# Mixed Moments for the Product of Ginibre Matrices 

Nick Halmagyi* ${ }^{\text {4 }}$, Shailesh Lal<br>( LPTHE, CNRS $\mathcal{G}$ Sorbonne Université, 4 Place Jussieu, F-75252, Paris, France<br>halmagyi@lpthe.jussieu.fr<br>, Faculdade de Ciencias, Universidade do Porto, 687 Rua do Campo Alegre, Porto 4169-007, Portugal.<br>slal@fc.up.pt


#### Abstract

We study the ensemble of a product of $n$ complex Gaussian i.i.d. matrices. We find this ensemble is Gaussian with a variance matrix which is averaged over a multi-Wishart ensemble. We compute the mixed moments and find that at large $N$, they are given by an enumeration of non-crossing pairings weighted by Fuss-Catalan numbers.


## 1 Introduction

The theory of random matrices (RMT) has a remarkably broad reach through physics, mathematics, biology, computer science and engineering. It is uncanny how fundamental contributions to RMT are regularly made by researchers in each of these domains. In the current work, with a view towards studying feedforward neural networks, we draw from results in these fields and study the mixed moments of products of complex i.i.d. Gaussian matrices ${ }^{11}$.

A defining feature of a Ginibre matrix, certainly in contrast to Hermitian matrices, is that they cannot be diagonalized; under the Schur decomposition of a complex matrix

$$
\begin{equation*}
X=U(\Lambda+T) U^{-1}, \quad \Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots \lambda_{N}\right), \quad U \in U(N) \tag{1.1}
\end{equation*}
$$

there is a non-trivial upper triangular component $T$ unless $X$ is Hermitian. Recall that diagonalization is at the heart of many results in Hermitian RMT [2], it allows a problem which a priori has $O\left(N^{2}\right)$ degrees of freedom to be reduced to an $O(N)$ computation. Ginibre managed to circumvent this obstacle by exploiting the fact that in the computation of the eigenvalue spectrum, the components of $T$ are decoupled Gaussian degrees of freedom and can be integrated. However the presence of $O\left(N^{2}\right)$ coupled degrees of freedom is not necessarily something one should shy away from; while the Hermitian matrix models have a single operator for a fixed number of insertions, the number of operators in the Ginibre ensemble grows exponentially with the number of insertions. It is thus evident that the space of Ginibre matrices is far richer that the space of Hermitian matrices and one should expect crucial features to be unattainable through methods which reduce the problem to $O(N)$ degrees of freedom.

There are numerous ways to generalize the Ginibre ensemble, two ideas which influenced the current work are:

1. Relax the i.i.d. condition while preserving the Gaussian structure of the distribution.
2. Consider products of Ginibre matrices.

To some extent both these generalizations are designed to produce solvable random matrix ensembles while remaining within, or at least in the vicinity of, the safe harbours of Gaussian ensembles. However we will in fact find, that products of Ginibre matrices remain Gaussian but are no longer identically distributed.

One way to relax explicitly the i.i.d. condition is to introduce a fixed variance matrix (or matrices), for exampl ${ }^{2}$ 2

$$
\begin{equation*}
P_{\Sigma}(X) \sim e^{-N \operatorname{Tr} X \Sigma^{-1} X^{\dagger}} \tag{1.3}
\end{equation*}
$$

and then when deploying Wick's theorem one uses the propogator

$$
\begin{equation*}
\left\langle X_{i j} X_{k l}^{\dagger}\right\rangle=\frac{1}{N} \delta_{i l} \Sigma^{j k} \tag{1.4}
\end{equation*}
$$

[^0]This particular deformation of the Ginibre ensemble has the effect of decorating the graphs which contribute to the moments of $X, X^{\dagger}$ with a product of traces of powers of $\Sigma$ and the precise enumeration is challenging. Nonetheless there are numerous results regarding computations of moments and limiting eigenvalue distributions reviewed in the nice books [5] 8], which we will use throughout this work.

The ensemble of products of random matrices has been an attractive research area for some time, much of the history is summarised nicely in the thesis [9]. In the last decade, there has however been a snowballing of results regarding this ensemble, yet strangely enough, this does not appear to be due to any particularly new technique which has come onling ${ }^{3}$ but rather it has been motivated by various potential applications. In that vein, we were drawn to this topic through the study of simple models of feed forward neural networks 12 and while the techniques we use are not particularly new, we certainly draw upon many recent results. We will find that the probability distribution of the product of Ginibre matrices is Gaussian but not identically distributed. The product structure introduces a variance matrix which is averaged (annealed) over a Wishart ensemble. Averaging over Gaussian random couplings is a feature of the SherringtonKirkpatrick model $[13$ and more recently the SYK model $[14-16]$, to the best of our knowledge, averaging over the Wishart ensemble is novel.

The connection between RMT and neural networks has a long history; the parameters of a feed forward neural network are neatly packaged into a chain of matrices which certainly hints that RMT could be useful a technique but it is probably fair to say that the current state of the art in this line of research is somewhat short of achieving practical applications with competitive precision. We have been particularly influenced by the recent works [12, 17, 20] but drawing a more precise bridge between our current results and neural networks will require further consideration. Since the number of active degrees of freedom in feed forward neural networks in principle scales greater than $O(N)$, we are particularly interested in results which go beyond the $O(N)$ degrees of freedom one has available in studies of eigenvalues and singular values. So whilst there has been impressive progress in understanding the eigenvalue spectrum of the product of $n$ Ginibre matrices, both at large $N[21]$ and finite $N[22,23]$ and also the singular values [24 26$]$ we would like to study the mixed moments in the product Gaussian ensemble.

Perhaps ones first reaction should be somewhat sceptical, these moments have yet to be computed even within the Ginibre ensemble [27] and as such it might seem inconceivable that this could then be accomplished in the product ensemble. While this is indeed true, we are optimistic that this problem can be overcome in the Ginibre ensemble and as additional motivation we ultimately find that the mixed moments in the product ensemble are closely related to the mixed moments in the Ginibre ensemble.

The structure of this paper is as follows: in section 2 we review some known facts about moments of complex Gaussian matrices, with and without a variance profile. In section 3 we review some basic and motivational results regarding the eigenvalue and singular value spectrum of the product ensemble. In section 4 we begin our study of the product Ginibre ensemble and derive the probability distribution function as well as our formula for the mixed moments for the case $n=2$. In section 5 we generalize our results to any positive integer $n$ and present our results for

[^1]the mixed moments. The main result of our paper is (5.18).

## 2 Moments of complex Gaussian matrices

### 2.1 The Ginibre model

The main difficulty encountered in computing expectation values of functions of Ginibre matrices $X$ is the strictly upper triangular term $T$ in the Schur decomposition (1.1). Of course when $X$ is Hermitian, $T$ vanishes and it follows that in many such cases the computation reduces to an integral over the $N$ eigenvalues $\Lambda$. The result of Ginibre [1], in computing the eigenvalue pdf of a complex Gaussian matrix, is an example of a scenario when the integral over the $O\left(N^{2}\right)$ upper triangular terms are all Gaussian and can be performed explicitly. In general however, even at large $N$, moments of Ginibre matrices cannot seemingly be reduced to a problem of $O(N)$ degrees of freedom as the components of $T$ couple non-trivally to the components of $\Lambda$. Nonetheless, there are still numerous exactly solvable results in the field of random complex matrices and we now mention a few which we have found to be relevant to the current work.

Expectation values of ratios of products of characteristic polynomials operators $4^{4}$

$$
\begin{equation*}
\left\langle\frac{\prod_{i=1}^{L_{1}} \operatorname{det}\left(\epsilon_{1}-X\right) \prod_{i=1}^{L_{2}} \operatorname{det}\left(\epsilon_{2}-X^{\dagger}\right)}{\prod_{i=1}^{M_{1}} \operatorname{det}\left(\eta_{1}-X\right) \prod_{i=1}^{M_{2}} \operatorname{det}\left(\eta_{2}-X^{\dagger}\right)}\right\rangle_{G} \tag{2.1}
\end{equation*}
$$

have been progressively computed in [28 32$]$ and the structure of these results follows closely the work done in the Hermitian case [33,34]. From these results one can derive expectation values for multi-trace operators of the form

$$
\begin{equation*}
\left\langle\operatorname{Tr} X^{k_{1}} \operatorname{Tr}\left(X^{\dagger}\right)^{l_{1}} \ldots \operatorname{Tr} X^{k_{n}} \operatorname{Tr}\left(X^{\dagger}\right)^{l_{n}}\right\rangle_{G} \tag{2.2}
\end{equation*}
$$

but not non-trivial expectation values of single trace operators, since the expectation values of (anti)-holomorphic single trace operators vanish by Wick's theorem:

$$
\begin{equation*}
\left\langle\operatorname{Tr} X^{k}\right\rangle_{G}=\left\langle\operatorname{Tr}\left(X^{\dagger}\right)^{k}\right\rangle_{G}=0 . \tag{2.3}
\end{equation*}
$$

Indeed, the non-trivial single-trace correlators in the Ginibre ensemble are of the form

$$
\begin{equation*}
C_{\mathbf{i}, \mathbf{j}}=\left\langle\mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right)\right\rangle_{G} \tag{2.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{O}_{\mathrm{i}, \mathbf{j}}\left(X, X^{\dagger}\right)=\operatorname{Tr} X^{i_{1}}\left(X^{\dagger}\right)^{j_{1}} \ldots X^{i_{k}}\left(X^{\dagger}\right)^{j_{k}} \tag{2.5}
\end{equation*}
$$

and

$$
\begin{align*}
\mathbf{i} & =\left(i_{1}, \ldots, i_{k}\right), \quad \mathbf{j}=\left(j_{1}, \ldots, j_{k}\right)  \tag{2.6}\\
\sum_{a} i_{a} & =\sum_{a} j_{a} \equiv m \tag{2.7}
\end{align*}
$$

[^2]At large $N$, to leading order, Wick's Theorem reduces the computation of $C_{\mathrm{i}, \mathrm{j}}$ to the enumeration of non-crossing pairings [35] and the state of the art regarding their enumeration can be found in [27,36]. The complete enumeration of non-crossing pairings and thus the evaluation of mixed moments in the Ginibre ensemble is currently unsolved, but for example in [27] one can find the following explicit result for a subset of mixed moment $5^{5}$

$$
\begin{equation*}
\left\langle\operatorname{Tr}\left(X^{s}\left(X^{\dagger}\right)^{s}\right)^{r}\right\rangle_{G}=F C_{s}(r) . \tag{2.8}
\end{equation*}
$$

which are the Fuss-Catalan numbers (see appendix $\bar{B}$ ). When $s=1$, these moments are equivalent to moments of the Marchenko-Pastur distribution of singular values of $X$ and are equal to the Catalan numbers.

### 2.2 Enumerating operators in the Ginibre model

It is interesting to count the number of operators (2.5) in the Ginibre model with $m$ insertions of both $X$ and $X^{\dagger}$ and see the growth for large $m$. The counting of operators in the product Ginibre model is identical. Since the trace induces an invariance under cyclic rotation of the $X, X^{\dagger}$ insertions, this is equivalent to the number of 2-ary necklaces with $m$-beads of each color [38] and one can compute this from the cycle index for the cyclic group [39]:

$$
\begin{equation*}
Z_{C_{2 m}}=\frac{1}{2 m} \sum_{d \mid 2 m} \varphi(d) a_{d}^{2 m / d} \tag{2.9}
\end{equation*}
$$

where $\varphi(d)$ is Eulers totient function. We can extract the number of necklaces with $m$ beads of each color from (2.9) by extracting the coefficient of $x^{m} y^{m}$ from (2.9) with $a_{d}$ replaced by $x^{d}+y^{d}$ :

$$
\begin{align*}
N_{2 m} & =\left.\frac{1}{2 m} \sum_{d \mid 2 m} \varphi(d)\left(x^{d}+y^{d}\right)^{2 m / d}\right|_{x^{m} y^{m}}  \tag{2.10}\\
& =\frac{1}{2 m} \sum_{d \mid m} \varphi(d)\binom{2 m / d}{m / d} \tag{2.11}
\end{align*}
$$

The largest contribution to $N_{2 m}$ comes from term $d=1$ and this term alone gives the bound

$$
\begin{equation*}
N_{2 m} \geq \frac{1}{2 m}\binom{2 m}{m} \tag{2.12}
\end{equation*}
$$

which for large $m$ approximates the Catalan number $C_{m}$ and grows as

$$
\begin{equation*}
C_{m} \sim \frac{4^{m}}{\pi^{1 / 2} m^{3 / 2}} \tag{2.13}
\end{equation*}
$$

From this counting we quantify the obvious fact that the number of inequivalent operators in the Ginibre model scales much larger than the Hermitian model. We see this as motivation for studying these moments in more detail.

[^3]
### 2.3 Non-trivial variance profile

An interesting direction of research is to generalize the Ginibre ensemble such that elements in $X$ are no longer i.i.d. but rather there is a non-trivial variance profile ${ }^{6}$. When this variance profile is set by some fixed, Hermitian positive definite matrix $W$ we denote these moments by ${ }^{7}$

$$
\begin{align*}
\widetilde{C}_{\mathbf{i}, \mathbf{j}}(W) & =\left\langle\mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X W^{1 / 2}, W^{1 / 2} X^{\dagger}\right)\right\rangle_{G}  \tag{2.15}\\
& =\frac{1}{N^{m}} \sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}}\left[N^{m-k} \tilde{c}_{\mathbf{i}, \mathbf{j}}\left(m_{1}, \ldots, m_{k}\right)\right] \operatorname{Tr} W^{m_{1}} \ldots \operatorname{Tr} W^{m_{k}} \tag{2.16}
\end{align*}
$$

for some set of coefficients $\tilde{c}_{\mathbf{i}, \mathbf{j}}$ and where we have defined the summation

$$
\begin{equation*}
\sum_{\mathbf{m} \in \mathcal{M}_{m, k}}=\sum_{\substack{m_{1}+\ldots+m_{k}=m \\ m_{1} \leq m_{2} \leq \ldots m_{k}}} . \tag{2.17}
\end{equation*}
$$

We have introduced factors of $N$ such that the $\tilde{c}_{\mathrm{i}, \mathrm{j}}$ are finite at large $N$. We note that (2.16) defines the coefficients $\tilde{c}_{\mathrm{i}, \mathrm{j}}$ and is exact in $N$. In this work we will need one particular result found in [5, 40, 41] where these authors have considered the expectation value of the operator

$$
\begin{equation*}
\mathcal{O}_{1^{m}, 1^{m}}\left(X W^{1 / 2}\right) \tag{2.18}
\end{equation*}
$$

and found that to leading order in $\frac{1}{N}$

$$
\begin{equation*}
\tilde{c}_{1^{m}, 1^{m}}\left(m_{1}, \ldots, m_{k}\right)=\frac{m!}{(m-k+1)!f\left(m_{1}, \ldots, m_{k}\right)} \tag{2.19}
\end{equation*}
$$

where

$$
\begin{equation*}
f\left(m_{1}, \ldots, m_{k}\right)=\prod_{j=1}^{m} f_{j}! \tag{2.20}
\end{equation*}
$$

and $f_{j}$ is the number of elements of $\left[m_{1}, \ldots, m_{k}\right]$ which equal $j$. In finite $N$ there will be corrections to $\tilde{c}_{1^{m}, 1^{m}}\left(m_{1}, \ldots, m_{k}\right)$ from non-planar diagrams.

We note that while in the absence of the variance profile $W$, the order at which a particular ribbon diagram contributes is dictated by its Euler characteristic [2, 42], the combinatorics giving rise to (2.19) goes beyond enumerating the number of faces, edges and vertices.

## 3 Product of Ginibre matrices

Much is known about the eigenvalue and singular value spectrum of the product of Ginibre matrices and we summarize some of these results here.

[^4]
### 3.1 Eigenvalues

Let $\left\{A_{i} \mid i=1, \ldots n\right\}$ be $N \times N$ Ginibre matrices where the real and imaginary parts of each entry are sampled from $\mathcal{N}\left(0, \frac{\sigma_{i}^{2}}{2 N}\right)$ so that

$$
\begin{equation*}
\left.\left\langle A_{i, j k}\right\rangle=0,\left.\quad\langle | A_{i, j k}\right|^{2}\right\rangle=\frac{\sigma_{i}^{2}}{N} \tag{3.1}
\end{equation*}
$$

or more precisely

$$
\begin{equation*}
P\left(A_{i}\right)=\left(\frac{N}{\pi \sigma_{i}^{2}}\right)^{N^{2}} \int d A_{i} d A_{i}^{\dagger} e^{-\frac{N}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} \tag{3.2}
\end{equation*}
$$

We want to study the matrix product of the $A_{i}$

$$
\begin{equation*}
X_{(n)}=A_{1} \ldots A_{n} \tag{3.3}
\end{equation*}
$$

The fundamental result in the theory of random complex matrices is due to Ginibre [1] and known as the circular law; to leading order in an expansion in $\frac{1}{N}$, the eigenvalues of a complex Gaussian $N \times N$ matrix are uniformly distributed on the disk. This has been extended to $X_{(n)}$ in 8 [21. This remarkably simple result for the distribution of eigenvalues of $X_{n}$ at large $N$ is

$$
\rho_{n}(z, \bar{z})= \begin{cases}\frac{1}{n \pi} \sigma^{-\frac{2}{n}}|z|^{-2+\frac{2}{n}} & |z| \leq \sigma  \tag{3.4}\\ 0 & |z|>\sigma\end{cases}
$$

and one can see in figure 1 that the for $n>1$, the eigenvalues are no longer uniformly distributed but are more dense near the origin.

At finite $N$, the eigenvalue pdf of $X_{(n)}$ has been computed in [23] using an inspired matrix factorization of $X_{(n)}$ and the large $N$ limit is shown to agree with (3.4).

(a) Eigenvalue distribution for $X=A_{1}$

(b) Eigenvalue distribution for $X=A_{1} A_{2}$

(c) Eigenvalue distribution for

$$
X=A_{1} A_{2} A_{3}
$$

Figure 1: The distribution of eigenvalues, computed for $N=2000$

[^5]
### 3.2 Singular values

The probability density of singular values of $X_{(n)}$ has been computed at large $N$ using a variety of methods $24,45,48]$ and at finite in $N$ in [26], once again using an ingenious matrix factorization and orthogonal polynomials. The moments $\left\langle\left(X_{(n)} X_{(n)}^{\dagger}\right)^{m}\right\rangle$ are integrals of this density of singular values and were in fact computed somewhat earlier in $24,45-47$. The upshot for the current work is that the moments are given by Fuss-Catalan numbers (see appendix B)

$$
\begin{equation*}
\frac{1}{N}\left\langle\left(X_{(n)} X_{(n)}^{\dagger}\right)^{m}\right\rangle=F C_{n}(m) \tag{3.5}
\end{equation*}
$$

Interestingly, comparing (2.8) we see the following equality

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr}\left(X_{(n)} X_{(n)}^{\dagger}\right)^{m}\right\rangle=\frac{1}{N}\left\langle\operatorname{Tr}\left(X_{(1)}^{n}\left(X_{(1)}^{n}\right)^{\dagger}\right)^{m}\right\rangle \tag{3.6}
\end{equation*}
$$

so that these moments are equal if we sample the $A_{i}$ identically, which is certainly not true of more general mixed moments.

## 4 Product of two Ginibre matrices

We will first derive our results for $n=2$, then in section 5 we will work with $n>2$, where additional features arise.

To derive the probability distribution function (pdf) of $X=A_{1} A_{2}$, we use a matrix delta function:

$$
\begin{equation*}
P_{(2)}(X, X)=\left(\frac{N^{2}}{\pi^{2} \sigma_{1}^{2} \sigma_{2}^{2}}\right)^{N^{2}} \int \mathcal{D} A \delta\left(X-A_{1} A_{2}\right) e^{-\frac{N}{\sigma_{1}^{2}} \operatorname{Tr} A_{1} A_{1}^{\dagger}-\frac{N}{\sigma_{2}^{2}} \operatorname{Tr} A_{2} A_{2}^{\dagger}} \tag{4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{D} A=\prod_{i=1}^{2} d A_{i} d A_{i}^{\dagger} \tag{4.2}
\end{equation*}
$$

Using the integral form of $\delta(X)$

$$
\begin{equation*}
\delta(X)=\frac{1}{\pi^{2 N^{2}}} \int d T d T^{\dagger} e^{i \operatorname{Tr} T X+i \operatorname{Tr} T^{\dagger} X^{\dagger}} \tag{4.3}
\end{equation*}
$$

we complete the square for $A_{2}$ and integrate it out to find

$$
\begin{align*}
P_{(2)}\left(X, X^{\dagger}\right) & =\left(\frac{N}{\pi^{3} \sigma_{2}^{2}}\right)^{N^{2}} \int d^{2} T d^{2} A_{1} e^{i \operatorname{Tr} T X+i \operatorname{Tr} T^{\dagger} X^{\dagger}} e^{-\frac{N}{\sigma_{1}^{2}} \operatorname{Tr} A_{1} A_{1}^{\dagger}\left(\mathbb{1}+\frac{\sigma^{2}}{N^{2}} T^{\dagger} T\right)}  \tag{4.4}\\
& =\frac{1}{\pi^{2 N^{2}}} \int d T d T^{\dagger} \frac{e^{i \operatorname{Tr} T X+i \operatorname{Tr} T^{\dagger} X^{\dagger}}}{\operatorname{det}\left(1+\frac{\sigma^{2}}{N^{2}} T^{\dagger} T\right)^{N}} \tag{4.5}
\end{align*}
$$

where we have defined

$$
\begin{equation*}
\sigma=\sigma_{1} \sigma_{2} \tag{4.6}
\end{equation*}
$$

We define the moment generating function to be

$$
\begin{equation*}
\varphi_{(2)}\left(T, T^{\dagger}\right)=\pi^{2 N^{2}} \int d T d T^{\dagger} e^{-i \operatorname{Tr} T X-i \operatorname{Tr} T^{\dagger} X^{\dagger}} P\left(X, X^{\dagger}\right) \tag{4.7}
\end{equation*}
$$

so we can generate the moments by derivatives on $\varphi\left(T, T^{\dagger}\right)$

$$
\begin{equation*}
\left\langle X_{i_{1} \bar{\imath}_{1}} \ldots X_{i_{m} \bar{\imath}_{m}} X_{\bar{\jmath}_{1} j_{1}}^{\dagger} X_{\bar{\jmath}_{n} j_{n}}^{\dagger}\right\rangle=\left.(-i)^{m+n} \partial_{T_{i_{1} \bar{\imath}_{1}}} \ldots \partial_{T_{i_{m} \bar{\imath}_{m}}} \partial_{T_{\bar{J}_{1} j_{1}}^{\dagger}} \ldots \partial_{T_{\bar{\jmath}_{n} j_{n}}^{\dagger}} \phi\left(T, T^{\dagger}\right)\right|_{T=T^{\dagger}=0} \tag{4.8}
\end{equation*}
$$

From (4.5) we have an explicit expression

$$
\begin{equation*}
\varphi_{(2)}\left(T, T^{\dagger}\right)=\frac{N^{N^{2}}}{\Gamma_{N}(N)} \int_{W>0} d W e^{-N \operatorname{Tr} W} e^{-\frac{\sigma^{2}}{N} \operatorname{Tr} T^{\dagger} T W} \tag{4.9}
\end{equation*}
$$

where we are using

$$
\begin{equation*}
\frac{1}{\operatorname{det} \Sigma^{a}}=\frac{1}{\Gamma_{N}(a)} \int_{W>0} d W \operatorname{det} W^{a-N} e^{-\operatorname{Tr} \Sigma W} \tag{4.10}
\end{equation*}
$$

The expression (4.10) gives the determinant of an $N \times N$ positive definite Hermitian matrix in terms of the complex multivariable gamma function $\Gamma_{N}(a)$ (see appendix Dfor details) and might be thought of as a multi-variable Schwinger trick. The $W$ appearing in Equation 4.10) is scaled by $N$ to obtain (4.9), and the integration is over the space of positive definite Hermitian matrices. We can then perform the Fourier transform of the moment generating function and integrate out $T$ to obtain an integral expression for the pdf of $X$ :

$$
\begin{equation*}
P_{(2)}\left(X, X^{\dagger}\right)=\frac{N^{N^{2}}}{\Gamma_{N}(N)}\left(\frac{N}{\pi \sigma^{2}}\right)^{N^{2}} \int_{W>0} d W \frac{e^{-N \operatorname{Tr} W}}{\operatorname{det} W^{N}} e^{-\frac{N}{\sigma^{2}} \operatorname{Tr} X X^{\dagger} W^{-1}} \tag{4.11}
\end{equation*}
$$

It is straightforward to verify that this pdf is normalized to 1 .

### 4.1 Moments in the product Ginibre model

Our expression for the pdf 4.11) might be recognized as a matrix Bessel function but we will find it more effective to view it as Gaussian in $X, X^{\dagger}$ with a variance matrix which is averaged over a Wishart measure. So we see that, as mentioned in the introduction, the product of two Ginibre matrices remains Gaussian but is no longer identically distributed.

We can compute the mixed moments of $X_{(2)}$

$$
\begin{align*}
C_{\mathbf{i}, \mathbf{j}}^{(2)} & =\left\langle\mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right)\right\rangle_{(2)}  \tag{4.12}\\
& =\frac{N^{N^{2}}}{\Gamma_{N}(N)}\left(\frac{N}{\pi \sigma^{2}}\right)^{N^{2}} \int_{W>0} d W \frac{e^{-N \operatorname{Tr} W}}{\operatorname{det} W^{N}} \int d^{2} X \mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right) e^{-\frac{N}{\sigma^{2}} \operatorname{Tr} X X^{\dagger} W^{-1}} \tag{4.13}
\end{align*}
$$

in two steps:

1. Compute the corresponding moment $\widetilde{C}_{\mathrm{i}, \mathrm{j}}(W)$ in the Ginibre model

$$
\begin{equation*}
\widetilde{C}_{\mathbf{i}, \mathbf{j}}(W)=\left(\frac{N}{\pi \sigma^{2}}\right)^{N^{2}} \frac{1}{\operatorname{det} W^{N}} \int d^{2} X \mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right) e^{-\frac{N}{\sigma^{2}} \operatorname{Tr} X X^{\dagger} W^{-1}} \tag{4.14}
\end{equation*}
$$

in terms of products of traces of $W$.
2. Average the resulting traces against the Wishart measur $\underbrace{9}$

$$
\begin{equation*}
P_{W}(W)=\frac{N^{N^{2}}}{\Gamma_{N}(N)} e^{-N \operatorname{Tr} W} \tag{4.15}
\end{equation*}
$$

It follows from Wick's theorem that there are $m=\sum_{a} i_{a}=\sum_{a} j_{a}$ insertions of $W$ but determining the trace structure in 2.16 by evaluating the coefficients $\tilde{c}_{\mathrm{i}, \mathrm{j}}$ is harder than computing $C_{\mathrm{i}, \mathrm{j}}$ which is already unsolved in the general case. The result from the first step is of the form

$$
\begin{equation*}
\widetilde{C}_{\mathbf{i}, \mathbf{j}}(W)=\frac{\sigma^{2 m}}{N^{m}} \sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}}\left[N^{m-k} \tilde{c}_{\mathbf{i}, \mathbf{j}}\left(m_{1}, \ldots, m_{k}\right)\right] \operatorname{Tr} W^{m_{1}} \ldots \operatorname{Tr} W^{m_{k}} \tag{4.16}
\end{equation*}
$$

The second step of averaging over the Wishart distribution is, to leading order in $N$, quite straightforward. The multi-trace expectation value factorizes into a product of single trace expectation values

$$
\begin{equation*}
\frac{1}{N^{k}}\left\langle\operatorname{Tr} W^{m_{1}} \ldots \operatorname{Tr} W^{m_{k}}\right\rangle_{W}=\frac{1}{N^{k}}\left\langle\operatorname{Tr} W^{m_{1}}\right\rangle_{W} \ldots\left\langle\operatorname{Tr} W^{m_{k}}\right\rangle_{W} \tag{4.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W^{l}\right\rangle=C_{l} \tag{4.18}
\end{equation*}
$$

where $C_{l}$ are the Catalan numbers. This enables us to write the following expression, valid in the large $N$ limit:

$$
\begin{equation*}
C_{\mathbf{i}, \mathbf{j}}^{(2)}=\sigma^{2 m} \sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}} \tilde{c}_{\mathbf{i}, \mathbf{j}}\left(m_{1}, \ldots, m_{k}\right) C_{m_{1}} \ldots C_{m_{k}}, \tag{4.19}
\end{equation*}
$$

where $\tilde{c}_{\mathrm{i}, \mathrm{j}}\left(m_{1}, \ldots, m_{k}\right)$ should be computed by enumerating particular planar diagrams. As remarked previously, this enumeration has been carried out in the literature for certain operators $\mathcal{O}\left(X, X^{\dagger}\right)$.

### 4.2 Examples

Here we compute by hand some mixed moments of $X_{(2)}, X_{(2)}^{\dagger}$ at leading order in $\frac{1}{N}$ using our prescription as well as some examples of the computation of 4.16) at finite $N 4.3$. We use Wick's theorem with the propagator

$$
\begin{equation*}
{\widetilde{X_{i j}}}_{k l}^{\dagger}=\frac{\sigma^{2}}{N} \delta_{i l} W_{j k} \tag{4.20}
\end{equation*}
$$

to evaluate the expectation value $(4.14$ in the Ginibre ensemble. In this way for example one finds

$$
\begin{equation*}
\widetilde{C}_{\mathbf{1}, \mathbf{1}}(W)=\frac{1}{N} \overleftarrow{X}_{i j} X_{j i}^{\dagger}=\frac{\sigma^{2}}{N} \operatorname{Tr} W \tag{4.21}
\end{equation*}
$$

[^6]
### 4.2.1 $\frac{1}{N}\left\langle\operatorname{Tr}\left(X_{(2)} X_{(2)}^{\dagger}\right)^{m}\right\rangle$

As mentioned in section 3.2 , expectation values of $\left\langle\operatorname{Tr}\left(X_{(2)} X_{(2)}^{\dagger}\right)^{m}\right\rangle$ are already known from the study of singular values of $X_{(n)}$. Here we will re-compute them as an exposition and check of our formalism.
$\mathbf{m}=1$
From (4.16) we find

$$
\begin{equation*}
\frac{1}{\sigma^{2}} \widetilde{C}_{1,1}(W)=\frac{1}{N} \tilde{c}_{1,1}(1) \operatorname{Tr} W \tag{4.22}
\end{equation*}
$$

with

$$
\begin{equation*}
\tilde{c}_{1,1}(1)=1 \tag{4.23}
\end{equation*}
$$

and thus $\widetilde{C}_{1,1}(W)=\sigma^{2} \operatorname{Tr} W$ and trivially agrees with 4.21. We then have

$$
\begin{align*}
\frac{1}{\sigma^{2}} C_{1,1}^{(2)} & =\tilde{c}_{1,1}(1) C_{1}  \tag{4.24}\\
& =1 \tag{4.25}
\end{align*}
$$

This agrees with (3.5) since $F C_{2}(1)=1$.
$\mathrm{m}=2$
From (4.16) we find

$$
\begin{equation*}
\frac{1}{\sigma^{4}} \widetilde{C}_{1^{2}, 1^{2}}(W)=\frac{1}{N} \tilde{c}_{1^{2}, 1^{2}}(2) \operatorname{Tr} W^{2}+\frac{1}{N^{2}} \tilde{c}_{1^{2}, 1^{2}}(1,1)(\operatorname{Tr} W)^{2} \tag{4.26}
\end{equation*}
$$

where from (2.19) we have

$$
\begin{equation*}
\tilde{c}_{1^{2}, 1^{2}}(2)=1, \quad \tilde{c}_{1^{2}, 1^{2}}(1,1)=1 . \tag{4.27}
\end{equation*}
$$

It might be instructive to be painfully explicit and demonstrate how (4.26) arises from non-crossing pairings, the two diagrams are in figures 2. The Ginibre moment with explicit indices is

$$
\begin{equation*}
\frac{1}{\sigma^{4}} \widetilde{C}_{1^{2}, 1^{2}}(W)=\frac{1}{N}\left\langle X_{i j} X_{j k}^{\dagger} X_{k l} X_{l i}^{\dagger}\right\rangle \tag{4.28}
\end{equation*}
$$

which by Wick's theorem gives

$$
\begin{align*}
\frac{1}{\sigma^{4}} \widetilde{C}_{1^{2}, 1^{2}}(W) & =\frac{1}{N^{3}} \delta_{i k} W_{j j} \delta_{k i} W_{l l}+\frac{1}{N^{3}} \delta_{i i} W_{j l} \delta_{k k} W_{l j}  \tag{4.29}\\
& =\frac{1}{N^{2}}(\operatorname{Tr} W)^{2}+\frac{1}{N} \operatorname{Tr} W^{2} \tag{4.30}
\end{align*}
$$

With red dots representing $X$ and blue dots representing $X^{\dagger}$, the diagram 2a gives the first term in (4.30) and 2 b gives the second. This example is slightly too simple in that all pairings are
non-crossing but this is most certainly not true in the general case. We then obtain $C_{1^{2}, 1^{2}}^{(2)}$ by averaging over the Wishart ensemble

$$
\begin{align*}
\frac{1}{\sigma^{4}} C_{1^{2}, 1^{2}}^{(2)} & =\frac{1}{N^{2}}\langle\operatorname{Tr} W\rangle^{2}+\frac{1}{N}\left\langle\operatorname{Tr} W^{2}\right\rangle  \tag{4.31}\\
& =C_{1}^{2}+C_{2}  \tag{4.32}\\
& =3 \tag{4.33}
\end{align*}
$$

which agrees with (3.5) since $F C_{2}(2)=3$.

(a) Pairing which gives $\operatorname{Tr} W^{2}$

(b) Pairing which gives $(\operatorname{Tr} W)^{2}$

Figure 2: Pairings for $\left\langle X_{(2)} X_{(2)}^{\dagger} X_{(2)} X_{(2)}^{\dagger}\right\rangle$, both are non-crossing

## $\mathbf{m}=\mathbf{3}$

We will present one more example in full detail as it exhibits some richer features than $m=2$ above. The Ginibre moment is given by

$$
\begin{align*}
\frac{1}{\sigma^{6}} \widetilde{C}_{1^{3}, 1^{3}}(W)= & \frac{1}{N}\left\langle\operatorname{Tr} X X^{\dagger} X X^{\dagger} X X^{\dagger}\right\rangle  \tag{4.34}\\
= & \frac{1}{N}\left\langle X_{i j} X_{j k}^{\dagger} X_{k l} X_{l m}^{\dagger} X_{m n} X_{n i}^{\dagger}\right\rangle  \tag{4.35}\\
= & \frac{1}{N^{4}} W_{j j} \delta_{i k}\left(W_{l l} \delta_{k m} W_{n n} \delta_{m i}+W_{l n} \delta_{k i} W_{n l} \delta_{m m}\right) \\
& +\frac{1}{N^{4}} W_{j l} \delta_{i m}\left(W_{l j} \delta_{k k} W_{n n} \delta_{m i}+W_{l n} \delta_{k i} W_{n j} \delta_{m k}\right) \\
& +\frac{1}{N^{4}} W_{j n} \delta_{i i}\left(W_{l l} \delta_{k m} W_{n j} \delta_{m k}+W_{l j} \delta_{k k} W_{n l} \delta_{m m}\right)  \tag{4.36}\\
= & \frac{1}{N^{3}}(\operatorname{Tr} W)^{3}+\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2} \\
& +\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2}+\frac{1}{N^{3}} \operatorname{Tr} W^{3} \\
& +\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2}+\frac{1}{N} \operatorname{Tr} W^{3} \tag{4.37}
\end{align*}
$$

In figures 3 we have the pairings which give rise to (4.37), hopefully it is clear from their arrangement which diagram corresponds to which contractions. We find that the trace structure which weights each diagram is not obvious from the diagram itself, in particular it is a finer structure than enumerating vertices, edges and faces on its corresponding fatgraph.

(a) Pairing which gives $\frac{1}{N^{3}}(\operatorname{Tr} W)^{3}$

(c) Pairing which gives $\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2}$

(e) Pairing which gives $\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2}$

(b) Pairing which gives $\frac{1}{N^{2}} \operatorname{Tr} W \operatorname{Tr} W^{2}$

(d) Pairing which gives the subleading

(f) Pairing which gives $\frac{1}{N} \operatorname{Tr} W^{3}$

Figure 3: Pairings which arise from $\frac{1}{N}\left\langle\operatorname{Tr}\left(X X^{\dagger}\right)^{3}\right\rangle$. Diagram 3d is crossing and thus contributes only at subleading order.

We have from (4.16)

$$
\begin{equation*}
\frac{1}{\sigma^{6}} C_{1^{3}, 1^{3}}^{(2)}=\frac{1}{N} \tilde{c}_{1^{3}, 1^{3}}(3) \operatorname{Tr} W^{3}+\frac{1}{N^{2}} \tilde{c}_{1^{3}, 1^{3}}(1,2) \operatorname{Tr} W \operatorname{Tr} W^{2}+\frac{1}{N^{3}} \tilde{c}_{1^{3}, 1^{3}}(1,1,1)(\operatorname{Tr} W)^{3} \tag{4.38}
\end{equation*}
$$

and obtain the large $N$ results from (2.19)

$$
\begin{equation*}
\tilde{c}_{1^{3}, 1^{3}}(1,1,1)=1, \quad \tilde{c}_{1^{3}, 1^{3}}(1,2)=3, \quad \tilde{c}_{1^{3}, 1^{3}}(3)=1 . \tag{4.39}
\end{equation*}
$$

We can see from (4.37) that $\tilde{c}_{1^{3}, 1^{3}}(1,1,1)$ counts the number of diagrams which give $(\operatorname{Tr} W)^{3}$, $\tilde{c}_{1^{3}, 1^{3}}(1,2)$ counts the diagrams which give $\operatorname{Tr} W \operatorname{Tr} W^{2}$ and $\tilde{c}_{1^{3}, 1^{3}}(3)$ counts the diagrams which give $\operatorname{Tr} W^{3}$. We finally have

$$
\begin{align*}
\frac{1}{\sigma^{6}} C_{1^{2}, 1^{2}}^{(2)} & =C_{3}+3 C_{2}+C_{1}  \tag{4.40}\\
& =12 \tag{4.41}
\end{align*}
$$

which agrees with (3.5) since $F C_{2}(3)=12$.

### 4.2.2 $\frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle$

This family of moments is manageable by hand since there is a single non-crossing pairing (see figure (4) and this evaluates to

$$
\begin{align*}
\frac{1}{\sigma^{2 m}} C_{m, m}^{(2)} & =\frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle  \tag{4.42}\\
& =\frac{1}{N^{m}}\langle\operatorname{Tr} W\rangle^{m}  \tag{4.43}\\
& =1 \tag{4.44}
\end{align*}
$$



Figure 4: The sole non-crossing pairing for $\left\langle X_{(2)}^{6}\left(X_{(2)}^{\dagger}\right)^{6}\right\rangle$

### 4.2.3 $\quad \frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m} X_{(2)}^{\dagger} X_{(2)}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle$

This family of moments is also manageable by hand since there are just two non-crossing pairings (see figures 5) and we find that this evaluates to


Figure 5: Non-crossing pairings for $\left\langle X_{(2)}^{5} X_{(2)}^{\dagger} X_{(2)}\left(X_{(2)}^{\dagger}\right)^{5}\right\rangle$

$$
\begin{align*}
\frac{1}{\sigma^{2 m}} C_{[m-1,1],[1, m-1]}^{(2)} & =\frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m} X_{(2)}^{\dagger} X_{(2)}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle  \tag{4.45}\\
& =\frac{1}{N^{m+1}}\langle\operatorname{Tr} W\rangle^{m+1}+\frac{1}{N^{m}}\left\langle\operatorname{Tr} W^{2}\right\rangle\langle\operatorname{Tr} W\rangle^{m-1}  \tag{4.46}\\
& =3 \tag{4.47}
\end{align*}
$$

Our numerical experiments are summarized in table 1 and give excellent agreement.

| $m$ | $\frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m} X_{(2)}^{\dagger} X_{(2)}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle$ |
| :---: | :---: |
| 1 | 2.998 |
| 2 | 2.998 |
| 3 | 2.998 |
| 4 | 2.997 |

Table 1: Numerical values of $\frac{1}{N}\left\langle\operatorname{Tr} X_{(2)}^{m} X_{(2)}^{\dagger} X_{(2)}\left(X_{(2)}^{\dagger}\right)^{m}\right\rangle$ with $N=500$ and averaged from 1500 samples.

### 4.3 Computing moments at finite $N$

While we have mostly focussed computing the moments in the product model at leading order in $N$ so far, there is no conceptual difficulty in implementing our prescription at finite $N$ on a case-by-case basis.

As an illustration, we shall compute the expectation value of $\operatorname{Tr}\left(X_{(2)} X_{(2)}^{\dagger}\right)^{3}$ without invoking the large $N$ approximation. The computation in the Ginibre ensemble by enumerating all pairs, both non-crossing and crossing, has already been done in 4.37). We are therefore left with computing the expectation values in the Wishart ensemble i.e. step 2 in our prescription. A straightforward way of computing these is from the integral

$$
\begin{equation*}
\mathcal{I}(N, T)=\frac{N^{2}}{\Gamma_{N}(N)} \int_{W>0} d W e^{-\operatorname{Tr} N W T}=\frac{1}{|T|^{N}} \tag{4.48}
\end{equation*}
$$

in terms of which

$$
\begin{equation*}
\left\langle W_{i_{1} j_{1}} W_{i_{2} j_{2}} \ldots W_{i_{m} j_{m}}\right\rangle_{W}=\left.\left(-\frac{1}{N}\right)^{m} \partial_{T_{j_{1} i_{1}}} \partial_{T_{j_{2} i_{2}}} \ldots \partial_{T_{j_{m i m}}} \frac{1}{|T|^{N}}\right|_{T_{i j}=\delta_{i j}} \tag{4.49}
\end{equation*}
$$

For the moment of interest, it suffices to compute terms up to $m=3$. We find

$$
\begin{gather*}
\left\langle W_{i j}\right\rangle=\delta_{i j}, \quad\left\langle W_{i j} W_{k l}\right\rangle=\delta_{i j} \delta_{k l}+\frac{1}{N} \delta_{i l} \delta_{k j}  \tag{4.50}\\
\left\langle W_{i j} W_{k l} W_{p q}\right\rangle=\delta_{p q} \delta_{k l} \delta_{i j}+\frac{1}{N}\left(\delta_{p q} \delta_{i l} \delta_{k j}+\delta_{k q} \delta_{p l} \delta_{i j}+\delta_{k l} \delta_{i q} \delta_{p j}\right)+\frac{1}{N^{2}}\left(\delta_{i q} \delta_{p l} \delta_{k j}+\delta_{i l} \delta_{k q} \delta_{p j}\right)
\end{gather*}
$$

Hence

$$
\begin{equation*}
\frac{1}{N}\langle\operatorname{Tr} W\rangle=1, \quad \frac{1}{N}\left\langle\operatorname{Tr} W^{2}\right\rangle=2, \quad \frac{1}{N}\left\langle\operatorname{Tr} W^{3}\right\rangle=5+\frac{1}{N^{2}} \tag{4.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{N^{2}}\left\langle\operatorname{Tr} W^{2} \operatorname{Tr} W\right\rangle=2+\frac{2}{N}+\frac{2}{N^{2}} \tag{4.52}
\end{equation*}
$$

We then use these expectation values with Equation (4.37) to find that the moment in the product ensemble is given by

$$
\begin{align*}
\frac{1}{\sigma^{6}} C_{1^{3}, 1^{3}} & =1+3\left(2+\frac{2}{N}+\frac{2}{N^{2}}\right)+\frac{1}{N^{2}}\left(5+\frac{1}{N^{2}}\right)+\left(5+\frac{1}{N^{2}}\right) \\
& =12+\frac{6}{N}+\frac{12}{N^{2}}+\frac{1}{N^{4}} \simeq 12+\frac{6}{N} \tag{4.53}
\end{align*}
$$

The $O(1)$ term is 12 , the result previously obtained in the large $N$ limit in Equation (4.40).
To summarize the corrections to the large $N$ formula in this case,

1. The contribution of the crossing pair was suppressed by $\frac{1}{N^{2}}$ with respect to the non-crossing pairs in the Ginibre moment,
2. The expectation values $\frac{1}{N^{k}} \operatorname{Tr}\left\langle W^{k}\right\rangle$ were given by $C_{k}+\mathcal{O}\left(\frac{1}{N^{2}}\right)$, and,
3. We found that

$$
\begin{equation*}
\left\langle\operatorname{Tr} W^{2} \operatorname{Tr} W\right\rangle=\left\langle\operatorname{Tr} W^{2}\right\rangle\langle\operatorname{Tr} W\rangle+\mathcal{O}\left(\frac{1}{N}\right) \tag{4.54}
\end{equation*}
$$

i.e. there is a non-trivial $O\left(\frac{1}{N}\right)$ correction to large $N$ factorization in the Wishart ensemble. From this it appears that the first subleading-in- $N$ correction comes from the fact that expectation values of multi-trace operators in $W$ no longer factorize into products of expectation values of single trace operators.

## 5 Product of multiple Ginibre matrices

In this section we generalize our results from section 4 to the product of $n$ Ginibre matrices

$$
\begin{equation*}
X_{(n)}=A_{1} A_{2} \ldots A_{n} \tag{5.1}
\end{equation*}
$$

We will again be able to reduce the computation of the mixed moments (denoted by $C_{\mathrm{i}, \mathrm{j}}^{(n)}$ below) to a computation of the mixed moments in the Ginibre model with deterministic variance profile $\widetilde{C}_{\mathbf{i}, \mathbf{j}}(W)$ but whereas the Catalan numbers played a central role in 4.19), they are now replaced with the Fuss-Catalan numbers defined in Equation (B.1).

The probablility distribution function of the $X_{(n)}$ is given by

$$
\begin{equation*}
P_{(n)}\left(X, X^{\dagger}\right)=\left(\frac{N^{n}}{\pi^{n} \sigma}\right)^{N^{2}} \int \mathcal{D}^{(n)} A \delta\left(X-\prod_{i=1}^{n} A_{1}\right) e^{-\sum_{i=1}^{n} \frac{N}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} \tag{5.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma=\sigma_{1} \ldots \sigma_{n}, \quad \mathcal{D}^{(n)} A=\prod_{i=1}^{n} d A_{i} d A_{i}^{\dagger} \tag{5.3}
\end{equation*}
$$

After some algebra (see appendix ED, we find that this reduces to

$$
\begin{equation*}
P_{(n)}\left(X, X^{\dagger}\right)=\frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}}\left(\frac{N}{\pi \sigma^{2}}\right)^{N^{2}} \int_{W_{i}>0} \prod_{i=1}^{n-1}\left[d W_{i} \frac{1}{\operatorname{det} W_{i}^{N}} e^{-N \operatorname{Tr} W_{i}}\right] e^{-\frac{N}{\sigma^{2}} \operatorname{Tr} W_{(n-1)}^{-1} X X^{\dagger}}, \tag{5.4}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{(n-1)}=W_{1} \ldots W_{n-1} \tag{5.5}
\end{equation*}
$$

and the integral is over positive definite Hermitian $W_{i}$. It is again straightforward to check that the pdf (5.4) is unit normalized.

### 5.1 Mixed moments at large $N$

To obtain an expression for the moments we will work to leading order in $\frac{1}{N}$. The mixed moments in this ensemble are then given by

$$
\begin{align*}
C_{\mathbf{i}, \mathbf{j}}^{(n)} & =\left\langle\mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right)\right\rangle_{(n)}  \tag{5.6}\\
& =\frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}}\left(\frac{N}{\pi \sigma^{2}}\right)^{N^{2}} \int_{W_{i}>0} \prod_{i=1}^{n-1}\left[d W_{i} \frac{1}{\operatorname{det} W_{i}^{N}} e^{-\operatorname{Tr} W_{i}}\right] \int d^{2} X \mathcal{O}_{\mathbf{i}, \mathbf{j}}\left(X, X^{\dagger}\right) e^{-\frac{N}{\sigma^{2}} \operatorname{Tr} W_{(n-1)}^{-1} X X^{\dagger}} \tag{5.7}
\end{align*}
$$

where $\langle\cdot\rangle_{(n)}$ denotes the expectation value in the product Ginibre ensemble (5.4). Using (2.16) we can obtain from (5.7)

$$
\begin{align*}
C_{\mathbf{i}, \mathbf{j}}^{(n)} & =\left\langle\widetilde{C}_{\mathbf{i}, \mathbf{j}}\left(W_{(n-1)}\right)\right\rangle_{W, n-1}  \tag{5.8}\\
& =\frac{\sigma^{2 m}}{N^{m}} \sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}}\left[N^{m-k} \tilde{c}_{\mathbf{i}, \mathbf{j}}\left(m_{1}, \ldots, m_{k}\right)\right]\left\langle\operatorname{Tr} W_{(n-1)}^{m_{1}}\right\rangle_{W, n-1} \ldots\left\langle\operatorname{Tr} W_{(n-1)}^{m_{k}}\right\rangle_{W, n-1} \tag{5.9}
\end{align*}
$$

where $\langle\cdot\rangle_{W, n-1}$ denotes the expectation value in the multi-Wishart ensemble, whose pdf is given by

$$
\begin{equation*}
P_{W, n-1}=\frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}} \prod_{i=1}^{n-1} e^{-N \operatorname{Tr} W_{i}} \tag{5.10}
\end{equation*}
$$

To proceed, we must for all $m>0$, compute the mixed moment in the multi-Wishart ensemble to leading order in $\frac{1}{N}$ :

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1}=\frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}} \int_{W_{i}>0}\left[\prod_{i=1}^{n-1} d W_{i} e^{-N \operatorname{Tr} W_{i}}\right] \frac{1}{N} \operatorname{Tr}\left(W_{1} \ldots W_{n-1}\right)^{m} \tag{5.11}
\end{equation*}
$$

which for $n=2$ is given by the Catalan numbers. This is where the result 2.16 and (2.19) is crucial; from the form of the operator insertion, we see that we can solve this moment recursively by first treating $W_{1} \ldots W_{n-2}=W_{(n-2)}$ as a variance matrix for $X_{n-1}$ where

$$
\begin{equation*}
W_{n-1}=X_{n-1} X_{n-1}^{\dagger} \tag{5.12}
\end{equation*}
$$

So we have

$$
\begin{align*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1}= & \frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}} \int_{W_{i}>0}\left[\prod_{i=1}^{n-2} d W_{i} e^{-N \operatorname{Tr} W_{i}}\right] \\
& \times\left(\frac{N}{\pi}\right)^{N^{2}} \int d^{2} X_{n-1} e^{-N \operatorname{Tr} X_{n-1} X_{n-1}^{\dagger}} \frac{1}{N} \operatorname{Tr}\left(W_{(n-2)} X_{n-1} X_{n-1}^{\dagger}\right)^{m} \tag{5.13}
\end{align*}
$$

then using $(2.16)$ and $(2.19)$ we find the recursion relation

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1}=\sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}} \tilde{c}_{1}{ }^{m}, 1^{m}\left(m_{1}, \ldots, m_{k}\right) \frac{1}{N}\left\langle\operatorname{Tr} W_{(n-2)}^{m_{1}}\right\rangle_{W, n-2} \ldots \frac{1}{N}\left\langle\operatorname{Tr} W_{(n-2)}^{m_{k}}\right\rangle_{W, n-2} \tag{5.14}
\end{equation*}
$$

Comparing (5.14) with (5.9) for $(\mathbf{i}, \mathbf{j})=\left(1^{m}, 1^{m}\right)$ we see that

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n)}^{m}\right\rangle_{W, n}=C_{1^{m}, 1^{m}}^{(n)} \tag{5.15}
\end{equation*}
$$

We note that (5.15) gives the equality of the following moments in the multi-Gaussian ensemble (see also the equality (3.6)

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr}\left(A_{1} A_{1}^{\dagger} \ldots A_{n} A_{n}^{\dagger}\right)^{m}\right\rangle=\frac{1}{N}\left\langle\operatorname{Tr}\left(A_{1} \ldots A_{n} A_{n}^{\dagger} \ldots A_{1}^{\dagger}\right)^{m}\right\rangle \tag{5.16}
\end{equation*}
$$

Finally, from (5.15) and (3.5) we have

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n)}^{m}\right\rangle_{W, n}=F C_{n}(m) \tag{5.17}
\end{equation*}
$$

and in appendix F we have checked (5.17) explicitly for low values of $m$. Our final expression for the general mixed moment in the product Ginibre ensemble is the main result of this paper:

$$
\begin{equation*}
C_{\mathbf{i}, \mathbf{j}}^{(n)}=\sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}} \tilde{c}_{\mathrm{i}, \mathrm{j}}\left(m_{1}, \ldots, m_{k}\right) F C_{n-1}\left(m_{1}\right) \ldots F C_{n-1}\left(m_{k}\right) \tag{5.18}
\end{equation*}
$$

As a mathematical aside, we note that it follows from (5.14) and (5.17) that the Fuss-Catalan numbers satisfy the following recursion relation ${ }^{10}$

$$
\begin{equation*}
F C_{n-1}(m)=\sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}} \tilde{c}_{1^{m}, 1^{m}}\left(m_{1}, \ldots, m_{k}\right) F C_{n-2}\left(m_{1}\right) \ldots F C_{n-2}\left(m_{k}\right) \tag{5.19}
\end{equation*}
$$

with $\tilde{c}_{1^{m}, 1^{m}}$ given by 2.19 .

### 5.2 Examples

The examples from section 4.2 .1 for general $n$ are verified in appendix $F$ when we check the recursion relation (5.17). We can also compute the examples from sections 4.2 .2 and 4.2 .3 as follows (using the same diagrammatica as before):

$$
\begin{align*}
\frac{1}{\sigma^{2 m}} C_{m, m}^{(n)} & =\frac{1}{N}\left\langle\operatorname{Tr} X_{(n)}^{m}\left(X_{(n)}^{\dagger}\right)^{m}\right\rangle  \tag{5.20}\\
& =\left(\frac{1}{N}\left\langle\operatorname{Tr} W_{n-1}\right\rangle_{W, n-1}\right)^{m}  \tag{5.21}\\
& =\left(F C_{n-1}(1)\right)^{m}  \tag{5.22}\\
& =1 \tag{5.23}
\end{align*}
$$

[^7]and
\[

$$
\begin{align*}
\frac{1}{\sigma^{2 m}} C_{[m-1,1],[1, m-1]}^{(n)} & =\frac{1}{N}\left\langle\operatorname{Tr} X_{(n)}^{m} X_{(n)}^{\dagger} X_{(n)}\left(X_{(n)}^{\dagger}\right)^{m}\right\rangle  \tag{5.24}\\
& =\frac{1}{N^{m+1}}\left\langle\operatorname{Tr} W_{n-1}\right\rangle_{W, n-1}^{m+1}+\frac{1}{N^{m}}\left\langle\operatorname{Tr} W_{n-1}^{2}\right\rangle_{W, n-1}\left\langle\operatorname{Tr} W_{n-1}\right\rangle_{W, n-1}^{m-1}  \tag{5.25}\\
& =F C_{n-1}(1)^{m+1}+F C_{n-1}(2) F C_{n-1}(1)^{m-1}  \tag{5.26}\\
& =n+1 \tag{5.27}
\end{align*}
$$
\]

Our numerical experiments are summarized in tables 2 and give excellent agreement with (5.27).

| m | $\frac{1}{N}\left\langle\operatorname{Tr} X_{(3)}^{m} X_{(3)}^{\dagger} X_{(3)}\left(X_{(3)}^{\dagger}\right)^{m}\right\rangle$ | m | $\frac{1}{N}\left\langle\operatorname{Tr} X_{(4)}^{m} X_{(4)}^{\dagger} X_{(4)}\left(X_{(4)}^{\dagger}\right)^{m}\right\rangle$ |
| :---: | :---: | :---: | :---: |
| 2 | 4.000 | 2 | 4.999 |
| 3 | 4.001 | 3 | 4.998 |
| 4 | 4.004 | 4 | 4.998 |

Table 2: Numerical values of $\frac{1}{N}\left\langle\operatorname{Tr} X_{(n)}^{m} X_{(n)}^{\dagger} X_{(n)}\left(X_{(n)}^{\dagger}\right)^{m}\right\rangle$ with $N=500$ and averaged from 1500 samples.

## 6 Discussion

In this work we have studied the ensemble of products of Ginibre matrices. We have computed the probability distribution function and then the mixed moments, our main result being (5.18). The slogan we have drawn from our investigation is that the product Ginibre ensemble remains Gaussian but not i.i.d. since the elements are no longer identically distributed. Another point of view is that the variance profile is randomly sampled from the Wishart ensemble; random couplings are certainly well known from models such a the Sherrington-Kirkpatrick model [13] and more recently the SYK model [14-16] but the product Ginibre model is somewhat novel in that the random couplings are Wishart not Gaussian distributed.

There are several directions for future research, most pressing for the problems considered in this paper is the evaluation of mixed moments for complex Gaussian matrices with a deterministic variance profile. There are several closely related and slightly more general themes to explore, in particular the $\beta=1,4$ versions of the current work (which has $\beta=2$ in Dyson's classification). Certainly $\beta=1$ is necessary for applications to neural networks but our explorations so far indicate the results are essentially identical $\beta=2$. There is some rather more quantitative changes in the results for the mixed moments when considering rectangular matrices and it would be interesting to pursue this line of inquiry.

Finally, one of our central goals was to understand the first subleading correction in $\frac{1}{N}$ to ensembles of products of matrices, we have only scratched the surface of this in section 4.3 and plan to look deeper into this. From our initial investigations in section 4.3, it seems suggestive that while moments in the Ginibre ensemble are corrected only at order $\frac{1}{N^{2}}$, moments in the product Ginibre ensemble receives corrections at order $\frac{1}{N}$ coming from the expectation value of a multitrace operator in the Wishart ensemble. This would be interesting to establish as such corrections are likely more straightforward than enumerating non-planar pairings in the Ginibre ensemble.

Acknowledgements: We would like to thank Jean-Bernard Zuber for collaboration at early stages of this project and numerous discussions throughout. NH would like also to thank Satya Majumdar for useful conversations, Surya Ganguli for interesting discussions which stimulated his interest in products of random matrices and Ruth Corran for her patient explanations. SL's work is supported by the Simons Foundation grant 488637 (Simons Collaboration on the Nonperturbative bootstrap) and the project CERN/FIS-PAR/0019/2017. Centro de Fisica do Porto is partially funded by the Foundation for Science and Technology of Portugal (FCT) under the grant UID-04650-FCUP.

## Appendices

## A Notation

Our notation for various expectation values is as follows:

$$
\begin{align*}
\left\langle\mathcal{O}\left(X, X^{\dagger}\right)\right\rangle_{G} & =\int d^{2} X \mathcal{O}\left(X, X^{\dagger}\right) P\left(X, X^{\dagger}\right)  \tag{A.1}\\
\left\langle\mathcal{O}\left(X, X^{\dagger}\right)\right\rangle_{(n)} & =\int d^{2} X \mathcal{O}\left(X, X^{\dagger}\right) P_{(n)}\left(X, X^{\dagger}\right)  \tag{A.2}\\
\langle\mathcal{O}(W)\rangle_{W} & =\int d W \mathcal{O}(W) P_{W}(W)  \tag{A.3}\\
\left\langle\mathcal{O}\left(W_{1}, \ldots, W_{n}\right)\right\rangle_{W, n} & =\int \prod_{i=1}^{n} d W_{i} \mathcal{O}\left(W_{1}, \ldots, W_{n}\right) P_{W, n} \tag{A.4}
\end{align*}
$$

where the probability distributions are defined in (3.2), (5.4), (4.15), (5.10). The measure $d^{2} X$ for a complex matrix $X$ is defined by

$$
\begin{equation*}
d^{2} X=d X d X^{\dagger}=\prod_{i, j=1}^{N} \operatorname{Re}\left(X_{i j}\right) \operatorname{Im}\left(X_{i j}\right) \tag{A.5}
\end{equation*}
$$

We also denote the integral over Hermitian positive-definite matrices $W$ by

$$
\begin{equation*}
\int_{W>0} d W \tag{A.6}
\end{equation*}
$$

## B Fuss-Catalan numbers

There is a small amount of discrepancy with what are referred to as Fuss-Catalan numbers. There are references which refer to a three-parameter sequence as the Fuss-Catalan numbers, we will use the definition from section 7.5 of [49] albeit with slightly different notation:

$$
\begin{equation*}
F C_{n}(m)=\frac{1}{n m+1}\binom{(n+1) m}{m} . \tag{B.1}
\end{equation*}
$$

When $n=1$ these reduce to the Catalan numbers

$$
\begin{equation*}
F C_{1}(m)=C_{m} . \tag{B.2}
\end{equation*}
$$

The Fuss-Catalan numbers appear in numerous places in the text and we have derived a novel recursion relation which they satisfy (5.19).

## C Product of Gaussian scalars

We include here an exposition of the ensemble obtained from the product of two complex Gaussian scalars $a$ and $b$ distributed in $\mathcal{N}(0,1)$. This is a simple and well known computation but we include it here as it served to enlighten us on how to think about the probability distribution function of the product of Ginibre matrices.

Given the probability distribution function of a complex scalar in $\mathcal{N}(0,1)$, viz.

$$
\begin{equation*}
P\left(a, a^{*}\right)=\frac{1}{2 \pi} \int d^{2} a e^{-\frac{1}{2}|a|^{2}} \tag{C.1}
\end{equation*}
$$

the probability distribution function for the product ensemble is

$$
\begin{align*}
P\left(z, z^{*}\right) & =\frac{1}{4 \pi^{2}} \int d^{2} a d^{2} b \delta^{(2)}(z-a b) e^{-|a|^{2} / 2-|b|^{2} / 2} \\
& =\frac{1}{4 \pi^{4}} \int d^{2} a d^{2} b d^{2} t e^{i t(z-a b)+i t^{*}(z-a b)^{*}} e^{-\frac{1}{2}|a|^{2}-\frac{1}{2}|b|^{2}} \\
& =\frac{1}{4 \pi^{4}} \int d^{2} t e^{i t z+i t^{*} z^{*}} \int d^{2} a d^{2} b e^{-\frac{1}{2}\left(a-i 2 t^{*} b^{*}\right)\left(a^{*}-i 2 t b\right)-2 t t^{*} b b^{*}} e^{-\frac{1}{2} b b^{*}} \\
& =\frac{1}{\pi^{2}} \int d^{2} t \frac{e^{i t z+i t^{*} z^{*}}}{1+4 t t^{*}} . \tag{C.2}
\end{align*}
$$

where we now see that the moment generating function $\phi_{z}\left(t, t^{*}\right)$ is given by

$$
\begin{equation*}
\phi_{z}\left(t, t^{*}\right)=\pi^{2} \int d^{2} z e^{-i t z-i t^{*} z^{*}} P\left(z, z^{*}\right)=\frac{1}{1+4 t t^{*}}, \tag{C.3}
\end{equation*}
$$

where we have used the normalization conventions of the main text. Continuing on and using the Schwinger trick in (C.2), we get

$$
\begin{align*}
P\left(z, z^{*}\right) & =\frac{1}{\pi^{2}} \int d^{2} t \int_{0}^{\infty} d u e^{i t z+i t^{*} z^{*}} e^{-u\left(1+4 t t^{*}\right)}  \tag{C.4}\\
& =\frac{1}{2 \pi} \int d u \frac{1}{u} e^{-u-\frac{z z^{*}}{4 u}}  \tag{C.5}\\
& =\frac{1}{2 \pi} K_{0}\left(|z|^{2}\right) \tag{C.6}
\end{align*}
$$

The reader may compare Equation (C.5) derived above with the Equation (4.11) obtained in the main text, setting $N=1$ there.

## D Multi-variable gamma functions

One of the key inputs to our analysis is the expression in Equation (4.10) for the determinant of a Hermitian positive definite matrix in terms of an integral over Hermitian positive definite matrices. This follows from the definition of the complex multivariate Gamma function. In this appendix we will prove this relation and provide some additional details about this function.

The complex multivariate Gamma function is given by 50

$$
\begin{align*}
\Gamma_{N}(a) & =\int_{W=W^{\dagger}>0} d W \operatorname{det} W^{a-N} e^{-\operatorname{Tr} W}  \tag{D.1}\\
& =\pi^{N(N-1) / 2} \prod_{k=1}^{N} \Gamma(a-k+1) \tag{D.2}
\end{align*}
$$

where $\Gamma$ is the usual gamma function and $\operatorname{Re}(a)>N-1$. The integration is over $N \times N$ Hermitian positive definite matrices $W$.

We will use Equation (D.1) to provide an integral representation for the determinant of a Hermitian positive definite matrix. This may usefully be thought of as a multi-variable or matrix generalization of the Schwinger trick, viz.

$$
\begin{align*}
\Gamma(a) & =\int_{0}^{\infty} d u u^{a-1} e^{-u}=\lambda^{a} \int_{0}^{\infty} d v v^{a-1} e^{-\lambda v} \\
\Rightarrow \frac{1}{\lambda^{a}} & =\frac{1}{\Gamma(a)} \int_{0}^{\infty} d v v^{a-1} e^{-\lambda v} \tag{D.3}
\end{align*}
$$

where we defined $u=\lambda v$.
With Equation (D.3) in mind, we now return to Equation (D.1). Consider a positive definite Hermitian matrix $\Sigma$, it has a unique square root $\Sigma_{0}$ which is also positive definite Hermitian:

$$
\begin{equation*}
\Sigma_{0} \Sigma_{0}=\Sigma \tag{D.4}
\end{equation*}
$$

As in the univariate case (D.3), we do the change of variables ${ }^{11}$

$$
\begin{equation*}
W=\Sigma_{0} V \Sigma_{0}, \quad d W=\operatorname{det} W=\operatorname{det} \Sigma \operatorname{det} V, \quad(\operatorname{det} \Sigma)^{N} d V \tag{D.7}
\end{equation*}
$$

in Equation (D.1) to obtain

$$
\begin{equation*}
\Gamma_{N}(a)=(\operatorname{det} \Sigma)^{a} \int_{V=V^{\dagger}>0} d V \operatorname{det} V^{a-N} e^{-\operatorname{Tr} \Sigma V} \tag{D.8}
\end{equation*}
$$

[^8]and hence
\[

$$
\begin{equation*}
\frac{1}{\operatorname{det} \Sigma^{a}}=\frac{1}{\Gamma_{N}(a)} \int_{W>0} d W \operatorname{det} W^{a-N} e^{-\operatorname{Tr} \Sigma W} \tag{D.9}
\end{equation*}
$$

\]

where we relabeled $V$ as $W$. The integral $\int_{W>0} d W$ is the integral over the space of positive definite Hermitian $W$. This is the main identity we use in the text. It is apparent that the $N=1$ case of this identity is Equation (D.3). For this reason, we refer to this as the matrix generalization of the Schwinger trick.

## E Moment generating function for the product of multiple Gaussians

In this Appendix we will prove the Equation (5.4) in the main text for the probability distribution function for the product of $n$ Ginibre matrices by computing the moment generating function. We start with the probability distribution of the product of $n$ Ginibre matrices in the form

$$
\begin{align*}
P_{(n)} & \left(X, X^{\dagger}\right)=\left(\frac{N^{n}}{\pi^{n} \sigma^{2}}\right)^{N^{2}} \int \mathcal{D}^{(n)} A \delta\left(X-\prod_{i=1}^{n} A_{1}\right) e^{-N \sum_{i=1}^{n} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}}  \tag{E.1}\\
& =\frac{1}{\pi^{2 N^{2}}}\left(\frac{N^{n}}{\pi^{n} \sigma^{2}}\right)^{N^{2}} \int d^{2} T e^{i \operatorname{Tr} T X+i \operatorname{Tr} T^{\dagger} X^{\dagger}} \int \mathcal{D}^{(n)} A e^{-i \operatorname{Tr} T A-i \operatorname{Tr}(T A)^{\dagger}} e^{-N \sum_{i=1}^{n} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} .
\end{align*}
$$

Therefore the moment generating function is

$$
\begin{equation*}
\varphi_{(n)}\left(T, T^{\dagger}\right)=\left(\frac{N^{n}}{\pi^{n} \sigma^{2}}\right)^{N^{2}} \int \mathcal{D}^{(n)} A e^{-i \operatorname{Tr} T A-i \operatorname{Tr}(T A)^{\dagger}} e^{-N \sum_{i=1}^{n} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} \tag{E.2}
\end{equation*}
$$

We will first integrate out $A_{n}$, then $A_{n-1}$, and so on, finally integrating out $A_{1}$. Doing the $A_{n}$ and $A_{n-1}$ integrals is straightforward, and follows the $n=2$ computations. We eventually find ${ }^{12}$

$$
\begin{align*}
\varphi_{(n)}\left(T, T^{\dagger}\right)= & \left(\frac{N^{n}}{\pi^{n} \sigma^{2}}\right)^{N^{2}}\left(\frac{\pi^{2}\left(\sigma_{n} \sigma_{n-1}\right)^{2}}{N^{2}}\right)^{N^{2}} \int \mathcal{D}^{(n-2)} A e^{-N \sum_{i=1}^{n-2} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} \\
& \frac{\operatorname{det}\left(\mathbb{1}+\frac{\left(\sigma_{n-1} \sigma_{n}\right)^{2}}{N^{2}}\left|T A_{1} \ldots A_{n-2}\right|^{2}\right)^{N}}{} \\
= & \left(\frac{N^{n}}{\pi^{n} \sigma^{2}}\right)^{N^{2}}\left(\frac{\pi^{2}\left(\sigma_{n} \sigma_{n-1}\right)^{2}}{N^{2}}\right)^{N^{2}} \int \mathcal{D}^{(n-2)} A e^{-N \sum_{i=1}^{n-2} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}} \\
& \int_{W_{1}>0} d W_{1} e^{-\operatorname{Tr} W_{1}} e^{-\frac{\left(\sigma_{n-1} \sigma_{n}\right)^{2}}{N^{2}} \operatorname{Tr}\left|T A_{1} \ldots A_{n-2}\right|^{2} W_{1}} . \tag{E.4}
\end{align*}
$$

[^9]where $\left|T A_{1} \ldots A_{n-2}\right|^{2}=T A_{1} \ldots A_{n-2} A_{n-2}^{\dagger} \ldots A_{1}^{\dagger} T^{\dagger}$. We can now integrate out $A_{n-2}$ and beyond. It is useful to write
\[

$$
\begin{equation*}
\operatorname{Tr}\left|T A_{1} \ldots A_{n-2}\right|^{2} W_{1}=\operatorname{Tr} A_{n-2} A_{n-2}^{\dagger}\left|W_{1}^{1 / 2} T A_{1} \ldots A_{n-3}\right|^{2} . \tag{E.5}
\end{equation*}
$$

\]

We may now integrate out $A_{n-2}$ to find

$$
\begin{align*}
\varphi_{(n)}\left(T, T^{\dagger}\right)= & \frac{1}{\pi^{2 N^{2}}} \frac{N^{n N^{2}}}{\sigma^{2 N^{2}}}\left(\frac{\left(\sigma_{n} \sigma_{n-1} \sigma_{n-2}\right)^{2}}{N^{3}}\right)^{N^{2}} \int \mathcal{D}^{(n-3)} A e^{-N \sum_{i=1}^{n-3} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}}  \tag{E.6}\\
& \int_{W_{1}>0} d W_{1} e^{-\operatorname{Tr} W_{1}} \frac{1}{\operatorname{det}\left(\mathbb{1}+\frac{\left(\sigma_{1} \sigma_{2} \sigma_{3}\right)^{2}}{N^{3}}\left|W_{1}^{1 / 2} T A_{1} \ldots A_{n-3}\right|^{2}\right)^{N}}  \tag{E.7}\\
= & \frac{1}{\pi^{2 N^{2}}} \frac{N^{n N^{2}}}{\sigma^{2 N^{2}}}\left(\frac{\left(\sigma_{n} \sigma_{n-1} \sigma_{n-2}\right)^{2}}{N^{2}}\right)^{N^{2}} \int \mathcal{D}^{(n)} A e^{-N \sum_{i=1}^{n-3} \frac{1}{\sigma_{i}^{2}} \operatorname{Tr} A_{i} A_{i}^{\dagger}}  \tag{E.8}\\
& \int_{W_{i}>0} d W_{1} d W_{2} e^{-\sum_{i=1}^{2} W_{i}} e^{-\frac{\left(\sigma_{n} \sigma_{n-1} \sigma_{n-2}\right)^{2}}{N^{3}} \operatorname{Tr}\left|W_{2}^{1 / 2} W_{1}^{1 / 2} T A_{1} \ldots A_{n-3}\right|^{2}} \tag{E.9}
\end{align*}
$$

We may similarly integrate out all $A_{i}$ to get

$$
\begin{equation*}
\varphi_{(n)}\left(T, T^{\dagger}\right)=\frac{1}{\Gamma_{N}(N)^{n-1}} \int_{W_{i}>0} \prod_{i=1}^{n-1} d W_{i} e^{-\sum_{i=1}^{n-1} W_{i}} e^{-\frac{\sigma^{2}}{N^{n}} \operatorname{Tr} T T^{\dagger} W_{1} W_{2} \ldots W_{n-1}} \tag{E.10}
\end{equation*}
$$

Finally we should rescale the $W_{i}$ by $N$ such that the action is $N W_{i}$ :

$$
\begin{equation*}
\varphi_{(n)}\left(T, T^{\dagger}\right)=\frac{N^{(n-1) N^{2}}}{\Gamma_{N}(N)^{n-1}} \int_{W_{i}>0} \prod_{i=1}^{n-1} d W_{i} e^{-N \sum_{i=1}^{n-1} \operatorname{Tr} W_{i}} e^{-\frac{\sigma^{2}}{N} \operatorname{Tr} T T^{\dagger} W_{1} W_{2} \ldots W_{n-1}} \tag{E.11}
\end{equation*}
$$

It is straightforward to Fourier transform this expression and obtain $P_{(n)}$ in eq (5.4).

## F Exact computation of $\operatorname{Tr} W_{(n)}^{m}$

In this section we will explicitly verify the expression 5.17 for the moments $\operatorname{Tr} W_{(n)}^{m}$ for some low values of $m$. For the reader's convenience, we reproduce (5.17) here

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n)}^{m}\right\rangle=F C_{n}(m) \tag{F.1}
\end{equation*}
$$

We will use the recursion relation (5.14)

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1}=\sum_{k=1}^{m} \sum_{\mathbf{m} \in \mathcal{M}_{m, k}} \tilde{c}_{1^{m}, 1^{m}}\left(m_{1}, \ldots, m_{k}\right) \frac{1}{N}\left\langle\operatorname{Tr} W_{(n-2)}^{m_{1}}\right\rangle_{W, n-2} \ldots \frac{1}{N}\left\langle\operatorname{Tr} W_{(n-2)}^{m_{k}}\right\rangle_{W, n-2}, \tag{F.2}
\end{equation*}
$$

alongwith the coefficients (2.19). We will also need the boundary condition

$$
\begin{equation*}
\left\langle W_{(1)}^{m}\right\rangle=C_{m} \tag{F.3}
\end{equation*}
$$

where $C_{m}$ is the Catalan number.
$\operatorname{Tr} \mathbf{W}_{(\mathbf{n}-\mathbf{1})}$
The first moment is simple, the only contribution to the recursion relation is

$$
\begin{equation*}
\tilde{c}_{1,1}(1)=1 . \tag{F.4}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1}=\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-2)}^{m}\right\rangle_{W, n-2}=\frac{1}{N}\left\langle\operatorname{Tr} W_{(1)}^{m}\right\rangle_{W, 1}=1 \tag{F.5}
\end{equation*}
$$

where we have repeatedly used the first equality to arrive at the second, and subsequently used $C_{1}=1$. Therefore

$$
\begin{align*}
\frac{1}{N}\left\langle\operatorname{Tr} W_{(n-1)}^{m}\right\rangle_{W, n-1} & =1  \tag{F.6}\\
& =F C_{n-1}(1) \tag{F.7}
\end{align*}
$$

$\operatorname{Tr} \mathbf{W}_{(\mathbf{n}-\mathbf{1})}^{\mathbf{2}}$
We first recall that

$$
\begin{equation*}
\tilde{c}_{1^{2}, 1^{2}}(2)=\tilde{c}_{1^{2}, 1^{2}}(1,1)=1 \tag{F.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(1)}^{2}\right\rangle=2 \tag{F.9}
\end{equation*}
$$

then

$$
\begin{align*}
\left\langle\operatorname{Tr} W_{(n-1)}^{2}\right\rangle & =\left\langle\operatorname{Tr} W_{(n-2)}^{2}\right\rangle+1  \tag{F.10}\\
& =n  \tag{F.11}\\
& =F C_{n-1}(2) \tag{F.12}
\end{align*}
$$

$\operatorname{Tr} \mathbf{W}_{(\mathbf{n}-\mathbf{1})}^{\mathbf{3}}$
We have

$$
\begin{equation*}
\tilde{c}_{1^{3}, 1^{3}}(1,1,1)=1, \quad \tilde{c}_{1^{3}, 1^{3}}(1,2)=3, \quad \tilde{c}_{1^{3}, 1^{3}}(3)=1 \tag{F.13}
\end{equation*}
$$

then

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(n-1)}^{3}\right\rangle=\left\langle\operatorname{Tr} W_{(n-2)}^{3}\right\rangle+3 F C_{n-2}(2)+1 \tag{F.14}
\end{equation*}
$$

which is solved by

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(n-1)}^{3}\right\rangle=F C_{n-1}(3) . \tag{F.15}
\end{equation*}
$$

$\operatorname{Tr} \mathbf{W}_{(\mathbf{n}-\mathbf{1})}^{4}$
We have

$$
\begin{gather*}
\tilde{c}_{1^{4}, 1^{4}}(1,1,1,1)=1 \quad, \quad \tilde{c}_{1^{4}, 1^{4}}(1,1,2)=5, \quad \tilde{c}_{1^{4}, 1^{4}}(1,3)=4 \\
\tilde{c}_{1^{4}, 1^{4}}(2,2)=2, \quad \tilde{c}_{1^{4}, 1^{4}}(4)=1 \tag{F.16}
\end{gather*}
$$

As a result,

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(n-1)}^{4}\right\rangle=\left\langle\operatorname{Tr} W_{(n-2)}^{4}\right\rangle+2 F C_{n-2}(2)^{2}+F C_{n-2}(4)+4 F C_{n-2}(3)+6 F C_{n-2}(2)+1 \tag{F.17}
\end{equation*}
$$

which is solved by

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(n-1)}^{3}\right\rangle=F C_{n-1}(4) \tag{F.18}
\end{equation*}
$$

$\operatorname{Tr} \mathbf{W}_{(\mathbf{n}-\mathbf{1})}^{\mathbf{5}}$
Starting with

$$
\begin{align*}
& \tilde{c}_{1^{5}, 1^{5}}(1,1,1,1,1)=1, \quad \tilde{c}_{1^{5}, 1^{5}}(1,1,1,2)=10, \quad \tilde{c}_{1^{5}, 1^{5}}(1,1,3)=10  \tag{F.19}\\
& \tilde{c}_{1^{5}, 1^{5}}(1,4)=5, \quad \tilde{c}_{1^{5}, 1^{5}}(1,2,2)=10, \quad \tilde{c}_{1^{5}, 1^{5}}(2,3)=5, \quad \tilde{c}_{1^{5}, 1^{5}}(5)=1 \tag{F.20}
\end{align*}
$$

we have

$$
\begin{align*}
\left\langle\operatorname{Tr}\left(W_{(n-1)}\right)^{5}\right\rangle= & \left\langle\operatorname{Tr} W_{(n-2)}^{5}\right\rangle+10 F C_{n-2}(2)+10 F C_{n-2}(3)+5 F C_{n-2}(4)  \tag{F.21}\\
& +10 F C_{n-2}(2)^{2}+5 F C_{n-2}(2) F C_{n-2}(3)+1 \tag{F.22}
\end{align*}
$$

which is solved by

$$
\begin{equation*}
\left\langle\operatorname{Tr} W_{(n-1)}^{5}\right\rangle=F C_{n-1}(5) \tag{F.23}
\end{equation*}
$$

## References

[1] J. Ginibre, "Statistical Ensembles of Complex, Quaternion, and Real Matrices," Journal of Mathematical Physics 6 (March, 1965) 440-449.
[2] E. Brezin, C. Itzykson, G. Parisi, and J. Zuber, "Planar Diagrams," Commun.Math.Phys. 59 (1978) 35.
[3] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, "Spectrum of Large Random Asymmetric Matrices," Phys. Rev. Lett. 60 (May, 1988) 1895-1898.
[4] B. A. Khoruzhenko and H. J. Sommers, "Non-hermitian random matrix ensembles," in The Oxford Handbook of Random Matrix Theory. Oxford University Press, 2011. 0911.5645.
[5] A. Tulino and S. Verd, "Random Matrix Theory and Wireless Communications," Foundations and Trends in Communications and Information Theory 1 (2004).
[6] P. Forrester, Log-Gases and Random Matrices (LMS-34). London Mathematical Society Monographs. Princeton University Press, 2010.
[7] Z. Bai and J. Silverstein, Spectral Analysis of Large Dimensional Random Matrices. Springer Series in Statistics. Springer-Verlag New York, 2010.
[8] R. Couillet and M. Debbah, Random Matrix Methods for Wireless Communications. Cambridge University Press, New York, NY, USA, 2011.
[9] J. R. Ipsen, "Products of Independent Gaussian Random Matrices," 1510.06128 .
[10] B. Eynard and N. Orantin, "Invariants of algebraic curves and topological expansion," math-ph/0702045.
[11] S. Dartois and P. J. Forrester, "Schwinger-Dyson and loop equations for a product of square Ginibre random matrices," Journal of Physics A: Mathematical and Theoretical 53 (2020), no. 17, 175201, 1906.04390.
[12] A. M. Saxe, J. L. McClelland, and S. Ganguli, "Exact solutions to the nonlinear dynamics of learning in deep linear neural networks," 1312.6120.
[13] D. Sherrington and S. Kirkpatrick, "Solvable model of a spin-glass," Physical Review Letters $35(12,1975) 1792+$.
[14] A. Kitaev, "KITP strings seminar and Entanglement 2015 program,". http://online.kitp.ucsb.edu/online/entangled15/.
[15] S. Sachdev and J.-w. Ye, "Gapless spin fluid ground state in a random, quantum Heisenberg magnet," Phys.Rev.Lett. 70 (1993) 3339, cond-mat/9212030.
[16] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," Phys. Rev. D94 (2016), no. 10, 106002, 1604.07818.
[17] Louart, Cosme and Liao, Zhenyu and Couillet, Romain, "A Random Matrix Approach to Neural Networks," Ann. Appl. Probab. $28(04,2018)$ 1190-1248, 1702.05419.
[18] J. Pennington and P. Worah, "Nonlinear random matrix theory for deep learning," in Advances in Neural Information Processing Systems 30, I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, eds., pp. 2637-2646. Curran Associates, Inc., 2017.
[19] B. Adlam, J. Levinson, and J. Pennington, "A Random Matrix Perspective on Mixtures of Nonlinearities for Deep Learning," arXiv e-prints (Dec, 2019) arXiv:1912.00827, 1912.00827.
[20] S. Yaida, "Non-gaussian processes and neural networks at finite widths," 1910.00019.
[21] Z. Burda, R. A. Janik, and B. Waclaw, "Spectrum of the product of independent random Gaussian matrices," Phys. Rev. E 81 (Apr, 2010) 041132, 0912.3422.
[22] J. C. Osborn, "Universal Results from an Alternate Random-Matrix Model for QCD with a Baryon Chemical Potential," Physical Review Letters 93 (Nov, 2004) hep-th/0403131.
[23] G. Akemann and Z. Burda, "Universal microscopic correlation functions for products of independent Ginibre matrices," Journal of Physics A: Mathematical and Theoretical 45 (Oct, 2012) 465201, 1208.0187.
[24] K. A. Penson and K. Zyczkowski, "Product of Ginibre matrices: Fuss-Catalan and Raney distributions," Phys. Rev. E 83 (Jun, 2011) 061118, 1103.3453.
[25] Z. Burda, R. A. Janik, and M. A. Nowak, "Multiplication Law and S-Transform for non-Hermitian Random Matrices," Physical Review E 84 (Dec, 2011) 1104.2452.
[26] G. Akemann, M. Kieburg, and L. Wei, "Singular value correlation functions for products of Wishart random matrices," Journal of Physics A Mathematical General 46 (July, 2013) 275205, 1303.5694.
[27] T. Kemp, K. Mahlburg, A. Rattan, and C. Smyth, "Enumeration of non-crossing pairings on bit strings," Journal of Combinatorial Theory, Series A 118 (2011), no. 1, 129 - 151, 0906.2183
[28] G. Akemann, "Microscopic correlations for non-Hermitian Dirac operators in three-dimensional QCD," Physical Review D 64 (Nov, 2001) hep-th/0106053.
[29] G. Akemann and G. Vernizzi, "Characteristic polynomials of complex random matrix models," Nuclear Physics B 660 (Jun, 2003) 532556, hep-th/0212051.
[30] G. Akemann and A. Pottier, "Ratios of characteristic polynomials in complex matrix models," Journal of Physics A: Mathematical and General 37 (Sep, 2004) L453L459, math-ph/0404068.
[31] M. C. Bergre, "Biorthogonal Polynomials for Potentials of two Variables and External Sources at the Denominator," hep-th/0404126.
[32] M. C. Bergre, "Correlation functions of complex matrix models," Journal of Physics A: Mathematical and General 39 (Jun, 2006) 87498773, hep-th/0511019.
[33] E. Brézin and S. Hikami, "Characteristic Polynomials of Random Matrices," Communications in Mathematical Physics 214 (Jan., 2000) 111-135, math-ph/9910005.
[34] Y. V. Fyodorov and E. Strahov, "An exact formula for general spectral correlation function of random Hermitian matrices," Journal of Physics A: Mathematical and General 36 (Mar, 2003) 32033213.
[35] A. Nica, R. Speicher, and L. M. Society, Lectures on the Combinatorics of Free Probability. No. vol. 13 in Lectures on the combinatorics of free probability. Cambridge University Press, 2006.
[36] P. Schumacher and C. Yan, "On the Enumeration of Non-crossing Pairings of Well-balanced Binary Strings," Annals of Combinatorics 17 (2013) 379-391.
[37] N. Alexeev, F. Götze, and A. Tikhomirov, "On the Asymptotic Distribution of the Singular Values of Powers of Random Matrices," Lithuanian Mathematical Journal 50 (April, 2010) 121-132, 1012.2743.
[38] "The On-Line Encyclopedia of Integer Sequences." https://oeis.org/A003239.
[39] A. Tucker, Applied Combinatorics. John Wiley and Sons, Inc., USA, 2006.
[40] Y. Yin, "Limiting spectral distribution for a class of random matrices," Journal of Multivariate Analysis 20 (1986), no. 1, 50-68.
[41] L. Li, A. Tulino, and S. Verdu, "Asymptotic Eigenvalue Moments for Linear Multiuser Detection," Communications in Information and Systems 1 (09, 2002).
[42] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B72 (1974) 461.
[43] F. Gtze and A. Tikhomirov, "On the Asymptotic Spectrum of Products of Independent Random Matrices," 1012.2710.
[44] S. O'Rourke and A. Soshnikov, "Products of Independent non-Hermitian Random Matrices," Electron. J. Probab. 16 (2011) 2219-2245.
[45] T. Banica, S. T. Belinschi, M. Capitaine, and B. Collins, "Free Bessel Laws," Canadian Journal of Mathematics 63 (Feb, 2011) 337.
[46] F. Benaych-Georges, "On a surprising relation between the MarchenkoPastur law, rectangular and square free convolutions," Ann. Inst. H. Poincar Probab. Statist. 46 (2010), no. 3, 644-652, 0808.3938.
[47] R. R. Muller, "On the Asymptotic Eigenvalue Distribution of Concatenated Vector-valued Fading Channels," IEEE Transactions on Information Theory 48 (2002), no. 7, 2086-2091.
[48] Z. Burda, A. Jarosz, G. Livan, M. A. Nowak, and A. Swiech, "Eigenvalues and singular values of products of rectangular Gaussian random matrices," Physical Review E 82 (Dec, 2010).
[49] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science. Addison-Wesley Longman Publishing Co., Inc., USA, 1989.
[50] A. T. James, "Distributions of matrix variates and latent roots derived from normal samples," Ann. Math. Statist. $35(06,1964) 475-501$.


[^0]:    ${ }^{1}$ The archetypical result in this field is due to Ginibre [1] who computed the eigenvalue distribution of these matrices and found it to be uniform on the disk, we will refer to complex i.i.d. Gaussian matrices as Ginibre matrices.
    ${ }^{2}$ Another interesting way to relax the i.i.d condition intiated in [3] (see 4] for a concise review), is to study Gaussian ensembles of the form

    $$
    \begin{equation*}
    P\left(X, X^{\dagger}\right) \sim e^{-\operatorname{Tr} X X^{\dagger}+(\tau \operatorname{Tr} X X+c . c)} \tag{1.2}
    \end{equation*}
    $$

    where $\tau$ is a complex scalar which one might call the elliptical parameter.

[^1]:    ${ }^{3}$ An interesting more modern development in RMT is the development of the Schwinger-Dyson (loop) equations into topological recursion [10] and there has been a recent work investigating such modern techniques to the product Ginibre ensemble 11.

[^2]:    ${ }^{4}$ We use $\left.\langle\rangle.\right\rangle_{G}$ to denote the expectation value in the Ginibre ensemble.

[^3]:    ${ }^{5}$ see also 37 where this is obtained from the distribution of singular values of $X^{s}$

[^4]:    ${ }^{6}$ Some excellent reviews are 5 [ 8 .
    ${ }^{7}$ In 2.15 the deterministic matrix $W$ appears within the operator insertion itself but using the identity

    $$
    \begin{equation*}
    \int d^{2} X f\left(X W^{1 / 2}\right) e^{-N \operatorname{Tr} X X^{\dagger}}=\frac{1}{\operatorname{det} W^{N}} \int d^{2} X f(X) e^{-N \operatorname{Tr} X W^{-1} X^{\dagger}} \tag{2.14}
    \end{equation*}
    $$

    we see that up to a normalization, this is equivalent to sampling $X$ from a Gaussian distribution with variance profile given by $W$.

[^5]:    ${ }^{8}$ See 25 for an interesting derivation using free probability and also 43 44

[^6]:    ${ }^{9}$ There are more general Wishart measures which would arise from the product of rectangular Gaussian matrices but we will not consider them here.

[^7]:    ${ }^{10}$ In example 5. section 7.5 of 49 we find that Fuss-Catalan numbers satisfy a similar but different recursion relation. Our recursion relation relates Fuss-Catalan numbers with different values of $n$ whereas the relation in (49) closes for a given value of $n$.

[^8]:    ${ }^{11}$ It is straightforward to check that $V$ is also hermitian positive definite. It is quite straightforward to verify these relations when $\Sigma$ is proportional to the identity matrix, in which case

    $$
    \begin{align*}
    \Sigma & =b \mathbb{1}, \quad \operatorname{det} \Sigma=b^{N}  \tag{D.5}\\
    d\left(\Sigma_{0}^{-1 / 2} V \Sigma_{0}^{-1 / 2}\right) & =b^{-N^{2}} d V=\operatorname{det} \Sigma^{-N} d V . \tag{D.6}
    \end{align*}
    $$

[^9]:    ${ }^{12}$ We have defined the quantity $|M|^{2}=M^{\dagger} M=M M^{\dagger}$ in the determinant for a complex matrix $M$. This is $a$ priori not well-defined as $M$ and $M^{\dagger}$ don't commute. However, under the determinant

    $$
    \begin{equation*}
    \operatorname{det}\left(1+M^{\dagger} M\right)=e^{\operatorname{tr} \ln \left(1+M^{\dagger} M\right)}=e^{\operatorname{tr} \ln \left(1+M M^{\dagger}\right)}=\operatorname{det}\left(1+M M^{\dagger}\right), \tag{E.3}
    \end{equation*}
    $$

    as we may readily verify from the series expansion of $\ln$. Hence $\operatorname{det}\left(1+|M|^{2}\right)$, which is what we work with, is indeed well defined.

