elements are also binary tree posets, the goal is to avoid using argyle posets with at least three minimal elements. By Theorems 4.8 and 4.9, this is possible if and only if w does not contain an inflation of a bivincular pattern $\underline{\overline{p}}$, where $\mathcal{P}(p)$ is such an argyle poset. By Proposition 4.4, this is the case if and only if p is $123\cdots$ or $(123\cdots)^R$. □

Binary interval posets can also be characterized by patterns, this time by classical pattern avoidance.

Theorem 6.3. $\mathcal{P}(w)$ is binary if and only if w avoids the patterns 2143 and 3142.

Proof. Suppose that w is not binary, so there is an interval $I \in \mathcal{P}(w)$ covering intervals $H_1, \ldots, H_k$ for $k > 3$. Let these be indexed from left to right in the canonical embedding of $\mathcal{P}(w)$. For the four values $i \in \{1, 2, k-1, k\}$, pick some $x_i \in H_i$. Thus $x_1 < x_2 < x_3 < x_4$. Then, by Lemma 2.1 and Proposition 4.3, the letters $\{x_i\}$ appear in w in one of the two following orders:

$$\cdots x_2 \cdots x_4 \cdots x_1 \cdots x_3 \cdots \quad \text{or} \quad \cdots x_3 \cdots x_1 \cdots x_4 \cdots x_2 \cdots ,$$

and so w contains either 2413 or 3142.

Now suppose that w contains a 2413-pattern as:

$$\cdots x_2 \cdots x_4 \cdots x_1 \cdots x_3 \cdots ,$$

where $x_1 < x_2 < x_3 < x_4$. Consider the joins of these elements: for all $i \neq j$,

$$x_i \vee x_j = X$$

for some interval X, and $[x_1, x_4] \subseteq X$. In particular, that $X \in \mathcal{P}(w)$ covers at least four distinct elements: one containing each x_i. Therefore $\mathcal{P}(w)$ is not a binary interval poset.

The argument for w containing 3142 is analogous. $\square$

The class of permutations that avoid the patterns 3142 and 2413, listed in Theorem 6.3, is also called *separable permutations* (see [9, P0013]). Separable permutations were defined in [2] using the language of "separating trees." These trees bear some resemblance to the interval posets we define here, but they are not the same. For example, the interval poset of the the separable permutation 1234 is not a tree. Separable permutations are enumerated by the large Schroeder numbers [5, A006318]. Corollary 5.3 allows us to count binary interval posets (whether or not they are trees).

Corollary 6.4. Fix $n > 1$. The number of binary interval posets with n minimal elements is $(S_n)/2$, where S_n is the nth large Schroeder number.

The final special family that we study can be classified by a simple observation.

Corollary 6.5. $\mathcal{P}(w)$ is a binary tree interval poset if and only if w avoids 2413, 3142, and any inflation of the bivincular patterns $\overline{\underline{123}}$ or $\overline{\underline{321}}$.

Proof. An interval poset is a binary tree interval poset if and only if it is both a tree and a binary interval poset. $\square$

A binary tree interval poset can be constructed by INT-POSET in a single step. This allows us to enumerate the permutations described in Corollary 6.5.

Corollary 6.6. Fix $n > 1$.

$$\left|\begin{Bmatrix}\text{binary tree interval posets} \\ \text{with } n \text{ minimal elements}\end{Bmatrix}\right| = \left|\left\{w \in \mathfrak{S}_n : \begin{matrix} w \text{ avoids } 2413, 3142 \text{ and} \\ \text{inflations of } \overline{\underline{123}} \text{ and } \overline{\underline{321}}\end{matrix}\right\}\right| = 2C_{n-1}$$

where C_n is the nth Catalan number.

Proof. There are C_{n-1} full binary trees with n leaves (see, for example, [8]). By Proposition 4.7 and Corollary 5.3, each binary tree interval poset is generated by exactly 2 permutations. The result then follows from Corollary 6.5. $\square$

7. Directions for further research

There are many directions in which to continue the study of interval posets for permutations. We highlight a selection of such questions here.

In Section 6, binary interval posets and binary tree interval posets were enumerated. An enumeration of tree interval posets, remains elusive.

Question 7.1. How many tree interval posets have n minimal elements?

Corollary 5.3 describes the interval posets with exactly two interval generators (necessarily w and w^R for some w).

Question 7.2. How many interval posets have exactly two interval generators?

When a poset has exactly two interval generators, we know how those two generators are related: they are reverses of each other. Beyond this restricted setting, though, we wonder what other relations might occur.

Question 7.3. What properties are shared among the interval generators of a poset P? That is, what can be said about the set $\mathcal{I}(P)$, besides that it is closed under reversal?

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