# A Bijection Between Two Different Classes of Partitions Enumerated by $p_{\nu}(n)$ 

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#### Abstract

In this paper, we give a purely bijective proof that two different partition classes that are both combinatorial interpretations of the partition function $p_{\nu}(n)$, a partition function related to the third order mock theta function $\nu(q)$, are equinumerous. In doing so, we give a partial solution to a combinatorial problem proposed in a paper by Andrews.


## 1 Introduction and Notation

Consider the third order mock theta function $\nu(q)$, which was first defined by Watson [4] and may be defined as follows:

$$
\begin{equation*}
\nu(q):=\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{\left(-q ; q^{2}\right)_{n+1}}, \tag{1}
\end{equation*}
$$

where the q -Pochhammer symbol $(a ; q)_{n}$ is defined as usual

$$
\begin{equation*}
(a ; q)_{n}:=\prod_{k=0}^{n-1}\left(1-a q^{k}\right) \tag{2}
\end{equation*}
$$

The partition function $p_{\nu}(n)$ may be defined as the partition function for which $\nu(-q)$ is the generating function, and a number of combinatorial interpretations have been given for this partition function. Among these is the number of self-conjugate odd Ferrers graphs of $2 n+1$ and the number of selfconjugate partitions of $4 n+1$ into odd parts [2], [3]. Odd Ferrers graphs, introduced by Andrews in [1], may be defined as Ferrers graphs in which a 2 is placed in every box, except the surrounding border, where 1s are placed in each box. For example, the following odd Ferrers graph represents the partition $7+7+3+1$ :

. Let $\mathcal{O}_{2 n+1}$ be the set of self-conjugate odd Ferrers graphs for $2 n+1$, let $\mathcal{S}_{4 n+1}$ be the set of selfconjugate partitions of $4 n+1$ into odd parts, let $\mathcal{O}=\cup_{n>0} \mathcal{O}_{2 n+1}$ and let $\mathcal{S}=\cup_{n>0} \mathcal{S}_{4 n+1}$. The following theorem has previously been proven through non-bijective means [2]:

Theorem $1\left|\mathcal{O}_{2 n+1}\right|=\left|\mathcal{S}_{4 n+1}\right|$ for all $n$.

We will give a purely bijective proof of this theorem by describing a bijection $\phi$ such that $\phi(\lambda)=\mu$, where $\lambda$ and $\mu$ are both partitions, $\lambda \in \mathcal{O}_{2 n+1}$, and $\mu \in \mathcal{S}_{4 n+1}$, and use the case where $\lambda=3+5+3$, representable as the following odd Ferrers graph:

| 1 | 1 | 1 |
| :--- | :--- | :--- |
| 1 | 2 | 2 |
| 1 | 2 |  |

as an example (Note that in this example case, $\lambda \in \mathcal{O}_{11}$, and that $\mu \in \mathcal{S}_{21}$ ). In doing so, we give a partial solution to the combinatorial challenge proposed by Andrews [2] asking for bijections between the various classes of partitions enumerated by $p_{\nu}(n)$.

## 2 A Bijection Between $\mathcal{O}_{2 n+1}$ and $\mathcal{S}_{4 n+1}$

Consider the fact that the Ferrers diagrams of self-conjugate partitions may be thought of as being made up of "hooks" of other self-conjugate partitions in which every part other than the greatest part is equal to 1 . For example, the Ferrers digram of the self-conjugate partition $4+4+2+2$

can be thought of as consisting of the following "hooks":

and

. Let $h_{i}$ denote the $i$ th "hook" in a self-conjugate partition $\pi$, where $i>0$. Note that, where $|\pi|$ may denote the sum of the parts of $\pi$, where $\left|h_{i}\right|$ may denote the sum of the parts in each hook in the Ferrers digram of $\pi$, and where $n$ may denote the number of hooks in $\pi$, that $\sum_{i=1}^{n}\left|h_{i}\right|=|\pi|$. Additionally, for $\lambda \in \mathcal{O}$, let $t=\sum_{i=2}^{n}\left|h_{i}\right|$, or the sum of the 2 s in the odd Ferrers diagram. We will distinguish between the hooks in $\lambda$ and the hooks in $\mu$ by using $h_{i}$ to denote the $i$ th hook in the former and $\eta_{i}$ to denote the $i$ th hook in the latter. The map $\phi(\lambda)=\mu$ may be described as follows:

Step 1: Create $\eta_{1}$ by creating a hook with the largest part equal to $\left|h_{1}\right|$. Note that $\left|\eta_{1}\right|=2\left|h_{1}\right|-1$. For the example case for $\lambda$ given above, $\eta_{1}$ would be the following:


Step 2: For each $h_{i}$ where $i>1$, create $\eta_{2 i-2}$ such that $\left|\eta_{2 i-2}\right|=\left|h_{i}\right|+1$, and $\eta_{2 i-1}$ such that $\left|\eta_{2 i-1}\right|=\left|h_{i}\right|-1$. For example, in the example case of $\lambda$ given above, $\left|h_{2}\right|=6$, so we create $\eta_{2}$ and $\eta_{3}$ such that $\left|\eta_{2}\right|=7$ and $\left|\eta_{3}\right|=5$, and since the number of hooks in $\lambda$ is equal to 2 , the creation of these hooks completes the bijection resulting in the following partition:

or $5+5+5+3+3$. The map described evidently always results in a self-conjugate partition. The map described also always results in a partition of $4 n+1$, because in creating $\eta_{1}$ we create a partition of size $2 h_{1}-1$, and in adding every $\eta_{i}$ such that $i>1$, we add $2 t$ to this partition, thus making a partition of size $2\left(h_{1}+t\right)-1$. We know that $h_{1}+t=|\lambda|=2 n+1$, so substituting $2 n+1$ for $h_{1}+t$ in the previous expression reveals that the sum of the parts in the newly created partition is always equal to $4 n+1$. Additionally, we know that the newly created partition is always a partition into odd parts because it always creates a partition in which the greatest part of $\eta_{1}$ is odd, the number of hooks is odd, and in which the greatest part of each hook alternates in parity, where the greatest part of $\eta_{2 i-2}$ is always one greater than the greatest part of $\eta_{2 i-1}$. The inverse map is obvious, so $\phi$ is a bijection, and thus $\left|\mathcal{O}_{2 n+1}\right|=\left|\mathcal{S}_{4 n+1}\right|$ for all $n$.

## 3 Further Remarks

Recall the natural bijection that exists between the class of self-conjugate partitions of $n$ and the class of partitions of $n$ into distinct odd parts that maps a self-conjugate partition onto a partition into distinct odd parts by making the sum of the parts in each of the hooks in the self conjugate partition into a part in the newly created partition. Where $\mathcal{D}_{2 n+1}$ may denote the set of partitions of $2 n+1$ into distinct parts in which there is 1 odd part which is greater than half the greatest even part and every other part is even and is of the form $4 k+2$ where $k \in \mathbb{N}$, and where $\mathcal{D} \mathcal{O}_{4 n+1}$ may denote the set of partitions of $4 n+1$ into an odd number of distinct odd parts such that, when ordered from largest to smallest, the parts alternate between being of the form $4 k+1$ and being of the form $4 k+3$ where again $k \in \mathbb{N}$, an analogous bijection exists between $\mathcal{O}_{2 n+1}$ and $\mathcal{D}_{2 n+1}$ and between $\mathcal{S}_{4 n+1}$ and $\mathcal{D} \mathcal{O}_{4 n+1}$. Thus, the bijection given above induces one between $\mathcal{D}_{2 n+1}$ and $\mathcal{D} \mathcal{O}_{4 n+1}$.

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## References

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