# THE TYPE $B$ PERMUTOHEDRON AND THE POSET OF INTERVALS AS A TCHEBYSHEV TRANSFORM 

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#### Abstract

We show that the order complex of intervals of a poset, ordered by inclusion, is a Tchebyshev triangulation of the order complex of the original poset. Besides studying the properties of this transformation, we show that the dual of the type $B$ permutohedron is combinatorially equivalent to the suspension of the order complex of the poset of intervals of a Boolean algebra (with the minimum and maximum elements removed).


## Introduction

Inspired by Postnikov's seminal work [28], we have seen a surge in the study of root polytopes in recent years. A basic object in these investigations is the permutohedron. This paper connects permutohedra with a variant of the Tchebyshev transform of a poset, introduced by the present author [19, 20] and studied by Ehrenborg and Readdy [12], and with the (generalized) Tchebyshev triangulations of a simplicial complex, first introduced by the present author in [21] and studied in collaboration with Nevo in [22]. The key idea of a Tchebyshev triangulation may be summarized as follows: we add the midpoint to each edge of a simplicial complex, and perform a sequence of stellar subdivisions, until we obtain a triangulation containing all the newly added vertices. Regardless of the order chosen, the face numbers of the triangulation will be the same, and may be obtained from the face numbers $f_{j}$ of the original complex by replacing the powers of $x$ with Tchebyshev polynomials of the first kind if we work with the appropriate generating function. The appropriate generating function in this setting is the polynomial $F(x)=\sum_{j} f_{j-1}((x-1) / 2)^{j}$. It is also known that the links of the original vertices in a Tchebyshev triangulation from a multiset of simplicial complexes, called a Tchebyshev triangulation of the second kind, whose face numbers are also the same for all Tchebyshev triangulations, and may be computed by replacing the powers of $x$ with Tchebyshev polynomials of the second kind in $F(x)$.

The formula connecting the face numbers of the type $A$ and type $B$ permutohedra is identical to computing the face numbers of a Tchebyshev triangulation. These permutohedra are simple polytopes, their duals are simplicial polytopes, their boundary complexes are called the type $A$ resp. type $B$ Coxeter complexes. The suspicion arises that the type $B$ Coxeter complex is a Tchebyshev triangulation of the type $A$ Coxeter complex.

[^0]The present work contains the verification of this conjecture. The type $A$ Coxeter complex is known to be the order complex of the Boolean algebra, and the type $B$ Coxeter complex turns out to be the suspension of an order complex, namely of the partially ordered set of intervals of the Boolean algebra, ordered by inclusion. We show that the operation of associating the poset of intervals to a partially ordered sets always induces a Tchebyshev triangulation at the level of order complexes. This observation may be helpful in constructing "type $B$ analogues" of other polytopes and partially ordered sets. Furthermore it inspires further study of the poset of intervals of a poset, initiated by Walker [34], and continued by Athanasiadis [2], Athanasiadis and Savvidou [5] and Jojić [24] among others.

This paper is structured as follows. After the Preliminaries, we introduce the poset of intervals in Section 2 and show that the order complex of the poset of intervals is always a Tchebyshev triangulation of the order complex of the original poset. We also introduce a graded variant of this operation that takes a graded poset into a graded poset. In Section 3 we show that the type $B$ Coxeter complex is the order complex of the graded poset of intervals of the Boolean algebra. In Section 4 we review how to compute the flag $f$-vector of graded a poset of intervals. This topic was first studied by Jojić [24], and we provide new proofs to some of his key formulas. Section 5 introduces interval transforms of the second kind. The corresponding multiset of order complexes is the Tchebyshev triangulation of the second kind corresponding to the Tchebyshev triangulation induced by taking the order complex of the graded poset of intervals of a graded poset. We find explicit flag $f$-vector formulas in terms of the mixing operator introduced by Ehrenborg and Readdy [10]. Inspired by the work of Ehrenborg and Readdy [12], we make the first steps towards describing all eigenvectors of the linear operator on the flag $f$-vectors, induced by taking the interval transforms of the second kind. In Section 6 we consider the special case of Eulerian posets, cite a formula by Jojić [24] and an analogous recurrence found by Ehrenborg and Fox [9] for the mixing operator, which may be used to compute the effect on the $c d$-index of taking the interval transform of the second kind. The latter result is used in Section 7 to compute the $c d$-index of the interval transform of the second kind of the ladder poset (the same calculation was already performed by Jojić [24] for the interval transform of the first kind of these posets). As part of the proof of our formula, we develop a weighted lattice path enumeration model to express the values $M\left(c^{i}, c^{j}\right)$ for the mixing operator of Ehrenborg and Readdy [10]. The other special example considered in this section is the Boolean lattice, where known results of Purtill [29], Hetyei [18] and of Ehrenborg and Readdy [11] come into play. These results use André permutations, first studied by Foata, Strehl and Schützenberger [14, 15], and their signed generalizations.

## 1. Preliminaries

1.1. Graded Eulerian posets. A partially ordered set is graded if it contains a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$ and a rank function $\rho$ satisfying $\rho(\widehat{0})=0$ and $\rho(y)=\rho(x)+1$ for each $x$ and $y$ such that $y$ covers $x$. The number of chains containing elements of fixed sets of ranks in a graded poset $P$ of rank $n+1$ is encoded by the flag $f$-vector $\left(f_{S}(P): S \subseteq\{1, \ldots, n\}\right)$. The entry $f_{S}$ in the flag $f$-vector is the number of chains $x_{1}<x_{2}<\cdots<x_{|S|}$ such that their set of ranks
$\left\{\rho\left(x_{i}\right): i \in\{1, \ldots,|S|\}\right\}$ is $S$. Inspired by Stanley 31] we introduce the upsilon invariant of a graded poset $P$ of rank $n+1$ by

$$
\Upsilon_{P}(a, b)=\sum_{S \subseteq\{1, \ldots, n\}} f_{S} u_{S}
$$

where $u_{S}=u_{1} \cdots u_{n}$ is a monomial in noncommuting variables $a$ and $b$ such that $u_{i}=b$ for all $i \in S$ and $u_{i}=a$ for all $i \notin S$. It should be noted that the term upsilon invariant is not used elsewhere in the literature, most sources switch to the ab-index $\Psi_{P}(a, b)$ defined as $\Upsilon_{P}(a-b, b)$. The $a b$-index may be also written as a linear combination of monomials in $a$ and $b$, the coefficients of these monomials form the flag h-vector. A graded poset $P$ is Eulerian if every nontrivial interval of $P$ has the same number of elements of even rank as of odd rank. All linear relations satisfied by the flag $f$-vectors of Eulerian posets were found by Bayer and Billera [6]. A very useful and compact rephrasing of the Bayer-Billera relations was given by Bayer and Klapper in [7]: they proved that satisfying the Bayer-Billera relations is equivalent to stating that the $a b$ index may be rewritten as a polynomial of $c=a+b$ and $d=a b+b a$. The resulting polynomial in noncommuting variables $c$ and $d$ is called the $c d$-index.

As an immediate consequence of the above cited results we obtain the following.
Corollary 1.1. The cd-index of a graded Eulerian poset $P$ may be obtained by rewriting $\Upsilon_{P}(a, b)$ as a polynomial of $c=a+2 b$ and $d=a b+b a+2 b^{2}$.

Note that this statement is a direct consequence of $\Upsilon_{P}(a-b, b)=\Psi_{P}(a, b)$ which is equivalent to $\Upsilon_{P}(a, b)=\Psi_{P}(a+b, b)$.
1.2. Tchebyshev triangulations and Tchebyshev transforms. A finite simplicial complex $\triangle$ is a family of subsets of a finite vertex set $V$. The elements of $\triangle$ are called faces, subject to the following rules: a subset of any face is a face and every singleton is a face. The dimension of a face is one less than the number of its elements, the dimension $d-1$ of the complex $\triangle$ is the maximum of the dimension of its faces. The number of $j$-dimensional faces is denoted by $f_{j}(\triangle)$ and the vector $\left(f_{-1}, f_{0}, \ldots, f_{d-1}\right)$ is the $f$-vector of the simplicial complex. We define the $F$-polynomial $F_{\Delta}(x)$ of a finite simplicial complex $\triangle$ as

$$
\begin{equation*}
F_{\triangle}(x)=\sum_{j=0}^{d} f_{j-1}(\triangle) \cdot\left(\frac{x-1}{2}\right)^{j} \tag{1.1}
\end{equation*}
$$

The join $\triangle_{1} * \triangle_{2}$ of two simplicial complexes $\triangle_{1}$ and $\triangle_{2}$ on disjoint vertex sets is the simplicial complex $\triangle_{1} * \triangle_{2}=\left\{\sigma \cup \tau: \sigma \in \triangle_{1}, \tau \in \triangle_{2}\right\}$. It is easy to show that the $F$-polynomials satisfy $F_{\triangle_{1} * \Delta_{2}}(x)=F_{\triangle_{1}}(x) \cdot F_{\triangle_{2}}(x)$. A special instance of the join operation is the suspension operation: the suspension $\triangle * \partial\left(\triangle^{1}\right)$ of a simplicial complex $\triangle$ is the join of $\triangle$ with the boundary complex of the one dimensional simplex. (A $(d-1)$-dimensional simplex is the family of all subsets of a $d$-element set, its boundary is obtained by removing its only facet from the list of faces.) The link of a face $\sigma$ is the subcomplex $\operatorname{link}_{\Delta}(\sigma)=\{\tau \in \triangle: \sigma \cap \tau=\emptyset, \sigma \cup \tau \in K\}$. A special type of simplicial complex we will focus on is the order complex $\triangle(P)$ of a finite partially ordered set $P$ : its vertices are the elements of $P$ and its faces are the increasing chains. The order
complex of a finite poset is a flag complex: its minimal non-faces are all two-element sets (these are the pairs of incomparable elements).

Every finite simplicial complex $\triangle$ has a standard geometric realization in the vector space with a basis $\left\{e_{v}: v \in V\right\}$ indexed by the vertices, where each face $\sigma$ is realized by the convex hull of the basis vectors $e_{v}$ indexed by the elements of $\sigma$.
Definition 1.2. We define a Tchebyshev triangulation $T(\triangle)$ of a finite simplicial complex $\triangle$ as follows. We number the edges $e_{1}, e_{2}, \ldots, e_{f_{1}(\Delta)}$ in some order, and we associate to each edge $e_{i}=\left\{u_{i}, v_{i}\right\}$ a midpoint $w_{i}$. We associate a sequence $\triangle_{0}:=$ $\triangle, \triangle_{1}, \triangle_{2} \ldots, \triangle_{f_{1}(\Delta)}$ of simplicial complexes to this numbering of edges, as follows. For each $i \geq 1$, the complex $\triangle_{i}$ is obtained from $\triangle_{i-1}$ by replacing the edge $e_{i}$ and the faces contained therein with the one-dimensional simplicial complex $L_{i}$, consisting of the vertex set $\left\{u_{i}, v_{i}, w_{i}\right\}$ and edge set $\left\{\left\{u_{i}, w_{i}\right\},\left\{w_{i}, v_{i}\right\}\right\}$, and by replacing the family of faces $\left\{e_{i} \cup \tau: \tau \in \operatorname{link}_{\Delta_{i-1}}\left(e_{i}\right)\right\}$ containing $e_{i}$ with the family of faces $\left\{\sigma^{\prime} \cup \tau: \sigma^{\prime} \in L_{i}\right\}$. In other words, we subdivide the edge $e_{i}$ into a path of length 2 by adding the midpoint $w_{i}$ and we also subdivide all faces containing $e_{i}$, by performing a stellar subdivision.

As it is defined by a sequence of a stellar subdivisions, it is clear that any Tchebyshev triangulation of $\triangle$ as defined above is indeed a triangulation of $\triangle$ in the following sense: if we consider the standard geometric realization of $\triangle$ and associate to each midpoint $w$ the midpoint of the line segment realizing the corresponding edge $\{u, v\}$ then the convex hulls of the vertex sets representing the faces of $T(\triangle)$ represent a triangulation of the geometric realization of $\triangle$. Furthermore, the following statement is a special case of [22, Theorem 3.3] and can also be derived from [3, Example 2.8], combined with Stanley's locality formula [32, Theorem 3.2].
Theorem 1.3. All Tchebyshev triangulations of a simplicial complex have the same $f$-vector.
Remark 1.4. [22, Theorem 3.3] allows replacing the operation of taking the midpoint of each edge with higher dimensional analogues. On the other hand, every Tchebyshev triangulation of $\triangle$ dissects each $k$-dimensional face into exactly $2^{k}$ faces of the same dimension. This property is shared by other triangulations of $\triangle$, such as the second edgewise subdivision, introduced by Freudenthal [17]. All triangulations of $\triangle$ with this property have the same $f$-vector by [3, Example 2.8], combined with [32, Theorem 3.2]. Thus, the formulas obtained by Brenti and Welker [8] for the $h$-vector of the second edgewise subdivision of $\triangle$ apply to Tchebyshev triangulations as well. See also Remark 1.7 below.

Tchebyshev triangulations of the second kind were first introduced in [21] in connection with some special Tchebyshev triangulations of the second kind. The idea was generalized to arbitrary generalized Tchebyshev triangulations in [22]. Here we specialize the definition introduced in [22] to Tchebyshev triangulations as follows. Recall that the link $\operatorname{link}_{\Delta}(\tau)$ of a face $\tau$ in a simplicial complex $\triangle$ is the set of faces $\{\sigma-\tau: \sigma \in \triangle, \tau \subseteq \sigma\}$.

Definition 1.5. Let $\triangle$ be an arbitrary simplicial complex with vertex set $V$ and $T(\triangle)$ a Tchebyshev triangulation. We define the corresponding Tchebyshev triangulation of the second kind $U(\triangle)$ as the collection of the links $\operatorname{link}_{T(\Delta)}(\{v\})$ for all vertices $v \in V$.

Note that $U(\triangle)$ is not a simplicial complex, but a multiset of simplicial complexes. We define its $f$-vector ( $F$-polynomial) as the sum of the $f$-vectors ( $F$-polynomials) of the complexes $\operatorname{link}_{T(\Delta)}(\{v\})$ for all $v \in V$. The following result is a direct consequence of [22, Theorem 3.3].
Theorem 1.6 (Hetyei and Nevo). All Tchebyshev triangulations of the second kind of a simplicial complex have the same $f$-vector.


Figure 1. A Tchebyshev triangulation and a second edgewise triangulation
Remark 1.7. While Tchebyshev triangulations have the same face numbers as the second edgewise triangulation, this result cannot be extended to Tchebyshev triangulations of the second kind. Figure 1 shows a simplicial complex with 4 (black) vertices, 5 edges and 2 two-dimensional faces. A Tchebyshev triangulation (shown in the middle, obtained by performing the first stellar subdivision at the midpoint of the edge $\left\{v_{1}, v_{2}\right\}$ ) has the same face numbers as the second edgewise triangulation (on the right). However, the sum of the $f$-vectors of the links of the original vertices is different in the two triangulations.

The following result has been shown in [21, Propositions 3.3 and 4.4] for a specific Tchebyshev triangulation. By the preceding theorems it holds for all Tchebyshev triangulations and motivates the choice of the terminology. The Tchebyshev transform $T(U)$ of the first (second) kind of polynomials used in the next result is the linear map $\mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ sending $x^{n}$ into the Tchebyshev polynomial of the first kind $T_{n}(x)$ (second kind $U_{n}(x)$ ).

Theorem 1.8. For any finite simplicial complex $\triangle$, the F-polynomial of any Tchebyshev triangulation $T(\triangle)$ is the Tchebyshev transform of the first kind of the F-polynomial of $\triangle$ :

$$
F_{T(\Delta)}(x)=T\left(F_{\Delta}(x)\right)
$$

Similarly, the F-polynomial of any Tchebyshev triangulation $U(\triangle)$ of the second kind is half of the Tchebyshev transform of the second kind of the F-polynomial of $\triangle$ :

$$
F_{U(\Delta)}(x)=\frac{1}{2} \cdot U\left(F_{\Delta}(x)\right)
$$

The notion of the Tchebyshev triangulation of a simplicial complex was motivated by a poset operation, first considered in [19] and formally introduced in [20] .

Definition 1.9. Given a locally finite poset $P$, its Tchebyshev transform of the first kind $T(P)$ is the poset whose elements are the intervals $[x, y] \subset P$ satisfying $x \neq y$, ordered by the following relation: $\left[x_{1}, y_{1}\right] \leq\left[x_{2}, y_{2}\right]$ if either $y_{1} \leq x_{2}$ or both $x_{1}=x_{2}$ and $y_{1} \leq y_{2}$ hold.

A geometric interpretation of this operation may be found in [20, Theorem 1.10]. The graded variant of this poset operation is defined in [12]. Given a graded poset $P$ with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1}<\widehat{0}$ and a new maximum element $\widehat{2}$. The graded Tchebyshev transform of the first kind of a graded poset $P$ is then the interval $[(\widehat{-1}, \widehat{0}),(\widehat{1}, \widehat{2})]$ in $T(P \cup\{\widehat{-1}, \widehat{2}\})$. By abuse of notation we also denote the graded Tchebyshev transform of a graded poset $P$ by $T(P)$. It is easy to show that $T(P)$ is also a graded poset, whose rank is one more than that of $P$. The following result may be found in [21, Theorem 1.5].

Theorem 1.10. Let $P$ be a graded poset and $T(P)$ its graded Tchebyshev transform. Then the order complex $\triangle(T(P) \backslash\{(\widehat{-1}, \widehat{0}),(\widehat{1}, \widehat{2})\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P \backslash\{\widehat{0}, \widehat{1}\})$.

As a consequence of Theorem 1.10, we have

$$
\begin{equation*}
F_{\Delta(T(P) \backslash\{(\widehat{-1}, \widehat{0}),(\hat{1}, \widehat{2})\})}=T\left(x \cdot F_{\Delta(P \backslash\{\hat{0}, \widehat{1}\})}\right) . \tag{1.2}
\end{equation*}
$$

It has been shown by Ehrenborg and Readdy [12] that there is a linear transformation assigning to the flag $f$-vectors of each graded poset $P$ of rank $n+1$ the flag $f$-vector of its Tchebyshev transform of the first kind $T(P)$. For Eulerian posets, they also compute the effect on the $c d$-index of taking the Tchebyshev transform of the first kind. They also studied the corresponding Tchebyshev transforms of the second kind.
1.3. Permutohedra of type $A$ and $B$. Permutohedra of type $A$ and $B$ have a vast literature, the results cited here may be found in [13] and in [33].

The type $A$ permutohedron $\operatorname{Perm}\left(A_{n-1}\right)$ is the convex hull of the $n$ ! vertices $(\pi(1), \ldots, \pi(n)) \in \mathbb{R}^{n}$, where $\pi$ is any permutation of the set $[1, n]:=\{1,2, \ldots, n\}$. The type $B$ permutohedron $\operatorname{Perm}\left(B_{n}\right)$ is the convex hull of all points of the form $( \pm \pi(1), \pm \pi(2) \ldots, \pm \pi(n)) \in \mathbb{R}^{n}$. Combinatorially equivalent polytopes may be obtained by taking the $A_{n-1}$-orbit, respectively $B_{n}$ orbit, of any sufficiently generic point in an $(n-1)$-dimensional (respectively $n$-dimensional) space, and the convex hull of the points in the orbit. [13, Section 2].

The type $A$ and $B$ permutohedra are simple polytopes, their duals are simplicial polytopes. The boundary complexes of these duals are combinatorially equivalent to the Coxeter complexes of the respective Coxeter groups. The Coxeter complex of the symmetric group $A_{n-1}$ on $[1, n]$ is the order complex of $\left.P([1, n])-\{\emptyset,[1, n]\}\right)$, where $P([1, n])$ is the Boolean algebra of rank $n$. The Coxeter complex of the Coxeter group $B_{n}$ is the order complex of the face lattice of the $n$-dimensional crosspolytope [33, Lecture 1]. In either case we consider the order complexes of the respective graded posets without their unique minimum and maximum elements: adding these would make the order complex contractible, whereas the boundary complexes of simplicial polytopes are homeomorphic to spheres. The standard $n$-dimensional crosspolytope is the convex hull of the vertices $\left\{ \pm e_{i}: i \in[1, n]\right\}$, where $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ is the standard basis of
$\mathbb{R}^{n}$. Each nontrivial face of the crosspolytope is the convex hull of a set of vertices of the form $\left\{e_{i}, i \in K^{+}\right\} \cup\left\{-e_{i}, i \in K^{-}\right\}$, where $K^{+}$and $K^{-}$is are disjoint subsets of $[1, n]$ and their union is not empty. Keeping in mind that each face of a polytope is the intersection of all the facets containing it, we have the following consequence.

Corollary 1.11. Each facet of $\operatorname{Perm}\left(B_{n}\right)$ is uniquely labeled with a pair of sets $\left(K^{+}, K^{-}\right)$ where $K^{+}$and $K^{-}$is are subsets of $[1, n]$, satisfying $K^{+} \subseteq[1, n]-K^{-}$and $K^{+}$and $K^{-}$ cannot be both empty. For a set of valid labels

$$
\left\{\left(K_{1}^{+}, K_{1}^{-}\right),\left(K_{2}^{+}, K_{2}^{-}\right), \ldots,\left(K_{m}^{+}, K_{m}^{-}\right)\right\}
$$

the intersection of the corresponding set of facets is a nonempty face of $\operatorname{Perm}\left(B_{n}\right)$ if and only if

$$
K_{1}^{+} \subseteq K_{2}^{+} \subseteq \cdots \subseteq K_{m}^{+} \subseteq[1, n]-K_{m}^{-} \subseteq[1, n]-K_{m-1}^{-} \subseteq \cdots \subseteq[1, n]-K_{1}^{-} \quad \text { holds }
$$

The triangle of $f$-vectors of the type $B$ Coxeter complexes is given in sequence A145901 in [27].

## 2. The poset of intervals as a Tchebyshev transform

We will use Corollary 1.11 to represent the type $B$ Coxeter complex using the poset of intervals of a Boolean algebra. In this section we review this construction and show that taking the poset of intervals induces a Tchebyshev triangulation.

Definition 2.1. An interval $[u, v]$ in a partially ordered set $P$ is the set of all elements $w \in P$ satisfying $u \leq w \leq v$. For a finite partially ordered set $P$ we define the poset $I(P)$ of the intervals of $P$ as the set of all intervals $[u, v] \subseteq P$, ordered by inclusion.

We may identify the singleton intervals $[u, u]$ in $I(P)$ with the elements of $P$. This subset of elements forms an antichain in $I(P)$, however, under this identification, the order complex of $I(P)$ looks like a triangulation of the order complex of $P$, see Figures 2 and 3. Figure 2 shows a partially ordered set and its order complex. The poset of its intervals and the order complex thereof may be seen in Figure 3 .


Figure 2. A partially ordered set $P$ and its order complex $\triangle(P)$
In Figure 3 we marked the vertices of the order complex associated to non-singleton intervals with white circles.

The following result is a generalization of [24, Remark 10], and an equivalent restatement of Walker's result [34, Theorem 4.1].

Theorem 2.2. For any finite partially ordered set $P$ the order complex $\triangle(I(P))$ of its poset of intervals is isomorphic to a Tchebyshev triangulation of $\triangle(P)$ as follows. For each $u \in P$ we identify the vertex $[u, u] \in \triangle(I(P))$ with the vertex $u \in \triangle(P)$ and for


Figure 3. The poset $I(P)$ of intervals of $P$ and its order complex
each nonsingleton interval $[u, v] \in I(P)$ we identify the vertex $[u, v] \in \triangle(I(P))$ with the midpoint of the edge $\{[u, u],[v, v]\}$. We number the midpoints $\left[u_{1}, v_{1}\right],\left[u_{2}, v_{2}\right], \ldots$ in such an order that $i<j$ holds whenever the interval $\left[u_{i}, v_{i}\right]$ contains the interval $\left[u_{j}, v_{j}\right]$.

Proof. We illustrate the Tchebyshev triangulation process with the poset shown in Figure 3. We list its nonsingleton intervals in the following order: [ $u_{1}, u_{3}$ ], $\left[u_{1}, u_{2}\right.$ ], $\left[u_{2}, u_{3}\right],\left[u_{1}, u_{4}\right]$. Figure 4 shows the stage of the process when we already added $\left[u_{1}, u_{3}\right]$ and $\left[u_{1}, u_{2}\right]$ but none of the remaining nonsingleton intervals. The following statement


Figure 4. The second step of the Tchebyshev triangulation process
may be shown by induction on the number of stages in the process: in each stage, the resulting complex is a flag complex, whose minimal nonfaces are the following:
(1) Pairs of singletons $\{[u, u],[v, v]\}$ such that $u$ and $v$ are not comparable in $P$.
(2) Pairs of singletons $\{[u, u],[v, v]\}$ such that $u<v$ holds in $P$, but the interval [ $u, v]$ has already been added to the triangulation.
(3) Pairs of intervals from $I(P)$ such that neither one contains the other.

In each stage of the process, the nonsingleton interval $[u, v]$ added is the first midpoint of any edge whose endpoints are contained in the interval $[u, v]$ of $P$. At the beginning of the stage the restriction of the current complex to intervals contained in $[u, v]$ only contains singleton intervals, and it is isomorphic to the order complex of $[u, v]$. Subdividing the edge $\{[u, u],[v, v]\}$ and all faces containing this edge results in a complex where both $[u, u]$ and $[v, v]$ can not appear in the same face any more, each such face is replaced with 2 faces: one containing $\{[u, u],[u, v]\}$ the other containing $\{[v, v],[u, v]\}$. All intervals $\left[u^{\prime}, v^{\prime}\right]$ containing $[u, v]$ have already been added in a previous stage, and now we add the edge $\left\{[u, v],\left[u^{\prime}, v^{\prime}\right]\right\}$. The cumulative effect of all these changes is that we obtain a new flag complex satisfying the listed criteria.

Remark 2.3. Walker's proof is a direct geometric argument. The proof above uses the more general result stated in [22, Theorem 3.3]. It also directly implies the face counting formula that holds for all Tchebyshev transforms.

When $P$ is a graded poset then $\left[u^{\prime}, v^{\prime}\right]$ covers $[u, v]$ in $I(P)$ exactly when the rank function $\rho$ of $P$ satisfies $\rho\left(v^{\prime}\right)-\rho\left(u^{\prime}\right)=\rho(v)-\rho(u)+1$. Hence we may define the following graded variant of the operation $P \mapsto I(P)$.
Definition 2.4. For a graded poset $P$ we define its graded poset of intervals $\widehat{I}(P)$ as the poset of all intervals of $P$, including the empty set, ordered by inclusion.


Figure 5. The graded poset of intervals of a chain

Remark 2.5. Figure 5 represents the graded poset of intervals of a chain of rank 3. It is worth comparing this illustration with [20, Figure 2] where the Tchebyshev transform of a chain of rank 3 is represented. The two posets are not isomorphic, not even after taking the dual of the Tchebyshev transform to make the number of elements at the same rank equal.

The following statement is straightforward.
Proposition 2.6. If $P$ is a graded poset of rank $n$ with rank function $\rho$ then $\widehat{I}(P)$ is a graded poset of rank $n+1$, in which the rank of a nonempty interval $[u, v]$ is $\rho(v)-\rho(u)+1$.

In analogy to Theorem 1.10 we have the following result.
Proposition 2.7. Let $P$ be a graded poset and $\widehat{I}(P)$ its graded poset of intervals. Then the order complex $\triangle(\widehat{I}(P)-\{\emptyset,[\widehat{0}, \widehat{1}]\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P-\{\widehat{0}, \widehat{1}\})$.
Proof. By Theorem 2.2, the order complex $\triangle(\widehat{I}(P)-\{\emptyset\})$ is a Tchebyshev triangulation of $\triangle(P)$. The order complex $\triangle(P)$ is the join of $\triangle(P-\{\widehat{0}, \widehat{1}\})$ with the one-dimensional simplex on the vertex set $\{\widehat{0}, \widehat{1}\}$. Performing the Tchebyshev triangulation results in subdividing every simplex containing the edge $\{\widehat{0}, \widehat{1}\}$ into two simplices. The removal of the midpoint $[\widehat{0}, \widehat{1}]$ leaves us exactly with those faces which are contained in a face of
$\triangle(P)$ that does not contain the edge $\{\widehat{0}, \widehat{1}\}$. Hence we obtain a Tchebyshev triangulation of a suspension of $\triangle(P-\{\widehat{0}, \widehat{1}\})$ : the suspending vertices are $\widehat{0}$ and $\widehat{1}$.

We conclude this section with the following observations regarding the direct product of two graded posets. Recall that the direct product $P \times Q$ of two graded posets $P$ and $Q$ is defined as the set of all ordered pairs $(u, v)$ where $u \in P$ and $v \in Q$, subject to the partial order $\left(u_{1}, v_{1}\right) \leq\left(u_{2}, v_{2}\right)$ holding exactly when $u_{1} \leq u_{2}$ holds in $P$ and $v_{1} \leq v_{2}$ holds in $Q$.

Proposition 2.8. If $P$ and $Q$ are graded posets then $I(P \times Q)$ is isomorphic to $I(P) \times$ $I(Q)$.

The straightforward verification is left to the reader. Proposition 2.8 may be immediately generalized to the graded poset of intervals using the diamond product introduced by Ehrenborg and Readdy in [12].

Definition 2.9. Given two graded posets $P$ and $Q$, their diamond product $P \diamond Q$ is defined as $\left(P-\left\{\widehat{0}_{P}\right\}\right) \times\left(Q-\left\{\widehat{0}_{Q}\right\}\right) \cup \widehat{0}$. In other words, to obtain the diamond product we remove the unique minimum elements of $P$ and $Q$ respectively, we take the direct product of the resulting posets and we add a new unique minimum element.

Corollary 2.10. If $P$ and $Q$ are graded posets then $\widehat{I}(P \times Q)$ is isomorphic to $\widehat{I}(P) \diamond$ $\widehat{I}(Q)$.

A special case of Corollary 2.10 may be found in [24, Proposition 4 (iv)].
Remark 2.11. It is worth comparing Corollary 2.10above with [12, Theorem 9.1] where it is stated that the Tchebyshev transform of the Cartesian product of two posets is the diamond product of their Tchebyshev transforms.

## 3. The type $B$ Coxeter complex as a Tchebyshev triangulation

After introducing $X:=K^{+}$and $Y:=[1, n]-K^{-}$, we may rephrase Corollary 1.11 as follows.

Corollary 3.1. We may label each facet of the type $B$ permutohedron $\operatorname{Perm}\left(B_{n}\right)$ with a nonempty interval $[X, Y]$ of the Boolean algebra $P([1, n])$ that is different from $P([1, n])=$ $[\emptyset,[1, n]]$. The set $\left\{\left[X_{1}, Y_{1}\right],\left[X_{2}, Y_{2}\right], \ldots,\left[X_{m}, Y_{m}\right]\right\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n]))-$ $\{\emptyset,[\emptyset,[1, n]]\}$.

The representation of each face of $\operatorname{Perm}\left(B_{n}\right)$ as an intersection of facets is unique, hence we obtain the following result.

Proposition 3.2. The dual of $\operatorname{Perm}\left(B_{n}\right)$ is a simplicial polytope whose boundary complex is combinatorially equivalent to the order complex $\triangle(\widehat{I}(P([1, n]))-\{\emptyset,[\emptyset,[1, n]]\})$.

As a consequence of this statement and of Proposition 2.7, we obtain the following result.

Corollary 3.3. The dual of $\operatorname{Perm}\left(B_{n}\right)$ is a simplicial polytope whose boundary complex is combinatorially equivalent to a Tchebyshev triangulation of the suspension of $\triangle(P([1, n])-\{\emptyset,[1, n]\})$.

The order complex $\triangle(\widehat{I}(P([1, n]))-\{\emptyset,[\emptyset,[1, n]]\})$ has been studied by Athanasiadis and Savvidou [5]. It is worth noting that the order complex $\triangle(P([1, n])-\{\emptyset,[1, n]\})$ is known to be combinatorially equivalent to the dual of the boundary complex of the permutohedron $\operatorname{Perm}\left(A_{n-1}\right)$. We may also think of this complex as the barycentric subdivision of the boundary of an $(n-1)$-dimensional simplex.


Figure 6. Half of the dual of $\operatorname{Perm}\left(B_{3}\right)$
Figure 6 represents "half" of the dual of Perm $\left(B_{3}\right)$. The boundary of the triangle whose vertices are labeled with singleton intervals $[\{i\},\{i\}]$ is shown in bold. (In general, the reader should imagine the boundary of a simplex, whose vertices are labeled with $[\{i\},\{i\}]$. .) The vertices of the barycentric subdivision of the boundary are marked with black circles. These correspond to singleton intervals of the form $[X, X]$, where $X$ is a subset of $[1,3]$. (In general, $X$ is a subset of $[1, n]$.) The suspending vertex $\emptyset$ is marked with a black square. The other suspending vertex $[1,3]$ (in general: $[1, n]$ ) is not shown in the picture. One would need to make another picture showing the boundary of the triangle with the suspending vertex, and "glue" the two pictures along the boundary of the triangle. The midpoints of the edges are marked with white circles. These are labeled with intervals $[X, Y]$ such that $X$ is properly contained in $Y$. The edges arising when we take the appropriate Tchebyshev triangulation are indicated with dashed lines. Note that this part of the picture is different on the "other side" of the dual of Perm $\left(B_{3}\right)$ : on the side shown the largest intervals labeling midpoints are of the form $[\emptyset,[1,3]-\{i\}]$ (in general $[\emptyset,[1, n]-\{i\}]$ ) whereas on the other side the largest such intervals are of the form $[\{i\},[1,3]]$ (in general: $[\{i\},[1, n]]$ ). We leave to the reader as a challenge to draw the other side of the dual of $\operatorname{Perm}\left(B_{3}\right)$.

Remark 3.4. By Proposition 3.2, the work of Anwar and Nazir [1] implies that the $h$-polynomial of the type $B$ Coxeter complex has only real roots. As a consequence of

Corollary 3.3 we know that this is a Tchebyshev triangulation and we may compute its $F$-polynomial using $\sqrt{1.2}$, and obtain that these polynomials have the same coefficients (up to sign) as the derivative polynomials for secant. Taking the signs into account we obtain the derivative polynomials for hyperbolic secant. For the Tchebyshev transform of a Boolean algebra this was first observed in [20, Corollary 9.3], but at the level of counting faces in the order complex of a graded poset there is no difference between considering the operator $P \mapsto T(P)$ and the operator $P \mapsto \widehat{I}(P)$. It has been shown in [21] that the derivative polynomials for hyperbolic tangent and hyperbolic secant have interlaced real roots in the interval $[-1,1]$. As noted in the same paper, the $F$-polynomial $F_{\Delta}(t)$ and the $h$-polynomial $h_{\Delta}(t)$ of a $(d-1)$-dimensional simplicial complex $\triangle$ are connected by the formula

$$
(1-t)^{d} \cdot F_{\triangle}\left(\frac{1+t}{1-t}\right)=(1-t)^{d} \sum_{j=0}^{d} f_{j}\left(\frac{t}{1-t}\right)^{j}=h_{\triangle}(t)
$$

Hence the real-rootedness of the $h$-polynomial of the type $B$ Coxeter complex is also a consequence of the fact that the derivative polynomials for the hyperbolic secant have real roots.

## 4. Computing the flag $f$-vector of the graded poset of intervals

In this section we review how for any graded poset $P$, the flag $f$-vector of its graded poset of intervals $\widehat{I}(P)$ may be obtained from the flag $f$-vector of $P$ by a linear transformation. Such formulas were first found by Jojić [24]. At the end of the section we will also present a more direct proof of his key formulas. By "chain" in this section we always mean a chain containing the unique minimum element and the unique maximum element. This treatment is equivalent to excluding both of these elements from all chains.

Definition 4.1. Given a chain $\emptyset \subset\left[u_{1}, v_{1}\right] \subset\left[u_{2}, v_{2}\right] \subset \cdots \subset\left[u_{k}, v_{k}\right] \subset\left[u_{k+1}, v_{k+1}\right]=$ $[\widehat{0}, \widehat{1}]$ in the graded poset of intervals $\widehat{I}(P)$ of a graded poset $P$, we call the set

$$
\left\{u_{1}, v_{1}, u_{2}, v_{2}, \ldots, u_{k+1}, v_{k+1}\right\}
$$

the support of the chain.
Obviously the support of a chain in $\widehat{I}(P)$ is a chain in $P$ containing the minimum element $\widehat{0}$ and the maximum element $\widehat{1}$.

The next statement expresses the number of chains in $\widehat{I}(P)$ having the same support in terms of the Pell numbers $P(n)$. These numbers are given by the initial conditions $P(1)=1$ and $P(2)=2$ and by the recurrence $P(n)=2 \cdot P(n-1)+P(n-2)$ for $n \geq 3$. A detailed bibliography on the Pell numbers may be found at sequence A000129 of [27].

Proposition 4.2. Let $P$ be a graded poset and let $c: \widehat{0}=z_{0}<z_{1}<\cdots<z_{m-1}<z_{m}=$ $\widehat{1}$ be a chain in it. Then the number of chains $\emptyset \subset\left[u_{1}, v_{1}\right] \subset\left[u_{2}, v_{2}\right] \subset \cdots \subset\left[u_{k}, v_{k}\right] \subset$ $\left[u_{k+1}, v_{k+1}\right]=[\widehat{0}, \widehat{1}]$ whose support is $c$ is the sum $P(m)+P(m+1)$ of two adjacent Pell numbers.

Proof. We proceed by induction on $m$. For $m=1$ there are three chains: $\emptyset \subset[\widehat{0}, \widehat{1}]$, $\emptyset \subset[\widehat{0}, \widehat{0}] \subset[\widehat{0}, \widehat{1}]$ and $\emptyset \subset[\widehat{1}, \widehat{1}] \subset[\widehat{0}, \widehat{1}]$. For $m=2$, there are the following seven chains with support $\widehat{0}<z_{1}<\widehat{1}$ :
(1) $\emptyset \subset\left[\widehat{0}, z_{1}\right] \subset[\widehat{0}, \widehat{1}]$,
(2) $\emptyset \subset[\widehat{0}, \widehat{0}] \subset\left[\widehat{0}, z_{1}\right] \subset[\widehat{0}, \widehat{1}]$,
(3) $\emptyset \subset\left[z_{1}, z_{1}\right] \subset\left[\widehat{0}, z_{1}\right] \subset[\widehat{0}, \widehat{1}]$,
(4) $\emptyset \subset\left[z_{1}, \widehat{1}\right] \subset[\widehat{0}, \widehat{1}]$,
(5) $\emptyset \subset[\widehat{1}, \widehat{1}] \subset\left[z_{1}, \widehat{1}\right] \subset[\widehat{0}, \widehat{1}]$,
(6) $\emptyset \subset\left[z_{1}, z_{1}\right] \subset\left[z_{1}, \widehat{1}\right] \subset[\widehat{0}, \widehat{1}]$, and
(7) $\emptyset \subset\left[z_{1}, z_{1}\right] \subset[\widehat{0}, \widehat{1}]$.

Let us list the elements of the chain in $\widehat{I}(P)$ in decreasing order. The largest element of the chain must be $[\widehat{0}, \widehat{1}]$, the unique maximum element. The next element is either the interval $\left[z_{1}, \widehat{1}\right]$ or the interval $\left[\widehat{0}, z_{m}\right]$ or the interval $\left[z_{1}, z_{m}\right]$. We can not make the minimum of this next interval larger than $z_{1}$ because that would force skipping $z_{1}$ in the support, similarly the maximum of this next interval is at least $z_{m}$. Applying the induction hypothesis to the intervals $\left[z_{1}, \widehat{1}\right],\left[\widehat{0}, z_{m}\right]$ and $\left[z_{1}, z_{m}\right]$, respectively, we obtain that the number of chains is

$$
2 \cdot(P(m)+P(m+1))+(P(m-1)+P(m))=P(m+1)+P(m+2)
$$

Remark 4.3. The numbers $P(n)+P(n+1)$ are listed as sequence A001333 in [27]. They are known as the numerators of the continued fraction convergents to $\sqrt{2}$, and have many combinatorial interpretations. The even, respectively odd indexed entries in this sequence may also be obtained by substitutions into the Tchebyshev polynomials of the first, respectively second kind.

It is transparent in the proof of Proposition 4.2 that the contributions of chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a, b)$ depends only on the contribution of their support to $\Upsilon_{P}(a, b)$. This observation motivates the following definition.

Definition 4.4. Given an ab-word $w$ of degree $n$, we define $\iota(w)$ as the contribution of all chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a, b)$, whose support is the same chain of $P$, contributing the word $w$ to $\Upsilon_{P}(a, b)$.

Theorem 4.5. The operator $\iota$ may be recursively computed using the following formulas.
(1) $\iota\left(a^{n}\right)=(a+2 b) a^{n}$ holds for $n \geq 0$. In particular, for the empty word $\varepsilon$ we have $\iota(\varepsilon)=(a+2 b)$.
(2) $\iota\left(a^{i} b a^{j}\right)=(a+2 b)\left(a^{i} b a^{j}+a^{j} b a^{i}\right)+b a^{i+j+1}$ holds for $i, j \geq 0$.
(3) $\iota\left(a^{i} b w b a^{j}\right)=\iota\left(a^{i} b w\right) b a^{j}+\iota\left(w b a^{j}\right) b a^{i}+\iota(w) b a^{i+j+1}$ holds for $i, j \geq 0$ and any ab-word $w$.

Proof. The only chain that contributes $a^{n}$ to the $a b$-index of a graded poset is the chain $\widehat{0}<\widehat{1}$ in a graded poset $P$ of rank $n+1$. As seen in the proof of Proposition 4.2, there
are 3 chains in $\widehat{I}(P)$ whose support is $\widehat{0}<\widehat{1}$, and their contribution is to the $a b$-index of $\widehat{I}(P)$ is $(a+2 b) a^{n}$.

Similarly, the only chains that contribute $a^{i} b a^{j}$ to the $a b$-index of a graded poset are the chains $\widehat{0}<z_{1}<\widehat{1}$ in a graded poset $P$ of rank $i+j+2$, where the rank of $z_{1}$ is $i+1$. As seen in the proof of Proposition 4.2 there are 7 chains in $\widehat{I}(P)$ whose support is $\widehat{0}<z_{1}<\widehat{1}$, and their contribution is to the $a b$-index of $\widehat{I}(P)$ is $(a+2 b)\left(a^{i} b a^{j}+a^{j} b a^{i}\right)+b a^{i+j}$.

Finally, consider a chain $c: \widehat{0}<z_{1}<z_{2}<\cdots<z_{k}<z_{k+1}=\widehat{1}$ that contributes $a^{i} b w b a^{j}$ to the $a b$-index of a graded poset $P$ of rank $n+1$. In such a chain the rank of $z_{1}$ is $i+1$ and the rank of of $z_{k}$ is $n-j$. The largest element below [ $\left.\widehat{0}, \widehat{1}\right]$ of any chain in $\widehat{I}(P)$ with support $c$ is either $\left[\widehat{0}, z_{k}\right]$ (of rank $n-j+1$ ) or $\left[z_{1}, \widehat{1}\right]$ (of rank $n+1-i$ ) or $\left[z_{1}, z_{k}\right]$ (of rank $n-i-j+1$ ). The three terms correspond to the contributions of the chains of these three types.

Corollary 4.6. There is a linear map $I_{n}: \mathbb{R}^{2^{n}} \rightarrow \mathbb{R}^{2^{n+1}}$ sending the flag $f$-vector of each graded poset $P$ of rank $n+1$ into the flag $f$-vector of its graded poset of intervals $\widehat{I}(P)$. This linear map may be obtained by encoding flag $f$-vectors with the corresponding upsilon invariants, and extending the map $\iota$ by linearity.

Example 4.7. Using Theorem 4.5 we obtain the following formulas.

$$
\begin{gathered}
n=1: \iota(a)=a^{2}+2 b a, \iota(b)=(a+2 b)(b+b)+b a=4 b^{2}+2 a b+b a . \\
n=2: \iota\left(a^{2}\right)=a^{3}+2 b a^{2}, \iota(a b)=(a+2 b)(a b+b a)+b a^{2}=a^{2} b+a b a+2 b a b+2 b^{2} a+b a^{2}, \\
\iota(b a)=a^{2} b+a b a+2 b a b+2 b^{2} a+b a^{2}=\iota(a b), \text { and } \\
\iota\left(b^{2}\right)=2 \iota(b) b+\iota(\varepsilon) b a=2\left(4 b^{2}+2 a b+b a\right) b+(a+2 b) b a \\
=8 b^{3}+4 a b^{2}+2 b a b+a b a+2 b^{2} a .
\end{gathered}
$$

We conclude this section by a shorter proof of two key formulas found by Jojić [24]. These express the connection between the $a b$-indices of $P$ and $\widehat{I}(P)$ using a coproduct operation first introduced by Ehrenborg and Readdy [10].

Definition 4.8. The coproduct $\Delta$ is defined on the algebra $\mathbb{Q}\langle a, b\rangle$, whose basis is the set of all ab-words, as follows. Given an ab-word $u=u_{1} u_{2} \cdots u_{n}$, we set

$$
\Delta(u)=\sum_{i=1}^{n} u_{1} \cdots u_{i-1} \otimes u_{i+1} \cdots u_{n}
$$

Note that the algebra $\mathbb{Q}\langle a, b\rangle$ also includes the empty word 1 as a basis vector. In the next theorem we will use Sweedler notation

$$
\Delta(u)=\sum_{u} u_{(1)} \otimes u_{(2)}
$$

to refer to the coproduct and the notation $u^{*}$ as a shorthand for $u^{*}=u_{n} u_{n-1} \cdots u_{1}$, obtained by reversing the word $u=u_{1} u_{2} \cdots u_{n}$.

Theorem 4.9 (Jojić). Given a graded poset $P$ of rank $n+1$, we have $\Psi_{\widehat{I}(P)}(a, b)=$ $\mathcal{I}\left(\Psi_{P}(a, b)\right)$, where the linear operator $\mathcal{I}: \mathbb{Q}\langle a, b\rangle \rightarrow \mathbb{Q}\langle a, b\rangle$ is defined by the following
recursive formulas on the basis of ab-words:

$$
\begin{align*}
& \mathcal{I}(u \cdot a)=\mathcal{I}(u) \cdot a+(a b+b a) \cdot u^{*}+\sum_{u} \mathcal{I}\left(u_{(2)}\right) \cdot a b \cdot u_{(1)}^{*}  \tag{4.1}\\
& \mathcal{I}(u \cdot b)=\mathcal{I}(u) \cdot b+(a b+b a) \cdot u^{*}+\sum_{u} \mathcal{I}\left(u_{(2)}\right) \cdot b a \cdot u_{(1)}^{*} . \tag{4.2}
\end{align*}
$$

Proof. We prove the theorem by showing the following, linearly equivalent formulas for the operator $\iota$ :

$$
\begin{align*}
& \iota(u a)=\iota(u) a+\sum_{u}^{\prime} \iota\left(u_{(2)}\right) \cdot(a b-b a) \cdot u_{(1)}^{*}  \tag{4.3}\\
& \iota(u b)=\iota(u) b+\left(a b+b a+2 b^{2}\right) \cdot u^{*}+\sum_{u}^{\prime} \iota\left(u_{(2)}\right) \cdot b(a+b) \cdot u_{(1)}^{*} . \tag{4.4}
\end{align*}
$$

Here the symbol $\sum_{u}^{\prime}$ is Sweedler notation for the coproduct $\Delta^{\prime}$ defined on the algebra $\mathbb{Q}\langle a, b\rangle$ by the rule

$$
\Delta^{\prime}\left(a^{i_{1}} b a^{i_{2}} b \cdots a^{i_{k}} b a^{i_{k+1}}\right)=\sum_{j=1}^{k} a^{i_{1}} b a^{i_{2}} b \cdots a^{i_{j}} \otimes a^{i_{j+1}} \cdots a^{i_{k}} b a^{i_{k+1}} .
$$

Equations (4.3) and (4.4) are linearly equivalent to the formulas stated in the theorem because $a b$-index $\Psi_{P}(a, b)$ is obtained by substituting $e=a-b$ into $\Upsilon_{P}(a, b)$, and the definition of $\Delta^{\prime}$ corresponds to the rule

$$
\Delta\left(e^{i_{1}} b e^{i_{2}} b \cdots e^{i_{k}} b e^{i_{k+1}}\right)=\sum_{j=1}^{k} e^{i_{1}} b e^{i_{2}} b \cdots e^{i_{j}} \otimes e^{i_{j+1}} \cdots e^{i_{k}} b e^{i_{k+1}}
$$

whose verification is left to the reader. After substituting $a-b$ into $a$, equations (4.3) and (4.4) become

$$
\begin{aligned}
\mathcal{I}(u(a-b)) & =\mathcal{I}(u)(a-b)+\sum_{u} \mathcal{I}\left(u_{(2)}\right) \cdot(a b-b a) \cdot u_{(1)}^{*} \\
\mathcal{I}(u b) & =\mathcal{I}(u) b+(a b+b a) \cdot u^{*}+\sum_{u} \mathcal{I}\left(u_{(2)}\right) \cdot b a \cdot u_{(1)}^{*} .
\end{aligned}
$$

The sum of these equations is the first equation stated in the theorem, while the second equation is the same in both pairs.

To prove (4.4), note that $\iota(u b)$ is the sum of all $a b$-words associated to chains of intervals supported by a fixed chain $\widehat{0}<x_{1}<\cdots<x_{k}<x_{k+1}<\widehat{1}$ where the rank of $x_{k+1}$ is one less than the rank of $\widehat{1}$ and the set of ranks of $x_{1}<\cdots<x_{k}$ is marked by the $a b$-word $u$. The summand $\iota(u) b$ is contributed by all chains of intervals containing $\left[\widehat{0}, x_{k+1}\right]$. These chains can not contain any interval containing $\widehat{1}$, except for $[\widehat{0}, \widehat{1}]$, so the remaining intervals in all such chains are contained in $\left[\widehat{0}, x_{k+1}\right]$. The sum $\sum_{u}^{\prime} \iota\left(u_{(2)}\right) \cdot$ $b(a+b) \cdot u_{(1)}^{*}$ is contributed by all chains of intervals, containing some intervals of the form $\left[x_{j}, x_{k+1}\right]$, for some $j \geq 1$ but not containing $\left[\widehat{0}, x_{k+1}\right]$. Let $i \geq 1$ be the least index for which $\left[x_{i}, x_{k+1}\right]$ belongs to the chain. the factor $\iota\left(u_{(2)}\right) b$ is contributed by the intervals contained in this interval and by the interval $\left[x_{i}, x_{k+1}\right]$ itself. For each $j<i$ the interval $\left[x_{j}, \widehat{1}\right]$ must belong to the chain, these contribute the factor $a u_{(1)}^{*}$. The factor
$(a+b)$ right after $\iota\left(u_{(2)}\right) b$ reflects the possibility of adding to the chain $\left[x_{j}, \widehat{1}\right]$ or omitting it. This choice may be done independently of everything else. The remaining terms are contributed by chains not containing any interval of the form $\left[\widehat{0}, x_{k+1}\right]$ or $\left[x_{i}, x_{k+1}\right]$. We claim that all these remaining chain contribute the term $\left(a b+b a+2 b^{2}\right) \cdot u^{*}$. Since $x_{k+1}$ must be part of the support, these chains contain at least one of $\left[x_{k+1}, \widehat{1}\right]$ and $\left[x_{k+1}, x_{k+1}\right]$. The intersection of such a chain of intervals with the set $\left\{[\hat{1}, \widehat{1}],\left[x_{k+1}, \widehat{1}\right],\left[x_{k+1}, x_{k+1}\right]\right\}$ can be $\left\{\left[x_{k+1}, \widehat{1}\right],\right\},\left\{\left[x_{k+1}, x_{k+1}\right]\right\},\left\{[\widehat{1}, \widehat{1}],\left[x_{k+1}, \widehat{1}\right],\right\}$, or $\left\{\left[x_{k+1}, \widehat{1}\right],\left[x_{k+1}, x_{k+1}\right]\right\}$. These four possibilities account for the presence of a factor $\left(a b+b a+2 b^{2}\right)$. For all $i \leq k$ the interval $\left[x_{i}, \widehat{1}\right]$ must be present, this explains the presence of the factor $u^{*}$.

To prove (4.3), note that $\iota(u a)$ is the sum of all $a b$-words associated to chains of intervals supported by a fixed chain $C: \widehat{0}<x_{1}<\cdots<x_{k}<\widehat{1}$ where the difference between the rank of $x_{k}$ and the rank of $\widehat{1}$ is at least 2 and the set of ranks of $x_{1}<\cdots<x_{k}$ is marked by the $a b$-word $u$. Let us fix a coatom $x_{k+1}$ in the interval $\left[x_{k}, \widehat{1}\right]$. There is an obvious bijection between the chains of intervals supported $C$ and the chains of intervals supported by the chain $C^{\prime}: \widehat{0}<x_{1}<\cdots<x_{k}<x_{k+1}$. This bijection is induced by replacing each occurrence of $\widehat{1}$ by $x_{k+1}$. Clearly the sum of the $a b$-words of all intervals supported by $C^{\prime}$ is $\iota(u)$, let us multiply this sum by $a$ on the right and compare the contribution of a chain of intervals supported by $C$ to $\iota(u a)$ with the contribution of the corresponding chain of intervals supported by $C^{\prime}$ to $\iota(u) a$.
Case 1: The chain of intervals supported by $C$ contains no interval of the form $\left[x_{i}, \widehat{1}\right]$. The contribution of such a chain of intervals to $\iota(u a)$ is the same as the contribution of the corresponding chain of intervals supported by $C^{\prime}$ to $\iota(u) a$.
Case 2: The chain of intervals supported by $C$ contains an interval of the form $\left[x_{i}, \widehat{1}\right]$. Let us take the largest $i$ with this property, that is, the least interval of this form, and let us break the $a b$ word corresponding to chain of intervals at the letter $b$ corresponding to this interval. The intervals of the chain of intervals contained in $\left[x_{i}, \widehat{1}\right]$ do not contain $\widehat{1}$ and they are the same in the corresponding chain of intervals supported by $C^{\prime}$. Their contribution is $\iota\left(u_{(2)}\right)$. The remaining intervals of the original chain of intervals supported by $C$ are all intervals of the form $\left[x_{j}, \widehat{1}\right]$ where $j<i$. In the corresponding chain of intervals each $\left[x_{j}, \widehat{1}\right]$ needs to be replaced by $\left[x_{j}, x_{k+1}\right]$, the rank of the corresponding interval is one less: the contribution of such a chain of intervals is thus $a b u_{(1)}^{*}$ to $\iota(u a)$, and the contribution of the corresponding intervals to $\iota(u) a$ is thus $b a u_{(1)}^{*}$.

## 5. Interval transforms of the second kind

In Section 22 we have seen that for any poset $P$, the order complex of the poset of intervals $I(P)$ is a Tchebyshev triangulation of the order complex of $P$. In this setting, the elements of the original poset $P$ are identified with the singleton intervals in $I(P)$. Hence we make the following definition.

Definition 5.1. Given a partially ordered set $P$ we define its interval transform of the second kind $I_{2}(P)$ the multiset of subposets of $I(P)$ defined as follows: for each $x \in P$ we take the subposets of $I(P)$ formed by all elements $[y, z] \in I(P)$ containing $[x, x]$.

It is a direct consequence of the definition and Theorem 2.2 we obtain the following.

Corollary 5.2. For any finite poset $P$ the order complex of $I_{2}(P)$ is a Tchebyshev triangulation of the second kind of the order complex of $P$, associated to the Tchebyshev triangulation of the first kind that is the order complex of $I(P)$.

Corollary 5.3. If $P$ is a graded poset then its interval transform of the second kind is the collection of all intervals of the form $[[x, x],[\widehat{0}, \widehat{1}]] \subset I(P)$ for each $x \in P$.
Definition 5.4. Let $\left(P_{1}, \ldots, P_{m}\right)$ be a list of graded posets. We extend the notions of the upsilon invariant and ab-index by linearity, i.e., we set

$$
\Upsilon_{\left(P_{1}, \ldots, P_{m}\right)}(a, b)=\sum_{i=1}^{m} \Upsilon_{P_{i}}(a, b) \quad \text { and } \quad \Psi_{\left(P_{1}, \ldots, P_{m}\right)}(a, b)=\sum_{i=1}^{m} \Psi_{P_{i}}(a, b)
$$

For a graded poset $P$ we then define the total ab-index $\Psi_{\widehat{I}_{2}(P)}(a, b)$ of $\widehat{I}_{2}(P)$ by

$$
\Psi_{\widehat{I}_{2}(P)}(a, b)=\sum_{u \in P} \Psi_{[[x, x],[\hat{0}, \hat{1}]]}(a, b) .
$$

The following statement is straightforward.
Proposition 5.5. Let $P$ be a graded poset. For each $x \in P$, the set of intervals $[y, z]$ contained in $[[x, x], \widehat{0}, \widehat{1}]] \subset \widehat{I}(P)$ and ordered by inclusion is isomorphic to the direct product $[\widehat{0}, x]^{*} \times[x, \widehat{1}]$. Here $[\widehat{0}, x]^{*}$ is the dual of the poset $[\widehat{0}, x]$, obtained by reversing the order of $[\widehat{0}, x]$.

Proposition 5.5 allows us to compute the effect of taking the interval transform on the second kind on the $a b$-index of a poset using the mixing operator introduced by Ehrenborg and Readdy [10, Definition 9.1]
Definition 5.6. The mixing operator $M$ is a bilinear operator defined on the noncommutative algebra $\mathbb{Q}\langle a, b\rangle$ as the follows. For each pair of ab-words $(u, v)$ we set

$$
M(u, v)=\sum_{r=1}^{2} \sum_{s=1}^{2} \sum_{n-r-s-1} M_{\text {is even }} M_{r, s, n}(u, v) .
$$

Here the operators $M_{r, s, n}(u, v)$ are recursively defined by

$$
\begin{aligned}
M_{1,2,2}(u, v) & =u \cdot a \cdot v \\
M_{2,1,2}(u, v) & =u \cdot b \cdot v \\
M_{1, s, n+1}(u, v) & =\sum_{u} u_{(1)} \cdot a \cdot M_{2, s, n}\left(u_{(2)}, v\right) \quad \text { and } \\
M_{2, s, n+1}(u, v) & =\sum_{v} v_{(1)} \cdot b \cdot M_{1, s, n}\left(u, v_{(2)}\right)
\end{aligned}
$$

It has been shown by Ehrenborg and Readdy [10, Theorem 9.2] that the $a b$-index of the direct product of the graded posets $P$ and $Q$ is given by

$$
\begin{equation*}
\Psi_{P \times Q}(a, b)=M\left(\Psi_{P}(a, b), \Psi_{Q}(a, b)\right) . \tag{5.1}
\end{equation*}
$$

Combining Equation (5.1) with Corollary 5.3 we may compute the total $a b$-index of an interval transform of a second kind as follows.

Theorem 5.7. Given a graded poset $P$ of rank $n+1$, we have $\Psi_{\widehat{I}_{2}(P)}(a, b)=\mathcal{I}_{2}\left(\Psi_{P}(a, b)\right)$, where the linear operator $\mathcal{I}_{2}: \mathbb{Q}\langle a, b\rangle \rightarrow \mathbb{Q}\langle a, b\rangle$ is given by the formula

$$
\mathcal{I}_{2}(u)=u+u^{*}+\sum_{u} M\left(u_{(1)}^{*}, u_{(2)}\right) .
$$

Proof. By Definition 5.4 and Proposition 5.5 we have

$$
\Psi_{\widehat{I}_{2}(P)}(a, b)=\sum_{x \in P} \Psi_{\widehat{0}, x]^{*} \times[x, \widehat{1}]}(a, b)
$$

By Equation (5.1) this may be rewritten as

$$
\begin{aligned}
\Psi_{\widehat{I_{2}(P)}}(a, b) & =\sum_{x \in P} M\left(\Psi_{\widehat{0}, x]^{*}}(a, b), \Psi_{[x, \widehat{1}]}(a, b)\right) \\
& =\Psi_{\widehat{[0, \widehat{1}}}(a, b)^{*}+\Psi_{\widehat{[0, \widehat{1}}}(a, b)+\sum_{\widehat{0}<x<\widehat{1}} M\left(\Psi_{\widehat{0}, x]}(a, b)^{*}, \Psi_{[x, \widehat{1}]}(a, b)\right) .
\end{aligned}
$$

The statement is now a direct consequence of [10, Equation (3.1)], stating

$$
\Delta \Psi_{P}(a, b)=\sum_{\widehat{0}<x<\widehat{1}} \Psi_{[\widehat{0}, x]}(a, b) \otimes \Psi_{[x, \widehat{1}]}(a, b)
$$

In analogy to [12, Theorem 10.10] we may find many eigenvalues and eigenvectors of the operator $I_{2}: \mathbb{Q}\langle a, b\rangle \rightarrow \mathbb{Q}\langle a, b\rangle$. The quest to find the eigenvalues of $I_{2}$ is complicated by the fact that this linear operator has a nontrivial kernel. To find part of this kernel, we first extend the operator $u \mapsto u^{*}$ by linearity to all $a b$-polynomials.

Corollary 5.8. If the homogeneous ab-polynomial $u \in \mathbb{Q}\langle a, b\rangle_{n}$ satisfies $u^{*}=-u$ then $I_{2}(u)=0$

Corollary 5.8 is a direct consequence of Theorem 5.7. It inspires decomposing the vectorspace $\mathbb{Q}\langle a, b\rangle_{n}$ of $a b$-polynomials of degree $n$ into a direct sum of the vector spaces of symmetric and antisymmetric ab-polynomials.

Definition 5.9. A homogeneous ab-polynomial $u \in \mathbb{Q}\langle a, b\rangle_{n}$ of degree $n$ is symmetric if it satisfies $u^{*}=u$ and antisymmetric if it satisfies $u^{*}=-u$. We denote the vectorspace of symmetric, respectively antisymmetric ab-polynomials of degree $n$ by $S_{\mathbb{Q}}\langle a, b\rangle_{n}$, respectively $A_{\mathbb{Q}}\langle a, b\rangle_{n}$.
Proposition 5.10. The vectorspace $\mathbb{Q}\langle a, b\rangle_{n}$ may be written as the direct sum

$$
\mathbb{Q}\langle a, b\rangle_{n}=S_{\mathbb{Q}}\langle a, b\rangle_{n} \oplus A_{\mathbb{Q}}\langle a, b\rangle_{n} .
$$

Here $\operatorname{dim} A_{\mathbb{Q}}\langle a, b\rangle_{n}=2^{n-1}-2^{\lfloor(n-1) / 2\rfloor}$ and $\operatorname{dim} S_{\mathbb{Q}}\langle a, b\rangle_{n}=2^{n-1}+2^{\lfloor(n-1) / 2\rfloor}$.
Proof. Clearly $S_{\mathbb{Q}}\langle a, b\rangle_{n} \cap A_{\mathbb{Q}}\langle a, b\rangle_{n}=0$ and each $u \in \mathbb{Q}\langle a, b\rangle_{n}$ may be written as

$$
u=\frac{1}{2} \cdot\left(u+u^{*}\right)+\frac{1}{2} \cdot\left(u-u^{*}\right)
$$

where $u+u^{*} \in S_{\mathbb{Q}}\langle a, b\rangle_{n}$ and $u-u^{*} \in A_{\mathbb{Q}}\langle a, b\rangle_{n}$. The dimension formulas are direct consequences of the fact that the number of symmetric ab-words $w$ satisfying $w=w^{*}$ is $2^{\lfloor(n+1) / 2\rfloor}$ and hence the number of asymmetric ab-words $w$ satisfying $w \neq w^{*}$ is
$2^{n}-2^{\lfloor(n+1) / 2\rfloor}$. Asymmetric $a b$-words $w$ form pairs $\left\{w, w^{*}\right\}$, and we may associate to each such unordered pair a vector $w-w^{*}$, these vectors form a basis of $A_{\mathbb{Q}}\langle a, b\rangle_{n}$.

Next we show the following analogue of [12, Proposition 10.9].
Lemma 5.11. If the homogeneous ab-polynomial $u \in \mathbb{Q}\langle a, b\rangle_{n}$ of degree $n$ is an eigenvector of the linear operator $I_{2}: \mathbb{Q}\langle a, b\rangle \rightarrow \mathbb{Q}\langle a, b\rangle$ then so is the homogeneous abpolynomial $\mathcal{L}(u):=(a-b) u+u(a-b) \in \mathbb{Q}\langle a, b\rangle_{n+1}$. Both eigenvectors have the same eigenvalue.
Proof. Assume $I_{2}(u)=\lambda \cdot u$ holds. By Theorem 5.7 we may write

$$
\begin{aligned}
I_{2}((a-b) u+u(a-b)) & =(a-b) u+u^{*}(a-b)+u(a-b)+(a-b) u^{*} \\
& +(a-b) \sum_{u} M\left(u_{(1)}^{*}, u_{(2)}\right)+\sum_{u} M\left(u_{(1)}^{*}, u_{(2)}\right)(a-b) \\
& =(a-b) I_{2}(u)+I_{2}(u)(a-b)=\lambda \cdot((a-b) u+u(a-b)) .
\end{aligned}
$$

Note that the restriction of the operator $\mathcal{L}$ to $S_{\mathbb{Q}}\langle a, b\rangle_{n}$ takes $S_{\mathbb{Q}}\langle a, b\rangle_{n}$ into $S_{\mathbb{Q}}\langle a, b\rangle_{n+1}$. The analogue of [12, Proposition 10.8] may be stated in more general terms, as follows.
Proposition 5.12. For any pair of graded partially ordered sets $P$ and $Q$,

$$
I_{2}(P \times Q)=I_{2}(P)(\times) I_{2}(Q) \quad \text { holds }
$$

Here $I_{2}(P)(\times) I_{2}(Q)$ denotes the multiset of posets $\left\{P_{1} \times Q_{1}: P_{1} \in I_{2}(P), Q_{1} \in I_{2}(Q)\right\}$.
Proof. The set $I_{2}(P \times Q)$ is the multiset of all intervals of $P \times Q$, of the form

$$
\left[[(p, q),(p, q)],\left[\left(\widehat{0}_{P}, \widehat{0}_{Q}\right),\left(\widehat{1}_{P}, \widehat{1}_{Q}\right)\right]\right]
$$

ordered by inclusion. Here $p$ ranges over all elements of $P$ and $Q$ independently ranges over all elements of $Q$. The statement follows from the obvious isomorphism

$$
\left[[(p, q),(p, q)],\left[\left(\widehat{0}_{P}, \widehat{0}_{Q}\right),\left(\widehat{1}_{P}, \widehat{1}_{Q}\right)\right]\right] \cong\left[[p, p],\left[\widehat{0}_{P}, \widehat{1}_{P}\right]\right] \times\left[[q, q],\left[\widehat{0}_{Q}, \widehat{1}_{Q}\right]\right]
$$

Corollary 5.13. Assume the homogeneous ab-polynomial $u_{i}$ is an eigenvector with eigenvalue $\lambda_{i}$ of the interval transform of the second kind $I_{2}$ for $i=1,2$. Then $M\left(u_{1}, u_{2}\right)$ is an eigenvector with eigenvalue $\lambda_{1} \cdot \lambda_{2}$.

As a consequence of [9, Corollary 4.3], if $u_{i} \in S_{\mathbb{Q}}\langle a, b\rangle_{n_{i}}$ holds for $i=1,2$ then $M\left(u_{1}, u_{2}\right) \in S_{\mathbb{Q}}\langle a, b\rangle_{n_{1}+n_{2}}$.

In [12] find all eigenvalues and eigenvectors of the Tchebyshev operator of the second kind, by repeated use of the analogues of Lemma 5.11 and Proposition 5.12. More precisely, [12, Theorem 10.10] states that a basis of eigenvectors may be generated by repeatedly using the pyramid operator Pyr : $u \mapsto M(1, u)$ and their variant of the lifting operator $\mathcal{L}$ which sends $u$ into $(a-b) u$. A key ingredient of their proof is the use of the fact that the intersection of the ranges of the pyramid operator and their lifting operator is zero. In our case this is not true any more, furthermore our operator $I_{2}$ has a nontrivial kernel. That said, we make the following conjectures.

Conjecture 5.14. For each $n \geq 1$, the kernel of $I_{2}: \mathbb{Q}\langle a, b\rangle_{n} \rightarrow \mathbb{Q}\langle a, b\rangle_{n+1}$ is $A_{\mathbb{Q}}\langle a, b\rangle_{n}$.
Conjecture 5.15. For each $n \geq 1$ a generating set of $S_{\mathbb{Q}}\langle a, b\rangle_{n}$, consisting of eigenvectors only may be found by taking all possible $n$-fold compositions of the pyramid operator Pyr and of the lift operator $\mathcal{L}$, and applying it to 1.

## 6. The graded poset of intervals of an Eulerian poset

The following result is due to C. Athanasiadis [2, Proposition 2.5], it is also stated in a special case by Jojić [24, Remark 10].

Proposition 6.1. If a graded poset $P$ is Eulerian then the same holds for the graded poset of its intervals $\widehat{I}(P)$.

Indeed, it is well known consequence of Phillip Hall's theorem (see [30, Propositition 3.8.5]) that a graded poset is Eulerian if and only if the reduced characteristic of the order complex of each open interval $(u, v)$ is $(-1)^{\rho(v)-\rho(u)}$ where $\rho$ is the rank function. Since taking the graded poset of intervals results in taking a triangulation of the suspension of each such order complex, the reduced Euler characteristic remains unchanged.

As a consequence of Proposition 6.1, the linear map $I_{n}$ takes the flag $f$-vector of any graded Eulerian poset of rank $n+1$ into the flag $f$-vector of a graded Eulerian poset of rank $n+2$. As a direct consequence of Theorem 4.9 the following formulas hold, see [24, Corollary 7]:

Corollary 6.2 (Jojić). Given a graded poset $P$ of rank $n+1$, we have $\Psi_{\widehat{I}(P)}(c, d)=$ $\mathcal{I}\left(\Psi_{P}(c, d)\right)$, where the linear operator $\mathcal{I}: \mathbb{Q}\langle c, d\rangle \rightarrow \mathbb{Q}\langle c, d\rangle$ is defined by the following recursive formulas on the basis of ab-words:

$$
\begin{aligned}
\mathcal{I}(u \cdot c) & =\mathcal{I}(u) \cdot c+2 d \cdot u^{*}+\sum_{u} \mathcal{I}\left(u_{(2)}\right) \cdot c \cdot u_{(1)}^{*} \\
\mathcal{I}(u \cdot d) & =\mathcal{I}(u) \cdot d+(d c+c d) \cdot u^{*}+d \cdot u^{*} \cdot c \\
& +\sum_{u}\left(\mathcal{I}\left(u_{(2)}\right) \cdot d \cdot \operatorname{Pyr}\left(u_{(1)}^{*}\right)+d \cdot u_{(2)}^{*} \cdot d \cdot u_{(1)}^{*}\right) .
\end{aligned}
$$

Here Pyr is the the linear operator defined by Ehrenborg and Readdy [10] associating to the $c d$ index of each poset $P$ the $c d$-index of $P \times B_{1}$ where $B_{1}$ is the Boolean algebra of rank 1 .

Proposition 6.1 has the following easy consequence.
Corollary 6.3. The interval transform of the second kind ${\widehat{I_{2}}}_{2}(P)$ of any graded Eulerian poset $P$ of rank $n+1$ is a multiset of Eulerian posets of rank $n+1$. Hence $\Psi_{\widehat{I}_{2}(P)}(a, b)$ is a polynomial of $c$ and $d$

We will use the notation $\Psi_{\widehat{I_{2}(P)}}(c, d)$ to stand for the polynomial $\Psi_{\widehat{I}_{2}(P)}(a, b)$ rewritten as an expression of $c$ and $d$. In analogy to Theorem4.9 and Corollary 6.2, the restriction of the map $\mathcal{I}_{2}$ to $c d$-polynomials allows us to compute $\Psi_{\widehat{I}_{2}(P)}(c, d)$. In doing so, the
following formulas of Ehrenborg and Fox [9, Theorem 5.1] are helpful. For two cd monomials $u$ and $w$ we have

$$
\begin{align*}
& M(u, v \cdot c)=v \cdot d \cdot u+M(u, v) \cdot c+\sum_{u} M\left(u_{(1)}, v\right) \cdot d \cdot u_{(2)} \quad \text { and }  \tag{6.1}\\
& M(u, v \cdot d)=v \cdot d \cdot \operatorname{Pyr}(u)+M(u, v) \cdot d+\sum_{u} M\left(u_{(1)}, v\right) \cdot d \cdot \operatorname{Pyr}\left(u_{(2)}\right) . \tag{6.2}
\end{align*}
$$

Remark 6.4. To avoid introduction a second coproduct denoted $\Delta^{*}$ and the counit $\epsilon$ used to state [9, Theorem 5.1], we rewrote formulas (6.1) and (6.2) using $M(\epsilon, v)=v$, $d \epsilon=c$ and $\operatorname{Pyr}(\epsilon)=1$.

Equations (6.1) and (6.2), together with the obvious

$$
\begin{equation*}
M(u, v)=M(v, u) \tag{6.3}
\end{equation*}
$$

the initial condition

$$
\begin{equation*}
M(1,1)=c \tag{6.4}
\end{equation*}
$$

and the obvious

$$
\begin{equation*}
\operatorname{Pyr}(u)=M(1, u) \tag{6.5}
\end{equation*}
$$

allow to compute the function $M(u, v)$ in a recursive fashion.

## 7. Special cases

In this section we compute the $c d$-indices of the poset of intervals and of the interval transform of the second kind of two special posets: the "ladder" poset $L_{n}$ and the Boolean algebra $P([1, n])$ of rank $n$.

The poset $L_{n}$ has exactly 2 elements: $-i$ and $i$ for each rank $i$ satisfying $0<i<n+1$, and any pair of elements at different ranks are comparable. The poset $L_{2}$ of rank 3 is shown in Figure 7. To simplify our notation in the proof of Theorem 7.1 below, we write the unique minimum element of $L_{n}$ as 0 and the unique maximum element as $n+1$. It is well, known that $\Psi_{L_{n}}(c, d)=c^{n}$.


Figure 7. The "ladder" poset $L_{2}$ of rank 3
The following result is due to Jojić [24, Theorem 9]
Theorem 7.1 (Jojić). Assume that the finite vector $\left(k_{0}, \ldots, k_{r}\right)$ of nonnegative integers satisfies $2 r+k_{0}+k_{2}+\cdots+k_{r}=n$. Then the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r}} d c^{k_{r}}$ in $\mathcal{I}\left(c^{n}\right)$ is $2^{r}\left(k_{1}+1\right)\left(k_{2}+1\right) \cdots\left(k_{r}+1\right)$.

Remark 7.2. As noted by Jojić, this formula is the dual of the one obtained for the other Tchebyshev transform see [19, Theorem 7.1] and [12, Corollary 6.6] (see also [19, Table 1], although the two poset operations are very different. This observation also suggests that, when we comparing it to the Tchebyshev transform, one would want to consider the dual of the poset of intervals, ordered by reverse inclusion.

Next we compute $I_{2}\left(c^{n}\right)$. To do so, the following consequence of Equation (6.1) will be useful:

$$
\begin{equation*}
M\left(c^{i}, c^{j+1}\right)=c^{j} \cdot d \cdot c^{i}+M\left(c^{i}, c^{j}\right) \cdot c+2 \sum_{k=1}^{i} M\left(c^{k-1}, c^{j}\right) \cdot d \cdot c^{i-k} . \tag{7.1}
\end{equation*}
$$

Lemma 7.3. The expressions $M\left(c^{i}, c^{j}\right)$ satisfy the recurrence

$$
M\left(c^{i+1}, c^{j+1}\right)=\left(M\left(c^{i}, c^{j+1}\right)+M\left(c^{i+1}, c^{j}\right)\right) c+M\left(c^{i}, c^{j}\right) \cdot\left(2 d-c^{2}\right)
$$

for $i, j \geq 0$
Proof. Replacing $i$ with $i+1$ in (7.1) yields

$$
\begin{align*}
M\left(c^{i+1}, c^{j+1}\right) & =c^{j} \cdot d \cdot c^{i+1}+M\left(c^{i+1}, c^{j}\right) \cdot c+2 \sum_{k=1}^{i} M\left(c^{k-1}, c^{j}\right) \cdot d \cdot c^{i+1-k}  \tag{7.2}\\
& +2 M\left(c^{i}, c^{j}\right) \cdot d
\end{align*}
$$

By multiplying both sides of (7.1) by $c$ on the right we obtain

$$
\begin{equation*}
M\left(c^{i}, c^{j+1}\right) c=c^{j} \cdot d \cdot c^{i+1}+M\left(c^{i}, c^{j}\right) \cdot c^{2}+2 \sum_{k=1}^{i} M\left(c^{k-1}, c^{j}\right) \cdot d \cdot c^{i+1-k} . \tag{7.3}
\end{equation*}
$$

The statement now follows after subtracting (7.3) from (7.2.).
Using Lemma 7.3 it is easy to show the following statement. Recall that a Delannoy path is a lattice path consisting of East steps $(1,0)$, North steps $(0,1)$ and Northeast steps $(1,1)$.

Theorem 7.4. $M\left(c^{i}, c^{j}\right)$ is the half of the total weight of all Delannoy paths from $(-1,0)$ or $(0,-1)$ to $(i, j)$ where each East step and North step has weight $c$ and each Northeast step has weight $\left(2 d-c^{2}\right)$. The weight of each Delannoy path is obtained by multiplying the weight of its steps, left to right, in the order from $(-1,0)$ or $(0,-1)$ to $(i, j)$.
Proof. Let us define the function $\widetilde{M}(i, j)$ as follows:

$$
\widetilde{M}(i, j)= \begin{cases}\frac{1}{2} c^{j} & \text { if } i=-1 \text { and } j \geq 0 \\ \frac{1}{2} c^{i} & \text { if } j=-1 \text { and } i \geq 0 \\ M\left(c^{i}, c^{j}\right) & \text { if } i, j \geq 0\end{cases}
$$

Note that the function $\widetilde{M}(i, j)$ is defined for all pairs of integers $(i, j)$ satisfying $i, j \geq$ -1 , except for $i=j=-1$. It suffices to show that the value of $\widetilde{M}(i, j)$ may be computed as half of the total weight of the Delannoy paths stated above.

This statement is certainly true for $\widetilde{M}(i,-1)$ for $i \geq 0$ : there is no Delannoy path from $(-1,0)$ to $(i,-1)$ and the only Delannoy path from $(0,-1)$ to $(i,-1)$ is the lattice
path consisting of $i$ East steps. Similarly, there is only one Delannoy path form $(-1,0)$ to $(-1, j)$, consisting of $j$ North steps and the statement holds for $\widetilde{M}(-1, j)$.

Observe next that there are exactly two Delannoy paths from $(-1,0)$ or $(0,-1)$ to $(0,0)$ : the first consists of a single East step the second consists of a single North step, their total weight is $M(1,1)=c$, as expected.

Next we show the validity of the statement for $\widetilde{M}(i, 0)$ when $i>0$. Note that the last step of any Delannoy path ending at $(i, 0)$ is either an East step from $(i-1,0)$ or a North step from $(i,-1)$ or a Northeast step from $(i-1,-1)$. We want to show that

$$
\begin{gathered}
\widetilde{M}(i, 0)=\widetilde{M}(i-1,0) \cdot c+\widetilde{M}(i,-1) \cdot c+\widetilde{M}(i-1,-1) \cdot\left(2 d-c^{2}\right), \quad \text { that is, } \\
M\left(c^{i}, 1\right)=M\left(c^{i-1}, 1\right) \cdot c+\frac{1}{2} \cdot c^{i+1}+c^{i-1} \cdot d-\frac{1}{2} \cdot c^{i+1},
\end{gathered}
$$

which is equivalent to

$$
M\left(c^{i}, 1\right)=M\left(c^{i-1}, 1\right) \cdot c+c^{i-1} \cdot d
$$

This last equation is a direct consequence of (6.3) and (6.1). The proof of the statement for $\widetilde{M}(0, j)$ when $j>0$ is completely analogous.

It remains to show the statement when both $i$ and $j$ are positive. For these the last step of every Delannoy path ending at $(i, j)$ is either an East step from $(i-1, j)$ or a North step from $(i, j-1)$ or a Northeast step from $(i-1, j-1)$. The statement is a direct consequence of Lemma 7.3 .

Recall that Stanley [31] introduced $e=a-b$ and noted that the existence of the $c d$ index of an Eulerian poset is equivalent to stating that the $a b$-index is a polynomial of $c$ and $e^{2}=c^{2}-2 d$. In terms of the resulting ce-index, Theorem 7.4 may be restated as follows.

Theorem 7.5. The coefficient of $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots e^{2} c^{k_{r}}$ in $M\left(c^{i}, c^{j}\right)$ is

$$
\frac{(-1)^{r}}{2} \cdot\binom{i+j+2-2 r}{i+1-r}
$$

if $k_{0}+k_{1}+\cdots+k_{r}+2 r=i+j+1$ and 0 otherwise.
Proof. By Theorem 7.4, a lattice path from $(-1,0)$ or $(0,-1)$ to $(i, j)$ contributes a term $(-1)^{r} / 2 \cdot c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots e^{2} c^{k_{r}}$ exactly when there are $r$ Northeast steps, there are $k_{0}$ North or East steps before the first Northeast step, $k_{r}$ North or East steps after the last Northeast step and there are exactly $k_{i}$ Northeast steps between the $i$ th and $(i+1)$ st Northeast step for $i=1, \ldots, r-1$. Hence each term contributed must satisfy $k_{0}+k_{1}+\cdots+k_{r}+2 r=i+j+1$.

Let us count first the number of lattice paths from $(-1,0)$ to $(i, j)$ contributing a term $(-1)^{r} / 2 \cdot c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots e^{2} c^{k_{r}}$. The parameters $i$ and $j$ must satisfy $i+1 \geq r$ and $j \geq r$, as each Northeast step increases both coordinates by 1 . Out of the $i+j+1-2 r$ North or East steps we must select $i+1-r$ East steps and $j-r$ North steps. This may be performed $\binom{i+j+1-2 r}{i+1-r}$ ways.

Similarly, the number of lattice paths from $(0,-1)$ to $(i, j)$ contributing a term $(-1)^{r} / 2 \cdot c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots e^{2} c^{k_{r}}$ is $\binom{i+j+1-2 r}{j+1-r}=\binom{i+j+1-2 r}{i-r}$. The stated result follows by Pascal's identity.

Remark 7.6. It is worth pointing out that the coefficient of $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots e^{2} c^{k_{r}}$ in $M\left(c^{i}, c^{j}\right)$ depends only on $i, j$ and $r$. For a fixed expression $M\left(c^{i}, c^{j}\right)$ the coefficient of a $c e$-word depends only on the number of factors $e^{2}$ in it.

Remark 7.7. For the somewhat similar diamond product, N.B. Fox gave a more general lattice path interpretation [16, Theorem 5.4]. It would be interesting to see whether a similar approach could also help express $M(u, v)$ in general as a total weight of lattice paths.

Using Theorem 7.5 we may express $I_{2}\left(c^{n}\right)$ as follows.
Proposition 7.8. Assume that the finite vector $\left(k_{0}, \ldots, k_{r}\right)$ of nonnegative integers satisfies $2 r+k_{0}+k_{2}+\cdots+k_{r}=n-1$. Then the coefficient of $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r-1}} e^{2} c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$, written as a ce-polynomial, is $(-1)^{r} \cdot 2^{n+1-2 r}$.
Proof. Observe first that by the definition of the coproduct, the relation $\Delta(c)=2 \cdot 1 \otimes 1$ and by Theorem 5.7 we have

$$
\mathcal{I}_{2}\left(c^{n}\right)=2 \cdot c^{n}+2 \sum_{i=0}^{n-1} M\left(c^{i}, c^{n-1-i}\right)
$$

For positive $r$, by Theorem 7.5 we get that the coefficient of $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r-1}} e^{2} c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$ is

$$
\sum_{i=r-1}^{n-r}(-1)^{r}\binom{n+1-2 r}{i+1-r}
$$

and the result follows by the binomial theorem. For $r=0$, we must take into account the term $2 c^{n}$ in front of the sum of terms of the form $M\left(c^{i}, c^{n-1-i}\right)$ and we must also note the summation limits. We obtain that the coefficient of $c^{n}$ in $I_{2}\left(c^{n}\right)$ is

$$
2+\sum_{i=0}^{n-1}\binom{n+1}{i+1}=2+2^{n+1}-2=2^{n+1}
$$

Theorem 7.9. Assume that the finite vector $\left(k_{0}, \ldots, k_{r}\right)$ of nonnegative integers satisfies $2 r+k_{0}+k_{2}+\cdots+k_{r}=n-1$. Then the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$, written as a cd-polynomial, is $2^{r+1}\left(k_{0}+1\right)\left(k_{1}+1\right) \cdots\left(k_{r}+1\right)$.

Proof. We may obtain the $c d$-index by substituting $e^{2}=c^{2}-2 d$ into the $c e$-index. Hence the $c d$ word $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$ is contributed by all ce-words that are obtained from $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r}} e^{2} c^{k_{r}}$ by replacing some factors $c^{2}$ by $e^{2}$. By Proposition 7.8, the coefficient of $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r}} e^{2} c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$ is $(-1)^{r} \cdot 2^{n+1-2 r}$, but when we replace all factors $e^{2}$ in this word by $\left(c^{2}-2 d\right)$, we have to multiply by $(-2)^{r}$. Hence, the contribution of the term $(-1)^{r} \cdot 2^{n+1-2 r} \cdot c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r}} e^{2} c^{k_{r}}$ to the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$ is $2^{n+1-r}$.

When we replace any factor $c^{2}$ in $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r}} e^{2} c^{k_{r}}$ by $e^{2}$, the coefficient of the resulting ce-word gets changed by a factor of $\left(-2^{-2}\right)=-1 / 4$. These additional factors $e^{2}$ contribute a factor of 1 when we replace $e^{2}$ with $\left(c^{2}-2 d\right)$ and consider the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$. To add up the contribution of all these other terms consider
first the special case when we compute the coefficient $\gamma_{n}$ of $c^{n}$ in $I_{2}\left(c^{n}\right)$, written as a $c d$-polynomial. For example, for $n=4$, by Proposition 7.8 we have

$$
I_{2}\left(c^{4}\right)=2^{5} \cdot c^{4}-2^{3} \cdot\left(c^{2} e^{2}+c e^{2} c+e^{2} c^{2}\right)+2^{1} \cdot e^{4},
$$

and if we rewrite this as a $c d$-polynomial, using $e^{2}=c^{2}-2 d$, we obtain

$$
\gamma_{4}=2^{5}-2^{3} \cdot 3+2^{1} \cdot 1=10
$$

For general $n$ we obtain

$$
\gamma_{n}=\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^{r} \cdot 2^{n+1-2 r} \cdot\binom{n-r}{n-2 r}
$$

It is easy to show (using for example the Fibonacci-type recurrence $\gamma_{n}=2 \gamma_{n-1}-\gamma_{n-2}$ ) that $\gamma_{n}=2(n+1)$. Let us compute next the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$ in $I_{2}\left(c^{n}\right)$, written as a $c d$-monomial. As noted above, the rewriting the term $(-1)^{r} \cdot 2^{n+1-2 r}$. $c^{k_{0}} e^{2} c^{k_{1}} e^{2} \cdots c^{k_{r}} e^{2} c^{k_{r}}$ contributes $2^{n+1-r}$ to the coefficient of $c^{k_{0}} d c^{k_{1}} d \cdots c^{k_{r-1}} d c^{k_{r}}$. To obtain the contribution of the other $c e$-terms, we may repeat the above reasoning to each factor $c^{k_{i}}$ for $i=0,1, \ldots, r$. We obtain the coefficient

$$
2^{n+1-r} \frac{\gamma_{k_{0}}}{2^{k_{0}+1}} \frac{\gamma_{k_{1}}}{2^{k_{1}+1}} \cdots \frac{\gamma_{k_{r}}}{2^{k_{1}+1}}=2^{n+1-r} \frac{\left(k_{0}+1\right)\left(k_{1}+1\right) \cdots\left(k_{r}+1\right)}{2^{n-2 r}} .
$$

As we have seen in Proposition 3.2, the order complex of the poset of intervals $\widehat{I}(P([1, n]))$ of the Boolean algebra $P([1, n])$ contains the type $B$ coxeter complex. Furthermore, the following statement is well-known.

Lemma 7.10. The poset of intervals $\widehat{I}(P([1, n]))$ of the Boolean algebra $P([1, n])$ is isomorphic to the face lattice $C_{n}$ of the $n$-dimensional cube.

Indeed, we may identify each vertex $\left(x_{1}, \ldots, x_{n}\right)$ of the standard cube $[0,1]^{n}$ with the subset $\sigma=\left\{i \in[1, n]: x_{i}=1\right\}$ of $[1, n]$. Each interval $[\sigma, \tau]$ corresponds to the face containing all vertices $\left(x_{1}, \ldots, x_{n}\right)$ satisfying $x_{i}=0$ for $i \in[1, n]-\tau$ and $x_{i}=1$ for $i \in \sigma$.

The $c d$-index of the cubical lattice has been expressed by Hetyei [18] and by Ehrenborg and Readdy [11] in terms of (different) signed generalizations of André-permutations. André permutations, first studied by Foata, Strehl and Schützenberger [14, 15] were used by Purtill [29] to express the $c d$-index of the Boolean algebra.

Purtill's approach may also be used to compute the interval transform of the second kind $I_{2}(P([1, n]))$ of a Boolean algebra, because of the following observation: the set of all faces containing a vertex of an $n$-dimensional hypercube, ordered by inclusion, form a lattice that is isomorphic to the Boolean algebra $P([1, n])$. (In other words, the link of a vertex in a cube is a simplex.) Hence we obtain

$$
\begin{equation*}
\Psi_{I_{2}(P([1, n]))}(c, d)=2^{n} \cdot \Psi_{P([1, n])}(c, d) \tag{7.4}
\end{equation*}
$$

Remark 7.11. Equation 7.4 exhibits a remarkable analogy to a result of Ehrenborg and Readdy [12, Theorem 10.10] completely describing all eigenvectors of the Tchebyshev transform of the second kind, discussed in their paper.

## 8. Concluding Remarks

It would be desirable to find more explicit formulas describing the $c d$-index of a graded poset of intervals of an Eulerian poset, but this seems harder than for the Tchebyshev transform studied in [19], [20] and [12]. The source of all difficulties seems that the operator $\iota$ recursively "rotates" the words involved: the recurrences call for cutting off certain initial segment of some words and placing their reverse at the end. That said, generalizations of permutohedra abound, and performing an analogous sequence of stellar subdivisions on their duals, respectively taking the graded poset of intervals for an associated poset may result in interesting geometric constructions, producing perhaps new type $B$ analogues. A first step in this direction may be found in the work of Athanasiadis [4] where the $r$-fold edgewise subdivision of the barycentric subdivision of a simplex is considered. Finally, applying the Tchebyshev transform studied in [19], [20] and [12] to a Boolean algebra creates a poset whose order complex has the same face numbers as the dual of a type $B$ permutohedron. It may be interesting to find out whether the resulting polytope also has a nice geometric representation.

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