Crossings and nestings over some Motzkin objects and q-Motzkin numbers

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Abstract

In this paper, we investigate on enumeration of some Motzkin objects according to the numbers of crossings and nestings. More precisely, we compute and express in terms of continued fractions the distributions of the statistics numbers of crossings and nestings over the sets of 4321- and 3412-involutions and the set of $(321, 3\bar{1}42)$-avoiding permutations. To get our results, we exploit the bijection of Biane restricted to the sets of 4321 and 3412-avoiding involutions which was characterized by Barnabei et al. and the bijection Chen et al. between $(321, 3\bar{1}42)$-avoiding permutations and Motzkin paths. Furthermore, we manipulate the obtained continued fractions to get the recursion formulas for the polynomial distributions of crossings and nestings and we discover that results involve two new q-Motzkin numbers.

Keywords: Crossing, nesting, Motzkin numbers, q-Motzkin numbers, restricted permutations, restricted involution.
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1 Introduction and result

We let S_n denote the set of all permutations of $[n] := \{1, \ldots, n\}$. A permutation $\sigma \in S_n$ is an *involution* whenever $\sigma(\sigma(i)) = i$ for all $i \in [n]$, and we write I_n to denote the set of all involutions in S_n. We refer $|\sigma|$ as the length of the permutation σ.

Suppose that $\sigma \in S_n$ and $\tau \in S_k$. We say a subsequence $s = \sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ is an *occurrence* of τ if and only if s and τ are in the same isomorphic order, i.e. $\sigma(i_x) < \sigma(i_y)$ if and only if $\tau(x) < \tau(y)$. Whenever σ contains no occurrence of τ, we say that σ *avoids* the pattern τ or simply σ is τ*-avoiding*. For any given set of permutations T, called set of patterns, we write $S_n(T)$ and $I_n(T)$ to denote respectively the sets of permutations and involutions in S_n which avoid every pattern in T, and we write $S(T)$ and $I(T)$ to denote respectively the sets of all permutations and involutions, including the empty permutation, which avoid every pattern in T. Usually, if $T = \{\tau_1, \ldots, \tau_k\}$, we write $S_n(T) = S_n(\tau_1, \ldots, \tau_k)$ and $S(T) = S(\tau_1, \ldots, \tau_k)$. For example, the subsequence 5297 of the permutation $\pi = 35142987$ is an occurrence of 2143. We can easily verify that $\pi \in I_9(4321)$.

A *barred permutation* is a permutation such that some of whose letters are barred. We let $\bar{S}_k$ denote the set of all barred permutation of length k. Example: $3\bar{2}4\bar{5}1$ is a barred permutation in $\bar{S}_5$. Whenever $\bar{\tau} \in \bar{S}_k$, we let τ denote the permutation obtained by unbarring $\bar{\tau}$, and τ' denote the reduction of the obtained permutation from $\bar{\tau}$ by removing the barred element (if any). Example: if $\bar{\tau} = 3\bar{2}4\bar{5}1$, we have $\tau = 32451$ and $\tau' = 231$. We say that a permutation σ avoids $\bar{\tau}$ if each occurrence of τ' in σ (if any) is part of an occurrence of τ in σ. For example, a permutation σ is $3\bar{1}42$-avoiding if and only if, for any occurrence $\sigma(i)\sigma(j)\sigma(k)$ of 321 in σ, there exists $i < l < j$ such that $\sigma(l) < \sigma(k)$.

A *statistic* over a given set E is a map $s : E \to \mathbb{N}$. The polynomial distribution of the statistic s over the E is the polynomial $\sum_{e \in E} q^{s(e)}$. For example, if we let $\mathcal{P}_n$ denote the set of all subsets of $[n]$ and $\text{card}(A)$ denote the cardinality of A for any $A \in \mathcal{P}_n$, then card is a statistic over $\mathcal{P}_n$ and its polynomial distribution is

$$\sum_{A \in \mathcal{P}_n} q^{\text{card}(A)} = (1+q)^n.$$

Furthermore, the generating function of the polynomial $\sum_{A \in \mathcal{P}_n} q^{\text{card}(A)}$ is

$$1 + \sum_{n \geq 1} \sum_{A \in \mathcal{P}_n} q^{\text{card}(A)} t^n = \frac{1}{1-(1+q)t}.$$

We will now define some statistics over S_n. For that, we let $\sigma \in S_n$. An *excedance* (resp. *fixed point*) of σ is an index i such that $\sigma(i) > i$ (resp. $\sigma(i) = i$). A *crossing* (resp. *nesting*, *inversion*) of σ is a pair of indices (i, j) such that $i < j < \sigma(i) < \sigma(j)$ or $\sigma(i) < \sigma(j) \leq i < j$ (resp. $i < j < \sigma(j) < \sigma(i)$ or $\sigma(j) < \sigma(i) \leq i < j$, $i < j$ and $\sigma(j) > \sigma(i)$). We let $\text{exc}(\sigma)$ (resp. $\text{cr}(\sigma)$, $\text{nes}(\sigma)$, $\text{inv}(\sigma)$) denote the number of excedances (resp. crossings, nestings, inversions) of σ. For example, the permutation $\pi = 4\ 6\ 2\ 9\ 8\ 1\ 7\ 3\ 10\ 5 \in S_{10}$ has the following properties: $\text{exc}(\pi) = 5$, $\text{cr}(\pi) = 7$, $\text{nes}(\pi) = 4$ and $\text{inv}(\pi) = 20$. Therefore, exc, cr, nes and inv are all statistics over S_n. Médicis and Viennot [14] and Randrianarivony [18] showed that these statistics are related by the following identity:

$$\text{inv}(\sigma) = \text{exc}(\sigma) + \text{cr}(\sigma) + 2\text{nes}(\sigma), \text{ for any } \sigma \in S_n. \tag{1.1}$$

In this paper, we are interested on the distribution of the statistics number of crossings and nestings over the sets $I_n(4321)$, $I_n(3412)$ and $S_n(321, 3\bar{1}42)$. These sets are all enumerated by the n-th Motzkin numbers (the sequence A001006 in [19]) and they are well studied in the literature (see [2, 6, 11] and reference therein).

The Motzkin numbers M_n are traditionally defined by the following recurrence relation:

$$M_0 = 1 \text{ and } M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k} \text{ for } n \geq 1. \tag{1.2}$$

The first values for M_n are $1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$. Using (1.2), the generating function for the Motzkin numbers, i.e. $M(t) = \sum_{n \geq 0} M_n t^n$, satisfies the following equivalent functional equations

$$M(t) = 1 + tM(t) + t^2 M(t)^2 \text{ and } M(t) = \frac{1}{1 - z - z^2 M(t)}. \tag{1.3}$$

Solving the first equation of (1.3) for $M(t)$, we obtain

$$M(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2}.$$

The second identity of (1.3) leads to the following continued fraction expansion for $M(t)$

$$M(t) = \cfrac{1}{1-z-\cfrac{z^2}{1-z-\cfrac{z^2}{1-z-\cfrac{z^2}{\ddots}}}}.$$

A q-Motzkin numbers is a polynomial in q which generalizes the Motzkin numbers. In other words, a q-Motzkin numbers is a polynomial satisfying the recurrence for the Motzkin numbers and equals to the Motzkin numbers when specializing $q = 1$. Some q-generalizations of Motzkin numbers are studied by Barcuccia et al. [1] and recently by Barnabei and al. [3]. Here, we are interested on two new q-Motzkin numbers $M_n(q)$ and $\tilde{M}_n(q)$. The first one $M_n(q)$ is defined

$$M_0(q) = 1, M_n(q) = M_{n-1}(q) + \sum_{k=0}^{n-2} q^k M_k(q) M_{n-2-k}(q) \text{ for } n \geq 1. \tag{1.4}$$

and the second one $\tilde{M}_n(q)$ by

$$\tilde{M}_0(q) = 1, \tilde{M}_n(q) = \tilde{M}_{n-1}(q) + \sum_{k=0}^{n-2} q^{k+1-(n-1)\delta_{k,n-2}} \tilde{M}_k(q) \tilde{M}_{n-2-k}(q) \text{ for } n \geq 1, \tag{1.5}$$

where δ is the usual Kronecker symbol. Hopping to extend the results on q-Motzkin numbers introduced in [1, 3], we will show using continued fractions how these q-Motzkin numbers are the distributions of crossings and nestings over restricted permutations.

The study of the crossings and nestings over permutations was introduced by Médicis and Viennot in [14] and extended in [5, 7, 8, 17, 18]. The study of these statistics over pattern avoiding permutations was introduced by the second author [15] in which he manipulated a known bijection of Elizalde and Pak [13] to find the equidistributions of the crossings over the sets of permutations avoiding the patterns 132, 213 and 321. Later, Rakotomamonjy et al. [16] enumerated the sets of permutations avoiding some pairs of patterns of length 3 according to cr and they find new combinatorial interpretations of some known sequences in [19]. In this work, we investigate on enumeration of some Motzkin objects, namely the sets $I_n(4321)$, $I_n(3412)$ and $S_n(321, 3\bar{1}42)$, according to the number of crossings and nestings (see Theorem 1.1). The use of some know bijections between these sets and Motzkin paths allows us to reach our goal.

A *Motzkin path* of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $u = (1,1)$, horizontal steps $h = (1,0)$, and down steps $d = (1,1)$. The set of Motzkin paths of length n will be denoted by $\mathcal{M}_n$.

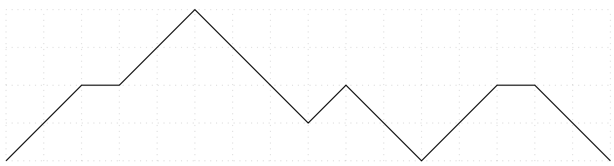


Figure 1: The Motzkin path $P = uuhuudddudduuhdd \in \mathcal{M}_{16}$.

It is well known that the cardinality of $\mathcal{M}_n$ is the n-th Motzkin number M_n. The bijections between our combinatorial objects and Motzkin paths are well studied in [2, 6]. Firstly, Baranabei et al. [2] studied the bijection of Biane [4] restricted to involutions. The bijection of Biane maps an involution into a Motzkin path whose down steps are labelled with an integer that does not exceed its height, while the other steps are unlabelled. Baranabei et al. proved the following characterizations:

- An involution τ avoids 4321 if and only if the label of any down step in the corresponding Motzkin path is 1.
- An involution τ avoids 3412 if and only if the label of any down step in the corresponding Motzkin path equals its height.

Consequently, when restricted to 4321- and 3412-avoiding involutions, we can therefore ignore the labels on Motzkin paths. Secondly, using reduced decomposition of Motzkin paths, Chen et al [6] exhibited a bijection between $S_n(321, 3\bar{1}42)$ and $\mathcal{M}_n$ which exchange the statistics inv and area $-$ sh$_u$. Throughout these two bijections, we interpret the statistics cr and nes in terms of statistics on Motzkin paths. Thus, as main results, we find the following combinatorial interpretations of $M_n(q)$ and $\tilde{M}_n(q)$.

Theorem 1.1. *We have the following identities*

$$M_n(q) = \sum_{\sigma \in I_n(4321)} q^{(\mathrm{cr}+\mathrm{nes})(\sigma)} = \sum_{\sigma \in I_n(3412)} q^{\mathrm{nes}(\sigma)},$$

$$\tilde{M}_n(q) = \sum_{\sigma \in S_n(321, 3\bar{1}42)} q^{\mathrm{cr}(\sigma)}.$$

We organize the rest of this paper in three sections. Section 2 is a preliminary section in which we define some statistics over Motzkin path and we compute the generating function of their joint distribution. In Section 3 and Section 4, by serving the obtained result in Section 2 throughout the bijections of Biane [4] and Chen et al. [6], we compute and express in terms of continued fractions the distributions of cr and nes over the sets $I_n(4321)$, $I_n(3412)$ and $S_n(321, 3\bar{1}42)$. Throughout the obtained continued fractions, we provide the proof of Theorem 1.1.

2 Motzkin paths and statistics

In this section, we define some statistics over Motzkin paths and we prove a preliminary result that is fundamental for the rest of the paper (see Theorem 2.4).

Let $P = P_1 P_2 \cdots P_n$ be a Motzkin path of length n. Let us first define the following sets:

$$Hor(P) := \{i | P_i = h\}, Up(P) := \{i | P_i = u\} \text{ and } Down(P) := \{i | P_i = d\}.$$

The *height* of each step s in P is the y-coordinate of the starting point of s. Therefore, the height of the i-th step of P is equal to

$$h_i(P) = \begin{cases} |P(1,i)|_u - |P(1,i)|_d - 1; & \text{if } P_i = u \\ |P(1,i)|_u - |P(1,i)|_d; & \text{if } P_i = h \\ |P(1,i)|_u - |P(1,i)|_d + 1; & \text{if } P_i = d \end{cases},$$

where $P(1,i) := P_1 P_2 \cdots P_i$ is the initial sub-path of length i of P and, for any word w, $|w|_a$ is the number of occurrences of the letter a in the word w. Here are some statistics over $\mathcal{M}_n$:

- area(P) the area between the path P and the x-axis,
- up(P) (resp. hor(P),down(P)) the number of up (resp. horizontal, down) steps of P,
- sh$_u(P)$ (resp. sh$_h(P)$, sh$_d(P)$) the sum of heights of all up (resp. horizontal, down) steps of P, i.e.

$$\text{sh}_u(P) := \sum_{i \in Up(P)} h_i(P), \text{sh}_h(P) := \sum_{i \in Hor(P)} h_i(P), \text{ and } \text{sh}_d(P) := \sum_{i \in Down(P)} h_i(P).$$

For example, the Motzkin path in Fig. 1 has the following properties: hor$(P) = 2$, up$(P) =$ down$(P) = 7$, sh$_u(P) = 8$, sh$_h(P) = 4$, sh$_d(P) = 15$ and area$(P) = 27$. We will show how these statistics are related.

Proposition 2.1. *For any Motzkin path P, we have*

$$\text{area}(P) = 2\text{sh}_d(P) + \text{sh}_h(P) - \text{down}(P).$$

Proof. The proof is simply based on the following properties.

- If the i-th step of P is a down step of height h, the area of the polygon $(i-1, h) - (i, h-1) - (i+1-h, 0) - (i-1-h, 0)$ is $2h - 1$.
- If the i-th step of P is a horizontal step of height h, the area of the polygon $(i-1, h) - (i, h) - (i+1-h, 0) - (i-h, 0)$ is h.

The area of the path P is equals to the sum of all area of such polygones corresponding to down and horizontal steps (see Figure 2). ☐

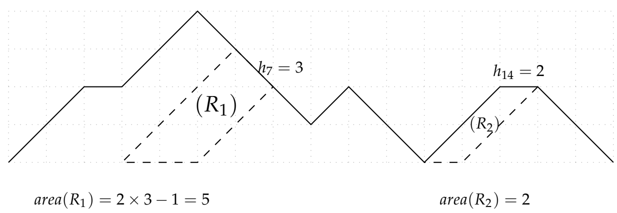


Figure 2: Decomposition of the area of a Motzkin path.

By the same way, when we decompose the area statistic according to up and horizontal steps, we obtain the following proposition.

Proposition 2.2. *For any Motzkin path P, we have*

$$\text{area}(P) = 2\text{sh}_u(P) + \text{sh}_h(P) + \text{up}(P). \tag{2.1}$$

Corollary 2.3. *For any Motzkin path P, we have* sh$_u(P) =$ sh$_d(P) -$ down(P).

Proof. Using both propositions 2.1 and 2.2 with the obvious fact that up$(P) =$ down(P) for any Motzkin path P, we obtain the corollary. ☐

We now consider the following polynomial

$$I_n(a,b,c,d) = \sum_{P\in\mathcal{M}_n} a^{\mathrm{hor}(P)} b^{\mathrm{up}(P)} c^{\mathrm{sh}_u(P)} d^{\mathrm{sh}_h(P)},$$

and we denote by $I(a,b,c,d;t) = \sum_{n\geq 0} I_n(a,b,c,d)t^n$ the generating function of $I_n(a,b,c,d)$. Before closing this section, we will prove a fundamental identity which allows us to get easily the main results of this paper.

Theorem 2.4. *The continued fraction expansion of $I(a,b,c,d;t)$ is*

$$\cfrac{1}{1-at-\cfrac{bt^2}{1-adt-\cfrac{bct^2}{1-ad^2t-\cfrac{bc^2t^2}{\ddots}}}}. \tag{2.2}$$

Proof. The proof is based on the usual decomposition of Motzkin paths. Let $P\in\mathcal{M}_n$. Using the first return decomposition, we obtain the following relations:

- if $P = hP'$ for some $P'\in\mathcal{M}_{n-1}$, then we have

$$(\mathrm{hor},\mathrm{up},\mathrm{sh}_u,\mathrm{sh}_h)P = (1+\mathrm{hor},\mathrm{up},\mathrm{sh}_u,\mathrm{sh}_h)P'.$$

- If $P = uP_1dP_2$ for some $k\geq 2$ such that $P_1\in\mathcal{M}_{k-2}$ and $P_2\in\mathcal{M}_{n-k}$, then we have

$$(\mathrm{hor},\mathrm{up},\mathrm{sh}_u,\mathrm{sh}_h)P = (\mathrm{hor},1+\mathrm{up},\mathrm{sh}_u+\mathrm{up},\mathrm{sh}_h+\mathrm{hor})P_1 + (\mathrm{hor},\mathrm{up},\mathrm{sh}_u,\mathrm{sh}_h)P_2.$$

These two points lead to the following recurrence for the polynomial $I_n(a,b,c,d)$:

$$I_n(a,b,c,d) = aI_{n-1}(a,b,c,d) + b\sum_{k=2}^{n} I_{k-2}(ad,bc,c,d)I_{n-k}(a,b,c,d), \tag{2.3}$$

with $I_0(a,b,c,d) = 1$. Thus, when we compute $I(a,b,c,d;t)$, we obtain from (2.3) the following relation

$$I(a,b,c,d;t) = \frac{1}{1-at-bt^2I(ad,bc,c,d;t)}.$$

Developing $I(ad,bc,c,d;t)$ in its turn, we obtain the desired continued fraction expansion for $I(a,b,c,d;t)$. ☐

3 Crossings and nestings over $I_n(4321)$ and $I_n(3412)$

In this section, we will establish the proof of the first part of Theorem 1.1 concerning the recursion for the polynomial distribution of crossings and nestings over the sets $I_n(4321)$ and $I_n(3412)$. For that, we first recall the bijection of Biane [4] restricted to the set of involutions that was studied by Baranabei et al. [2]. The bijection maps an involution into a Motzkin path whose down steps are labelled with an integer that does not exceed its height, while the other steps are unlabelled. Baranabei et al. [2] proved the following characterizations:

- An involution σ avoids 4321 if and only if the label of any down step in the corresponding Motzkin path is 1.

- An involution σ avoids 3412 if and only if the label of any down step in the corresponding Motzkin path equals its height.

Consequently, when restricted to 4321- and 3412-avoiding involutions, we can ignore the labels on Motzkin paths. Let us denote by Φ_1 (resp. Φ_2) the bijection from $\mathcal{M}_n$ to $I_n(4321)$ (resp. $I_n(3412)$). In this section, we manipulate the bijections Φ_1 and Φ_2, and we will show how these bijections exchange the statistics $(\text{fp}, \text{exc}, \text{cr}, \text{nes})$ and $(\text{hor}, \text{up}, a.\text{sh}_u, b.\text{sh}_h)$, where a and b are two arbitrary integers.

Since the construction of the Motzkin path from an involution throughout Φ_1 and Φ_2 are the same and obvious (see [2]), we just focus on how to get the corresponding involutions from Motzkin paths. Let $P \in \mathcal{M}_n$. We will construct $\sigma_1 = \Phi_1(P)$ by the following procedure (see Figure 3 for graphical illustration).

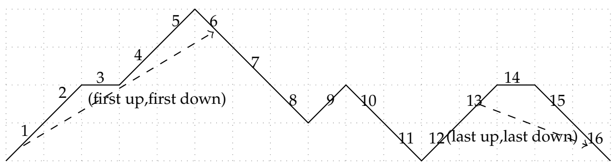
- From left to right, number the steps of P from 1 up to n.

- Read the steps of P from left to right and join the k-th up step with the k-th down step, $k = 1, 2, \ldots$.

- Then, we have

  (a) (i) is a 1-cycle of σ_1, i.e. $\sigma_1(i) = i$, if and only if the step numbered i is horizontal.

  (b) (i, j) is a 2-cycle of σ_1, i.e. $\sigma_1(i) = j$ and $\sigma_1(j) = i$, if and only if i and j are matched by the previous step.



$\Downarrow$

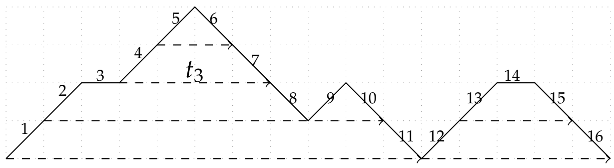
(1 6)(2 7)(3)(4 8)(5 10)(9 11)(12 15)(13 16)(14)

Figure 3: The corresponding 4321-avoiding involution from a Motzkin path.

Before giving the formulation of Φ_2, we first recall what a tunnel is. Notion of tunnels on Dyck paths (Dyck paths are Motzkin paths having no horizontal step) is introduced by Elizalde et al. [12, 13] and Barnabey et al. [3] extend the notion in terms of Motzkin paths. A *tunnel* of a Motzkin path P is a horizontal segment between two distinct lattice points of P that intersects P only at these two points and lying always below P. Since each tunnel t start form the starting point of an up u step to the ending point of a down step d. We say that the tunnel t matches u and d. For example, the Motzkin path in Figure 4 has 7 tunnels drown in right dashed arrows. In particular, the tunnel t_3 matches the up step numbered 4 and the down step numbered 7. For any Motzkin path P, we can construct $\sigma_2 = \Phi_2(P)$ by the following procedure (see Figure 4):

- From left to right, number the steps of P from 1 up to n.

- Then, we have
  (a) (i) is a 1-cycle of σ_2 if and only if the step numbered i is horizontal.
  (b) (i,j) is a 2-cycle of σ_2 if and only if the up step numbered i is matched with the down step numbered j throughout tunnel.



$\Downarrow$

(1 11)(2 8)(3)(4 7)(5 6)(9 10)(12 16)(13 15)(14)

Figure 4: The corresponding 3412-avoiding involution from a Motzkin path.

By these definitions, it is not difficult to show that the bijections Φ_1 and Φ_2 are well defined.

Proposition 3.1. *Let P be a Motzkin path. Assume that $\sigma_1 = \Phi_1(P)$ and $\sigma_2 = \Phi_2(P)$. We have the following properties*

(i) $(\mathrm{cr}, \mathrm{nes})(\sigma_1) = (2\mathrm{sh}_u, \mathrm{sh}_h)(P)$.

(ii) $\mathrm{nes}(\sigma_2) = (2\mathrm{sh}_u + \mathrm{sh}_h)(P)$.

Proof. Let σ be an involution. By definition, we have

$$\mathrm{nes}(\sigma) = 2\sum_{\sigma(i)>i} |\{j<i|\sigma(j)>\sigma(i)\}| + \sum_{\sigma(i)=i} |\{j>i|\sigma(j)<i\}|,$$
$$\text{and } \mathrm{cr}(\sigma) = 2\sum_{\sigma(i)>i} |\{i<j<\sigma(i)|\sigma(j)>\sigma(i)\}|.$$

Know first that the following properties are obvious:

(a) if σ is 4321-avoiding, then we have $|\{j<i|\sigma(j)>\sigma(i)\}| = 0$ for all i such that $\sigma(i)>i$.

(b) if σ is 3412-avoiding, then we have $\mathrm{cr}(\sigma) = 0$.

Let us suppose that $\sigma_1 = \Phi_1(P)$ for some Motzkin path P. According to (a), we have

$$\begin{aligned}
\mathrm{nes}(\sigma_1) &= \sum_{\sigma_1(i)=i} |\{j>i|\sigma_1(j)>i\}| \\
&= \sum_{\sigma_1(i)=i} |\{k<i|\sigma_1(k)>i\}| \\
&= \sum_{i\in Hor(P)} (|P(1,i)|_u - |P(1,i)|_d), \\
&= \mathrm{sh}_h(P).
\end{aligned}$$

Moreover, according to the given definition of Φ_1, we have $|\{i < j < \sigma_1(i)|\sigma_1(j) > \sigma_1(i)\}| = |P(1, \sigma_1(i))|_u - |P(1, \sigma_1(i))|_d = h_{\sigma_1(i)}(P) - 1$ (with $\sigma_1(i)$ as down step of σ_1). Consequently, we get

$$\begin{aligned} \mathrm{cr}(\sigma_1) &= 2 \sum_{\sigma_1(i)>i} |\{i < j < \sigma_1(i)|\sigma_1(j) > \sigma_1(i)\}| \\ &= 2 \sum_{\sigma_1(i)>i} (h_{\sigma_1(i)}(P) - 1) \\ &= 2 \sum_{l \in Down(P)} (h_l(P) - 1) \\ &= 2(\mathrm{sh}_d(P) - \mathrm{down}(P)) \\ &= 2\mathrm{sh}_u(P) \quad (\text{ see corollary } 2.3). \end{aligned}$$

This ends the proof of (i). To prove (ii), we now suppose that $\sigma_2 = \Phi_2(P)$ for some Motzkin path P. For any excedance i of σ_2, we have $|\{j < i|\sigma_2(j) > \sigma_2(i)\}| = |P(1, i))|_u - |P(1, i)|_d = h_i(P)$ and $i \in Up(P)$.

$$\begin{aligned} \mathrm{nes}(\sigma_2) &= 2 \sum_{\sigma_2(i)>i} |\{j < i|\sigma_2(j) > \sigma_2(i)\}| + \sum_{\sigma(i)=i} |\{j > i|\sigma_2(j) > i\}|, \\ &= 2 \sum_{i \in Up(P)} h_i(P) + \sum_{i \in Hor(P)} h_i(P), \\ &= 2\mathrm{sh}_u(P) + \mathrm{sh}_h(P). \end{aligned}$$

This also ends the proof of (ii). Thus, Proposition 3.1 follows. □

We have now all necessary tools to prove the following results concerning the joint distribution of the statistics fp, exc, cr and nes over our sets of 4321 and 3412-avoiding involutions.

Theorem 3.2. *We have the following identities*

$$\sum_{\sigma \in I(4321)} x^{\mathrm{fp}(\sigma)} y^{\mathrm{exc}(\sigma)} p^{\mathrm{cr}(\sigma)} q^{\mathrm{nes}(\sigma)} t^{|\sigma|} = \cfrac{1}{1 - xt - \cfrac{yt^2}{1 - xqt - \cfrac{yp^2t^2}{1 - xq^2t - \cfrac{yp^4t^2}{\ddots}}}}, \tag{3.1}$$

and

$$\sum_{\sigma \in I(3412)} x^{\mathrm{fp}(\sigma)} y^{\mathrm{exc}(\sigma)} q^{\mathrm{nes}(\sigma)} t^{|\sigma|} = \cfrac{1}{1 - xt - \cfrac{yt^2}{1 - xqt - \cfrac{yq^2t^2}{1 - xq^2t - \cfrac{yq^4t^2}{\ddots}}}}. \tag{3.2}$$

Proof. It is obvious to see from their definitions that the bijections Φ_1 and Φ_2 exchange the statistics (fp, exc) and (hor, up) (or (hor, down)). So, using Proposition 3.1, we get

- $(\mathrm{hor}, \mathrm{up}, 2\mathrm{sh}_u, \mathrm{sh}_h) \xrightarrow{\Phi_1} (\mathrm{fp}, \mathrm{exc}, \mathrm{cr}, \mathrm{nes})$

- and $(\mathrm{hor}, \mathrm{up}, 2\mathrm{sh}_u + \mathrm{sh}_h) \xrightarrow{\Phi_2} (\mathrm{fp}, \mathrm{exc}, \mathrm{nes})$

Consequently, we have

$$\sum_{\sigma \in I_n(4321)} x^{\mathrm{fp}(\sigma)} y^{\mathrm{exc}(\sigma)} p^{\mathrm{cr}(\sigma)} q^{\mathrm{nes}(\sigma)} = I_n(x, y, p^2, q),$$
$$\text{and} \sum_{\sigma \in I_n(3412)} x^{\mathrm{fp}(\sigma)} y^{\mathrm{exc}(\sigma)} q^{\mathrm{nes}(\sigma)} = I_n(x, y, q^2, q).$$

So, we obtain easily identity (3.1) (resp. (3.2)) from (2.2), by setting $a = x$, $b = y$, $c = p^2$ (resp. $c = q^2$) and $d = q$. This ends the proof of Theorem 3.2. □

Now, to close this section, we will prove the first identities of the main result presented in introduction (see. Theorem 1.1).

Theorem 3.3. *We have* $\sum_{\sigma \in I_n(4321)} q^{(\mathrm{cr}+\mathrm{nes})(\sigma)} = \sum_{\sigma \in I_n(4312)} q^{\mathrm{cr}(\sigma)} = M_n(q).$

Proof. Let us denote by

$$A(q;t) = \cfrac{1}{1 - t - \cfrac{t^2}{1 - qt - \cfrac{q^2t^2}{1 - q^2t - \cfrac{q^4t^2}{\ddots}}}}.$$

Simple manipulation of the continued fraction expansion of $A(q;t)$ leads to the following functional equation

$$A(q;t) = \frac{1}{1 - t - t^2 A(q;qt)}.$$

This is also equivalent to the following one

$$A(q;t) = 1 + tA(q;t) + t^2 A(q;t) A(q;qt).$$

When we extract the coefficient $A_n(q)$ of t^n on both sides of this equation, we obtain

$$A_0(q) = 1, A_n(q) = A_{n-1}(q) + \sum_{k=0}^{n-2} q^k A_k(q) A_{n-2-k}(q) \text{ for } n \geq 1.$$

Since the polynomials $A_n(q)$ and $M_n(q)$ have the same recurrence (see recursion (1.2)), so they are the same. Furthermore, from (3.1) and (3.2), we obtain

$$\sum_{\sigma \in I(4321)} q^{(\mathrm{cr}+\mathrm{nes})(\sigma)} t^{|\sigma|} = \sum_{\sigma \in I(3412)} q^{\mathrm{cr}(\sigma)} t^{|\sigma|} = A(q;t).$$

Thus, we have Theorem 3.3. □

4 Crossings over $S_n(321, 3\bar{1}42)$

The main goal of this section is to prove the second part of Theorem 1.1 concerning the distribution of crossing over $S_n(321, 3\bar{1}42)$. For that, we will use the bijection of Chen et al. [6] based on reduced decomposition of permutations and strip decomposition of Motzkin paths.

A *transposition* $s_i = (i\ i+1)$ (i.e. 2-cycle) is a map from S_n to itself which interchanges the numbers in the ith and $(i+1)$th position in a permutation. For example, $s_3(623514) = 625314$. Every permutation σ in S_n can be represented as a sequence of transpositions $s_{i_1}s_{i_2}\cdots s_{i_k}$ which, when applied from right to left to the identity permutation $123\cdots n$, results in σ. Such representation is called the *canonical reduced decomposition* of σ and it is not necessarily unique. For example, 321 can be written as both $s_1s_2s_1$ and $s_2s_1s_2$. In fact, the product of simple transpositions satisfies the Braid relations:

$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \text{ and } s_is_j = s_js_i, \text{ where } |i-j| \neq 1.$$

Let $\sigma \in S_n$. We now determine the unique canonical reduced decomposition corresponding to σ by a specific procedure. For that, we denote by $\mathcal{R}(\sigma)$ the set of the pairs resulting of the following procedure and initiate it to the empty set.

1. If $\sigma^{(0)} := \sigma$ is the identity, then the corresponding canonical reduced decomposition of σ is the identity. Otherwise, we go to the next step.

2. Locate the pair (n_1, i_1), where n_1 is the greatest excedance value of $\sigma^{(0)}$ and i_1 its position. Move n_1 from i_1 to position n_1 by applying right to left the sequence $s_{n_1-1}s_{n_1-2}\cdots s_{i_1+1}s_{i_1}$ and leaves the relative order of the other numbers unchanged. Let $\sigma^{(1)}$ be the resulting permutation. Then, add the pair (n_1-1, i_1) to $\mathcal{R}(\sigma)$ and pass to the next step.

3. Return to the first step and applied the procedure to $\sigma^{(1)}$.

By this manner, if we have $\mathcal{R}(\sigma) = \{(h_1, t_1), (h_2, t_2), \ldots, (h_k, t_k)\}$, i.e. $(h_{k+1-j}, t_{k+1-j}) = (n_j - 1, i_j)$ for any $1 \leq j \leq k$, then the canonical reduced decomposition of σ is $\sigma_1\sigma_2\cdots\sigma_k$, where $\sigma_j = s_{h_j}s_{h_j-1}\cdots s_{t_j}$ for any $1 \leq j \leq k$. It is known that such decomposition is unique. In Chen et al. [6], the set $\mathcal{R}(\sigma)$ is known as the set of pairs (*head,tail*) and they showed the following characterization.

Theorem 4.1. [6, Thm. 2.3] *Let $\sigma \in S_n$ and $\mathcal{R}(\sigma) = \{(h_1, t_1), (h_2, t_2), \ldots, (h_k, t_k)\}$. We have $\sigma \in S_n(321, 3\bar{1}42)$ if and only if $t_{j-1} + 2 \leq t_j$ for $1 < j \leq k$.*

Here are some example,

- the canonical reduced decomposition of the identity $id = 12\cdots n$ is the identity since $\mathcal{R}(id) = \emptyset$;

- the canonical reduced decomposition of $\pi = n\ n-1\cdots 21$ is π is $\pi = s_1/s_2s_1/s_3s_2s_1/\cdots/s_{n-1}s_{n-2}\cdots s_1$ since $\mathcal{R}(\pi) = \{(1,1), (2,1), \ldots, (n-1,1)\}$;

- if $\sigma = 6\,1\,7\,2\,3\,8\,4\,10\,5\,11\,9\,15\,12\,16\,13\,14$, we have

  $$\mathcal{R}(\sigma) = \{(5,1), (6,3), (7,6), (9,8), (10,10), (14,12), (15,14)\}.$$

  Thus, the canonical reduced decomposition of σ is $\sigma_1\sigma_2\cdots\sigma_7$, with $\sigma_1 = s_5s_4s_3s_2s_1$, $\sigma_2 = s_6s_5s_4s_3$, $\sigma_3 = s_7s_6$, $\sigma_4 = s_9s_8$, $\sigma_5 = s_{10}$, $\sigma_6 = s_{14}s_{13}s_{12}$, and $\sigma_7 = s_{15}s_{14}$.

The following theorem is an extension of Theorem 2.4 proved by Chen et al. .

Theorem 4.2. *For any* $\sigma \in S_n(321, 3\bar{1}42)$*, if* $\mathcal{R}(\sigma) = \{(h_r, t_r)\}_{1\le r\le k}$*, then we have* $Des(\sigma) = Exc(\sigma) = \{t_1, t_2, \dots, t_k\}$*.*

Proof. Let $\sigma \in S_n(321, 3\bar{1}42)$. Since σ is 321-avoiding, then it is nonnesting (see. [15, Lem. 5.1]). Consequently, if $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{n-k}$ are respectively excedances and non-excedances of σ, then we have $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)$ and $\sigma(j_1) < \sigma(j_2) < \cdots < \sigma(j_{n-k})$. It is clear that, when we apply the above procedure, we get $\mathcal{R}(\sigma) = \{(\sigma(i_1) - 1, i_1), (\sigma(i_2) - 1, i_2), \dots, (\sigma(i_k) - 1, i_k)\}$. So, we have $Exc(\sigma) = \{i_1, i_2, \dots, i_k\}$. For any $1 \le j \le k$, since $i_j + 2 < i_{j+1}$ (see Theorem 4.1), then $i_j + 1$ is a non-excedance of σ and consequently we have $\sigma(i_j) > \sigma(i_j + 1)$. Thus, excedances of σ are all descents of σ. Furthermore, for some x, a non-excedance j_x of σ can not be descent of σ because j_x is followed by, either an excedance and we have $\sigma(j_x) < j_x + 1 < \sigma(j_x + 1)$, or a non-excedance and we have $\sigma(j_x) < \sigma(j_x + 1) < j_x + 1$. Thus, we get $Des(\sigma) = Exc(\sigma)$. □

Now, we recall the (x+y)-labeling and the strip decomposition of a Motzkin path. We call the *(x+y)-labeling* of a Motzkin path P the action of labeling all cells between P and the x-axis, except the triangular cells containing an up-step, by the sum of the coordinates of their left bottom coin. We now define the strip decomposition of a Motzkin path. Suppose $P = P_{n,k}$ is a Motzkin path of length n that contains k up steps. If $k = 0$, then the strip decomposition of $P_{n,0}$ is simply the empty set. For any $P_{n,k} \in \mathcal{M}_n$, let $A \to B$ be the last up step and $E \to F$ the last down step on $P_{n,k}$. Then we define the strip of $P_{n,k}$ as the path from B to F along the path $P_{n,k}$. Now we move the points from B to E one layer lower, namely, subtract the y-coordinate by 1, and denote the adjusted points by $B_0, \dots, E_0$. We form a new Motzkin path by using the path $P_{n,k}$ up to the point A, then joining the point A to B_0 and following the adjusted segment until we reach the point E_0, then continuing with the points on the x-axis to reach the destination $(n, 0)$. Denote this Motzkin path by $P_{n,k-1}$, which may end with some horizontal steps. From the strip of $P_{n,k}$, we may define the value h_k as the label of the cell containing the step $E \to F$. Clearly, we have $h_k \le n - 1$. The value t_k is defined as the label of the cell containing the step starting from the point B.

Iterating the above procedure, we obtain the set $\mathcal{S}(P) := \{(h_i, t_i)\}_{1\le i\le k}$, called the *strip decomposition* of P, satisfying

$$h_j < h_{j+1} \text{ and } t_j + 2 < t_{j+1} \text{ for any } 1 \le i < k. \tag{4.1}$$

The strip decomposition of the Motzkin path $P = uuhduudddudduuhdd$ drawn in Figure 5 is $\mathcal{S}(P) = \{(5, 1), (6, 3), (7, 6), (9, 8), (10, 10), (14, 12), (15, 14)\}$. It is known that every Motzkin path P can be determined from the set $\mathcal{S}(P)$ satisfying (4.1) by inversing the above procedure (see [6, Fig. 2]).

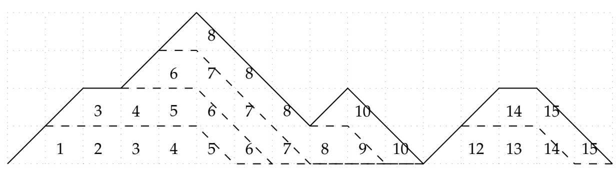


Figure 5: The $(x + y)$-labelling and the strip decomposition of a Motzkin path.

According to Theorem 4.1, each strip decomposition of a Motzkin path P, i.e. $\mathcal{S}(P) := \{(h_i, t_i)\}_{1\le i\le k}$, satisfying 4.1 is associated with a permutation $\sigma \in S_n(321, 3\bar{1}42)$ throughout its reduced decomposition deduced from $\mathcal{R}(\sigma) = \{(h_i, t_i)\}_{1 \le i \le k}$, i.e. $\sigma_1 \sigma_2 \cdots \sigma_k$, where $\sigma_j = s_{h_j} s_{h_j - 1} \cdots s_{t_j}$. In other words, the bijection Φ_3 of Chen et al. from $\mathcal{M}_n$ to $S_n(321, 3\bar{1}42)$ is defined as follows: for any $P \in \mathcal{M}_n$

$$\sigma = \Phi_3(P) \text{ if and only if } \mathcal{R}(\sigma) = \mathcal{S}(P).$$

Proposition 4.3. *Let $P \in \mathcal{M}_n$ and $\sigma = \Phi_3(P)$. We have*

$$(\mathrm{exc}, \mathrm{cr})(\sigma) = (\mathrm{up}, \mathrm{sh}_u + \mathrm{sh}_h)(P).$$

Proof. Since the strip decomposition of P is associated with up steps, we have $\mathrm{up}(P) = |\mathcal{S}(P)| = |\mathcal{R}(\sigma)|$. Moreover, using Theorem 4.2, we obtain $|\mathcal{R}(\sigma)| = \mathrm{exc}(\sigma)$. So, we get $\mathrm{up}(P) = \mathrm{exc}(\sigma)$. Chen et al. proved that we have $\mathrm{inv}(\sigma) = (\mathrm{area} - \mathrm{sh}_u)(P)$ (see [6, Thm. 3.2]). Finally, we get

$$\begin{aligned} \mathrm{cr}(\sigma) &= \mathrm{inv}(\sigma) - \mathrm{exc}(\sigma) \ \text{ using (1.1) with } \mathrm{nes}(\sigma) = 0 \\ &= \mathrm{area}(P) - \mathrm{sh}_u(P) - \mathrm{up}(P) \\ &= \mathrm{sh}_u(P) + \mathrm{sh}_h(P) \text{ using (2.1) of Proposition 2.2.} \end{aligned}$$

This ends the proof of our proposition. □

Theorem 4.4. *We have*

$$\sum_{\sigma \in S(321, 3\bar{1}42)} y^{\mathrm{exc}(\sigma)} q^{\mathrm{cr}(\sigma)} t^{|\sigma|} = \cfrac{1}{1 - t - \cfrac{yt^2}{1 - qt - \cfrac{yqt^2}{1 - q^2 t - \cfrac{yq^2t^2}{\ddots}}}}. \tag{4.2}$$

Proof. By the preceding proposition, we can get $\sum_{\sigma \in S_n(321, 3\bar{1}42)} y^{\mathrm{exc}(\sigma)} q^{\mathrm{cr}(\sigma)} = I_n(1, y, q, q)$. By setting $a = 1$, $b = y$ and $c = d = q$ in (2.2), we obtain the theorem. □

We now establish the proof of the second part of our main result concerning the recursion for the polynomials $\sum_\sigma q^{\mathrm{cr}(\sigma)}$, where the sum runs over all $\sigma \in S_n(321, 3\bar{1}42)$. For that, we recall the Stieltjes tableau $(h_{n,i})$ of two sequences (α_i) and (β_i) considered by Dumont [9] in his master course on the J-continued fractions and Motzkin paths. He defined the tableau $(h_{n,i})$ through the following recurrence

$$\begin{cases} h_{0,0} &= 1, \text{ and } h_{n,i} = 0 \text{ if } i < 0 \text{ or } i > n; \\ h_{n,i} &= \beta_i h_{n-1,i-1} + \alpha_{i+1} h_{n-1,i} + h_{n-1,i+1}, \text{ elswhere.} \end{cases} \tag{4.3}$$

Using recursion (4.3), Dumont sowed the following theorem. To prove it, we can inspired from the paper of Dumont and Randrianarivony [10].

Theorem 4.5. [9] *We have*

$$\cfrac{1}{1 - \alpha_1 t - \cfrac{\beta_1 t^2}{1 - \alpha_2 t - \cfrac{\beta_2 t^2}{1 - \alpha_3 t - \cfrac{\beta_3 t^2}{\ddots}}}} = h_{0,0} + h_{1,0} t + h_{2,0} t^2 + \cdots + h_{n,0} t^n + \cdots . \tag{4.4}$$

Here, we are interested on the particular Stieltjes tableau $(H_{n,i})$ obtained from (α_i) and (β_i) with $\alpha_i = \beta_i = q^{i-1}$ for any integer $i \geq 0$, i.e.

$$\begin{cases} H_{0,0} &= 1, \text{ and } H_{n,i} = 0 \text{ if } i < 0 \text{ or } i > n; \\ H_{n,i} &= q^{i-1}H_{n-1,i-1} + q^i H_{n-1,i} + H_{n-1,i+1}, \text{ elswhere.} \end{cases} \tag{4.5}$$

As direct corollary of Theorem 4.3, we have

$$\cfrac{1}{1 - t - \cfrac{t^2}{1 - qt - \cfrac{qt^2}{1 - q^2 t - \cfrac{q^2t^2}{\ddots}}}} = H_{0,0} + H_{1,0}t + H_{2,0}t^2 + \cdots + H_{n,0}t^n + \cdots . \tag{4.6}$$

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	$1+q$	q		
3	$3+q$	$2+2q+q^2$	$q+q^2+q^3$	q^3	
4	$5+3q+q^2$	$3+4q+3q^2+2q^3$	$2q+2q^2+3q^3+q^4+q^5$	$q^3+q^4+q^5+q^6$	q^6

Table 1: Values of the q-tableau $(H_{n,i})$ for $0 \leq n, i \leq 4$.

We will prove the following unexpected recursion satisfied by the sequence $(H_{n,i})$.
Theorem 4.6. *For any integers $n \geq 1$ and $1 \leq i \leq n$, we have*

$$H_{n,i} = q^{i-1}(H_{n-1,i-1} + \sum_{k=i-1}^{n-2} q^{1+k} H_{k,i-1} H_{n-1-k,0}). \tag{4.7}$$

Proof. To prove (4.7), we will proceed by induction on n. For that, we can easily verify that (4.7) holds for $n = 1, 2, 3$. Suppose that it holds until some integer n. To prove that it still holds for $n+1$, we treat separately the cases $i = 1$ and $i \geq 2$. By definition, we have

$$H_{n+1,1} = H_{n,0} + qH_{n,1} + H_{n,2}. \tag{4.8}$$

For $H_{n,2}$, we use (4.7) and the relationship $H_{k,1} = H_{k+1,0} - H_{k,0}$ for $k \geq 1$ to obtain

$$\begin{aligned} H_{n,2} &= q(H_{n-1,1} + \sum_{k=1}^{n-2} q^{1+k} H_{k,1} H_{n-1-k,0}) \\ &= q(H_{n,0} - H_{n-1,0} + \sum_{k=1}^{n-2} q^{1+k}(H_{k+1,0} - H_{k,0}) H_{n-1-k,0}) \\ &= q(H_{n,0} + \sum_{k=2}^{n-1} q^k H_{k,0} H_{n-k,0} - (H_{n-1,0} + \sum_{k=1}^{n-2} q^{1+k} H_{k,0} H_{n-1-k,0})). \end{aligned}$$

Using the recurrence hypothesis for $H_{n,1}$, we obtain

$$\begin{aligned} H_{n,2} &= q(H_{n,0} + \sum_{k=2}^{n-1} q^k H_{k,0} H_{n-k,0} - (H_{n,1} - qH_{0,0}H_{n-1,0})) \\ &= q(H_{0,0}H_{n,0} + qH_{1,0}H_{n-1,0} + \sum_{k=2}^{n-1} q^k H_{k,0}H_{n-k,0} - H_{n,1}) \text{ (since } H_{0,0} = H_{1,0} = 1) \\ &= \sum_{k=0}^{n-1} q^{1+k} H_{k,0}H_{n-k,0} - qH_{n,1}. \end{aligned}$$

Thus, when returning to (4.8), we obtain

$$\begin{aligned} H_{n+1,1} &= H_{n,0} + qH_{n,1} + \sum_{k=0}^{n-1} q^{1+k} H_{k,0}H_{n-k,0} - qH_{n,1} \\ &= H_{n,0} + \sum_{k=0}^{n-1} q^{1+k} H_{k,0}H_{n-k,0}. \end{aligned}$$

Now, we can treat by the same way the case $i \geq 2$ as follows. We start from the relation

$$H_{n+1,i} = q^{i-1}H_{n,i-1} + q^i H_{n,i} + H_{n,i+1}. \tag{4.9}$$

Then, using (4.7) for $H_{n,i+1}$, we get

$$H_{n,i+1} = q^{i-1}(H_{n-1,i} + \sum_{k=i}^{n-2} q^{1+k} H_{k,i} H_{n-1-k,0}).$$

Now, replace $H_{k,i}$ by $H_{k+1,i-1} - q^{i-2}H_{k,i-2} - q^{i-1}H_{k,i-1}$ for any integer $i \geq 2$ and manipulate the obtained identity to get

$$\begin{aligned} H_{n,i+1} = q^i(H_{n,i-1} + \sum_{k+1}^{n-1} q^k H_{k,i-1}H_{n-k,0} - \underbrace{q^{i-2}(H_{n-1,i-2} + \sum_{k=i}^{n-2} q^{1+k} H_{k,i-2}H_{n-1-k,0})}_{A} \\ \underbrace{- q^i(H_{n-1,i-1} + \sum_{k=i}^{n-2} q^{1+k} H_{k,i-1}H_{n-1-k,0}))}_{B} \end{aligned}$$

Using the recurrence hypothesis for $H_{n,i-1}$ and $H_{n,i}$, we get

$$\begin{aligned} A &= H_{n,i-1} - q^{i-2}(q^{i-1}H_{i-2,i-2}H_{n+1-i,0} + q^i H_{i-1,i-2}H_{n-i,0}) \\ &= H_{n,i-1} - q^{i-2}(H_{i-1,i-1}H_{n+1-i,0} + q^i H_{i-1,i-2}H_{n-i,0}) \text{ since } q^{i-1}H_{i-2,i-2} = H_{i-1,i-1}. \end{aligned}$$

and $B = H_{n,i} + q^{i-1}(q^i H_{i-1,i-1}H_{n-i,0})$. Thus, using also the fact that $q^{i-2}H_{i-1,i-2} + q^{i-1}H_{i-1,i-1} = H_{i,i-1}$, we get

$$A + B = H_{n,i} + H_{n,i-1} - q^{i-2}H_{i-1,i-1}H_{n+1-i,0} - q^i H_{i,i-1}H_{n-i,0}.$$

Consequently, we obtain

$$\begin{aligned} H_{n,i+1} &= q^i(q^{i-2}H_{i-1,i-1}H_{n+1-i,0} + q^i H_{i,i-1}H_{n-i,0} + \sum_{k=i+1}^{n-1} q^k H_{k,i-1}H_{n-k,0} - H_{n,i}) \\ &= q^i(\sum_{k=i-1}^{n-1} q^k H_{k,i-1}H_{n-k,0} - H_{n,i}) \\ &= q^{i-1}\sum_{k=i-1}^{n-1} q^{1+k}H_{k,i-1}H_{n-k,0} - q^i H_{n,i}. \end{aligned}$$

So, when returning to (4.9) and substitute $H_{n,i+1}$, we obtain

$$H_{n+1,i} = q^{i-1}(H_{n,i-1} + \sum_{k=i-1}^{n-1} q^{1+k}H_{k,i-1}H_{n-k,0}).$$

This also ends the proof of Theorem 4.6. □

Corollary 4.7. *For any positive integer n, we have $H_{n,0} = \tilde{M}_n(q)$.*

Proof. Using (4.5) with the fact that $H_{n,0} = H_{n-1,0} + H_{n-1,1}$ for any integer $n \geq 0$, we get

$$\begin{aligned} H_{n,0} &= H_{n-1,0} + H_{n-2,0} + \sum_{k=0}^{n-3} q^{1+k}H_{k,0}H_{n-1-k,0} \\ &= H_{n-1,0} + \sum_{k=0}^{n-2} q^{1+k-(n-1)\delta_{k,n-2}}H_{k,0}H_{n-1-k,0}. \end{aligned}$$

Since $H_{n,0}$ and $\tilde{M}_n(q)$ satisfy the same recursion (see (1.5)), then they are equal. □

Combining this last corollary with identity (4.6), we obtain our desired result.

Theorem 4.8. *For any positive integer n, we have*

$$\sum_{\sigma \in S_n(321,3\bar{1}42)} q^{\mathrm{cr}(\sigma)} = \tilde{M}_n(q).$$

We close this section with an interesting non trivial identity concerning equality of two continued fractions as another consequence of Corollary 4.7.

Theorem 4.9. *We have he following identity*

$$\cfrac{1}{1-\cfrac{t+t^2}{1-\cfrac{qt^2}{1-\cfrac{qt+(qt)^2}{1-\cfrac{q^3t^2}{\ddots}}}}} = \cfrac{1}{1-t-\cfrac{t^2}{1-qt-\cfrac{qt^2}{1-q^2t-\cfrac{q^2t^2}{\ddots}}}}.$$

Proof. Since the polynomial $\tilde{M}_n(q)$ satisfies the following recursion

$$\tilde{M}_0(q) = 1, \tilde{M}_n(q) = \tilde{M}_{n-1}(q) + \tilde{M}_{n-2}(q) + \sum_{k=0}^{n-3} q^{1+k}\tilde{M}_k(q)\tilde{M}_{n-2-k}(q) \text{ for } n \geq 1,$$

then its generating function, $\tilde{M}(q;t) := \sum_{n\geq 0} \tilde{M}_n(q)t^n$, satisfies

$$\tilde{M}(q;t) = \cfrac{1}{1 - \cfrac{t+t^2}{1 - qt^2\tilde{M}(q;qt)}}.$$

So, when we continue to develop $\tilde{M}(q;q^k t)$ for $k \geq 1$, we obtain the continued fraction expansion for $\tilde{M}(q;t)$

$$\tilde{M}(q;t) = \cfrac{1}{1 - \cfrac{t+t^2}{1 - \cfrac{qt^2}{1 - \cfrac{qt+(qt)^2}{1 - \cfrac{q^3t^2}{\ddots}}}}}.$$

Combining Corollary 4.7 with identity (4.6), we also have

$$\tilde{M}(q;t) = \cfrac{1}{1 - t - \cfrac{t^2}{1 - qt - \cfrac{qt^2}{1 - q^2t - \cfrac{q^2t^2}{\ddots}}}}.$$

This proves the desired identity of our theorem. □

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