A Bijection Between Two Different Classes of Partitions Enumerated by $p_{\nu}(n)$

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Abstract

In this paper, we give a purely bijective proof that two different partition classes that are both combinatorial interpretations of the partition function $p_{\nu}(n)$, a partition function related to the third order mock theta function $\nu(q)$, are equinumerous. In doing so, we give a partial solution to a combinatorial problem proposed in a paper by Andrews.

1 Introduction and Notation

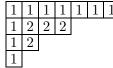
Consider the third order mock theta function $\nu(q)$, which was first defined by Watson [4] and may be defined as follows:

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},\tag{1}$$

where the q-Pochhammer symbol $(a;q)_n$ is defined as usual

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$
(2)

The partition function $p_{\nu}(n)$ may be defined as the partition function for which $\nu(-q)$ is the generating function, and a number of combinatorial interpretations have been given for this partition function. Among these is the number of self-conjugate odd Ferrers graphs of 2n+1 and the number of self-conjugate partitions of 4n+1 into odd parts [2], [3]. Odd Ferrers graphs, introduced by Andrews in [1], may be defined as Ferrers graphs in which a 2 is placed in every box, except the surrounding border, where 1s are placed in each box. For example, the following odd Ferrers graph represents the partition 7 + 7 + 3 + 1:



. Let \mathcal{O}_{2n+1} be the set of self-conjugate odd Ferrers graphs for 2n+1, let \mathcal{S}_{4n+1} be the set of selfconjugate partitions of 4n+1 into odd parts, let $\mathcal{O} = \bigcup_{n>0} \mathcal{O}_{2n+1}$ and let $\mathcal{S} = \bigcup_{n>0} \mathcal{S}_{4n+1}$. The following theorem has previously been proven through non-bijective means [2]:

Theorem 1 $|O_{2n+1}| = |S_{4n+1}|$ for all *n*.

We will give a purely bijective proof of this theorem by describing a bijection ϕ such that $\phi(\lambda) = \mu$, where λ and μ are both partitions, $\lambda \in \mathcal{O}_{2n+1}$, and $\mu \in \mathcal{S}_{4n+1}$, and use the case where $\lambda = 3 + 5 + 3$, representable as the following odd Ferrers graph:

1	1	1
1	2	2
1	2	

as an example (Note that in this example case, $\lambda \in \mathcal{O}_{11}$, and that $\mu \in \mathcal{S}_{21}$). In doing so, we give a partial solution to the combinatorial challenge proposed by Andrews [2] asking for bijections between the various classes of partitions enumerated by $p_{\nu}(n)$.

2 A Bijection Between \mathcal{O}_{2n+1} and \mathcal{S}_{4n+1}

Consider the fact that the Ferrers diagrams of self-conjugate partitions may be thought of as being made up of "hooks" of other self-conjugate partitions in which every part other than the greatest part is equal to 1. For example, the Ferrers digram of the self-conjugate partition 4 + 4 + 2 + 2

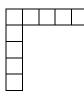


can be thought of as consisting of the following "hooks":

and

. Let h_i denote the *i*th "hook" in a self-conjugate partition π , where i > 0. Note that, where $|\pi|$ may denote the sum of the parts of π , where $|h_i|$ may denote the sum of the parts in each hook in the Ferrers digram of π , and where *n* may denote the number of hooks in π , that $\sum_{i=1}^{n} |h_i| = |\pi|$. Additionally, for $\lambda \in \mathcal{O}$, let $t = \sum_{i=2}^{n} |h_i|$, or the sum of the 2s in the odd Ferrers diagram. We will distinguish between the hooks in λ and the hooks in μ by using h_i to denote the *i*th hook in the former and η_i to denote the *i*th hook in the latter. The map $\phi(\lambda) = \mu$ may be described as follows:

Step 1: Create η_1 by creating a hook with the largest part equal to $|h_1|$. Note that $|\eta_1| = 2|h_1| - 1$. For the example case for λ given above, η_1 would be the following:



Step 2: For each h_i where i > 1, create η_{2i-2} such that $|\eta_{2i-2}| = |h_i| + 1$, and η_{2i-1} such that $|\eta_{2i-1}| = |h_i| - 1$. For example, in the example case of λ given above, $|h_2| = 6$, so we create η_2 and η_3 such that $|\eta_2| = 7$ and $|\eta_3| = 5$, and since the number of hooks in λ is equal to 2, the creation of these hooks completes the bijection resulting in the following partition:



or 5 + 5 + 5 + 3 + 3. The map described evidently always results in a self-conjugate partition. The map described also always results in a partition of 4n+1, because in creating η_1 we create a partition of size $2h_1 - 1$, and in adding every η_i such that i > 1, we add 2t to this partition, thus making a partition of size $2(h_1 + t) - 1$. We know that $h_1 + t = |\lambda| = 2n + 1$, so substituting 2n + 1 for $h_1 + t$ in the previous expression reveals that the sum of the parts in the newly created partition is always equal to 4n + 1. Additionally, we know that the newly created partition is always a partition into odd parts because it always creates a partition in which the greatest part of η_1 is odd, the number of hooks is odd, and in which the greatest part of each hook alternates in parity, where the greatest part of η_{2i-2} is always one greater than the greatest part of η_{2i-1} . The inverse map is obvious, so ϕ is a bijection, and thus $|\mathcal{O}_{2n+1}| = |\mathcal{S}_{4n+1}|$ for all n.

3 Further Remarks

Recall the natural bijection that exists between the class of self-conjugate partitions of n and the class of partitions of n into distinct odd parts that maps a self-conjugate partition onto a partition into distinct odd parts by making the sum of the parts in each of the hooks in the self conjugate partition into a part in the newly created partition. Where \mathcal{D}_{2n+1} may denote the set of partitions of 2n + 1into distinct parts in which there is 1 odd part which is greater than half the greatest even part and every other part is even and is of the form 4k + 2 where $k \in \mathbb{N}$, and where \mathcal{DO}_{4n+1} may denote the set of partitions of 4n + 1 into an odd number of distinct odd parts such that, when ordered from largest to smallest, the parts alternate between being of the form 4k + 1 and being of the form 4k + 3 where again $k \in \mathbb{N}$, an analogous bijection exists between \mathcal{O}_{2n+1} and \mathcal{D}_{2n+1} and \mathcal{DO}_{4n+1} .

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