Feynman checkers:
towards algorithmic quantum theory

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arXiv:2007.12879v1 [math-ph] 25 Jul 2020

Abstract

We survey and develop the most elementary model of electron motion introduced by R.Feynman. It is a game, in which a checker moves on a checkerboard by simple rules, and we count the turns. It is also known as a one-dimensional quantum walk or an Ising model at imaginary temperature. We solve mathematically a problem by R.Feynman from 1965, which was to prove that the model reproduces the usual quantum-mechanical free-particle kernel for large time, small average velocity, and small lattice step. We compute the small-lattice-step and the large-time limits, justifying heuristic derivations by J.Narlikar from 1972 and by A.Ambainis et al. from 2001. For the first time we observe and prove concentration of measure in the former limit. We perform the second quantization of the model. The main tools are the Fourier transform and the stationary phase method.

Keywords and phrases. Feynman checkerboard, quantum walk, Ising model, Young diagram, Dirac equation, stationary phase method

MSC2010: 82B20, 11L03, 68Q12, 81P68, 81T25, 81T40, 05A17, 11P82, 33C45.

Contents

1 Introduction

We survey and develop the most elementary model of electron motion introduced by R. Feynman (see Figure 1). It is a game, in which a checker moves on a checkerboard by simple rules, and we count the number of turns (see Definition 2). It can be viewed as a particular case of a 1-dimensional quantum walk, or an Ising model, or count of Young diagrams of certain type.

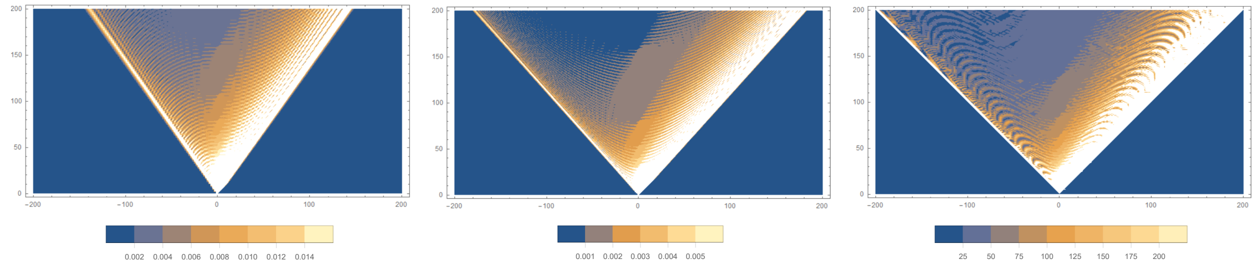


Figure 1: The probability to find an electron in a small square around a given point (white depicts strong oscillations of the probability). Left: in the basic model from §2 (cf. [45, Figure 6]). Middle: in the upgrade from §3 for smaller square side. Right: in continuum theory. For the latter, the relative probability density is depicted.

1.1 Motivation

The simplest way to understand what is the model about is the classical *double-slit experiment* (see Figure 2). In this experiment, a (*coherent*) beam of electrons is directed towards a plate pierced by two parallel slits, and the part of the beam passing through the slits is observed on a screen behind the plate. If one of the slits is closed, then the beam illuminates a spot on the screen. If both slits are open, one would expect a larger spot, but in fact one observes a sequence of bright and dark bands (*interferogram*).

This shows that electrons behave like a wave: the waves travel through both slits, and the contributions of the two paths either amplify or cancel each other depending on the final phases.

Further, if the electrons are sent through the slits one at time, then single dots appear on the screen, as expected. Remarkably, however, the same interferogram with bright and dark bands emerges when the electrons are allowed to build up one by one. One cannot predict where a particular electron hits the screen; all we can do is to compute the probability to find the electron at a given place.

The *Feynman sum-over-paths* (or *path integral*) method of computing such probabilities is to assign phases to all possible paths and to sum up the resulting waves (see [10, 11]). *Feynman checkers* (or *Feynman checkerboard*) is a particularly simple combinatorial rule for those phases in the case of an electron freely moving (or better jumping) in 1 space and 1 time dimension.

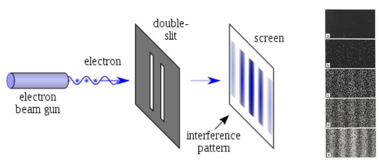


Figure 2: (from Wikipedia) Double-slit experiment

1.2 Background

The beginning. The checkers model was invented by R. Feynman in 1940s [39] and first published in 1965 [11]. In Problem 2.6 there, a function on a lattice of small step ε was constructed (called *kernel*; see (5)) and the following task was posed:

> If the time interval is very long ($t_b - t_a \gg \hbar/mc^2$) and the average velocity is small [$x_b - x_a \ll c(t_b - t_a)$], show that the resulting kernel is approximately the same as that for a free particle [given in Eq. (3-3)], except for a factor $\exp[(imc^2/\hbar)(t_b - t_a)]$.

Mathematically, this means that the kernel (divided by $2i\varepsilon \exp[(-imc^2/\hbar)(t_b - t_a)]$) asymptotically equals *free-particle kernel* (4) (this is Eq. (3-3) from [11]) in the triple limit when time tends to infinity, whereas the average velocity and the lattice step tend to zero (see Table 1 and Figure 3). Both scaling by the lattice step and tending it to zero were understood, otherwise the mentioned "exceptional" factor would be different (see the end of §2). We show that the assertion, although incorrect literally, holds under mild assumptions (see Corollary 4).

Although the Feynman problem might seem self-evident for theoretical physicists, even the first step of a mathematical solution (disproving the assertion as stated) is not found in literature. As usual, the main difficulty is to prove the convergence rather than to guess the limit. The same concerns the results below: previously they were derived heuristically rather than proved mathematically, by an approximate computation without estimating the error.

propagator	continuum	lattice	context	references
free-particle kernel	(4)	-	quantum mechanics	[11, (3-3)]
spin-1/2 retarded propagator	(20),(21)	(5)	relativistic quantum mechanics	cf. [22, (13)] and [11, (2-27)]
spin-1/2 Feynman propagator	(28),(29)	(26)	quantum field theory	cf. [3, §9F]

Table 1: Expressions for the *propagators* of a particle freely moving in 1 space and 1 time dimension. The meaning of the norm square of a propagator is the relative probability density to find the particle at a particular point, or alternatively, the charge density at the point.

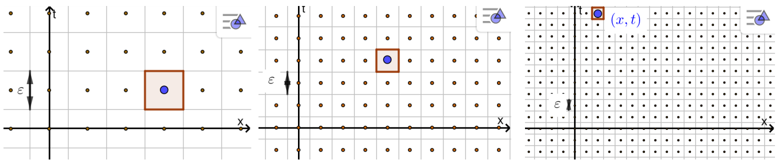


Figure 3: (by V. Skopenkova) The Feynman triple limit: $t \to +\infty$, $x/t \to 0$, $\varepsilon \to 0$

In 1972 J. Narlikar discovered that the above kernel reproduces the *spin-1/2 retarded propagator* in the different limit when the lattice step tends to zero but time stays fixed [33] (see Table 1, Figures 4 and 1, Corollary 5). In 1984 T.Jacobson–L.Schulman repeated this derivation, applied *stationary phase method* among other bright ideas, and found the probability of changing the movement direction [22] (cf. Theorem 5).

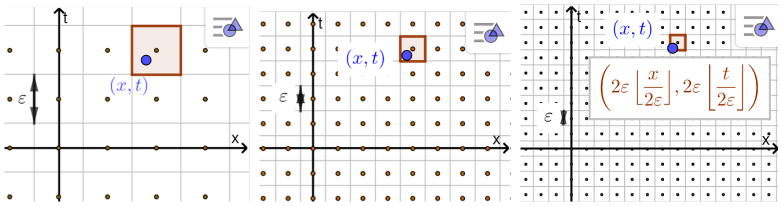


Figure 4: Continuum limit: the point (x, t) stays fixed while the lattice step ε tends to zero

Ising model. In 1981 H. Gersch noticed that Feynman checkers can be viewed as a 1-dimensional Ising model with *imaginary* temperature or edge weights (see §2.2 and [16], [22, §3]). Imaginary values of these quantities are usual in physics (e.g., in quantum field theory or in alternating current networks). Due to the imaginarity, contributions of most configurations cancel each other, which makes the model highly nontrivial in spite of being 1-dimensional. In particular, the model exhibits a phase transition (see Figures 1 and 5). Surprisingly, the latter seems to have never been reported before. Phase transitions were studied only in more complicated 1-dimensional Ising models [24, §III], [32, 19], in spite of a known equivalent result, which we are going to discuss now (see Theorem 1(B)).

Quantum walks. In 2001 A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous performed a breakthrough [2]. They studied Feynman checkers under names *one-dimensional quantum walk* and *Hadamard walk*; although those cleverly defined models were completely equivalent to Feynman's simple one. They computed the large-time limit of the model (see Theorem 3). They discovered several striking properties having sharp contrast with both continuum theory and the classical random walks. First, the most probable average electron velocity in the model equals $1/\sqrt{2}$ of the speed of light, and the probability of exceeding this value is very small (see Figures 5 and 1 to the left and Theorem 1(B)). Second, if an absorbing boundary is put immediately to the left of the starting position, then the probability that the electron is absorbed is $2/\pi$. Third, if an additional absorbing boundary is put at location $x > 0$, the probability that the electron is absorbed to the left actually increases, approaching $1/\sqrt{2}$ in the limit $x \to +\infty$. Recall that in the classical case both absorption probabilities are 1. In addition, they found many combinatorial identities and expressed the above kernel through the values of Jacobi polynomials at a particular point (see Remark 3; cf. [42, §2]).

N. Konno generalized those results to a *biased quantum walk* [28, 29], which is still essentially equivalent to Feynman checkers. He found the distribution of the electron position in the (weak) large-time limit (see Figure 5 and Theorem 1(B)). Later those results were generalized to quantum walks on graphs and applied to quantum algorithms. We refer to the surveys by N. Konno, J. Kempe, and S.E. Venegas-Andraca [29, 26, 44] for further details in this direction.

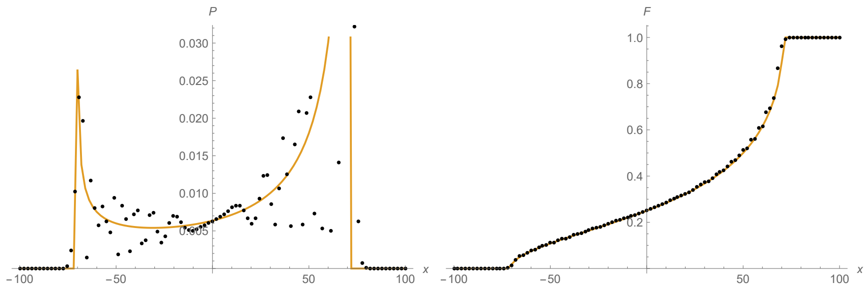


Figure 5: The distribution (left) and the cumulative distribution function (right) of the electron position x at time $t = 100$ in natural units for the basic model from §2 (dots). Their (weak) scaling limits as $t \to \infty$ (curves). The left curve is also the "limiting partition function norm squared" in the Ising model. The discontinuity of the function reflects a phase transition.

Lattice quantum field theories. In a more general context, this is a direction towards creation of *Minkowskian* lattice quantum field theory, with both space and time being discrete [3]. In 1970s F. Wegner and K. Wilson introduced *lattice gauge theory* as a computational tool for gauge theory describing all known interactions (except gravity); see [31] for a popular-science introduction. This culminated in determining the proton mass theoretically with error less than 2% in a sense. This theory is *Euclidean* in the sense that it involves *imaginary* time. Likewise, an asymptotic formula for the Green function for the (massless) *Euclidean* lattice Dirac equation [27, Theorem 4.3] played a crucial role in the continuum limit of the Ising model performed by D. Chelkak–S. Smirnov [8]. Similarly, asymptotic formulae for the *Minkowskian* one (Theorems 3–4) can be useful for missing Minkowskian lattice quantum field theory. Several authors argue that Feynman checkers has the advantage of *no fermion doubling* and avoids the Nielsen–Ninomiya no-go theorem [5, 14].

Several upgrades of Feynman checkers have been discussed. For instance, around 1990s B. Gaveau–L. Schulman and G. Ord added electromagnetic field to the model [15, 35]. That time they achieved neither exact charge conservation nor generalization to non-Abelian gauge fields; this is fixed in Definition 3. Another example is adding mass matrix by P. Jizba [23].

It is an old dream to incorporate also checker paths turning backwards in time or forming cycles [39, p. 481–483], [21]; this would mean creation of electron-positron pairs, celebrating a passage from quantum mechanics to quantum field theory. One looks for a combinatorial model reproducing the *Feynman propagator* rather than the *retarded* one in the continuum limit (see Table 1). So far, most of the known constructions just redefine the *same* model (e.g., the title of [36] is misleading: the Feynman propagator is not discussed there). In the *massless* case, a Feynman propagator on the lattice was constructed by C. Bender–L. Mead–K. Milton–D. Sharp in [3, §9F] and [4, §IV], but the construction was not combinatorial; cf. Definition 5.

Another long-standing open problem is to generalize the model to the 4-*dimensional* real world. In his Nobel prize lecture, R. Feynman mentioned his own unsuccessful attempts. There are several recent approaches, e.g., by B. Foster–T. Jacobson from 2017 [14]. Those are not yet as simple and beautiful as the original 2-dimensional model, as it is written in [14, §7.1] itself.

On physical and mathematical works. The physical literature on the subject is quite extensive [44], and we cannot mention all remarkable works in this brief overview. Many papers are well-written, insomuch that the *physical* theorems and proofs there could be carelessly taken for *mathematical* ones (see the end of §12). Conversely, we have found a paper [18] by G.R. Grimmett–S. Janson–P.F. Scudo in a *physical* journal, containing a *mathematical* proof of an important result (Theorem 1). Surprisingly, in the whole literature we have not found the shouting property of *concentration of measure* for lattice step tending to zero (see Corollary 6). We are also not aware of a single mathematical work on the subject before the present paper.

1.3 Contributions

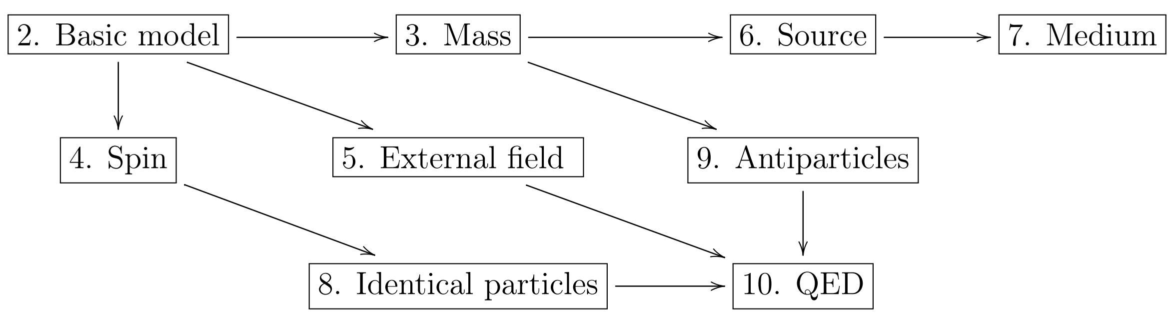
We solve mathematically a problem by R. Feynman from 1965, which was to prove that his model reproduces the usual quantum-mechanical free-particle kernel for large time, small average velocity, and small lattice step (see Corollary 4). We compute the large-time and small-lattice-step limits (see Corollaries 1 and 5) and asymptotic formulae (see Theorems 3–4). For the first time we observe and prove concentration of measure in the latter limit: the average velocity of an electron emitted by a point source is close to the speed of light with high probability (see Corollary 6). The results can be interpreted as asymptotic properties of Young diagrams (see Corollary 2) and Jacobi polynomials (see Remark 3).

All these results are proved mathematically for the first time. For their statements, just Definition 2 and Remark 1 are sufficient. In Definitions 3–5 we perform a coupling to lattice gauge theory and the second quantization of the model, promoting Feynman checkers to a full-fledged lattice quantum field theory.

1.4 Organization of the paper and further directions

First we give the definitions and precise statements of the results, and in the process provide a zero-knowledge examples for basic concepts of quantum theory. These are precisely those examples that Feynman presents first in his own books: Feynman checkers (see §3) is the first specific example in the whole book [11]. The thin-film reflection (see §7) is the first example in [10]; see Figures 10–11 there. Thus we hope that these examples could be enlightening to readers unfamiliar with quantum theory.

We start with the simplest (and rough) particular case of the model and upgrade it step by step in each subsequent section. Before each upgrade, we summarize which physical *question* does it address, which simplifying *assumptions* does it resolve or impose additionally, and which experimental or theoretical *results* does it explain. Some upgrades (§§7–9) are just announced to be discussed in a subsequent publication. Our aim is *(1+1)-dimensional lattice quantum electrodynamics* ("QED") but the last step on this way (mentioned in §10) is still undone. Open problems are collected in §11. For easier navigation, we present the upgrades-dependence chart:



Hopefully this is a promising path to making quantum field theory rigorous and algorithmic. An *algorithmic* quantum field theory would be a one which, given an experimentally observable quantity and a number $\delta > 0$, would provide a *precise statement* of an algorithm predicting a value for the quantity within accuracy δ. (Surely, the predicted value needs not to agree with the experiment for δ less than accuracy of theory itself.) See Algorithm 1 for a toy example. This would be an extension of *constructive* quantum field theory (currently far from being algorithmic). Application of quantum theory to computer science is in mainstream now, but the opposite direction could provide benefit as well. (Re)thinking algorithmically is a way to make a subject available to nonspecialists, as it is happening with, say, algebraic topology.

The paper is written in a mathematical level of rigor, in the sense that all the definitions, conventions, and theorems (including corollaries, propositions, lemmas) should be understood literally. Theorems remain true, even if cut out from the text. The proofs of theorems use the statements but not the proofs of the other ones. Most statements are much less technical than the proofs; hence the proofs are kept in a separate section (§12) and long computations are kept in [40]. In the process of the proofs, we give a zero-knowledge introduction to the main tools to study the model: combinatorial identities, the Fourier transform, the method of moments, the stationary phase method, contour integration, the Hardy–Littlewood circle method. Remarks are informal and usually not used elsewhere (hence skippable).

2 Basic model (Hadamard walk)

Question: what is the probability to find an electron in the square (x,t), if it was emitted from $(0,0)$?
Assumptions: no self-interaction, no creation of electron-positron pairs, fixed mass and lattice step, point source; the electron moves either in a plane "uniformly along the t-axis", or along a line (and then t is time).
Results: double-slit experiment (qualitative explanation), charge conservation, large-time limiting distribution.

2.1 Definition and examples

We first give an informal definition of the model in the spirit of [10] and then a precise one.

On an infinite checkerboard, a checker moves to the diagonal-neighboring squares, either upwards-right or upwards-left. To each path s of the checker, assign a vector $a(s)$ as follows. Take a stopwatch that can time the checker as it moves. Initially the stopwatch hand points upwards. While the checker moves straightly, the hand does not rotate, but each time when the checker changes the direction, the hand rotates through 90° clockwise (independently of the direction the checker turns). The final direction of the hand is the direction of the required vector $a(s)$. The length of the vector is set to be $1/2^{(t-1)/2}$, where t is the total number of moves (this is just a normalization). For instance, for the path in Figure 6 to the top-left, the vector $a(s) = (1/8, 0)$ points to the right and has length $1/8$.

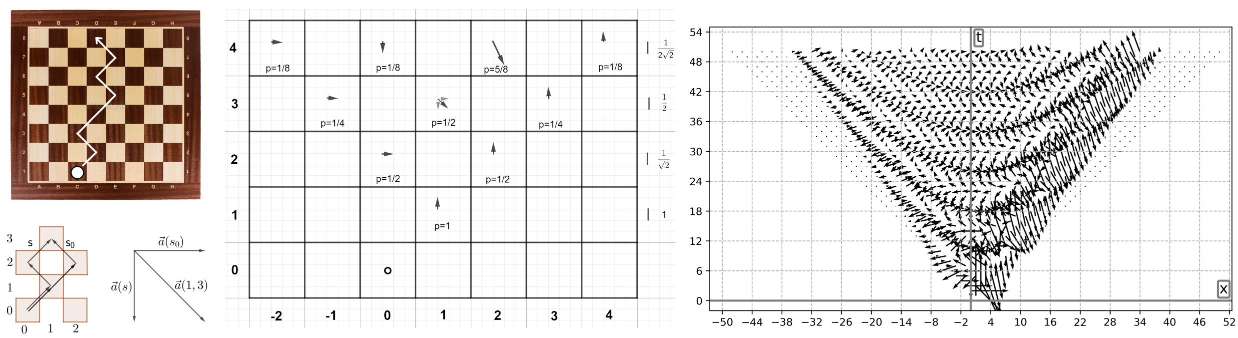


Figure 6: (by V. Skopenkova, M. Fedorov) Checker paths (left). The arrows $a(x,t)$ and $P(x,t)$ for small x, t (middle; the scale depends on the row). The arrows $10 \cdot a(x,t)$ for $t \leq 50$ (right).

Denote by $a(x,t) := \sum_s a(s)$ the sum over all the checker paths from the square $(0,0)$ to the square (x,t), *starting with the upwards-right move*. For instance, $a(1,3) = (0,-1/2)+(1/2,0) = (1/2,-1/2)$; see Figure 6 to the bottom-left. The length square of the vector $a(x,t)$ is called the *probability to find an electron in the square* (x,t), *if it was emitted from* $(0,0)$ (see §2.2 for a discussion of the terminology). The vector $a(x,t)$ itself is called the *arrow* [10, Figure 6].

Let us summarize this construction rigorously.

Definition 1. A *checker path* is a finite sequence of integer points in the plane such that the vector from each point (except the last one) to the next one equals either $(1,1)$ or $(-1,1)$. A *turn* is a point of the path (not the first and not the last one) such that the vectors from the point to the next and to the previous ones are orthogonal. The *arrow* is the complex number

$$a(x,t) := 2^{(1-t)/2}\, i \sum_{s} (-i)^{\text{turns}(s)},$$

where the sum over all checker paths s from $(0,0)$ to (x,t) with the first step to $(1,1)$, and $\text{turns}(s)$ is the number of turns in s. Hereafter an empty sum is 0 by definition. Denote

$$P(x,t) := |a(x,t)|^2, \qquad a_1(x,t) := \operatorname{Re} a(x,t), \qquad a_2(x,t) := \operatorname{Im} a(x,t).$$

Points (or squares) (x,t) with even and odd $x+t$ are called *black* and *white* respectively.

Figure 6 to the middle and right depicts the arrows $a(x,t)$ and the probabilities $P(x,t)$ for small x,t. Figure 7 depicts the graphs of $P(x,1000)$, $a_1(x,1000)$, and $a_2(x,1000)$ as functions in an even number x. We see that under variation of the final position x at a fixed large time t, right after the peak the probability falls to very small although still nonzero values. What is particularly interesting is the unexpected position of the peak, far from $x=1000$. In Figure 1 to the left, the color of a point (x,t) with even $x+t$ depicts the value $P(x,t)$. Notice that the sides of the apparent angle are *not* the lines $t=\pm x$, in contrast to what one could expect. The sides are $t=\pm\sqrt{2}x$ (see Theorem 1(A)).

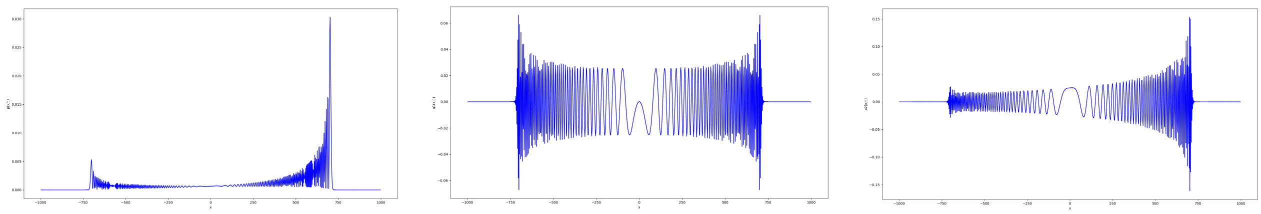


Figure 7: (by A.Daniyarkhodzhaev–F.Kuyanov) The plots of $P(x,1000)$, $a_1(x,1000)$, $a_2(x,1000)$

2.2 Physical interpretation

Let us comment on the physical interpretation of the model and ensure that it captures unbelievable behavior of electrons. In fact there are two different interpretations; see Table 2.

object	standard interpretation	spin-chain interpretation
s	path	configuration of "+" and "−" in a row
$\text{turns}(s)$	number of turns	half of the configuration energy
t	time	volume
x	position	difference between the number of "+" and "−"
x/t	average velocity	magnetization
$a(x,t)$	probability amplitude	partition function up to constant
$P(x,t)$	probability	partition function norm squared
$\dfrac{i\,a_2(x,t)}{a(x,t)}$	conditional probability amplitude of the last move upwards-right	"probability" of equal signs at the ends of the spin chain

Table 2: Physical interpretations of Feynman checkers

Standard interpretation. Here the x- and t-coordinates are interpreted as the electron position and time respectively. Sometimes (e.g., in Example 1) we a bit informally interpret both as position, and assume that the electron performs a "uniform classical motion" along the t-axis. We work in the natural system of units, where the speed of light, the Plank and the Boltzmann constants equal 1. Thus the lines $x = \pm t$ represent motion with the speed of light. Any checker path lies above both lines, i.e in the light cone, which means agreement with relativity: the speed of electron cannot exceed the speed of light.

To think of $P(x,t)$ as a probability, consider the t-coordinate as fixed, and the squares $(-t,t), (-t+2,t), \dots, (t,t)$ as all the possible outcomes of an experiment. For instance, the t-th horizontal might be a screen detecting the electron. We shall see that all the numbers $P(x,t)$ on one horizontal sum up to 1 (Proposition 2), thus indeed can be considered as probabilities. Notice that the probability to find the electron in a set $X \subset \mathbb{Z}$ is $P(X,t) := \sum_{x\in X} P(x,t) = \sum_{x\in X} |a(x,t)|^2$ rather than $\left|\sum_{x\in X} a(x,t)\right|^2$ (cf. [10, Figure 50]).

In reality, one cannot measure the electron position exactly. A fundamental limitation is the electron *reduced Compton wavelength* $\lambda = 1/m \approx 4\cdot 10^{-13}$ meters, where m is the electron mass. Physically, the basic model approximates the continuum by a lattice of step exactly λ. But that is still a rough approximation: one needs even smaller step to prevent accumulation of approximation error at larger distances and times. For instance, Figures 1 and 8 to the left show a finite-lattice-step effect: the average velocity x/t cannot exceed $1/\sqrt{2}$ of the speed of light with high probability. (An explanation in physical terms: lattice regularization cuts off distances smaller than the lattice step, hence small wavelengths, hence large momenta, and hence large velocities.) A more precise model is given in §3: compare the plots in Figure 1.

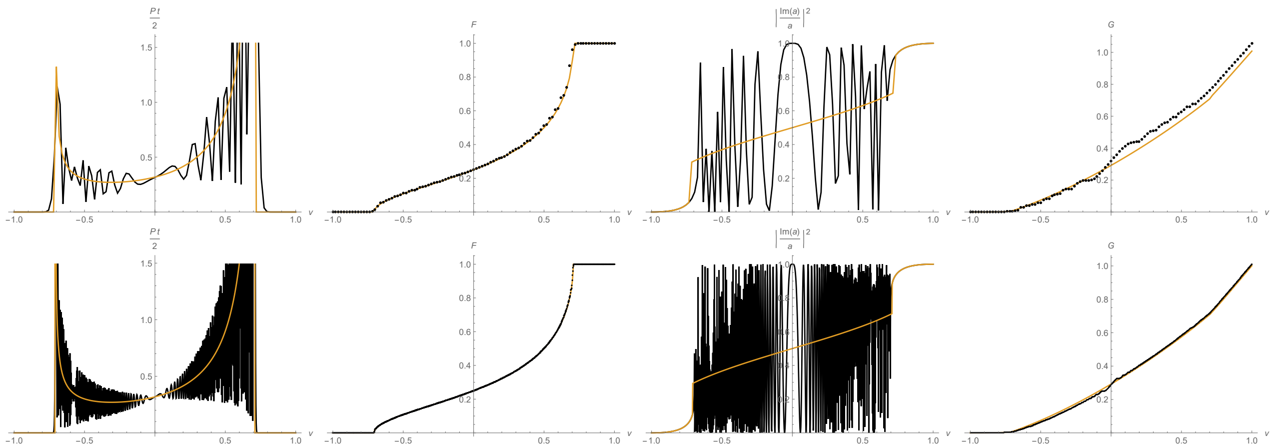


Figure 8: The plots of $\frac{t}{2}P(2\lceil\frac{vt}{2}\rceil,t)$, $F_t(v) := \sum\limits_{x\le vt} P(x,t)$, $\left|\frac{a_2(2\lceil vt/2\rceil,t)}{a(2\lceil vt/2\rceil,t)}\right|^2$, $G_t(v) := \sum\limits_{x\le vt} \frac{2}{t}\left|\frac{a_2(x,t)}{a(x,t)}\right|^2$ (dark) for $t = 100$ (top), $t = 1000$ (bottom), and their distributional limits as $t\to\infty$ (light).

As we shall see now, the model qualitatively captures unbelievable behavior of electrons. (For correct quantitative results, an upgrade involving a *coherent* source is required; see §6.)

The *probability to find an electron in the square* (x,t) *subject to absorption in the square* (x',t') is defined analogously to $P(x,t)$, only the summation is over those checker paths s that do not pass through (x',t'). The probability is denoted by $P(x,t \text{ bypass } x',t')$. Informally, this means an additional outcome of the experiment: the electron has been absorbed and has not reached the screen. For a while let us view the two black squares $(\pm1,1)$ on the horizontal $t=1$ as two slits in a horizontal plate (cf. Figure 2).

Example 1 (Double-slit experiment)**.** *Distinct paths cannot be viewed as "mutually exclusive":*

$$P(0,4) \neq P(0,4 \textit{ bypass } 2,2) + P(0,4 \textit{ bypass } 0,2).$$

Absorption might increase probabilities at some places: $P(0,4) = 1/8 < 1/4 = P(0,4 \textit{ bypass } 2,2)$.

The standard interpretation of Feynman checkers is also known as the *Hadamard walk*, the *1-dimensional quantum walk*, or *quantum lattice gas*. Those are all equivalent but lead to generalizations of the model in distinct directions [44, 29, 45].

Spin-chain interpretation. There is a *very different* physical interpretation of the same model: a 1-dimensional Ising model with imaginary temperature and fixed magnetization.

Recall that a *configuration* in the Ising model is a sequence $\sigma = (\sigma_1, \dots, \sigma_t)$ of ± 1 of fixed length. The *magnetization* and the *energy* of the configuration are $\sum_{k=1}^{t} \sigma_k / t$ and $H(\sigma) = \sum_{k=1}^{t-1} (1 - \sigma_k \sigma_{k+1})$ respectively. The *probability* of the configuration is $e^{-\beta H(\sigma)}/Z(\beta)$, where the *inverse temperature* $\beta = 1/T > 0$ is a parameter and the *partition function* $Z(\beta) := \sum_\sigma e^{-\beta H(\sigma)}$ is a normalization factor. Additional restrictions on configurations σ are usually imposed.

Now, moving the checker along a path s, write "+" for each upwards-right move, and "−" for each upwards-left one; see Figure 9 to the left. The resulting sequence of signs is a configuration in the Ising model, the number of turns in s is one half of the configuration energy, and the ratio of the final x- and t-coordinates is the magnetization. Thus $a(x,t) = \sum_s a(s)$ coincides up to constant with the partition function for the Ising model at the *imaginary* inverse temperature $\beta = i\pi/4$ under the fixed magnetization x/t:

$$a(x,t) = 2^{(1-t)/2}\, i \sum_{\substack{(\sigma_1,\dots,\sigma_t) \in \{+1,-1\}^t: \\ \sigma_1 = +1, \quad \sum_{k=1}^{t-1} \sigma_k = x}} \exp\left(\frac{i\pi}{4} \sum_{k=1}^{t-1} (\sigma_k \sigma_{k+1} - 1) \right).$$

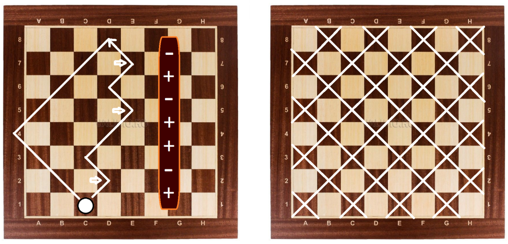


Figure 9: Young diagrams (the arrows point to steps) and the Ising model. The auxiliary grid.

Notice a crucial difference of the resulting spin-chain interpretation from both the usual Ising model and the above standard interpretation. In the latter two models, the magnetization x/t and the average velocity x/t were *random variables*; now the magnetization x/t (not to be confused with an external magnetic field) is an *external condition*. The configuration space in the spin-chain interpretation consists of sequences of "+" and "−" with *fixed* numbers of "+" and "−". Summation over configurations with different x or t would make no sense: e.g., recall that $P(X,t) = \sum_{x \in X} |a(x,t)|^2$ rather than $\left|\sum_{x \in X} a(x,t)\right|^2$.

Varying the magnetization x/t, viewed as an external condition, we observe a *phase transition*. So far it has been proved in a toy sense: the limiting *partition function* $a(x,t)$ is discontinuous when x/t passes through $\pm 1/\sqrt{2}$ (see Theorem 1(B)). The phase transition emerges as $t \to \infty$. Actually the number of oscillations of $a(x,t)$ increases with t; thus the limit exists only in the distributional (weak) sense, and $a(x,t)$ should be scaled by a suitable power of t.

More reasonable order parameters could be the *free energy density* $-\log a(x,t)/\beta t$ and the *"probability" $i\, a_2(x,t)/a(x,t)$ of equal signs at the ends of the spin chain.* These quantities are complex (and even multi-valued) just because the temperature is imaginary. Numerical experiments then confirm the phase transition in the sense of nonanalyticity of the order parameters at the same points $x/t = \pm 1/\sqrt{2}$; see Figures 10 and 8, and Problems 4–6.

(A comment for specialists: the phase transition is not related to accumulation of zeroes of the partition function in the plane of complex parameter β as in [24, §III] and [32]. In our situation, $\beta = i\pi/4$ is fixed, the *real* parameter x/t varies, and the partition function $a(x,t)$ has no zeroes at all [34, Theorem 1].)

Young-diagram interpretation. Our results have also a combinatorial interpretation.

The *number of steps* (or *inner corners*) in a Young diagram with w columns of heights $x_1, \dots, x_w$ is the number of elements in the set $\{x_1, \dots, x_w\}$; see Figure 9. Then the value $2^{(h+w-1)/2}\, a_1(h-w, h+w)$ is the difference between the number of Young diagrams with an odd and an even number of steps, having exactly w columns and h rows.

Interesting behaviour starts already for $h = w$ (see Proposition 4). For $h = w$ even, the difference vanishes. For $h = w = 2n+1$ odd, it is $(-1)^n \binom{2n}{n}$. Such 4-periodicity roughly

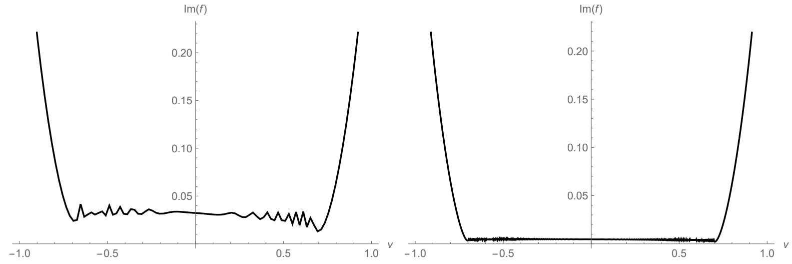


Figure 10: The graphs of the imaginary part of the free energy density $\operatorname{Im} f_t(v) := -\frac{4}{\pi t}\log\left|a(2\lceil\frac{vt}{2}\rceil,t)\right|$ for $t=100$ (left) and $t=1000$ (right)

remains for h close to w (see Theorem 2). For fixed half-perimeter $h+w$, the difference slowly oscillates as h/w increases, attains a peak at $h/w \approx 3+2\sqrt{2}$, and then harshly falls to very small values (see Corollary 2 and Theorem 3).

Similarly, $2^{(h+w-1)/2}a_2(h-w,h+w)$ is the difference between the number of Young diagrams with an even and an odd number of steps, having exactly w columns and *less* than h rows. The behaviour is similar. The upgrade in §3 is related to *Stanley character polynomials* [42, §2].

Discussion of the definition. Now compare Definition 1 with the ones in the literature.

The notation "a" comes from "*a*rrow" and "probability *a*mplitude"; other names are "wave-function", "kernel", "the Green function", "propagator". More traditional notations are "ψ", "K", "G", "Δ", "S" depending on the context. We prefer a neutral one suitable for all contexts.

The factor of i and the minus sign in the definition are irrelevant (and absent in the original definition [11, Problem 2.6]). They come from the ordinary starting direction and rotation direction of the stopwatch hand, and reduce the number of minus signs in what follows.

The normalization factor $2^{(1-t)/2}$ can be explained by analogy to the classical random walk. If the checker were performing just a random walk, choosing one of the two possible directions at each step (after the obligatory first upwards-right move), then $|a(s)|^2 = 2^{1-t}$ would be the probability of a path s. This analogy should be taken with a grain of salt: in quantum theory, the "probability of a path" has absolutely *no* sense (recall Example 1). The reason is that the path is not something one can measure: a measurement of the electron position at one moment t strongly affects the motion for all later moments t.

Conceptually, one should also fix the direction of the *last* move of the path s (see [11, bottom of p.35]). Luckily, this is not required in the present paper (and thus is not done), but becomes crucial in further upgrades (see §4 for an explanation).

One could ask where does the definition come from. Following Feynman, we do not try to explain or "derive" it physically. This quantum model cannot be obtained from a classical one by the standard Feynman sum-over-paths approach: there is simply *no* clear classical analogue of a spin 1/2 particle (cf. §4). An attempt to "derive" the model would appeal to much more complicated notions than the model itself, and would inevitably face the problem of absence of a true understanding of *spin* (however, see [2, 5, 33]). The true motivation for the introduced model is its simplicity, agreement with basic principles (like probability conservation), and with experiment (which here means the correct continuum limit; see Corollary 5).

2.3 Identities

Let us state several well-known basic properties of the model. The proofs are given in §12.1.

First, the arrow coordinates $a_1(x,t)$ and $a_2(x,t)$ satisfy the following recurrence relation.

Proposition 1 (Dirac equation)**.** *For each integer x and each positive integer t we have*

$$a_1(x,t+1) = \frac{1}{\sqrt{2}}a_2(x+1,t) + \frac{1}{\sqrt{2}}a_1(x+1,t);$$
$$a_2(x,t+1) = \frac{1}{\sqrt{2}}a_2(x-1,t) - \frac{1}{\sqrt{2}}a_1(x-1,t).$$

This mimics the (1+1)-dimensional Dirac equation in the Weyl basis [38, (19.4) and (3.31)]

$$\begin{pmatrix} m & \partial/\partial x - \partial/\partial t \\ \partial/\partial x + \partial/\partial t & m \end{pmatrix} \begin{pmatrix} a_2(x,t) \\ a_1(x,t) \end{pmatrix} = 0, \qquad (1)$$

only the derivatives are replaced by finite differences, m is set to 1, and the normalization factor $1/\sqrt{2}$ is added. For the upgrade in §3, this analogy becomes transparent (see Remark 2). The Weyl basis is not unique, thus there are several forms of equation (1); cf. [22, (1)].

The Dirac equation implies the conservation of probability.

Proposition 2 (Probability/charge conservation). *For each integer* $t \geq 1$ *we get* $\sum_{x\in\mathbb{Z}} P(x,t) = 1$.

For $a_1(x,t)$ and $a_2(x,t)$, there is an "explicit" formula (more ones are given in Appendix A).

Proposition 3 ("Explicit" formula). *For each integers* $|x| < t$ *such that* $x+t$ *is even we have*

$$a_1(x,t) = 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-1)^r \binom{(x+t-2)/2}{r} \binom{(t-x-2)/2}{r},$$

$$a_2(x,t) = 2^{(1-t)/2} \sum_{r=1}^{(t-|x|)/2} (-1)^r \binom{(x+t-2)/2}{r} \binom{(t-x-2)/2}{r-1}.$$

The following proposition is a straightforward corollary.

Proposition 4 (Particular values). *For each* $1 \leq k \leq t-1$ *the numbers* $a_1(-t+2k,t)$ *and* $a_2(-t+2k,t)$ *are the coefficients before* z^{t-k-1} *and* z^{t-k} *in the expansion of the polynomial* $2^{(1-t)/2}(1+z)^{t-k-1}(1-z)^{k-1}$. *In particular,*

$$a_1(0,4n+2) = \frac{(-1)^n}{2^{(4n+1)/2}}\binom{2n}{n}, \qquad a_1(0,4n) = 0,$$

$$a_2(0,4n+2) = 0, \qquad a_2(0,4n) = \frac{(-1)^n}{2^{(4n-1)/2}}\binom{2n-1}{n}.$$

In §3.1 we give more identities. The sequences $2^{(t-1)/2}a_1(x,t)$ and $2^{(t-1)/2}a_2(x,t)$ are present in the on-line encyclopedia of integer sequences [41, A098593 and A104967].

2.4 Asymptotic formulae

The following remarkable result was observed in [2, §4] (see Figures 5 and 8), stated precisely and derived heuristically in [28, Theorem 1], and proved mathematically in [18, Theorem 1]. See a short exposition of the latter proof in §12.2, and generalizations in §3.2.

Theorem 1 (Large-time limiting distribution; see Figure 8). (**A**) *For each* $v \in \mathbb{R}$ *we have*

$$\lim_{t\to\infty} \sum_{x \leq vt} P(x,t) = F(v) := \begin{cases} 0, & \text{if } v \leq -1/\sqrt{2}; \\ \frac{1}{\pi}\arccos\frac{1-2v}{\sqrt{2}(1-v)}, & \text{if } |v| < 1/\sqrt{2}; \\ 1, & \text{if } v \geq 1/\sqrt{2}. \end{cases}$$

(**B**) *We have the following convergence in distribution as* $t \to \infty$*:*

$$tP(\lceil vt \rceil, t) \xrightarrow{d} F'(v) = \begin{cases} \frac{1}{\pi(1-v)\sqrt{1-2v^2}}, & \text{if } |v| < 1/\sqrt{2}; \\ 0, & \text{if } |v| \geq 1/\sqrt{2}. \end{cases}$$

(**C**) *For each integer* $r \geq 0$ *we have* $\lim_{t\to\infty} \sum_{x\in\mathbb{Z}} \left(\frac{x}{t}\right)^r P(x,t) = \int_{-1}^{1} v^r F'(v)\, dv$.

Theorem 1(B) demonstrates a phase transition in Feynman checkers, if interpreted as an Ising model at imaginary temperature and fixed magnetization. Recall that then the magnetization v is an external condition (rather than a random variable) and $P(\lceil vt \rceil, t)$ is the norm square of the partition function (rather than a probability). The distributional limit of $tP(\lceil vt \rceil, t)$ is discontinuous at $v = \pm 1/\sqrt{2}$, reflecting a phase transition.

Our first new result is an analytic approximation of $a(x,t)$ accurate for small $|x|/t$ (see Figure 11). This solves an analogue of the Feynman problem for the basic model (cf. Corollary 4).

Theorem 2 (Large-time limit near the origin). *For each integers x, t such that $|x| < t^{3/4}$ and $x + t$ is even we have*

$$a(x,t) = i\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{i\pi t}{4} + \frac{ix^2}{2t}\right) + i\frac{2x}{\sqrt{\pi t^3}} \cos\left(-\frac{\pi(t+1)}{4} + \frac{x^2}{2t}\right) + O\left(\frac{\log^2 t}{t^{3/2}} + \frac{x^4}{t^{7/2}}\right), \tag{2}$$

$$P(x,t) = \frac{2}{\pi t}\left[1 + \frac{x}{t}\left(1 + \sqrt{2}\cos\left(\frac{\pi(2t+1)}{4} - \frac{x^2}{t}\right)\right) + O\left(\frac{\log^2 t}{t} + \frac{x^4}{t^3}\right)\right]. \tag{3}$$

Recall that $f(x,t) = O(g(x,t))$ means that there is a constant C (not depending on x, t) such that for each x, t satisfying the assumptions of the theorem we have $|f(x,t)| \le C\, g(x,t)$.

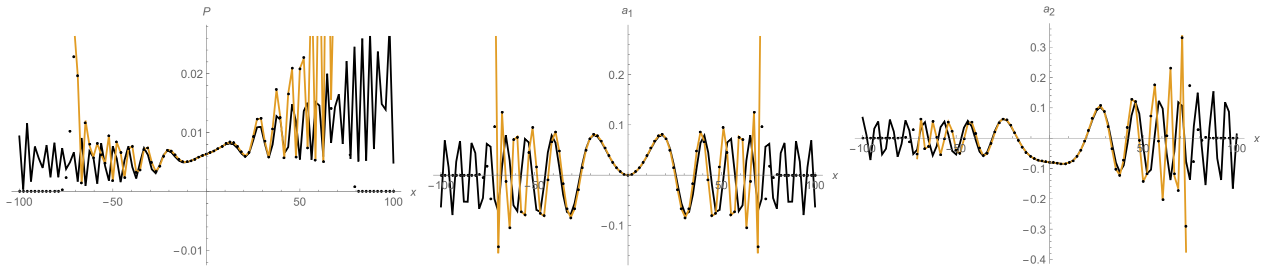


Figure 11: The graphs of $P(x,100)$, $a_1(x,100)$, $a_2(x,100)$ (dots) and their analytic approximations from Theorem 2 (dark) and Theorem 3 (light).

Thus for $x/t = o(t^{-1/4})$ the model asymptotically reproduces the *free-particle kernel*

$$K(x,t) = \sqrt{\frac{m}{2\pi t}} \exp\left(\frac{imx^2}{2t} - \frac{i\pi}{4}\right). \tag{4}$$

The first term in (2) is kernel (4) for $m = 1$ times $2i^{3/2}e^{-i\pi t/4}$. The latter "exceptional" factor essentially differs from the one in the Feynman problem by $\pi/4$ before t in the exponential; this additional $\pi/4$ is due to the finite lattice step $1/m$. The second term in (2) is a kind of relativistic correction responsible for the dependence of the probability $P(x,t)$ on x. The second term coincides with the one in asymptotic expansion of the Bessel functions in spin-1/2 retarded propagator (20) for $m = 1$, up to a factor of $\pi/4$ in the argument of cosine. We shall see that the assumption $|x| < t^{3/4}$ is essential in the Feynman problem (see Example 4). Although Theorem 2 is an easy corollary of Theorem 3, we give a direct alternative proof in §12.3.

3 Mass (biased quantum walk)

Question: what is the probability to find an electron of mass m in the square (x,t), if it was emitted from $(0,0)$?
Assumptions: the mass and the lattice step are now arbitrary.
Results: analytic expressions for the probability for large time or small lattice step, concentration of measure.

3.1 Identities

Definition 2. Fix $\varepsilon > 0$ and $m \ge 0$ called *lattice step* and *particle mass* respectively. Consider the lattice $\varepsilon\mathbb{Z}^2 = \{(x,t) : x/\varepsilon, t/\varepsilon \in \mathbb{Z}\}$ (see Figure 3). A *checker path* s is a finite sequence of lattice points such that the vector from each point (except the last one) to the next one equals either $(\varepsilon,\varepsilon)$ or $(-\varepsilon,\varepsilon)$. Denote by $\mathrm{turns}(s)$ the number of points in s (not the first and not the last one) such that the vectors from the point to the next and to the previous ones are orthogonal. For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>0$, denote by

$$a(x,t,m,\varepsilon):=(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}\,i\sum_s(-im\varepsilon)^{\mathrm{turns}(s)} \tag{5}$$

the sum over all checker paths s on $\varepsilon\mathbb{Z}^2$ from $(0,0)$ to (x,t) with the first step to $(\varepsilon,\varepsilon)$. Denote

$$P(x,t,m,\varepsilon):=|a(x,t,m,\varepsilon)|^2,\quad a_1(x,t,m,\varepsilon):=\operatorname{Re}a(x,t,m,\varepsilon),\quad a_2(x,t,m,\varepsilon):=\operatorname{Im}a(x,t,m,\varepsilon).$$

Remark 1. In particular, $P(x,t)=P(x,t,1,1)$ and $a(x,t)=a(x,t,1,1)=a(x\varepsilon,t\varepsilon,1/\varepsilon,\varepsilon)$.

One interprets $P(x,t,m,\varepsilon)$ as the probability to find an electron of mass m in the square $\varepsilon\times\varepsilon$ with the center (x,t), if the electron was emitted from the origin. Notice that the value $m\varepsilon$, hence $P(x,t,m,\varepsilon)$, is dimensionless in the natural units, where $\hbar=c=1$. In Figure 1 to the middle, the color of a point (x,t) depicts the value $P(x,t,1,0.5)$ (if $x+t$ is an integer). Recently I. Novikov elegantly proved that the probability vanishes nowhere inside the light cone: $P(x,t,m,\varepsilon)\neq0$ for $m>0$, $|x|<t$ and even $(x+t)/\varepsilon$ [34, Theorem 1].

Example 2 (Boundary values). *We have $a(t,t,m,\varepsilon)=i(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}$ and $a(2\varepsilon-t,t,m,\varepsilon)=m\varepsilon(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}$ for each $t\in\varepsilon\mathbb{Z}$, $t>0$, and $a(x,t,m,\varepsilon)=0$ for each $x>t$ or $x\le-t$.*

Example 3 (Massless and heavy particles). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>0$, we have*

$$P(x,t,0,\varepsilon)=\begin{cases}1, & \text{for } x=t;\\ 0, & \text{for } x\neq t.\end{cases}\quad\text{and}\quad\lim_{m\to\infty}P(x,t,m,\varepsilon)=\begin{cases}1, & \text{for } x=0 \text{ or } \varepsilon, \text{ and } \frac{x+t}{\varepsilon} \text{ even};\\ 0, & \text{otherwise.}\end{cases}$$

Let us list known combinatorial properties of the model [44, 29]; see §12.1 for simple proofs.

Proposition 5 (Dirac equation). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>0$, we have*

$$a_1(x,t+\varepsilon,m,\varepsilon)=\frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x+\varepsilon,t,m,\varepsilon)+m\varepsilon\,a_2(x+\varepsilon,t,m,\varepsilon)), \tag{6}$$

$$a_2(x,t+\varepsilon,m,\varepsilon)=\frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_2(x-\varepsilon,t,m,\varepsilon)-m\varepsilon\,a_1(x-\varepsilon,t,m,\varepsilon)). \tag{7}$$

Remark 2. This equation reproduces Dirac equation (1) in the continuum limit $\varepsilon\to0$: for C^2 functions $a_1,a_2\colon\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ satisfying (6)–(7) on $\varepsilon\mathbb{Z}^2$, the left-hand side of (1) is $O_m\left(\varepsilon\cdot(\|a_1\|_{C^2}+\|a_2\|_{C^2})\right)$.

Proposition 6 (Probability conservation). *For each $t\in\varepsilon\mathbb{Z}$, $t>0$, we get $\sum_{x\in\varepsilon\mathbb{Z}}P(x,t,m,\varepsilon)=1$.*

Proposition 7 (Klein–Gordon equation). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>\varepsilon$, we have*

$$\sqrt{1+m^2\varepsilon^2}\,a(x,t+\varepsilon,m,\varepsilon)+\sqrt{1+m^2\varepsilon^2}\,a(x,t-\varepsilon,m,\varepsilon)-a(x+\varepsilon,t,m,\varepsilon)-a(x-\varepsilon,t,m,\varepsilon)=0.$$

This equation reproduces the *Klein–Gordon equation* $\frac{\partial^2a}{\partial t^2}-\frac{\partial^2a}{\partial x^2}+m^2a=0$ in the limit $\varepsilon\to0$.

Proposition 8 (Symmetry). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>0$, we have*

$$a_1(x,t,m,\varepsilon)=a_1(-x,t,m,\varepsilon),\qquad (t-x)\,a_2(x,t,m,\varepsilon)=(t+x-2\varepsilon)\,a_2(2\varepsilon-x,t,m,\varepsilon),$$
$$a_1(x,t,m,\varepsilon)+m\varepsilon\,a_2(x,t,m,\varepsilon)=a_1(2\varepsilon-x,t,m,\varepsilon)+m\varepsilon\,a_2(2\varepsilon-x,t,m,\varepsilon).$$

Proposition 9 (Huygens' principle). *For each $x,t,t'\in\varepsilon\mathbb{Z}$, where $t>t'>0$, we have*

$$a_1(x,t,m,\varepsilon)=\sum_{x'\in\varepsilon\mathbb{Z}}\left[a_2(x',t',m,\varepsilon)a_1(x-x'+\varepsilon,t-t'+\varepsilon,m,\varepsilon)+a_1(x',t',m,\varepsilon)a_2(x'-x+\varepsilon,t-t'+\varepsilon,m,\varepsilon)\right],$$
$$a_2(x,t,m,\varepsilon)=\sum_{x'\in\varepsilon\mathbb{Z}}\left[a_2(x',t',m,\varepsilon)a_2(x-x'+\varepsilon,t-t'+\varepsilon,m,\varepsilon)-a_1(x',t',m,\varepsilon)a_1(x'-x+\varepsilon,t-t'+\varepsilon,m,\varepsilon)\right].$$

Informally, Huygens' principle means that each black square (x',t') on the t'-th horizontal acts like an independent point source, with the amplitude and phase determined by $a(x',t',m,\varepsilon)$.

Proposition 10 (Equal-time recurrence relation). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>0$, we have*

$$(x+\varepsilon)((x-\varepsilon)^2-(t-\varepsilon)^2)a_1(x-2\varepsilon,t,m,\varepsilon)+(x-\varepsilon)((x+\varepsilon)^2-(t-\varepsilon)^2)a_1(x+2\varepsilon,t,m,\varepsilon)=$$
$$=2x\left((1+2m^2\varepsilon^2)(x^2-\varepsilon^2)-(t-\varepsilon)^2\right)a_1(x,t,m,\varepsilon),\quad (8)$$
$$x((x-2\varepsilon)^2-t^2)a_2(x-2\varepsilon,t,m,\varepsilon)+(x-2\varepsilon)(x^2-(t-2\varepsilon)^2)a_2(x+2\varepsilon,t,m,\varepsilon)=$$
$$=2(x-\varepsilon)\left((1+2m^2\varepsilon^2)(x^2-2\varepsilon x)-t^2+2\varepsilon t\right)a_2(x,t,m,\varepsilon).$$

This allows to compute $a_1(x,t)$ and $a_2(x,t)$ quickly on far horizontals, starting from $x=2\varepsilon-t$ and $x=t$ respectively (see Example 2).

Proposition 11 ("Explicit" formula). *For each $(x,t)\in\varepsilon\mathbb{Z}^2$ with $|x|<t$ and $(x+t)/\varepsilon$ even,*

$$a_1(x,t,m,\varepsilon)=(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}\sum_{r=0}^{\frac{t-|x|}{2\varepsilon}}(-1)^r\binom{(x+t)/2\varepsilon-1}{r}\binom{(t-x)/2\varepsilon-1}{r}(m\varepsilon)^{2r+1},\quad (9)$$
$$a_2(x,t,m,\varepsilon)=(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}\sum_{r=1}^{\frac{t-|x|}{2\varepsilon}}(-1)^r\binom{(x+t)/2\varepsilon-1}{r}\binom{(t-x)/2\varepsilon-1}{r-1}(m\varepsilon)^{2r};\quad (10)$$

Remark 3. For each $|x|\ge t$ we have $a(x,t,m,\varepsilon)=0$ unless $0<t=x\in\varepsilon\mathbb{Z}$, which gives $a(t,t,m,\varepsilon)=(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}i$. Beware that the proposition is *not* applicable for $|x|\ge t$.

By the definition of Gauss hypergeometric function, we can rewrite the formula as follows:

$$a_1(x,t,m,\varepsilon)=m\varepsilon\left(1+m^2\varepsilon^2\right)^{(1-t/\varepsilon)/2}\cdot{}_2F_1\left(1-\frac{x+t}{2\varepsilon},1+\frac{x-t}{2\varepsilon};1;-m^2\varepsilon^2\right),$$
$$a_2(x,t,m,\varepsilon)=m^2\varepsilon^2\left(1+m^2\varepsilon^2\right)^{(1-t/\varepsilon)/2}\cdot{}_2F_1\left(2-\frac{x+t}{2\varepsilon},1+\frac{x-t}{2\varepsilon};2;-m^2\varepsilon^2\right)\cdot\left(1-\frac{x+t}{2\varepsilon}\right).$$

This gives a lot of identities. E.g., Gauss contiguous relations connect the values $a(x,t,m,\varepsilon)$ at any 3 neighboring lattice points; cf. Propositions 5 and 10. In terms of the Jacobi polynomials,

$$a_1(x,t,m,\varepsilon)=m\varepsilon(1+m^2\varepsilon^2)^{(x/\varepsilon-1)/2}P^{(0,-x/\varepsilon)}_{(x+t)/2\varepsilon-1}\left(\frac{1-m^2\varepsilon^2}{1+m^2\varepsilon^2}\right),$$
$$a_2(x,t,m,\varepsilon)=-m^2\varepsilon^2(1+m^2\varepsilon^2)^{(x/\varepsilon-3)/2}P^{(1,1-x/\varepsilon)}_{(x+t)/2\varepsilon-2}\left(\frac{1-m^2\varepsilon^2}{1+m^2\varepsilon^2}\right).$$

There is a similar expression through Kravchuk polynomials (cf. Propostion 4). In terms of Stanley character polynomials (defined in [42, §2]),

$$a_2(0,t,m,\varepsilon)=(-1)^{t/2\varepsilon-1}(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}\left(\tfrac{t}{2\varepsilon}-1\right)G_{t/2\varepsilon-1}(1;m^2\varepsilon^2).$$

Proposition 12 (Fourier integral). *Set $\omega_p:=\frac{1}{\varepsilon}\arccos(\frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}})$. Then for each $x,t\in\varepsilon\mathbb{Z}$ such that $t>0$ and $(x+t)/\varepsilon$ is even we have*

$$a_1(x,t,m,\varepsilon)=\frac{im\varepsilon^2}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\frac{e^{ipx-i\omega_p(t-\varepsilon)}\,dp}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}},$$
$$a_2(x,t,m,\varepsilon)=\frac{\varepsilon}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\left(1+\frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}}\right)e^{ip(x-\varepsilon)-i\omega_p(t-\varepsilon)}\,dp.$$

Fourier integral represents a wave emitted by a point source as a superposition of waves of wavelength $2\pi/p$ and frequency ω_p.

Proposition 13 (Full space-time Fourier transform). *Denote $\delta_{x\varepsilon}:=1$, if $x=\varepsilon$, and $\delta_{x\varepsilon}:=0$, if $x\ne\varepsilon$. For each $m>0$ and $(x,t)\in\varepsilon\mathbb{Z}$ such that $t>0$ and $(x+t)/\varepsilon$ is even we get*

$$a_1(x,t,m,\varepsilon)=\lim_{\delta\to+0}\frac{m\varepsilon^3}{4\pi^2}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\frac{e^{ipx-i\omega(t-\varepsilon)}\,d\omega dp}{\sqrt{1+m^2\varepsilon^2}\cos(\omega\varepsilon)-\cos(p\varepsilon)-i\delta},$$
$$a_2(x,t,m,\varepsilon)=\lim_{\delta\to+0}\frac{-i\varepsilon^2}{4\pi^2}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\frac{\sqrt{1+m^2\varepsilon^2}\sin(\omega\varepsilon)+\sin(p\varepsilon)}{\sqrt{1+m^2\varepsilon^2}\cos(\omega\varepsilon)-\cos(p\varepsilon)-i\delta}e^{ip(x-\varepsilon)-i\omega(t-\varepsilon)}\,d\omega dp+\delta_{x\varepsilon}\delta_{t\varepsilon}.$$

3.2 Asymptotic formulae

Large-time limit. Now we state the main theorem: an analytic approximation of $a(x,t,m,\varepsilon)$, very accurate for x/t not too close to the (approximate) peaks $\pm 1/\sqrt{1+m^2\varepsilon^2}$ (see Figure 12).

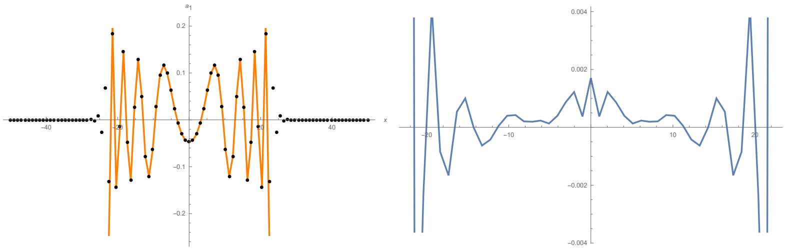


Figure 12: Graphs of $a_1(x,50,4,0.5)$ (left, dots), its analytic approximation given by Theorem 3 (left, curve), their difference (right)

Theorem 3 (Large-time asymptotic formula; see Figure 12)**.** (**A**) *For each $\delta > 0$ there is $C_\delta > 0$ such that for each $m,\varepsilon > 0$ and each $(x,t) \in \varepsilon\mathbb{Z}^2$ satisfying*

$$|x|/t < 1/\sqrt{1+m^2\varepsilon^2} - \delta, \qquad \varepsilon \le 1/m, \qquad t > C_\delta/m, \tag{11}$$

we have

$$a_1(x,t+\varepsilon,m,\varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi}}\left(t^2-(1+m^2\varepsilon^2)x^2\right)^{-1/4}\sin\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right), \tag{12}$$

$$a_2(x+\varepsilon,t+\varepsilon,m,\varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi}}\left(t^2-(1+m^2\varepsilon^2)x^2\right)^{-1/4}\sqrt{\frac{t+x}{t-x}}\cos\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right), \tag{13}$$

for $(x+t)/\varepsilon$ odd and even respectively, where

$$\theta(x,t,m,\varepsilon) := \frac{t}{\varepsilon}\arcsin\frac{m\varepsilon t}{\sqrt{(1+m^2\varepsilon^2)(t^2-x^2)}} - \frac{x}{\varepsilon}\arcsin\frac{m\varepsilon x}{\sqrt{t^2-x^2}} + \frac{\pi}{4}. \tag{14}$$

(**B**) *For each $m,\varepsilon,\delta > 0$ and each $(x,t) \in \varepsilon\mathbb{Z}^2$ satisfying*

$$|x|/t > 1/\sqrt{1+m^2\varepsilon^2} + \delta \qquad \textit{and} \qquad \varepsilon \le 1/m,$$

we have

$$a(x,t,m,\varepsilon) = O\left(\frac{\varepsilon}{mt^2\delta^3}\right).$$

Recall that notation $f(x,t,m,\varepsilon) = O_\delta(g(x,t,m,\varepsilon))$ means that there is a constant $C(\delta)$ (depending on δ but *not* on x,t,m,ε) such that for each x,t,m,ε,δ satisfying the assumptions of the theorem we have $|f(x,t,m,\varepsilon)| \le C(\delta)\,g(x,t,m,\varepsilon)$.

The main terms in Theorem 3(A) were computed in [2, Theorem 2] in the particular case $m=\varepsilon=1$. It was one of the key discoveries of that breakthrough paper. To be precise, there is a minor difference in the main terms in [2, Theorem 2] and (12); but that difference is within the error term. Practically, the approximation in (12) is better by several orders of magnitude. In [2, §4] two ways to compute the main terms were suggested: via Jacobi polynomials (see Remark 3) using the Darboux method and via the Fourier integral (see Proposition 12) using the stationary phase method. The former way is currently far from a mathematical proof (see the end of §12), although it allows to prove (12) in the particular case when $x=0$ and $1/m\varepsilon$ is bounded [7, Theorem 3]. The latter way is actually Step 1 of the proof of the theorem in §12.4.

Concerning Theorem 3(B), a much stronger estimate was heuristically derived in [2, Theorem 1] for $m=\varepsilon=1$. It is interesting to get a mathematical proof of that estimate.

Theorem 3 has several interesting corollaries. First, it allows to pass to the large-time distributional limit (see Figure 8). Compared to Theorem 1, it provides convergence in a stronger sense, not accessible by the method of moments.

Corollary 1 (Large-time limiting distribution)**.** *For each $m > 0$ and $\varepsilon \le 1/m$ we have*

$$\lim_{\substack{t\to\infty \\ t\in\varepsilon\mathbb{Z}}} \sum_{\substack{x\le vt \\ x\in\varepsilon\mathbb{Z}}} P(x,t,m,\varepsilon) \rightrightarrows F(v,m,\varepsilon) := \begin{cases} 0, & \text{if } v \le -1/\sqrt{1+m^2\varepsilon^2}; \\ \frac{1}{\pi}\arccos\frac{1-(1+m^2\varepsilon^2)v}{\sqrt{1+m^2\varepsilon^2}(1-v)}, & \text{if } |v| < 1/\sqrt{1+m^2\varepsilon^2}; \\ 1, & \text{if } v \ge 1/\sqrt{1+m^2\varepsilon^2} \end{cases}$$

as $t\to\infty$ uniformly in v.

Another result not accessible by known methods is stated in terms of Young diagrams.

Corollary 2 (Steps of Young diagrams; see Figure 9)**.** *Denote by $n_+(h\times w)$ and $n_-(h\times w)$ the number of Young diagrams with exactly h rows and w columns, having an even and an odd number of steps (defined in page 9) respectively. Then for almost every $r > 1$ we have*

$$\limsup_{w\to\infty} \frac{\sqrt{w}}{2^{(r+1)w/2}} \left|n_+(\lceil rw\rceil \times w) - n_-(\lceil rw\rceil \times w)\right| = \begin{cases} \frac{1}{\sqrt{\pi}}(6r-r^2-1)^{-1/4}, & \text{if } r < 3+2\sqrt{2}; \\ 0, & \text{if } r > 3+2\sqrt{2}. \end{cases}$$

Feynman triple limit. Theorem 3 allows to pass to the limit $(1/t, x/t, \varepsilon) \to 0$ as follows.

Corollary 3 (Simpler and rougher asymptotic formula)**.** *Under the assumptions of Theorem 3(A) we have*

$$a(x,t,m,\varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi t}} \exp\left(-im\sqrt{t^2-x^2} + \frac{i\pi}{4}\right)\left(1 + O_\delta\left(\frac{1}{mt} + \frac{|x|}{t} + m^3\varepsilon^2 t\right)\right). \qquad (15)$$

Corollary 4 (Feynman triple limit; see Figure 3)**.** *For each $m \ge 0$ and each sequence $(x_n, t_n, \varepsilon_n)$ such that $(x_n, t_n) \in \varepsilon_n\mathbb{Z}^2$, $(x_n+t_n)/\varepsilon_n$ is even, and*

$$1/t_n, \quad x_n/t_n^{3/4}, \quad \varepsilon_n t_n^{1/2} \quad \to \quad 0 \qquad \text{as} \qquad n\to\infty, \qquad (16)$$

we have the equivalence

$$\frac{1}{2i\varepsilon_n} a(x_n, t_n, m, \varepsilon_n) \sim \sqrt{\frac{m}{2\pi t_n}} \exp\left(-imt_n - \frac{i\pi}{4} + \frac{imx_n^2}{2t_n}\right) \qquad \text{as} \quad n\to\infty. \qquad (17)$$

For equivalence (17), assumptions (16) are essential and sharp, as the next example shows.

Example 4. *Equivalence* (17) *does not hold for $(x_n, t_n, \varepsilon_n) = (0, n, 1/\sqrt{n})$ or $(n^{3/4}, n, 1/n)$: by Theorem 3, the ratio of the left- and the right-hand side tends to $e^{-i\pi/6}$ and $e^{-im/8}$ respectively rather than to* 1.

Corollary 4 solves the Feynman problem (and moreover corrects the statement, by revealing the required sharp assumptions). The main difficulty here is that it concerns *triple* rather than *iterated* limit. We are not aware of any approach which could solve the problem without proving the whole Theorem 3. E.g., the Darboux asymptotic formula for the Jacobi polynomials (see Remark 3) is suitable for the iterated limit when first $t\to+\infty$, then $\varepsilon\to 0$, giving a (weaker) result already independent on x. Neither the Darboux nor Mehler–Heine nor more recent asymptotic formulae [30] are applicable when $1/m\varepsilon$ or x/ε is unbounded. Conversely, the next theorem is suitable for the iterated limit when first $\varepsilon\to 0$, then $x/t\to 0$, then $t\to+\infty$, but not for the triple limit because the remainder blows up as $t\to\infty$.

Continuum limit. The limit $\varepsilon\to 0$ involves the *Bessel functions of the first kind*:

$$J_0(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{2k}}{(k!)^2}, \qquad\qquad J_1(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{2k+1}}{k!(k+1)!}.$$

Theorem 4 (Asymptotic formula in the continuum limit)**.** *For each $m, \varepsilon, \delta > 0$ and $(x,t) \in \varepsilon\mathbb{Z}^2$ such that $(x+t)/\varepsilon$ even, $t - |x| \ge \delta$, and $\varepsilon < \delta\, e^{-3ms}/16$, where $s := \sqrt{t^2-x^2}$, we have*

$$a(x,t,m,\varepsilon) = m\varepsilon\left(J_0(ms) - i\,\frac{t+x}{s}\,J_1(ms) + O\left(\frac{\varepsilon}{\delta}\log^2\frac{\delta}{\varepsilon}\cdot e^{m^2t^2}\right)\right).$$

The main term in Theorem 4 was computed in [33, §1]. Numerical experiment shows that the error term decreases faster than asserted (see Table 3 computed in [40, §14]).

In the next corollary, we approximate a fixed point (x,t) in the plane by the lattice point $\left(2\varepsilon\left\lceil\frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil\frac{t}{2\varepsilon}\right\rceil\right)$ (see Figure 4). The factors of 2 make the latter accessible for the checker.

Corollary 5 (Uniform continuum limit; see Figure 4). *For each fixed $m \geq 0$ we have*

$$\frac{1}{2\varepsilon}\, a\left(2\varepsilon\left\lceil\frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil\frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right) \rightrightarrows \frac{m}{2}\, J_0(m\sqrt{t^2-x^2}) - i\,\frac{m}{2}\sqrt{\frac{t+x}{t-x}}\, J_1(m\sqrt{t^2-x^2}) \tag{18}$$

as $\varepsilon \to 0$ uniformly on compact subsets of the angle $|x| < t$.

The proof of *pointwise* convergence is simpler and is presented in Appendix B.

Corollary 6 (Concentration of measure). *For each $t, m, \delta > 0$ we have*

$$\sum_{x\in\varepsilon\mathbb{Z}\,:\,0\leq t-|x|\leq\delta} P(x,t,m,\varepsilon) \to 1 \qquad \text{as} \qquad \varepsilon\to 0 \qquad \text{so that} \qquad \frac{t}{2\varepsilon}\in\mathbb{Z}.$$

This result, although expected, is not found in literature. An elementary proof is given in §12.7. We remark a sharp contrast between the continuum and the large-time limit here: by Corollary 1, there is *no* concentration of measure as $t \to \infty$ for fixed ε.

ε	$5\varepsilon\log_{10}^2(5\varepsilon)$	$\max\limits_{x\in(-0.8,0.8)\cap 2\varepsilon\mathbb{Z}} \left\lvert\frac{1}{2\varepsilon}a(x,1,10,\varepsilon) - G^R_{11}(x,1) - iG^R_{12}(x,1)\right\rvert$
0.02	0.1	1.1
0.002	0.04	0.06
0.0002	0.009	0.006

Table 3: Approximation of spin-1/2 retarded propagator (20) by Feynman checkers ($m = 10$, $\delta = 0.2$, $t = 1$)

3.3 Physical interpretation

Let us discuss the meaning of the continuum limit. In this subsection we omit some technical definitions not used in the sequel.

Limit (18) reproduces the spin-1/2 retarded propagator describing motion of an electron along a line. More precisely, the *spin-1/2 retarded propagator*, or the *retarded Green function for Dirac equation* (1) is a matrix-valued tempered distribution $G^R(x,t) = (G^R_{kl}(x,t))$ on $\mathbb{R}^2$ vanishing for $t < |x|$ and satisfying

$$\begin{pmatrix} m & \partial/\partial x - \partial/\partial t \\ \partial/\partial x + \partial/\partial t & m \end{pmatrix} \begin{pmatrix} G^R_{11}(x,t) & G^R_{12}(x,t) \\ G^R_{21}(x,t) & G^R_{22}(x,t) \end{pmatrix} = \delta(x)\delta(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{19}$$

where $\delta(x)$ is the Dirac delta function. The propagator is given by (cf. [22, (13)], [38, (3.117)])

$$G^R(x,t) = \frac{m}{2}\begin{pmatrix} J_0(m\sqrt{t^2-x^2}) & -\sqrt{\frac{t+x}{t-x}}\, J_1(m\sqrt{t^2-x^2}) \\ \sqrt{\frac{t-x}{t+x}}\, J_1(m\sqrt{t^2-x^2}) & J_0(m\sqrt{t^2-x^2}) \end{pmatrix} \qquad \text{for } |x| < t. \tag{20}$$

In addition, $G^R(x,t)$ involves a generalized function supported on the lines $t = \pm x$, not observed in the limit (18) and not specified here. A more common expression is (cf. Proposition 13)

$$G^R(x,t) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lim_{\delta\to+0} \begin{pmatrix} m & -ip-i\omega \\ -ip+i\omega & m \end{pmatrix} \frac{e^{ipx-i\omega t}\,dp\,d\omega}{m^2+p^2-(\omega+i\delta)^2}, \tag{21}$$

where the limit is taken in the weak topology of matrix-valued tempered distributions and the integral is understood as the Fourier transform of tempered distributions (cf. [13, (6.47)]).

The propagator "square" $G^R_{11}(x,t)^2 + G^R_{12}(x,t)^2$ is ill-defined (because of the square of the Dirac delta-function supported on the lines $t = \pm x$ involved). Thus the propagator lacks probabilistic interpretation, and global charge conservation (Proposition 6) has no continuum analogue. For instance, $\int_{(-t,t)}(G^R_{11}(x,t)^2+G^R_{12}(x,t)^2)\,dx = t/2 \neq$ const paradoxically. A physical explanation: the line $t = x$ carries infinite charge flowing inside the angle $|x| < t$. One can interpret the propagator "square" for $|x| \neq t$ as a relative probability density or charge density (see Figure 1). In the spin-chain interpretation, the propagator is the limit of the partition function for one-dimensional Ising model at the inverse temperature $\beta = i\pi/4 - \log(m\varepsilon)/2$. Those are essentially the values of β for which phase transition is possible [32].

The normalization factor $1/2\varepsilon$ before "a" in (18) can be explained as division by the length associated to a black lattice point in the x-direction. On a deeper level, it comes from the normalization of $G^R(x,t)$ arising from (19).

Theorem 4 is a toy result in *algorithmic quantum field theory*: it determines the lattice step to compute the propagator with given accuracy. So far this is not a big deal, because the propagator has a known analytic expression and is not really experimentally-measurable; neither the efficiency of the algorithm is taken into account. But that is a first step.

Algorithm 1 (Approximation algorithm for spin-1/2 retarded propagator (20))**.** Input: mass $m > 0$, coordinates $|x| < t$, accuracy level Δ.

Output: an approximate value G_{kl} of $G^R_{kl}(x,t)$ within distance Δ from the true value (20).

Algorithm: compute $G_{kl} = \frac{(-1)^{(k-1)l}}{2\varepsilon}\, a_{(k+l) \bmod 2+1}\left(2\varepsilon\left\lceil \frac{(-1)^{(k-1)l}x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right)$ by (5) for

$$\varepsilon = (t-|x|)\min\left\{\frac{1}{16\,e^{3mt}}, \left(\frac{\Delta}{9C\,m\,e^{m^2t^2}}\right)^3\right\}, \qquad \text{where} \qquad C = 100.$$

Here we used an explicit estimate for the constant C understood in the big-O notation in Theorem 4; it is easily extracted from the proof. The theorem and the estimate remain true, if $a\,(x,t,m,\varepsilon)$ with $(x,t) \in \varepsilon\mathbb{Z}^2$ is replaced by $a\left(2\varepsilon\left\lceil\frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil\frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right)$ with arbitrary $(x,t) \in \mathbb{R}^2$.

4 Spin

Question: what is the probability to find a right electron at (x,t), if a right electron was emitted from $(0,0)$?
Assumptions: electron chirality is now taken into account.
Results: the probability of chirality flip.

A feature of the model is that the electron *spin* emerges naturally rather than is added artificially.

It goes almost without saying to view the electron as being in one of the two states depending on the last-move direction: *right-moving* or *left-moving* (or just '*right*' or '*left*' for brevity).

The *probability to find a right electron in the square* (x,t)*, if a right electron was emitted from the square* $(0,0)$, is the length square of the vector $\sum_s a(s)$, where the sum is over only those paths from $(0,0)$ to (x,t), which both start and finish with an upwards-right move. The *probability to find a left electron* is defined analogously, only the sum is taken over paths which start with an upwards-right move but finish with an upwards-left move. Clearly, these probabilities equal $a_2(x,t)^2$ and $a_1(x,t)^2$ respectively, because the last move is directed upwards-right if and only if the number of turns is even.

These right and left electrons are exactly the $(1+1)$-dimensional analogue of *chirality* states for a spin 1/2 particle [38, §19.1]. Indeed, it is known that the components $a_2(x,t)$ and $a_1(x,t)$ in Dirac equation in the Weyl basis (1) are interpreted as wave functions of right- and left-handed particles respectively. The relation to the movement direction becomes transparent for $m = 0$: a general solution of (1) is $(a_2(x,t), a_1(x,t)) = (a_2(x-t,0), a_1(x+t,0))$; thus the maxima of $a_2(x,t)$ and $a_1(x,t)$ (if any) move to the right and to the left respectively as t increases. Beware that in 3 or more dimensions, spin is *not* the movement direction and cannot be explained in nonquantum terms.

This gives a more conceptual interpretation of the model: an experiment outcome is a pair (final x-coordinate, last-move direction), whereas the final t-coordinate is fixed. The probabilities to find a right/left electron are the fundamental ones. In further upgrades, $a_1(x,t)$ and $a_2(x,t)$ become complex numbers and $P(x,t)$ should be *defined* as $|a_1(x,t)|^2 + |a_2(x,t)|^2$ rather than by the above formula $P(x,t) = |a(x,t)|^2 = |a_1(x,t)+ia_2(x,t)|^2$, being a coincidence.

Theorem 5 (Probability of chirality flip)**.** *For integer* $t > 0$ *we get* $\sum_{x\in\mathbb{Z}} a_1(x,t)^2 = \frac{1}{2\sqrt{2}} + O\left(\frac{1}{\sqrt{t}}\right)$.

See Figure 13 for an illustration and comparison with the upgrade from §5. The physical interpretation of the theorem is limited: in continuum theory, the probability of chirality flip (for an electron emitted by a point source) is ill-defined similarly to the propagator "square" (see §3.3). A related more reasonable quantity is studied in [22, p. 381] (cf. Problem 6). Recently I. Bogdanov has generalized the theorem to an arbitrary mass and lattice step (see Definition 2): if $0 \le m\varepsilon \le 1$ then $\lim_{t\to+\infty, t\in\varepsilon\mathbb{Z}} \sum_{x\in\varepsilon\mathbb{Z}} a_1(x,t,m,\varepsilon)^2 = \frac{m\varepsilon}{2\sqrt{1+m^2\varepsilon^2}}$ [7, Theorem 2]. This has confirmed a conjecture by I. Gaidai-Turlov–T. Kovalev–A. Lvov.

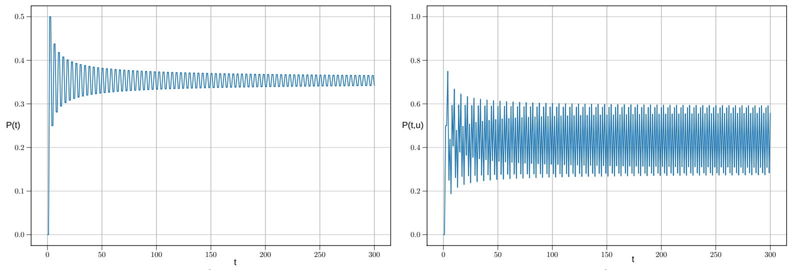


Figure 13: (by G. Minaev–I. Russkikh) The graphs of the probabilities $P(t) = \sum_{x\in\mathbb{Z}} a_1(x,t)^2$ and $P(t,u) = \sum_{x\in\mathbb{Z}} a_1(x,t,u)^2$ of chirality flip with magnetic field off and on respectively

5 External field

Question: what is the probability to find an electron at (x,t), if it moves in a given electromagnetic field u?
Assumptions: the electromagnetic field vanishes outside the xt-plane; it is not affected by the electron.
Results: "spin precession" in a magnetic field (qualitative explanation), charge conservation.

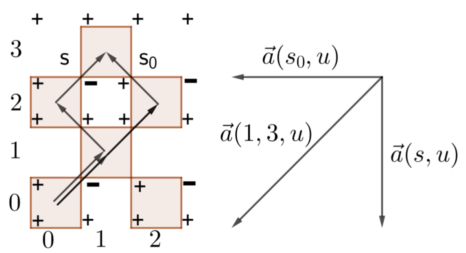


Figure 14: Paths in a field

Another feature of the model is that external electromagnetic field emerges naturally rather than is added artificially. We start with an informal definition, then give a precise one, and finally show exact charge conservation.

In the basic model, the stopwatch hand did not rotate while the checker moved straightly. It goes without saying to modify the model, rotating the hand uniformly during the motion. This does not change the model essentially: since all the paths from the initial to the final position have the same length, their vectors are rotated through the same angle, not affecting probabilities. A more interesting modification is when the current rotation angle depends on the checker position. This is exactly what electromagnetic field does. In what follows, the rotation angle assumes only the two values 0° and 180° for simplicity, meaning just multiplication by ± 1.

Thus an electromagnetic field is viewed as a fixed assignment u of numbers $+1$ and -1 to all the vertices of the squares. For instance, in Figure 14, the field equals -1 at the top-right vertex of each square (x,t) with both x and t even. Modify the definition of the vector $a(s)$ by reversing the direction each time when the checker passes through a vertex with the field -1. Denote by $a(s,u)$ the resulting vector. Define $a(x,t,u)$ and $P(x,t,u)$ analogously to $a(x,t)$ and $P(x,t)$ replacing $a(s)$ by $a(s,u)$ in the definition. For instance, if $u = +1$ identically, then $P(x,t,u) = P(x,t)$.

Let us slightly rephrase this construction, making the relation to lattice gauge theory more transparent. We introduce an auxiliary grid with the vertices at the centers of black squares (see Figure 9 to the right). It is the graph where the checker actually moves.

Definition 3. An *edge* is a segment joining nearest-neighbor integer points with even sum of the coordinates. Let u be a map from the set of all edges to $\{+1, -1\}$. Denote by

$$a(x, t, u) := 2^{(1-t)/2}\, i \sum_s (-i)^{\text{turns}(s)} u(s_0 s_1) u(s_1 s_2) \dots u(s_{t-1} s_t)$$

the sum over all checker paths $s = (s_0, s_1, \dots, s_t)$ with $s_0 = (0, 0)$, $s_1 = (1, 1)$, and $s_t = (x, t)$. Set $P(x, t, u) := |a(x, t, u)|^2$. Define $a_1(x, t, u)$ and $a_2(x, t, u)$ analogously to $a(x, t, u)$, only add the condition $s_{t-1} = (x + 1, t - 1)$ and $s_{t-1} = (x - 1, t - 1)$ respectively. For half-integers x, t denote by $u(x, t)$ the value of u on the edge with the midpoint (x, t).

Remark 4. Here the field u is a fixed external classical field not affected by the electron.

This definition is analogous to one of the first constructions of gauge theory by Weyl–Fock–London, and gives a coupling of Feynman checkers to the Wegner–Wilson $\mathbb{Z}/2\mathbb{Z}$ lattice gauge theory. In particular, it reproduces the correct spin 1 for the electromagnetic field: a function defined on the set of edges is a discrete analogue of a *vector* field, i.e., a *spin* 1 field. Although this way of coupling is classical, it has never been explicitly applied to Feynman checkers (cf. [15, p. 36]), and is very different from both the approach of [35] and Feynman-diagram intuition [10].

For an arbitrary *gauge group*, $a_1(x, t, u)$ and $a_2(x, t, u)$ are defined analogously, only u becomes a map from the set of edges to a matrix group, e.g., $U(1)$ or $SU(n)$. Then we set $P(x, t, u) := \sum_k \left(|(a_1(x, t, u))_{k1}|^2 + |(a_2(x, t, u))_{k1}|^2\right)$, where $(a_j)_{kl}$ are the entries of a matrix a_j.

Example 5 ("Spin precession" in a magnetic field)**.** *Let $u(x + 1/2, t + 1/2) = -1$, if both x and t even, and $u(x + 1/2, t + 1/2) = +1$ otherwise ("*homogeneous magnetic field*"; see Figure 14). Then the probability $P(t, u) := \sum_{x\in\mathbb{Z}} a_1(x, t, u)^2$ of detecting a left electron (see §4) is plotted in Figure 13 to the right. It apparently tends to a "periodic regime" as $t \to \infty$ (see Problem 11).*

The following propositions are proved analogously to Propositions 5–6, only a factor of $u(x \pm \frac{1}{2}, t + \frac{1}{2})$ is added due to the last step of the path passing through the vertex $(x \pm \frac{1}{2}, t + \frac{1}{2})$.

Proposition 14 (Dirac equation in electromagnetic field)**.** *For each integers x and $t \geq 1$,*

$$a_1(x, t + 1, u) = \frac{1}{\sqrt{2}} u\left(x + \frac{1}{2}, t + \frac{1}{2}\right)(a_1(x + 1, t, u) + a_2(x + 1, t, u)),$$
$$a_2(x, t + 1, u) = \frac{1}{\sqrt{2}} u\left(x - \frac{1}{2}, t + \frac{1}{2}\right)(a_2(x - 1, t, u) - a_1(x - 1, t, u)).$$

Proposition 15 (Probability/charge conservation)**.** *For each integer $t \geq 1$, $\sum\limits_{x\in\mathbb{Z}} P(x, t, u) = 1$.*

6 Source

Question: what is the probability to find an electron at (x, t), if it was emitted by a source of wavelength λ?
Assumptions: the source is now realistic.
Results: wave propagation, dispersion relation.

A realistic source produces a wave rather than electrons localized at $x = 0$ (as in the basic model). This means solving Dirac equation (6)–(7) with *(quasi-)periodic initial conditions.*

To state the result, it is convenient to rewrite Dirac equation (6)–(7) using the notation

$$\tilde{a}_1(x, t) = a_1(x, t + \varepsilon, m, \varepsilon), \qquad \tilde{a}_2(x, t) = a_2(x + \varepsilon, t + \varepsilon, m, \varepsilon),$$

so that it gets form

$$\tilde{a}_1(x, t) = \frac{1}{\sqrt{1 + m^2\varepsilon^2}}(\tilde{a}_1(x + \varepsilon, t - \varepsilon) + m\varepsilon\, \tilde{a}_2(x, t - \varepsilon)), \tag{22}$$
$$\tilde{a}_2(x, t) = \frac{1}{\sqrt{1 + m^2\varepsilon^2}}(\tilde{a}_2(x - \varepsilon, t - \varepsilon) - m\varepsilon\, \tilde{a}_1(x, t - \varepsilon)). \tag{23}$$

The following proposition is proved by direct checking (available in [40, §12]).

Proposition 16 (Wave propagation, dispersion relation)**.** *Equations* (22)–(23) *with the initial condition*

$$\tilde a_1(x,0)=\tilde a_1(0,0)e^{2\pi i x/\lambda},$$
$$\tilde a_2(x,0)=\tilde a_2(0,0)e^{2\pi i x/\lambda};$$

have the unique solution

$$\tilde a_1(x,t)= \ a\cos\tfrac{\alpha}{2}\, e^{2\pi i(x/\lambda+t/T)} + \ b\sin\tfrac{\alpha}{2}\, e^{2\pi i(x/\lambda-t/T)}, \tag{24}$$
$$\tilde a_2(x,t)= ia\sin\tfrac{\alpha}{2}\, e^{2\pi i(x/\lambda+t/T)} - ib\cos\tfrac{\alpha}{2}\, e^{2\pi i(x/\lambda-t/T)}, \tag{25}$$

where the numbers $T\ge 2$*,* $\alpha\in[0,\pi]$*, and* $a,b\in\mathbb{C}$ *are given by*

$$\cos(2\pi\varepsilon/T)=\frac{\cos(2\pi\varepsilon/\lambda)}{\sqrt{1+m^2\varepsilon^2}},\qquad \cot\alpha=\frac{\sin(2\pi\varepsilon/\lambda)}{m\varepsilon},\qquad \begin{aligned} a&=\tilde a_1(0,0)\cos\tfrac{\alpha}{2}-i\tilde a_2(0,0)\sin\tfrac{\alpha}{2},\\ b&=\tilde a_1(0,0)\sin\tfrac{\alpha}{2}+i\tilde a_2(0,0)\cos\tfrac{\alpha}{2}.\end{aligned}$$

Remark 5. General solution of (22)–(23) in appropriate functional space is obtained by replacing a and b by sufficiently general functions in λ and integration of (24)–(25) over $p=2\pi/\lambda$.

The solution of continuum Dirac equation (1) is given by the same expression (24)–(25), only $2\pi/T$ and α are redefined by $\frac{4\pi^2}{T^2}=\frac{4\pi^2}{\lambda^2}+m^2$ and $\cot\alpha=2\pi/m\lambda$ instead. In both continuum and discrete setup, these are the hypotenuse and the angle in a right triangle with one leg $2\pi/\lambda$ and another leg either m or $(\arctan m\varepsilon)/\varepsilon$ respectively, lying in the plane or a sphere of radius $1/\varepsilon$ respectively. This spherical-geometry interpretation is new and totally unexpected.

A comment for specialists: replacing a and b by creation and annihilation operators, i.e., the second quantization of the lattice Dirac equation, leads to the model from §9.

For the next upgrades, we just announce results to be discussed in subsequent publications.

7 Medium

Question: which part of light of given color is reflected from a glass plate of given width?
Assumptions: right angle of incidence, no polarization of light; mass now depends on x but not on the color.
Results: thin-film reflection (quantitative explanation).

Feynman checkers can be applied to describe propagation of light in transparent media such as glass. Light propagates as if it had acquired some nonzero mass plus potential energy (depending on the refractive index) inside the media, while both remain zero outside. In general the model is inappropriate to describe light; partial reflection is a remarkable exception. Notice that similar classical phenomena are described by quantum models [44, §2.7].

In Feynman checkers, we announce a rigorous derivation of the following well-known formula for the percentage P of light of wavelength λ reflected from a transparent plate of width L and refractive index n:

$$P=\frac{(n^2-1)^2}{(n^2+1)^2+4n^2\cot^2(2\pi Ln/\lambda)}.$$

This makes Feynman's popular-science discussion of partial reflection [10] completely rigorous and shows that his model has experimentally-confirmed predictions in the real world, not just a 2-dimensional one.

8 Identical particles

Question: what is the probability to find electrons at F and F', if they were emitted from A and A'?
Assumptions: motion of several electrons is now described.
Results: exclusion principle, locality, charge conservation.

We announce a simple-to-define upgrade describing the motion of several electrons, respecting exclusion principle, locality, and probability conservation (cf. [45, §4.2]).

Definition 4. Fix integer points $A = (0,0)$, $A' = (x_0, 0)$, $F = (x,y)$, $F' = (x', y)$ and their diagonal neighbors $B = (1,1)$, $B' = (x_0+1, 1)$, $E = (x-1, y-1)$, $E' = (x'-1, y-1)$, where $x_0 \neq 0, x' \geq x$. Denote

$$a(AB, A'B' \to EF, E'F') := \sum_{\substack{s: AB \to EF \\ s': A'B' \to E'F'}} a(s)a(s') - \sum_{\substack{s: AB \to E'F' \\ s': A'B' \to EF}} a(s)a(s'),$$

where the first sum is over all pairs consisting of a checker path s starting with the move AB and ending with the move EF, and a path s' starting with the move $A'B'$ and ending with the move $E'F'$, whereas in the second sum the final moves are interchanged.

The length square $P(AB, A'B' \to EF, E'F') := |a(AB, A'B' \to EF, E'F')|^2$ is called the *probability to find right electrons at F and F', if they are emitted from A and A'*. Define $P(AB, A'B' \to EF, E'F')$ analogously also for $E = (x \pm 1, y-1)$, $E' = (x' \pm 1, y-1)$. Here we require $x' \geq x$, if both signs are the same, and allow arbitrary x' and x, otherwise.

9 Antiparticles

Question: what is the expected charge in the square (x,t), if an electron was emitted from the square $(0,0)$?
Assumptions: electron-positron pairs now created and annihilated, the t-axis is time.
Results: spin-1/2 Feynman propagator in the continuum limit, an analytic expression for the large-time limit.

9.1 Identities

Finally, we introduce a completely new upgrade (*Feynman anti-checkers*), allowing creation and annihilation of electron-positron pairs during the motion. The upgrade is defined just by allowing odd $(x+t)/\varepsilon$ in the Fourier integral (Proposition 12), that is, computing the same integral in white checkerboard squares in addition to black ones. This is equivalent to the second quantization of lattice Dirac equation (22)–(23), which we do not need to work out (cf. [3, §9F] and [4, §IV] for the massless case). Anyway, the true motivation of the upgrade is a remarkable analogy with the initial model and appearance of spin-1/2 Feynman propagator (28) in the continuum limit (see Figure 15).

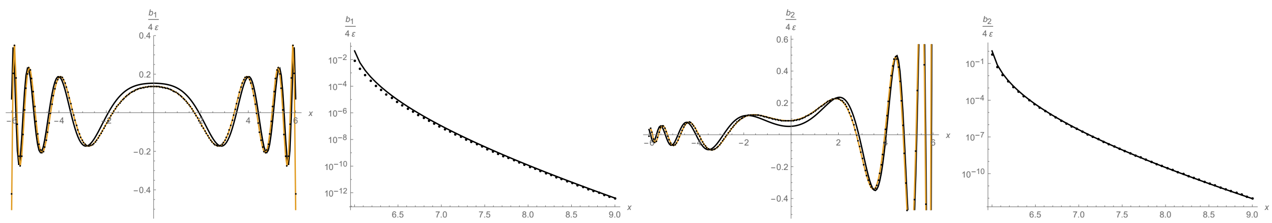


Figure 15: Plots of $b_1(x, 6, 4, 0.03)/0.12$ (left, dots), $b_2(x, 6, 4, 0.03)/0.12$ (right, dots), their analytic approximation from Theorem 6 (light), the imaginary part of the Feynman propagator $\operatorname{Im} G^F_{11}(x, 6)$ (left, dark) and $\operatorname{Im} G^F_{12}(x, 6)$ (right, dark) given by (28) for $m = 4$ and $t = 6$.

Definition 5. (Cf. Proposition 12, see Figure 15.) Fix $m \geq 0$ and $\varepsilon > 0$. For each $(x,t) \in \varepsilon\mathbb{Z}^2$, where $t > 0$, denote $\omega_p := \frac{1}{\varepsilon} \arccos(\frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}})$ and

$$\begin{aligned} A_1(x,t,m,\varepsilon) &:= \frac{im\varepsilon^2}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{ipx - i\omega_p(t-\varepsilon)}\, dp}{\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}}; \\ A_2(x,t,m,\varepsilon) &:= \frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \left(1 + \frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}}\right) e^{ip(x-\varepsilon) - i\omega_p(t-\varepsilon)}\, dp. \end{aligned} \tag{26}$$

For $t \leq 0$ the definition is the same, only the overall sign is changed for $(x+t)/\varepsilon$ even. In particular, $A_k(x,t,m,\varepsilon) = a_k(x,t,m,\varepsilon)$ for $(x+t)/\varepsilon$ even, $t > 0$, and $k = 1, 2$. Denote $A_k(x,t,m,\varepsilon) =: ib_k(x,t,m,\varepsilon)$ for $(x+t)/\varepsilon$ odd. Set $b_k(x,t,m,\varepsilon) := 0$ for $(x+t)/\varepsilon$ even.

Thus the real and the imaginary part "lives" on the black and white squares respectively, analogously to discrete analytic functions [8]. The sign convention for $t \le 0$ is dictated by the analogy to continuum theory (cf. (27) and (29)).

Example 6. *The value* $b_1(0,1,1,1) = \Gamma(\frac{1}{4})^2/(2\pi)^{3/2} = \frac{2}{\pi}K(i) =: G \approx 0.83463$ *is the Gauss constant and* $-b_2(0,1,1,1) = 2\sqrt{2\pi}/\Gamma(\frac{1}{4})^2 = \frac{2}{\pi}(E(i) - K(i)) = 1/\pi G =: L' \approx 0.38138$ *is the inverse lemniscate constant, where* $K(z)$ *and* $E(z)$ *are the complete elliptic integrals of the 1st and 2nd kind respectively (cf.* [12, §6.1]*).*

The other values are even more complicated irrationalities (see Table 4).

$b_1(x,t,1,1)$

2	$\frac{G-L'}{\sqrt{2}}$		$\frac{G-L'}{\sqrt{2}}$		$\frac{7G-15L'}{3\sqrt{2}}$
1		G		$G-2L'$	
0	$\frac{G-L'}{\sqrt{2}}$		$\frac{G-L'}{\sqrt{2}}$		$\frac{7G-15L'}{3\sqrt{2}}$
-1		$-L'$		$\frac{2G-3L'}{3}$	
t \ x	-1	0	1	2	3

$b_2(x,t,1,1)$

2	$\frac{G-3L'}{3\sqrt{2}}$		$\frac{-G-L'}{\sqrt{2}}$		$\frac{-G+3L'}{\sqrt{2}}$
1		$-L'$		L'	
0	$\frac{G-3L'}{\sqrt{2}}$		$\frac{G+L'}{\sqrt{2}}$		$\frac{-G+3L'}{3\sqrt{2}}$
-1		G		$\frac{G}{3}$	
t \ x	-1	0	1	2	3

Table 4: The values $b_1(x,t,1,1)$ and $b_2(x,t,1,1)$ for small x,t (see Definition 5 and Example 6)

We announce that the analogues of Propositions 5–10 remain true literally, if a_1 and a_2 are replaced by b_1 and b_2 respectively (the assumption $t > 0$ can then be dropped). As a consequence, $2^{(t-1)/2}b_1(x,t,1,1)$ and $2^{(t-1)/2}b_2(x,t,1,1)$ are all rational linear combinations of the Gauss constant G and the inverse lemniscate constant L' for each $(x,t) \in \mathbb{Z}^2$.

We also announce an "explicit" formula: for $m, \varepsilon > 0$, $(x,t) \in \varepsilon\mathbb{Z}^2$ with $(x+t)/\varepsilon$ odd we get

$$b_1(x,t,m,\varepsilon) = \left(1+m^2\varepsilon^2\right)^{\frac{1}{2}-\frac{t}{2\varepsilon}} \left(-m^2\varepsilon^2\right)^{\frac{t-|x|}{2\varepsilon}-\frac{1}{2}} \binom{\frac{t+|x|}{2\varepsilon}-1}{|x|/\varepsilon}$$
$$\cdot {}_2F_1\left(1+\frac{|x|-t}{2\varepsilon}, 1+\frac{|x|-t}{2\varepsilon}; 1+\frac{|x|}{\varepsilon}; -\frac{1}{m^2\varepsilon^2}\right),$$
$$b_2(x+\varepsilon,t+\varepsilon,m,\varepsilon) = \left(1+m^2\varepsilon^2\right)^{-\frac{t}{2\varepsilon}} (m\varepsilon)^{\frac{t-|x|}{\varepsilon}} (-1)^{\frac{t-|x|}{2\varepsilon}+\frac{1}{2}} \binom{\frac{t+|x|}{2\varepsilon}-1+\theta(x)}{|x|/\varepsilon}$$
$$\cdot {}_2F_1\left(\frac{|x|-t}{2\varepsilon}, 1+\frac{|x|-t}{2\varepsilon}; 1+\frac{|x|}{\varepsilon}; -\frac{1}{m^2\varepsilon^2}\right), \qquad \text{where } \theta(x) := \begin{cases} 1, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The idea of the proof is induction on $t/\varepsilon \ge 1$: the base is given by [17, 9.112, 9.131.1, 9.134.3] and the step is given by the analogue of (6)–(7) for b_1 and b_2 plus [17, 9.137].

Remark 6. (Cf. Remark 3) These expressions can be rewritten as the *Jacobi functions of the second kind* of half-integer order (see the definition in [43, (4.61.1)]). For instance, for each $(x,t) \in \varepsilon\mathbb{Z}^2$ such that $|x| > t$ and $(x+t)/\varepsilon$ is odd we have

$$b_1(x,t,m,\varepsilon) = \frac{2m\varepsilon}{\pi}\left(1+m^2\varepsilon^2\right)^{(t/\varepsilon-1)/2} Q^{(0,t/\varepsilon-1)}_{(|x|-t)/2\varepsilon}(1+2m^2\varepsilon^2).$$

Remark 7. The number $b_1(x,\varepsilon,m,\varepsilon)$ equals $(1+\sqrt{1+m^2\varepsilon^2})/m\varepsilon$ times the probability that a planar simple random walk over white squares dies at (x,ε), if it starts at $(0,\varepsilon)$ and dies with the probability $1 - 1/\sqrt{1+m^2\varepsilon^2}$ before each step. Nothing like that is known for $b_1(x,t,m,\varepsilon)$ and $b_2(x,t,m,\varepsilon)$ with $t \ne \varepsilon$ (see Problem 15).

The following results are proved almost literally as Proposition 13 and Theorem 3(A). (The only difference is negation of the summands involving $f_-(p)$ in (56), (61) (65), (67); the analogues of Lemmas 11 and 17 are then obtained by direct checking.)

Proposition 17 (Full space-time Fourier transform). *Denote $\delta_{x\varepsilon} := 1$, if $x = \varepsilon$, and $\delta_{x\varepsilon} := 0$, if $x \neq \varepsilon$. For each $m > 0$ and $(x,t) \in \varepsilon\mathbb{Z}^2$ we get*

$$A_1(x,t,m,\varepsilon) = \lim_{\delta\to+0} \frac{m\varepsilon^3}{4\pi^2} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{ipx-i\omega(t-\varepsilon)}\, d\omega dp}{\sqrt{1+m^2\varepsilon^2}\cos(\omega\varepsilon) - \cos(p\varepsilon) - i\delta},$$

$$A_2(x,t,m,\varepsilon) = \lim_{\delta\to+0} \frac{-i\varepsilon^2}{4\pi^2} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{\sqrt{1+m^2\varepsilon^2}\sin(\omega\varepsilon) + \sin(p\varepsilon)}{\sqrt{1+m^2\varepsilon^2}\cos(\omega\varepsilon) - \cos(p\varepsilon) - i\delta} e^{ip(x-\varepsilon)-i\omega(t-\varepsilon)}\, d\omega dp + \delta_{x\varepsilon}\delta_{t\varepsilon}. \tag{27}$$

Theorem 6 (Large-time asymptotic formula; see Figure 15). *For each $\delta > 0$ there is $C_\delta > 0$ such that for each $m, \varepsilon > 0$ and each $(x,t) \in \varepsilon\mathbb{Z}^2$ satisfying* (11) *we have*

$$b_1(x, t+\varepsilon, m, \varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi}} \left(t^2 - (1+m^2\varepsilon^2)x^2\right)^{-1/4} \cos\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right),$$

$$b_2(x+\varepsilon, t+\varepsilon, m, \varepsilon) = -\varepsilon\sqrt{\frac{2m}{\pi}} \left(t^2 - (1+m^2\varepsilon^2)x^2\right)^{-1/4} \sqrt{\frac{t+x}{t-x}} \sin\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right),$$

for $(x+t)/\varepsilon$ even and odd respectively, where $\theta(x,t,m,\varepsilon)$ is given by (14).

9.2 Physical interpretation

One interprets $\frac{1}{2}|A_1(x,t,m,\varepsilon)|^2 + \frac{1}{2}|A_2(x,t,m,\varepsilon)|^2$ as the *expected charge* in a square (x,t) with $t > 0$, in the units of electron charge. The numbers cannot be anymore interpreted as probabilities to find the electron in the square. The reason is that now the outcomes of the experiment are not mutually exclusive: one can detect an electron in two distinct squares simultaneously. There is nothing mysterious about that: Any measurement necessarily influences the electron. This influence might be enough to create an electron-positron pair from the vacuum. Thus one can detect a newborn electron in addition to the initial one; and there is no way to distinguish one from another. (A more formal explanation for specialists: the states in the Fock space representing the electron localized at distant regions are not mutually orthogonal; their inner product is essentially provided by the Feynman propagator.)

Numerical experiments confirm that the model reproduces the Feynman propagator rather than the retarded one in the continuum limit (see Figure 15 and Problem 13). The *spin*-1/2 *Feynman propagator* equals

$$G^F(x,t) = \begin{cases} \dfrac{m}{4}\begin{pmatrix} J_0(ms) - iY_0(ms) & -\frac{t+x}{s}\left(J_1(ms) - iY_1(ms)\right) \\ \frac{t-x}{s}\left(J_1(ms) - iY_1(ms)\right) & J_0(ms) - iY_0(ms) \end{pmatrix}, & \text{if } |x| < |t|; \\ \dfrac{im}{2\pi}\begin{pmatrix} K_0(ms) & \frac{t+x}{s} K_1(ms) \\ \frac{x-t}{s} K_1(ms) & K_0(ms) \end{pmatrix}, & \text{if } |x| > |t|; \end{cases} \tag{28}$$

where $Y_n(z)$ and $K_n(z)$ are Bessel functions of the 2nd kind and modified Bessel functions of the 2nd kind, and $s := \sqrt{|t^2 - x^2|}$. In addition, there is a generalized function supported on the lines $t = \pm x$ which we do not specify. The Feynman propagator satisfies (19). We see that it has additional imaginary part (and an overall factor of 1/2) compared to retarded one (20). In particular, it does not vanish for $|x| > |t|$: annihilation of electron at one point and creation at another one may result in apparent motion faster than light.

A more common expression is the Fourier transform of a weak limit (cf. (21) and [13, (6.51)])

$$G^F(x,t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \lim_{\delta\to+0} \begin{pmatrix} m & -ip - i\omega \\ -ip + i\omega & m \end{pmatrix} \frac{e^{ipx - i\omega t}\, dp d\omega}{m^2 + p^2 - \omega^2 - i\delta}. \tag{29}$$

Overall, a small correction introduced by the upgrade reflects some fundamental limitations on measurement rather than adds something meaningful to description of the motion. The upgrade should only be viewed as an ingredient for more realistic models with interaction.

10 Towards $(1+1)$-dimensional quantum electrodynamics

Question: what is the probability to find electrons (or an electron and a positron) with momenta q and q' in the far future, if they were emitted with momenta p and p' in the far past?
Assumptions: interaction now switched on; all simplifying assumptions removed except the default ones: no nuclear forces, no gravitation, electron moves along the x-axis only, and the t-axis is time.
Results: repulsion of like charges and attraction of opposite charges (qualitative explanation expected).

Construction of the required model is a widely open problem because in particular it requires the missing mathematically rigorous construction of the *Minkowskian* lattice gauge theory.

11 Open problems

We start with problems relying on Definition 1. The first one is simple-looking but open.

Problem 1. (A.Daniyarkhodzhaev–F.Kuyanov; see Figure 7 to the left) Denote by $x_{\max}(t)$ a point where $P(x,t)$ has a maximum for fixed t. Is $x_{\max}(t) - t/\sqrt{2}$ bounded as $t \to \infty$?

Problem 2. (S. Nechaev; see Figure 7 to the left) Find the positions of "wide gaps" in the plot of $P(x,t)$ for fixed large t. (Cf. asymptotic formulae (12)–(13).)

Problem 3. (A. Borodin) Find an asymptotic formula for $a(2\lceil \frac{vt}{2} \rceil, t)$ as $t \to \infty$ for $v > 1/\sqrt{2}$.

The aim of the next 3 problems is to prove the phase transition in a strong sense: the limiting free energy density and other order parameters are nonanalytic at $v = \pm 1/\sqrt{2}$ (see page 9).

Problem 4. (See Figure 10) For each $v \in [-1,1]$ find $\lim_{t\to\infty} \frac{1}{t} \log \left| a(2\lceil \frac{vt}{2} \rceil, t) \right|$ and prove that it has discontinuous derivative at $v = \pm 1/\sqrt{2}$. (Cf. the proof of Corollary 1 in §12.5.)

The next problem is on the "probability" of equal signs at the ends of the spin-chain.

Problem 5. (See Figure 8) Prove that for each $0 < v < 1/\sqrt{2}$ we have

$$\lim_{t\to\infty} \sum_{0 \le x \le vt} \frac{2}{t} \left| \frac{a_2(x,t)}{a(x,t)} \right|^2 = \frac{1}{2} \left(1 + v - \sqrt{1-v^2} + \log \frac{1+\sqrt{1-v^2}}{2} \right).$$

Compute the same limit for $1/\sqrt{2} < v < 1$. (Cf. the proof of Corollary 1 in §12.5.)

The next one is on the "probability" of equal signs at the ends and the middle of the chain.

Problem 6. (Cf. [22, p. 381].) Find the weak limit $\lim_{t\to\infty} \left| \sum_{x\in\mathbb{Z}} \frac{a_2(x,t)^2}{a_2(2\lceil vt \rceil - 1, 2t-1)} \right|^2$.

For a set $M \subset \mathbb{Z}^2$ define $P(x, t \text{ bypass } M)$ analogously to $P(x,t)$, only the summation is over checker paths disjoint with M. Denote by $P_{\text{hit}}(M) = \sum_{p\in M} P(p \text{ bypass } M \setminus \{p\})$ the *probability that the electron is absorbed in the set M*.

Problem 7. (G. Minaev–I. Russkikh; cf. [2, §5], [34, §4]) Do the following equations hold:

$$P_{\text{hit}}(\{(-1,t) : t \ge 3 \text{ odd}\}) = \frac{1}{2} P_{\text{hit}}(\{(3,t) : t \ge 3 \text{ odd}\}) = \frac{4}{\pi} - 1?$$

Notice that similar numbers appear in the simple random walk on $\mathbb{Z}^2$ [37, Table 2].
The following problem generalizes and specifies Problem 1 above; it relies on Definition 2.

Problem 8. (A.Daniyarkhodzhaev–F.Kuyanov, cf. [2, §4]) Denote by $x_{\max} = x_{\max}(t,m,\varepsilon)$ the point where $P(x) := P(x,t,m,\varepsilon)$ has a maximum. Is $x_{\max}/\varepsilon - t/\varepsilon\sqrt{1+m^2\varepsilon^2}$ uniformly bounded? Does $P(x)$ decrease for $x > x_{\max}$? Find an asymptotic formula for $a(x,t,m,\varepsilon)$ for x in a neighborhood of $t/\sqrt{1+m^2\varepsilon^2}$.

Problem 9. (M. Blank–S. Shlosman) Is the number of times the function $a_1(x) := a_1(x,t,m,\varepsilon)$ changes the sign on $[-t,t]$ bounded as $\varepsilon \to 0$ for fixed t,m?

Corollary 5 gives uniform limit on compact subsets of the angle $|x| < t$, hence misses the main contribution to the probability. Now we ask for the weak limit detecting the peak.

Problem 10. Find the weak limits $\lim\limits_{\varepsilon\to 0} \frac{1}{2\varepsilon}\, a\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right)$ and $\lim\limits_{\varepsilon\to 0} \frac{1}{4\varepsilon^2}\, P\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right)$ on the whole $\mathbb{R}^2$. Is the former limit equal to propagator (21) including the generalized function supported on the lines $t = \pm x$? What is the physical interpretation of the latter limit (providing a value to the ill-defined square of the propagator)?

The following problem relying on Definition 3 would demonstrate "spin precession".

Problem 11. (See Figure 13 to the right) Is $P(x) = \sum\limits_{x\in\mathbb{Z}} a_1(x,t,u)^2$ a periodic function asymptotically as $t \to \infty$ for $u(x+\frac{1}{2}, t+\frac{1}{2}) = (-1)^{(x-1)(t-1)}$?

Define $a(x,t,m,\varepsilon,u)$ analogously to $a(x,t,m,\varepsilon)$ and $a(x,t,u)$, unifying Definitions 2–3 and Remark 4. The next problem asks if this reproduces Dirac equation in electromagnetic field.

Problem 12. (Cf. [15]) Fix $A_0(x,t), A_1(x,t) \in C^2(\mathbb{R}^2)$. For each edge s_1s_2 set

$$u(s_1s_2) := \exp\left(-i\int_{s_1}^{s_2} (A_0(x,t)\,dt + A_1(x,t)\,dx)\right).$$

Denote $\psi_k(x,t) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\, a_k\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil, m, \varepsilon, u\right)$ for $k = 1, 2$. Does the limit satisfy

$$\begin{pmatrix} m & \partial/\partial x - \partial/\partial t + iA_0(x,t) - iA_1(x,t) \\ \partial/\partial x + \partial/\partial t - iA_0(x,t) - iA_1(x,t) & m \end{pmatrix} \begin{pmatrix} \psi_2(x,t) \\ \psi_1(x,t) \end{pmatrix} = 0 \quad \text{for } t > 0?$$

The next problem is to show that Definition 5 reproduces Feynman propagator (28).

Problem 13. (See Figure 15) Prove that

$$\frac{1}{4\varepsilon}\, b_1\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil + \varepsilon, m, \varepsilon\right) \rightrightarrows \operatorname{Im} G^F_{11}(x,t);$$
$$\frac{1}{4\varepsilon}\, b_2\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil + \varepsilon, m, \varepsilon\right) \rightrightarrows \operatorname{Im} G^F_{12}(x,t)$$

as $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^2 \setminus \{|t| = |x|\}$.

This problem can perhaps be approached by expressing b_1 and b_2 through Jacobi functions of the 2nd kind (see Remark 6), and applying the Liouville–Steklov method to their differential equation [43, (4.2.1)].

Problem 14. (Cf. Corollary 1) Prove that $\lim\limits_{\substack{t\to\infty \\ t\in\varepsilon\mathbb{Z}}} \sum\limits_{\substack{x\le vt \\ x\in\varepsilon\mathbb{Z}}} \dfrac{|A_1(x,t,m,\varepsilon)|^2 + |A_2(x,t,m,\varepsilon)|^2}{2} = F(v,m,\varepsilon)$.

Problem 15. (Cf. Remark 7) Find a "combinatorial definition" of numbers (26) as a sum over paths with common starting- and end-points, possibly with downwards-left and downwards-right moves, where each summand depends only on the number of turns of each possible "type" in a path.

The last problem is informal; it stands for half a century.

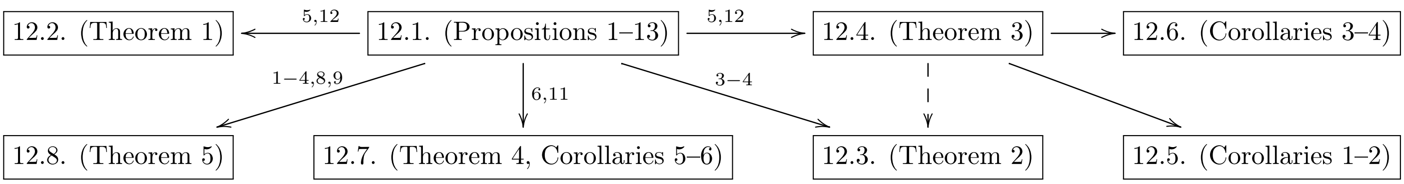
Problem 16. (Cf. [14]) Generalize the model to 4 dimensions so that the pointwise/weak limit

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\, a\left(2\varepsilon\left\lceil \frac{x}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{y}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{z}{2\varepsilon}\right\rceil, 2\varepsilon\left\lceil \frac{t}{2\varepsilon}\right\rceil, m, \varepsilon\right)$$

coincides with the spin-1/2 retarded propagator, now in 3 space- and 1 time-dimension.

12 Proofs

Let us present a chart showing the dependence of the above results and further subsections:



Particular proposition numbers are shown above the arrows. Propositions 7, 10, and 13 are not used in the main results. The dashed arrow depicts an alternative proof.

In the process of the proofs, we give a zero-knowledge introduction to the used methods. Some proofs are simpler than the original ones.

12.1 Identities: elementary combinatorics (Propositions 1–13)

Let us prove the identities from §3; the ones from §2 are the particular case $m = \varepsilon = 1$.

Proof of Propositions 1 and 5. Let us derive a recurrence for $a_2(x,t,m,\varepsilon)$. Take a path s on $\varepsilon\mathbb{Z}^2$ from $(0,0)$ to (x,t) with the first step to $(\varepsilon,\varepsilon)$. Set $a(s,m\varepsilon) := i(-im\varepsilon)^{\mathrm{turns}(s)}(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}$.

The last move in the path s is made either from $(x-\varepsilon,t-\varepsilon)$ or from $(x+\varepsilon,t-\varepsilon)$. If it is from $(x+\varepsilon,t-\varepsilon)$, then $\mathrm{turns}(s)$ must be odd, hence s does not contribute to $a_2(x,t,m,\varepsilon)$. Assume further that the last move in s is made from $(x-\varepsilon,t-\varepsilon)$. Denote by s' the path s without the last move. If the directions of the last moves in s and s' coincide, then $a(s,m\varepsilon) = \frac{1}{\sqrt{1+m^2\varepsilon^2}}a(s',m\varepsilon)$, otherwise $a(s,m\varepsilon) = \frac{-im\varepsilon}{\sqrt{1+m^2\varepsilon^2}}a(s',m\varepsilon) = \frac{m\varepsilon}{\sqrt{1+m^2\varepsilon^2}}(\mathrm{Im}\,a(s',m\varepsilon) - i\mathrm{Re}\,a(s',m\varepsilon))$.

Summation over all paths s' gives the required equation

$$
\begin{aligned}
a_2(x,t,m,\varepsilon) = \mathrm{Im}\sum_{s\ni(x-\varepsilon,t-\varepsilon)} a(s,m\varepsilon) &= \sum_{s'\ni(x-2\varepsilon,t-2\varepsilon)}\frac{\mathrm{Im}\,a(s',m\varepsilon)}{\sqrt{1+m^2\varepsilon^2}} - \sum_{s'\ni(x,t-2\varepsilon)}\frac{m\varepsilon\,\mathrm{Re}\,a(s',m\varepsilon)}{\sqrt{1+m^2\varepsilon^2}} \\
&= \frac{1}{\sqrt{1+m^2\varepsilon^2}}\left(a_2(x-\varepsilon,t-\varepsilon,m,\varepsilon) - m\varepsilon\, a_1(x-\varepsilon,t-\varepsilon,m,\varepsilon)\right).
\end{aligned}
$$

The recurrence for $a_1(x,t+\varepsilon,m,\varepsilon)$ is proved analogously. □

Proof of Propositions 2 and 6. The proof is by induction over t/ε. The base $t/\varepsilon = 1$ is obvious. The step of induction follows immediately from the following computation using Proposition 5:

$$
\begin{aligned}
&\sum_{x\varepsilon\in\mathbb{Z}} P(x,t+\varepsilon,m,\varepsilon) = \sum_{x\in\varepsilon\mathbb{Z}}\left[a_1(x,t+\varepsilon,m,\varepsilon)^2 + a_2(x,t+\varepsilon,m,\varepsilon)^2\right] \\
&= \tfrac{1}{1+m^2\varepsilon^2}\left(\sum_{x\in\varepsilon\mathbb{Z}}[a_1(x+\varepsilon,t,m,\varepsilon)+m\varepsilon\, a_2(x+\varepsilon,t,m,\varepsilon)]^2 + \sum_{x\in\varepsilon\mathbb{Z}}[a_2(x-\varepsilon,t,m,\varepsilon)-m\varepsilon\, a_1(x-\varepsilon,t,m,\varepsilon)]^2\right) \\
&= \tfrac{1}{1+m^2\varepsilon^2}\left(\sum_{x\in\varepsilon\mathbb{Z}}[a_1(x,t,m,\varepsilon)+m\varepsilon\, a_2(x,t,m,\varepsilon)]^2 + \sum_{x\in\varepsilon\mathbb{Z}}[a_2(x,t,m,\varepsilon)-m\varepsilon\, a_1(x,t,m,\varepsilon)]^2\right) \\
&= \sum_{x\in\varepsilon\mathbb{Z}}\left[a_1(x,t,m,\varepsilon)^2 + a_2(x,t,m,\varepsilon)^2\right] = \sum_{x\in\varepsilon\mathbb{Z}} P(x,t,m,\varepsilon).
\end{aligned}
$$

□

Lemma 1 (Conjugate Dirac equation)**.** *For each $(x,t)\in\varepsilon\mathbb{Z}^2$, where $t>\varepsilon$, we have*

$$
\begin{aligned}
a_1(x,t-\varepsilon,m,\varepsilon) &= \frac{1}{\sqrt{1+m^2\varepsilon^2}}(a_1(x-\varepsilon,t,m,\varepsilon) - m\varepsilon\, a_2(x+\varepsilon,t,m,\varepsilon)); \\
a_2(x,t-\varepsilon,m,\varepsilon) &= \frac{1}{\sqrt{1+m^2\varepsilon^2}}(m\varepsilon\, a_1(x-\varepsilon,t,m,\varepsilon) + a_2(x+\varepsilon,t,m,\varepsilon)).
\end{aligned}
$$

Proof of Lemma 1. The second equation is obtained from Proposition 5 by substituting (x,t) by $(x-\varepsilon, t-\varepsilon)$ and $(x+\varepsilon, t-\varepsilon)$ in (6) and (7) respectively and adding them with the coefficients $m\varepsilon/\sqrt{1+m^2\varepsilon^2}$ and $1/\sqrt{1+m^2\varepsilon^2}$. The first equation is obtained analogously. □

Proof of Proposition 7. The real part of the desired equation is the sum of the first equations of Lemma 1 and Proposition 5. The imaginary part the sum of the second ones. □

Proof of Proposition 8. Let us prove the first identity. For a path s denote by s' the reflection of s with respect to the t axis, and by s'' the path consisting of the same moves as s', but in the opposite order.

Take a path s from $(0,0)$ to (x,t) with the first move upwards-right such that $\mathrm{turns}(s)$ is odd (the ones with $\mathrm{turns}(s)$ even do not contribute to $a_1(x,t,m,\varepsilon)$). Then the last move in s is upwards-left. Therefore, the last move in s' is upwards-right, hence the first move in s'' is upwards-right. The endpoint of both s' and s'' is $(-x,t)$, because reordering of moves does not affect the endpoint. Thus $s \mapsto s''$ is a bijection between the paths to (x,t) and to $(-x,t)$ with $\mathrm{turns}(s)$ odd. Thus $a_1(x,t,m,\varepsilon) = a_1(-x,t,m,\varepsilon)$.

We prove the second identity by induction on t/ε (this proof was found and written by E. Kolpakov). The base of induction ($t/\varepsilon = 1$ and $t/\varepsilon = 2$) is obvious.

Step of induction: take $t \geq 3\varepsilon$. Applying the inductive hypothesis for the three points $(x-\varepsilon, t-\varepsilon), (x+\varepsilon, t-\varepsilon), (x, t-2\varepsilon)$ and the identity just proved, we get

$$\begin{aligned}
(t-x)a_2(x-\varepsilon,t-\varepsilon,m,\varepsilon) &= (x+t-4\varepsilon)a_2(3\varepsilon-x,t-\varepsilon,m,\varepsilon),\\
(t-x-2\varepsilon)a_2(x+\varepsilon,t-\varepsilon,m,\varepsilon) &= (x+t-2\varepsilon)a_2(\varepsilon-x,t-\varepsilon,m,\varepsilon),\\
(t-x-2\varepsilon)a_2(x,t-2\varepsilon,m,\varepsilon) &= (x+t-4\varepsilon)a_2(2\varepsilon-x,t-2\varepsilon,m,\varepsilon),\\
a_1(x-\varepsilon,t-\varepsilon,m,\varepsilon) &= a_1(\varepsilon-x,t-\varepsilon,m,\varepsilon).
\end{aligned}$$

Summing up the 4 equations with the coefficients $1, 1, -\sqrt{1+m^2\varepsilon^2}, -2m\varepsilon^2$ respectively, we get

$$\begin{aligned}
&(t-x)\left(a_2(x-\varepsilon,t-\varepsilon,m,\varepsilon)+a_2(x+\varepsilon,t-\varepsilon,m,\varepsilon)-\sqrt{1+m^2\varepsilon^2}\,a_2(x,t-2\varepsilon,m,\varepsilon)\right)\\
&\quad -2m\varepsilon^2\,a_1(x-\varepsilon,t-\varepsilon,m,\varepsilon)-2\varepsilon\,a_2(x+\varepsilon,t-\varepsilon,m,\varepsilon)+2\varepsilon\sqrt{1+m^2\varepsilon^2}\,a_2(x,t-2\varepsilon,m,\varepsilon)=\\
&= -2m\varepsilon^2\,a_1(\varepsilon-x,t-\varepsilon,m,\varepsilon)-2\varepsilon\,a_2(3\varepsilon-x,t-\varepsilon,m,\varepsilon)+2\varepsilon\sqrt{1+m^2\varepsilon^2}\,a_2(2\varepsilon-x,t-2\varepsilon,m,\varepsilon)\\
&+(t+x-2\varepsilon)\left(a_2(3\varepsilon-x,t-\varepsilon,m,\varepsilon)+a_2(\varepsilon-x,t-\varepsilon,m,\varepsilon)-\sqrt{1+m^2\varepsilon^2}\,a_2(2\varepsilon-x,t-2\varepsilon,m,\varepsilon)\right).
\end{aligned}$$

Here the 3 terms in the 2nd line, as well as the 3 terms in the 3rd line, cancel each other by Lemma 1. Applying the Klein–Gordon equation (Proposition 7) to the expressions in the 1st and 4th line and cancelling the common factor $\sqrt{1+m^2\varepsilon^2}$, we get the desired identity

$$(t-x)a_2(x,t,m,\varepsilon) = (t+x-2\varepsilon)a_2(2\varepsilon-x,t,m,\varepsilon).$$

The third identity follows from the first one and Proposition 5:

$$\begin{aligned}
a_1(x,t,m,\varepsilon)+m\varepsilon\,a_2(x,t,m,\varepsilon) &= \sqrt{1+m^2\varepsilon^2}\,a_1(x-\varepsilon,t+\varepsilon,m,\varepsilon) =\\
&= \sqrt{1+m^2\varepsilon^2}\,a_1(\varepsilon-x,t+\varepsilon,m,\varepsilon) = a_1(2\varepsilon-x,t,m,\varepsilon)+m\varepsilon\,a_2(2\varepsilon-x,t,m,\varepsilon).
\end{aligned}$$ □

The 1st and the 3rd identities can also be proved simultaneously by induction on t/ε using Proposition 5.

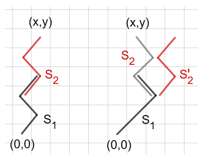


Figure 16: (by I. Bogdanov) The path cut into two parts (see the proof of Proposition 9).

Proof of Proposition 9. Take a checker path s from $(0,0)$ to (x,t). Denote by (x',t') the point where s intersects the line $t=t'$. Denote by s_1 the part of s that joins $(0,0)$ with (x',t'). Denote by s_2 the part starting at the intersection point of s with the line $t=t'-\varepsilon$ and ending at (x,t) (see Figure 16). Translate the path s_2 so that it starts at $(0,0)$. Set $a(s,m\varepsilon) := i(-im\varepsilon)^{\mathrm{turns}(s)}(1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2}$. Since $\mathrm{turns}(s)=\mathrm{turns}(s_1)+\mathrm{turns}(s_2)$, it follows that

$$\mathrm{Re}\, a(s,m\varepsilon) = \begin{cases} \mathrm{Re}\, a(s_1,m\varepsilon)\mathrm{Im}\, a(s_2,m\varepsilon), & \text{if the move to } (x',t') \text{ is upwards-left,} \\ \mathrm{Im}\, a(s_1,m\varepsilon)\mathrm{Re}\, a(s_2,m\varepsilon), & \text{if the move to } (x',t') \text{ is upwards-right.} \end{cases}$$

In the former case replace the path s_2 by the path s_2' obtained by the reflection with respect to the line $x=0$ (and starting at the origin). We have $\mathrm{Im}\, a(s_2',m\varepsilon)=\mathrm{Im}\, a(s_2,m\varepsilon)$. Therefore,

$$a_1(x,t,m,\varepsilon) = \sum_{s} \mathrm{Re}\, a(s,m\varepsilon) = \sum_{x'} \sum_{s\ni(x',t')} \mathrm{Re}\, a(s,m\varepsilon)$$

$$= \sum_{x'} \left(\sum_{s\ni(x',t'),(x'-\varepsilon,t'-\varepsilon)} \mathrm{Im}\, a(s_1,m\varepsilon)\mathrm{Re}\, a(s_2,m\varepsilon) + \sum_{s\ni(x',t'),(x'+\varepsilon,t'-\varepsilon)} \mathrm{Re}\, a(s_1,m\varepsilon)\mathrm{Im}\, a(s_2',m\varepsilon) \right)$$

$$= \sum_{x'} \left[a_2(x',t',m,\varepsilon)a_1(x-x'+\varepsilon,t-t'+\varepsilon,m,\varepsilon) + a_1(x',t',m,\varepsilon)a_2(x'-x+\varepsilon,t-t'+\varepsilon,m,\varepsilon)\right].$$

The formula for $a_2(x,t,m,\varepsilon)$ is proven analogously. □

Proof of Proposition 10. Denote by $f(x,t)$ the difference between the left- and the right-hand side of (8). Introduce the operator

$$[\Box_m f](x,t) := \sqrt{1+m^2\varepsilon^2}\, f(x,t+\varepsilon) + \sqrt{1+m^2\varepsilon^2}\, f(x,t-\varepsilon) - f(x+\varepsilon,t) - f(x-\varepsilon,t).$$

It suffices to prove that

$$[\Box_m^4 f](x,t) = 0 \qquad \text{for } t \geq 5\varepsilon. \tag{30}$$

Then (8) will follow by induction on t/ε: (30) expresses $f(x,t+4\varepsilon)$ as a linear combination of $f(x',t')$ with smaller t', and it remains to check that $f(x,t)=0$ for $t\leq 8\varepsilon$, which is done in [40, §11].

To prove (30), write

$$f(x,t) =: p_1(x,t)a(x-2\varepsilon,t,m,\varepsilon) + p_2(x,t)a(x+2\varepsilon,t,m,\varepsilon) + p_3(x,t)a(x,t,m,\varepsilon)$$

for certain cubic polynomials $p_k(x,t)$ (see (8)), and observe the *Leibnitz rule*

$$\Box_m(fg) = f\cdot\Box_m g + \sqrt{1+m^2\varepsilon^2}(\nabla_{t+}f\cdot T_{t+}g - \nabla_{t-}f\cdot T_{t-}g) - \nabla_{x+}f\cdot T_{x+}g + \nabla_{x-}f\cdot T_{x-}g,$$

where $[\nabla_{t\pm}f](x,t) := \pm(f(x,t\pm\varepsilon)-f(x,t))$ and $[\nabla_{x\pm}f](x,t) := \pm(f(x\pm\varepsilon,t)-f(x,t))$ are the finite difference operators, $[T_{t\pm}g](x,t) := g(x,t\pm\varepsilon)$ and $[T_{x\pm}g](x,t) := g(x\pm\varepsilon,t)$ are the translation operators. Since $\Box_m\, a(x,t,m,\varepsilon)=0$ by Proposition 7, each operator $\nabla_{t\pm},\nabla_{x\pm}$ decreases $\deg p_k(x,t)$, and all the above operators commute, by the Leibnitz rule we get (30). This proves the first identity in the proposition; the second one is proved analogously (the induction step is checked in [40, §11]). □

Alternatively, Proposition 10 can be derived by applying the Gauss contiguous relations to the hypergeometric expression from Remark 3 seven times.

Proof of Propositions 3 and 11. Let us find $a_1(x,t,m,\varepsilon)$. Consider a path with an odd number of turns; the other ones do not contribute to $a_1(x,t,m,\varepsilon)$. Denote by $2r+1$ the number of turns in the path. Denote by R and L the number of upwards-right and upwards-left moves respectively. Let $x_1,x_2,\ldots,x_{r+1}$ be the number of upwards-right moves *before* the first, the third, ..., the last turn respectively. Let $y_1, y_2, \dots, y_{r+1}$ be the number of upwards-left moves *after* the first, the third, ..., the last turn respectively. Then $x_k, y_k \geq 1$ for $1 \leq k \leq r+1$ and

$$R = x_1 + \dots + x_{r+1};$$
$$L = y_1 + \dots + y_{r+1}.$$

The problem now reduces to a combinatorial one: the number of paths with $2r+1$ turns equals the number of positive integer solutions of the resulting equations. For the first equation, this number equals to the number of ways to put r sticks between R coins in a row, that is, $\binom{R-1}{r}$. Thus

$$a_1(x,t,m,\varepsilon) = (1+m^2\varepsilon^2)^{(1-t/\varepsilon)/2} \sum_{r=0}^{\min\{R,L\}} (-1)^r \binom{R-1}{r}\binom{L-1}{r} (m\varepsilon)^{2r+1}.$$

Thus (9) follows from $L+R = t/\varepsilon$ and $R-L = x/\varepsilon$. Formula (10) is derived analogously. □

Proof of Proposition 12. The proof is by induction on t/ε. The base $t/\varepsilon = 1$ is obvious. The inductive step is the following computation and an analogous computation for $a_2(x, t+\varepsilon, m, \varepsilon)$:

$$\begin{aligned} a_1(x,t+\varepsilon,m,\varepsilon) &= \frac{1}{\sqrt{1+m^2\varepsilon^2}} \left(a_1(x+\varepsilon,t,m,\varepsilon) + m\varepsilon\, a_2(x+\varepsilon,t,m,\varepsilon)\right) \\ &= \frac{m\varepsilon^2}{2\pi\sqrt{1+m^2\varepsilon^2}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \left(\frac{ie^{ip\varepsilon}}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}} + 1 + \frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}} \right) e^{ipx - i\omega_p(t-\varepsilon)}\,dp \\ &= \frac{m\varepsilon^2}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{(i\cos(\omega_p\varepsilon) + \sin(\omega_p\varepsilon))\, e^{ipx-i\omega_p(t-\varepsilon)}\,dp}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}} = \frac{im\varepsilon^2}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{ipx-i\omega_p t}\,dp}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}}. \end{aligned}$$

Here the 1st equality is Proposition 5. The 2nd one is the inductive hypothesis. The 3rd one follows from $\cos\omega_p\varepsilon = \frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}}$ and $\sin\omega_p\varepsilon = \sqrt{1 - \frac{\cos^2 p\varepsilon}{1+m^2\varepsilon^2}} = \sqrt{\frac{m^2\varepsilon^2+\sin^2 p\varepsilon}{1+m^2\varepsilon^2}}$. □

Alternatively, Proposition 12 can be derived by integration of (24)–(25) over $p = 2\pi/\lambda$ for $\tilde{a}_1(0,0) = 0$, $\tilde{a}_2(0,0) = 1$.

Proof of Proposition 13. To prove the formula for $a_1(x,t,m,\varepsilon)$, we do the ω-integral:

$$\frac{\varepsilon}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{-i\omega(t-\varepsilon)}\,d\omega}{\sqrt{1+m^2\varepsilon^2}\cos(\omega\varepsilon) - \cos(p\varepsilon) - i\delta} \overset{(*)}{=} \frac{1}{2\pi i} \oint_{|z|=1} \frac{2\,z^{t/\varepsilon-1}\,dz}{\sqrt{1+m^2\varepsilon^2}z^2 - 2(\cos(p\varepsilon)+i\delta)z + \sqrt{1+m^2\varepsilon^2}}$$

$$\overset{(**)}{=} \frac{\left(\left(\cos p\varepsilon + i\delta - i\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon + \delta^2 - 2i\delta\cos p\varepsilon}\right)/\sqrt{1+m^2\varepsilon^2}\right)^{t/\varepsilon-1}}{-i\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon+\delta^2-2i\delta\cos p\varepsilon}} \overset{(***)}{\rightrightarrows} \frac{i\,e^{-i\omega_p(t-\varepsilon)}}{\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon}}$$

as $\delta \to 0$ uniformly in p. Here we assume that $m, t, \delta > 0$ and δ is sufficiently small. Equality $(*)$ is obtained by the change of variables $z = e^{-i\omega\varepsilon}$ and then the change of the contour direction to the counterclockwise one. To prove $(**)$, we find the roots of the denominator

$$z_\pm = \frac{\cos p\varepsilon + i\delta \pm i\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon+\delta^2-2i\delta\cos p\varepsilon}}{\sqrt{1+m^2\varepsilon^2}},$$

where $\sqrt{z}$ denotes the value of the square root with positive real part. Then $(**)$ follows from the residue formula: the expansion

$$z_\pm = \frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}} \left(1 \pm \frac{\delta}{\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon}}\right) + \frac{i}{\sqrt{1+m^2\varepsilon^2}} \left(\delta \pm \sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon}\right) + O_{m,\varepsilon}\left(\delta^2\right)$$

shows that z_- is inside the unit circle, whereas z_+ is outside, for sufficiently small $\delta > 0$. In $(***)$ we denote $\omega_p := \frac{1}{\varepsilon}\arccos(\frac{\cos p\varepsilon}{\sqrt{1+m^2\varepsilon^2}})$ so that $\sin\omega_p\varepsilon = \sqrt{\frac{m^2\varepsilon^2+\sin^2 p\varepsilon}{1+m^2\varepsilon^2}}$ and pass to the limit $\delta \to 0$ which is uniform in p by the assumption $m > 0$.

The resulting uniform convergence allows to interchange the limit and the p-integral, and we arrive at Fourier integral for $a_1(x,t,m,\varepsilon)$ in Proposition 12. The formula for $a_2(x,t,m,\varepsilon)$ is proved analogously, with the case $t = \varepsilon$ considered separately. □

12.2 Phase transition: the method of moments (Theorem 1)

In this subsection we give a simple exposition of the proof of Theorem 1 from [18] using the *method of moments*. The theorem also follows from Corollary 1 obtained by another method in §12.5. We rely on the following well-known result.

Lemma 2. (See [6, Theorems 30.1–30.2]) *Let $f_t\colon \mathbb{R} \to [0,+\infty)$, where $t = 0,1,2,\dots,$ be piecewise continuous functions such that $\alpha_{r,t} := \int_{-\infty}^{+\infty} v^r f_t(v)\,dv$ is finite and $\alpha_{0,t} = 1$ for each $r,t = 0,1,2,\dots$. If the series $\sum_{r=0}^{\infty} \alpha_{r,0} z^r/r!$ has positive radius of convergence and $\lim_{t\to\infty} \alpha_{r,t} = \alpha_{r,0}$ for each $r = 0,1,2,\dots,$ then f_t converges to f_0 in distribution.*

Proof of Theorem 1. Let us prove (C); then (A) and (B) will follow from Lemma 2 for $f_0(v) := F'(v)$ and $f_t(v) := P(\lceil vt\rceil, t)/t$ because those functions vanish for $|v| > 1$, hence $\alpha_{r,t} \le 1$.

Rewrite Proposition 12 in a form, valid for *each* $x,t \in \mathbb{Z}$ independently on the parity:

$$\begin{pmatrix} a_1(x,t) \\ a_2(x,t) \end{pmatrix} = \int_{-\pi}^{\pi} \begin{pmatrix} \hat{a}_1(p,t) \\ \hat{a}_2(p,t) \end{pmatrix} e^{ip(x-1)} \frac{dp}{2\pi} := \int_{-\pi}^{\pi} \begin{pmatrix} \hat{a}_{1+}(p,t) + a_{1-}(p,t) \\ \hat{a}_{2+}(p,t) + a_{2-}(p,t) \end{pmatrix} e^{ip(x-1)} \frac{dp}{2\pi}, \tag{31}$$

where

$$\begin{aligned} \hat{a}_{1\pm}(p,t) &= \mp \frac{ie^{ip}}{2\sqrt{1+\sin^2 p}} e^{\pm i\omega_p(t-1)}; \\ \hat{a}_{2\pm}(p,t) &= \frac{1}{2}\left(1 \mp \frac{\sin p}{\sqrt{1+\sin^2 p}}\right) e^{\pm i\omega_p(t-1)}; \end{aligned} \tag{32}$$

and $\omega_p := \arccos\frac{\cos p}{\sqrt{2}}$. Now (31) holds for each $x,t \in \mathbb{Z}$: Indeed, the identity

$$e^{-i\omega_{p+\pi}(t-1)+i(p+\pi)(x-1)} = e^{-i(\pi-\omega_p)(t-1)+ip(x-1)+i\pi(x-1)} = (-1)^{(x+t)} e^{i\omega_p(t-1)+ip(x-1)}$$

shows that the contributions of the two summands $\hat{a}_{k\pm}(p,t)$ to integral (31) are equal for $t+x$ even and cancel for $t+x$ odd. The summand $\hat{a}_{k-}(p,t)$ contributes $a_k(x,t)/2$ by Proposition 12.

By the derivative property of Fourier series and the Parseval theorem, we have

$$\sum_{x\in\mathbb{Z}} \frac{x^r}{t^r} P(x,t) = \sum_{x\in\mathbb{Z}} \begin{pmatrix} a_1(x,t) \\ a_2(x,t) \end{pmatrix}^{*} \frac{x^r}{t^r} \begin{pmatrix} a_1(x,t) \\ a_2(x,t) \end{pmatrix} = \int_{-\pi}^{\pi} \begin{pmatrix} \hat{a}_1(p,t) \\ \hat{a}_2(p,t) \end{pmatrix}^{*} \frac{i^r}{t^r} \frac{\partial^r}{\partial p^r} \begin{pmatrix} \hat{a}_1(p,t) \\ \hat{a}_2(p,t) \end{pmatrix} \frac{dp}{2\pi}. \tag{33}$$

The derivative is estimated as follows:

$$\frac{\partial^r}{\partial p^r} \hat{a}_{k\pm}(p,t) = \left(\pm i(t-1)\frac{\partial \omega_p}{\partial p}\right)^r \hat{a}_{k\pm}(p,t) + O_r(t^{r-1}) = \left(\pm \frac{i(t-1)\sin p}{\sqrt{1+\sin^2 p}}\right)^r \hat{a}_{k\pm}(p,t) + O_r(t^{r-1}). \tag{34}$$

Indeed, differentiate (32) r times using the Leibnitz rule. If we differentiate the exponential factor $e^{\pm i\omega_p(t-1)}$ each time, then we get the main term. If we differentiate a factor rather than the exponential $e^{\pm i\omega_p(t-1)}$ at least once, then we get less than r factors of $(t-1)$, hence the resulting term is $O_r(t^{r-1})$ by compactness because it is continuous and 2π-periodic in p.

Substituting (34) into (33) we arrive at

$$\begin{aligned} \sum_{x\in\mathbb{Z}} \frac{x^r}{t^r} P(x,t) &= \int_{-\pi}^{\pi} \begin{pmatrix} \hat{a}_1(p,t) \\ \hat{a}_2(p,t) \end{pmatrix}^{*} \left(\frac{\sin p}{\sqrt{1+\sin^2 p}}\right)^r \begin{pmatrix} (-1)^r \hat{a}_{1+}(p,t) + \hat{a}_{1-}(p,t) \\ (-1)^r \hat{a}_{2+}(p,t) + \hat{a}_{2-}(p,t) \end{pmatrix} \frac{dp}{2\pi} + O_r\left(\frac{1}{t}\right) \\ &= \int_{-\pi}^{\pi} \left(\frac{\sin p}{\sqrt{1+\sin^2 p}}\right)^r \frac{1}{2}\left((-1)^r\left(1 - \frac{\sin p}{\sqrt{1+\sin^2 p}}\right) + 1 + \frac{\sin p}{\sqrt{1+\sin^2 p}}\right) \frac{dp}{2\pi} + O_r\left(\frac{1}{t}\right) \\ &= \int_{-\pi/2}^{\pi/2} \left(\frac{\sin p}{\sqrt{1+\sin^2 p}}\right)^r \left(1 + \frac{\sin p}{\sqrt{1+\sin^2 p}}\right) \frac{dp}{\pi} + O_r\left(\frac{1}{t}\right) = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{v^r\,dv}{\pi(1-v)\sqrt{1-2v^2}} + O_r\left(\frac{1}{t}\right). \end{aligned}$$

Here the 2nd equality follows from $\hat{a}_{1\pm}(p,t)^*\hat{a}_{1\pm}(p,t) + \hat{a}_{2\pm}(p,t)^*\hat{a}_{2\pm}(p,t) = \frac{1}{2}\left(1 \mp \frac{\sin p}{\sqrt{1+\sin^2 p}}\right)$ and $\hat{a}_{1\pm}(p,t)^*\hat{a}_{1\mp}(p,t) + \hat{a}_{2\pm}(p,t)^*\hat{a}_{2\mp}(p,t) = 0$. The 3rd one is obtained by the changes of variables $p \mapsto -p$ and $p \mapsto \pi - p$ applied to the integral over $[-\pi/2, \pi/2]$. The 4th one is obtained by the change of variables $v = \sin p/\sqrt{1+\sin^2 p}$ so that $dp = d\arcsin \frac{v}{\sqrt{1-v^2}} = dv/(1-v^2)\sqrt{1-2v^2}$. □

12.3 Large-time limit near the origin: the circle method (Theorem 2)

In this subsection we prove Theorem 2. Although the theorem follows easily from Theorem 3 by the Taylor expansion of (14) in x up to order 4, we present a direct proof relying on Proposition 4 only. It uses the *Hardy–Littlewood circle method* in the simplest form, when the main contribution to an integral comes from just 4 stationary points.

Let us give the plan of the proof: introduce some notation and state some lemmas.

Lemma 3. *Take integers*

$$n \geq k \geq 0, \qquad x := 2k - n, \qquad \text{and} \qquad t := n + 2. \tag{35}$$

Then

$$a_1(x,t) = 2^{(n-1)/2} i^{-k} \hat{f}(-x), \tag{36}$$
$$a_2(x,t) = 2^{(n-1)/2} i^{-k} \hat{f}(2-x), \tag{37}$$

where

$$\hat{f}(q) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(p) e^{-ipq}\, dp \qquad \text{and} \qquad f(p) := \cos^{n-k} p \, \sin^k p. \tag{38}$$

Notice that $f(p)$ is *not* proportional to the Fourier series $\sum_{x\in\mathbb{Z}} a_1(x,t)e^{-ipx}$ because both x and $f(p)$ depend on k, and thus Fourier inversion formula is not applicable.

Throughout this subsection assume that

$$\frac{|x|}{t} < \frac{1}{20}. \tag{39}$$

This inequality follows from the assumption $|x| < t^{3/4}$ in Theorem 2 for sufficiently large t.

The Fourier integral $\hat{f}(q)$ is estimated in several steps. The main contribution comes from the sharp extrema of the function $f(p)$. The derivative is

$$f'(p) = f(p)(k \cot p - (n-k)\tan p).$$

Here $k \neq 0, n$ by (39); we ignore the points where $f(p) = 0$ because they cannot be global extrema. Thus the global extrema on $[-\pi, \pi]$ are the 4 points $\pm c$ and $\pm c \mp \pi$, where

$$c := \arctan\sqrt{\frac{k}{n-k}}. \tag{40}$$

Lemma 4. *Assume* (35)–(40)*. Then* $|c - \pi/4| < 1/9$ *and* $9/10 < \tan c < 10/9$.

Decompose $[-\pi, \pi] = E_0 \sqcup E_1 \sqcup \dots \sqcup E_4$, where $E_1, \dots, E_4$ are δ-neighborhoods of these extrema and $E_0 = [-\pi,\pi] \setminus (E_1 \cup \dots \cup E_4)$ is the remaining set. The parameter δ is going to be determined later. Write

$$\hat{f}(q) = \hat{f}_0(q) + \dots + \hat{f}_4(q), \tag{41}$$

where

$$\hat{f}_j(q) = \frac{1}{2\pi}\int_{E_j} f(p)e^{-ipq}\, dp \qquad \text{for each } j = 0,1,2,3,4. \tag{42}$$

We start with a rough estimate of the function $f(p)$ through the distance to the extremum c.

Lemma 5. *Assume* (35)–(42) *and* $0 < c + p < \pi/2$. *Then*

$$f(c+p) \le f(c)e^{-np^2/4}.$$

This immediately gives an estimate for the integral on the set E_0:

$$\hat{f}_0(q) = f(c)O(e^{-n\delta^2/4}). \tag{43}$$

Next we expand the function $f(p)$ in a neighborhood of c. We consider complex p as well.

Lemma 6. *Assume* (35)–(42). *Then for each complex number* p *with* $|p| < 1/2$ *we have*

$$f(c+p) = f(c)\exp\left(-np^2 + O(|x|\,|p|^3 + t|p|^4)\right). \tag{44}$$

Recall that notation $f(p,x,t) = \exp(O(g(p,x,t)))$ means that there is a number $C > 0$ (not depending on p, x, t) and a complex-valued function $h(p,x,t)$ such that $f(p,x,t) = e^{h(p,x,t)}$ and $|h(p,x,t)| \le Cg(p,x,t)$ for each p, x, t satisfying the assumptions of the lemma.

Next we substitute expansion (44) into (42) and apply the *method of steepest descent.*

Lemma 7. *Assume* (35)–(42) *and* $|q|/t < \delta/2 < 1/8$. *Then*

$$\hat{f}_1(q) = \frac{f(c)}{2\sqrt{\pi n}}\exp\left(-\frac{q^2}{4n} - icq + O(|x|\delta^3 + t\delta^4)\right)\left(1 + O\left(e^{-t\delta^2/4}\right)\right) \tag{45}$$

The integrals $\hat{f}_2(q), \hat{f}_3(q), \hat{f}_4(q)$ are estimated analogously. Finally, applying Lemma 6 for $p = \pi/4 - c$, we arrive at the following lemma.

Lemma 8. *Assume* (35)–(42). *Then*

$$f(c) = 2^{-n/2}\exp\left(\frac{x^2}{4n} + O\left(\frac{x^4}{t^3}\right)\right). \tag{46}$$

Substituting the resulting asymptotic formulae (45), (46), and estimate (43) into (41), we obtain an asymptotic formula for $\hat{f}(q)$ and thus for $a(x,t)$.

Now realize this plan.

Proof of Lemma 3. Applying the Cauchy formula to Proposition 4, we get

$$a_1(-n+2k, n+2) = \frac{2^{-(n+1)/2}}{2\pi i r}\int_\gamma \frac{(1+z)^{n-k}(1-z)^k}{z^{n-k+1}}dz,$$

where γ is a contour performing r counterclockwise turns around the origin. Taking the contour $z = e^{2ip}$, where $p \in [-\pi, \pi]$, performing $r = 2$ turns, we get

$$(1+z)^{n-k}(1-z)^k = 2^n i^{-k} e^{ipn} f(p) \qquad \text{and} \qquad z^{-n+k-1}\,dz = 2i\,e^{-2i(n-k)p}\,dp.$$

This gives (36); analogously one gets (37). □

Remark 8. A contour γ performing just one turn gives the formula $a_1(x,t) = \frac{1}{\pi}2^{(n-1)/2}i^{-k}\int_0^\pi f(p)e^{ipx}\,dp$. It can be alternatively used in the proof of Theorem 2; nothing changes essentially.

Proof of Lemma 4. By (39) we get $(n-2k)/n \le 2|x|/t < 1/10$, thus $9/11 < k/(n-k) < 11/9$, hence $9/10 < \tan c = \sqrt{\frac{k}{n-k}} < 10/9$. By the Lagrange theorem, $\tan c - \tan(\pi/4) = (c - \pi/4)/\cos^2\zeta$ for some $\zeta \in (c, \pi/4)$. Hence $|c - \pi/4| \le |\tan c - \tan(\pi/4)| < |10/9 - 1| = 1/9$. □

Proof of Lemma 5. Since the sine and the cosine are concave on the interval $[0, \pi/2]$, they can be estimated from above by a linear function representing the tangent line at the point c:

$$\sin \pi(c+p) \le \sin c(1 + p\cot c), \qquad \cos \pi(c+p) \le \cos c(1 - p\tan c).$$

Thus

$$|f(c+p)| \le f(c)(1+p\cot c)^k(1-p\tan c)^{n-k} = f(c)\exp\left(k\log(1+p\cot c) + (n-k)\log(1-p\tan c)\right).$$

To estimate the logarithms, we apply the inequality

$$\log(1+z) \le z - \frac{z^2}{4} \qquad \text{for } z \in (-1;1). \tag{47}$$

The inequality follows from

$$e^z e^{-z^2/4} \ge \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)\left(1 - \frac{z^2}{4}\right) = 1 + z + \frac{z^2}{4}\left(1 - \frac{z}{3} - \frac{z^2}{2} - \frac{z^3}{6}\right) \ge 1 + z.$$

We are going to apply inequality (47) at the points $z_1 = p\cot c$ and $z_2 = -p\tan c$. Let us check that indeed $z_1, z_2 \in (-1;1)$. Since $0 < c + p < \pi/2$, by Lemma 4 it follows that

$$|z_2| = |p|\tan c < \left(\frac{\pi}{4} + \left(\frac{\pi}{4} - c\right)\right)\tan c < \left(\frac{\pi}{4} + \frac{1}{9}\right)\cdot\frac{10}{9} < 1.$$

Analogously $|z_1| < 1$, as required.

By inequality (47) we get

$$f(c+p) \le f(c)\exp\left(k\left(z_1 - \frac{z_1^2}{4}\right) + (n-k)\left(z_2 - \frac{z_2^2}{4}\right)\right) =$$
$$= f(c)\exp\left(p\sqrt{k(n-k)} - p\sqrt{k(n-k)} - \frac{p^2}{4}(n-k) - \frac{p^2}{4}k\right) = f(c)e^{-np^2/4}. \qquad \square$$

Proof of Lemma 6. Take the Taylor expansions with remainders in the Lagrange form:

$$\sin(c+p) = \sin c + p\cos c - \frac{p^2}{2}\sin c - \frac{p^3}{6}\cos c + \frac{p^4}{24}\sin(c+\zeta p), \tag{48}$$
$$\cos(c+p) = \cos c - p\sin c - \frac{p^2}{2}\cos c + \frac{p^3}{6}\sin c + \frac{p^4}{24}\cos(c+\eta p), \tag{49}$$

where $0 \le \zeta, \eta \le 1$. Since $|p| < 1/2$ it follows that $|\sin(c+\zeta p)| \le e^{\zeta|p|} < 2$ and $|\cos(c+\eta p)| \le e^{\eta|p|} < 2$. By definition of c, we have

$$\sin c = \sqrt{\frac{k}{n}}, \quad \cos c = \sqrt{\frac{n-k}{n}}, \quad \tan c = \sqrt{\frac{k}{n-k}}, \quad \cot c = \sqrt{\frac{n-k}{k}}.$$

By (39) we get $|n - 2k| \le n/2$, thus $n/4 \le k \le 3n/4$, hence $\sin c \ge 1/2$ and $\cos c \ge 1/2$. Thus

$$\sin(c+p) = \sin c\left(1 + p\cot c - \frac{p^2}{2} - \frac{p^3}{6}\cot c + \frac{|p|^4}{6}\zeta'\right) =: \sin c\,(1+z_1), \tag{50}$$
$$\cos(c+p) = \cos c\left(1 - p\tan c - \frac{p^2}{2} + \frac{p^3}{6}\tan c + \frac{|p|^4}{6}\eta'\right) =: \cos c\,(1+z_2), \tag{51}$$

for some complex ζ', η' of absolute value < 1. Since $|p| < 1/2$ and $9/10 < \tan c < 10/9$ by Lemma 4, we get

$$|z_{1,2}| \le |p|\max\{\cot c, \tan c\} + \frac{|p|^2}{2} + \frac{|p|^3}{6}\max\{\cot c, \tan c\} + \frac{|p|^4}{6} < \frac{3}{4}.$$

Substituting expansions (50) and (51) into the Taylor expansion

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + O(z^4) \qquad \text{for } |z| < 3/4,$$

and then into (38), by a direct computation (available in [40, §10]) we get

$$\frac{f(c+p)}{f(c)} = \exp\left(-np^2 + \frac{(n-2k)np^3}{3\sqrt{k(n-k)}} + O(n|p|^4)\right) = \exp\left(-np^2 + O(|x||p|^3) + O(t|p|^4)\right).$$

Here we used that $n/3 < \sqrt{k(n-k)} < n$ by assumption (39). □

Proof of Lemma 7. Denote $g(p) := f(c+p)e^{np^2}/f(c)$, so that

$$\hat{f}_1(q) = \int_{-\delta}^{\delta} f(c+p)e^{-i(c+p)q}\frac{dp}{2\pi} = \frac{f(c)}{2\pi}e^{-icq}\int_{-\delta}^{\delta} e^{-np^2-ipq}g(p)\,dp = \frac{f(c)}{2\pi}e^{-q^2/4n-icq}\int_{-\delta}^{\delta} e^{-\left(\sqrt{n}p+iq/2\sqrt{n}\right)^2}g(p)\,dp.$$

Switch to complex variable $z = \sqrt{n}p + iq/2\sqrt{n}$. Then the limits of integration become $B_1 := -\delta\sqrt{n} + iq/2\sqrt{n}$ and $B_2 := \delta\sqrt{n} + iq/2\sqrt{n}$:

$$\hat{f}_1(q) = \frac{f(c)}{2\pi\sqrt{n}}e^{-q^2/4n-icq}\int_{B_1}^{B_2} e^{-z^2}g\left(\frac{z}{\sqrt{n}} - \frac{iq}{2n}\right)dz. \tag{52}$$

The function under the integral is analytic in the whole complex plane, thus the integral over the interval B_1B_2 can be replaced by the integral over the broken line $B_1A_1A_2B_2$ with the corners at $A_1 := -\delta\sqrt{n}$ and $A_2 := \delta\sqrt{n}$.

Estimate the contribution from the integrals over the intervals A_1B_1 and A_2B_2. Since $|q|/t < \delta/2 < 1/8$, it follows that for each $z \in B_1A_1A_2B_2$ we have

$$\left|\frac{z}{\sqrt{n}} - \frac{iq}{2n}\right| < 2\delta < \frac{1}{2}. \tag{53}$$

Then by Lemma 6 for each $z \in B_1A_1A_2B_2$ we get

$$g\left(\frac{z}{\sqrt{n}} - \frac{iq}{2n}\right) = \exp\left(O(|x|\delta^3 + t\delta^4)\right). \tag{54}$$

Then

$$\int_{A_\nu}^{B_\nu} e^{-z^2}g\left(\frac{z}{\sqrt{n}} - \frac{iq}{2n}\right)dz \le \frac{|q|}{2\sqrt{n}}\max_{z\in A_\nu B_\nu}\left|e^{-z^2}\right|\max_{z\in A_\nu B_\nu}\left|g\left(\frac{z}{\sqrt{n}} - \frac{iq}{n}\right)\right| =$$
$$= \frac{|q|}{2\sqrt{n}}\exp\left(-n\delta^2 + \frac{|q|^2}{4n} + O(|x|\delta^3 + t\delta^4)\right) = O\left(\exp\left(-\frac{n\delta^2}{4}\right)\right)\exp\left(O(|x|\delta^3 + t\delta^4)\right).$$

In the latter estimate we used the inequality $|q|/t < \delta/2$, which implies $|q| < n\delta$, so that

$$\frac{|q|}{2\sqrt{n}}\exp\left(-n\delta^2 + \frac{|q|^2}{4n}\right) \le \frac{\sqrt{n}\delta}{2}\exp\left(-\frac{n\delta^2}{2}\right) \le \exp\left(-\frac{n\delta^2}{4}\right).$$

Now compute the contribution from the integral over the interval A_1A_2. Using asymptotic formula (54), let us show that

$$\int_{A_1}^{A_2} e^{-z^2}g\left(\frac{z}{\sqrt{n}} - \frac{iq}{2n}\right)dz = \exp\left(O(|x|\delta^3 + t\delta^4)\right)\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-z^2}\,dz.$$

Indeed, if $|x|\delta^3 + t\delta^4 \le 2\pi$, then $\exp\left(O(|x|\delta^3 + t\delta^4)\right)$ is the same as $1 + O(|x|\delta^3 + t\delta^4)$, and the formula follows. If $|x|\delta^3 + t\delta^4 > 2\pi$, then $\exp\left(O(|x|\delta^3 + t\delta^4)\right)$ is the same as $O\left(\exp(|x|\delta^3 + t\delta^4)\right)$, and the formula follows again.

It remains to compute

$$\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} e^{-z^2}\,dz = \int_{-\infty}^{+\infty} e^{-z^2}\,dz - 2\int_{\delta\sqrt{n}}^{+\infty} e^{-z^2}\,dz = \sqrt{\pi} + O\left(e^{-n\delta^2}\right),$$

where we applied the estimate for the *complimentary error function* [1, Eq. 7.1.13]

$$\int_N^{\infty} e^{-z^2}\,dz \le \frac{e^{-N^2}}{N+1} = O\left(e^{-N^2}\right) \qquad \text{for } N > 0.$$

Combining the estimates for the integrals over A_1B_1, A_1A_2, A_2B_2, we get the desired result. □

Proof of Lemma 8. Take the Taylor expansion

$$\arcsin\sqrt{\frac{1}{2}+z} = \frac{\pi}{4} + z + O(|z|^3) \qquad \text{for } z \in \left(-\frac{1}{4}, \frac{1}{4}\right).$$

Substituting $z = x/2n$ so that $|z| = |x|/2n \le |x|/t < 1/4$ by (39), we get

$$c = \arcsin\sqrt{\frac{k}{n}} = \frac{\pi}{4} + \frac{x}{2n} + O\left(\frac{|x|^3}{n^3}\right). \tag{55}$$

Apply Lemma 6 for $p = \pi/4 - c$. Then by (44) we get

$$f\left(\frac{\pi}{4}\right) = f(c)\exp\left(-n\left(\frac{x}{2n} + O\left(\frac{|x|^3}{n^3}\right)\right)^2 + O\left(\frac{x^4}{n^3}\right)\right) = f(c)\exp\left(-\frac{x^2}{4n} + O\left(\frac{x^4}{t^3}\right)\right).$$

It remains to notice that

$$f\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^k \left(\cos\frac{\pi}{4}\right)^{n-k} = 2^{-n/2}.$$

□

Proof of Theorem 2. We may assume $t > 10^6$ because only finitely many pairs (x,t) with $t \le 10^6$ satisfy the assumptions of the theorem. For $t > 10^6$ the assumption $|x| < t^{3/4}$ implies inequality (39).

We need to find asymptotic formulae for the integral $\hat{f}(q)$ for $q = -x$ and $q = 2 - x$ (see Lemma 3). Thus assume further that $|q| \le |x| + 2$ and $q + t$ is even. To guarantee the inequality $|q|/t < \delta/2$ required for Lemma 7, we need to take $\delta > 2(|x|+2)/t$. To guarantee that the right-hand side of (43) is small enough, we need to take $\delta > 4\sqrt{\frac{\log n}{n}}$. The remainder under the first exponential in (45) increases with δ, thus we assign the least possible value

$$\delta := \max\left\{4\sqrt{\frac{\log n}{n}}, 2\frac{|x|+2}{t}\right\}.$$

Then clearly $\delta < 1/4$ by the inequalities $t > 10^6$ and (39). The assumption $|x| < t^{3/4}$ implies that $|x|\delta^3 + t\delta^4 = O(1)$, hence

$$\exp\left(O(|x|\delta^3 + t\delta^4)\right) = 1 + O(R(t)), \qquad \text{where} \qquad R(t) = t^{-1}\log^2 t + x^4 t^{-3}.$$

For such choice of δ, formula (45) becomes

$$\hat{f}_1(q) = \frac{f(c)}{2\sqrt{\pi n}}\exp\left(-\frac{q^2}{4n} - icq\right)(1 + O(R(t))),$$

Analogous formulae hold for $\hat{f}_2(q), \hat{f}_3(q), \hat{f}_4(q)$ with c replaced by $-c$ and $\pm\pi \mp c$. Estimate (43) becomes

$$\hat{f}_0(q) = f(c)O(e^{-n\delta^2/4}) = \frac{f(c)}{2\sqrt{\pi n}}e^{-q^2/4n}O\left(\sqrt{n}\exp\left(\frac{q^2}{4n} - \frac{n\delta^2}{4}\right)\right) = \frac{f(c)}{2\sqrt{\pi n}}e^{-q^2/4n}O(R(t)),$$

because $\sqrt{n}\exp\left(\frac{q^2}{4n}-\frac{n\delta^2}{4}\right)\le\sqrt{n}\exp\left(-\frac{n\delta^2}{8}\right)\le n^{-3/2}$ by the inequalities $|q|\le|x|+2<\delta t/2<\delta\sqrt{2}n/2$ for $t>10^6$ and $\delta>4\sqrt{\frac{\log n}{n}}$.

Substituting the resulting formulae for $\hat{f}_0(q),\dots,\hat{f}_4(q)$ into (41), then applying Lemma 8 and substituting (55), and using that $q+n$ even and $|q|\le|x|+2$, we find

$$\hat{f}(q)=\frac{f(c)}{\sqrt{\pi n}}e^{-\frac{q^2}{4n}}\left(e^{-icq}+(-1)^k e^{icq}+O(R(t))\right)=2i^k\frac{f(c)}{\sqrt{\pi n}}e^{-\frac{q^2}{4n}}\left(\cos\left(cq+\frac{\pi k}{2}\right)+O(R(t))\right)$$
$$=i^k\frac{2^{\frac{2-n}{2}}}{\sqrt{\pi n}}e^{\frac{x^2-q^2}{4n}}\left(\cos\left(cq+\frac{\pi k}{2}\right)+O(R(t))\right)=i^k\frac{2^{\frac{2-n}{2}}}{\sqrt{\pi n}}e^{\frac{x^2-q^2}{4n}}\left(\cos\left(\frac{\pi(2k+q)}{4}+\frac{xq}{2n}\right)+O\left(R(t)\right)\right).$$

Substituting the resulting expression into equations (36)–(37), we get

$$a_1(x,t)=\sqrt{\frac{2}{\pi n}}\left(\sin\left(\frac{\pi t}{4}-\frac{x^2}{2n}\right)+O\left(R(t)\right)\right);$$
$$a_2(x,t)=\sqrt{\frac{2}{\pi n}}e^{(x-1)/n}\left(\cos\left(\frac{\pi t}{4}-\frac{x^2}{2n}+\frac{x}{n}\right)+O\left(R(t)\right)\right).$$

Let us simplify these expressions further. Denote $\theta:=\pi t/4-x^2/2n$. By the Cauchy inequality, $x^2/t^2\le 1/t+x^4/t^3\le R(t)$ for all x. Thus $e^{(x-1)/n}=1+x/n+O(R(t))$ and

$$\cos\left(\theta+\frac{x}{n}\right)=\cos\theta\cos\frac{x}{n}-\sin\theta\sin\frac{x}{n}=\cos\theta-\frac{x}{n}\sin\theta+O\left(R(t)\right).$$

This allows to rewrite

$$a_2(x,t)=\sqrt{\frac{2}{\pi n}}\left(1+\frac{x}{n}\right)\left(\cos\theta-\frac{x}{n}\sin\theta+O\left(R(t)\right)\right)=\sqrt{\frac{2}{\pi n}}\left(\cos\theta+\frac{\sqrt{2}x}{n}\cos\left(\theta+\frac{\pi}{4}\right)+O\left(R(t)\right)\right).$$

Here we can replace n by $t=n+2$ (inside θ as well) because

$$\frac{1}{\sqrt{n}}=\frac{1}{\sqrt{t}}\sqrt{\frac{n+2}{n}}=\frac{1}{\sqrt{t}}\left(1+O\left(\frac{1}{n}\right)\right)=\frac{1}{\sqrt{t}}\left(1+O\left(R(t)\right)\right),$$
$$\frac{x}{n}=\frac{x}{t}\cdot\frac{n+2}{n}=\frac{x}{t}\left(1+O\left(R(t)\right)\right),\qquad\text{and}\qquad\frac{x^2}{n}=\frac{x^2}{t}+\frac{2x^2}{nt}=\frac{x^2}{t}+O\left(R(t)\right).$$

We arrive at the required result. □

12.4 The main result: the stationary phase method (Theorem 3)

In this subsection we prove Theorem 3. First we outline the plan of the argument, then prove the theorem modulo some technical lemmas, and finally the lemmas themselves.

The plan is to apply the Fourier transform and the *stationary phase method* to the resulting oscillatory integral. The proof consists of 5 steps, with the first one known before [2, §4]:

Case (A): $|x|/t<1/\sqrt{1+m^2\varepsilon^2}-\delta$.

Step 1: computing the main term in the asymptotic formula;

Step 2: estimating approximation error arising from neighborhoods of stationary points;

Step 3: estimating approximation error arising from a neighborhood of the origin;

Step 4: estimating error arising from the complements to those neighborhoods.

Case (B): $|x|/t>1/\sqrt{1+m^2\varepsilon^2}+\delta$.

Step 5: estimating approximation error without stationary points.

Proof of Theorem 3 modulo some lemmas. Derive the asymptotic formula for $a_1(x, t+\varepsilon, m, \varepsilon)$; the derivation for $a_2(x+\varepsilon, t+\varepsilon, m, \varepsilon)$ is analogous and is discussed at the end of the proof. By Proposition 12 and the identity $e^{i\omega_{p+\pi/\varepsilon}t-i(p+\pi/\varepsilon)x} = e^{i(\pi/\varepsilon-\omega_p)t-ipx-i\pi x/\varepsilon} = -e^{-i\omega_p t-ipx}$ for $(t+x)/\varepsilon$ odd, we get

$$a_1(x,t+\varepsilon,m,\varepsilon) = \frac{m\varepsilon^2}{2\pi i}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{e^{i\omega_p t-ipx}}{\sqrt{m^2\varepsilon^2+\sin(p\varepsilon)}}\,dp = \int_{-\pi/2\varepsilon}^{\pi/2\varepsilon} g(p)(e(f_+(p)-e(f_-(p)))\,dp, \quad (56)$$

where $e(z) := e^{2\pi i z}$ and

$$f_\pm(p) = \frac{1}{2\pi}\left(-px \pm \frac{t}{\varepsilon}\arccos\frac{\cos(p\varepsilon)}{\sqrt{1+m^2\varepsilon^2}}\right), \quad (57)$$

$$g(p) = \frac{m\varepsilon^2}{2\pi i\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}}. \quad (58)$$

Case (A): $|x|/t < 1/\sqrt{1+m^2\varepsilon^2} - \delta$.

Step 1. We estimate oscillatory integral (56) using the following known technical result.

Lemma 9 (Weighted stationary phase integral)**.** [20, Lemma 5.5.6] *Let $f(p)$ be a real function, four times continuously differentiable for $\alpha \le p \le \beta$, and let $g(p)$ be a real function, three times continuously differentiable for $\alpha \le p \le \beta$. Suppose that there are positive parameters M, N, T, U, with*

$$M \ge \beta - \alpha, \qquad N \ge M/\sqrt{T},$$

and positive constants C_r such that, for $\alpha \le p \le \beta$,

$$\left|f^{(r)}(p)\right| \le C_r T/M^r, \qquad \left|g^{(s)}(p)\right| \le C_s U/N^s,$$

for $r = 2, 3, 4$ and $s = 0, 1, 2, 3$, and

$$f''(p) \ge T/C_2 M^2.$$

Suppose also that $f'(p)$ changes sign from negative to positive at a point $p = \gamma$ with $\alpha < \gamma < \beta$. If T is sufficiently large in terms of the constants C_r, then we have

$$\int_\alpha^\beta g(p)e(f(p))\,dp = \frac{g(\gamma)e(f(\gamma)+1/8)}{\sqrt{f''(\gamma)}} + \frac{g(\beta)e(f(\beta))}{2\pi i f'(\beta)} - \frac{g(\alpha)e(f(\alpha))}{2\pi i f'(\alpha)}$$
$$+ O_{C_0,\dots,C_4}\left(\frac{M^4 U}{T^2}\left(1+\frac{M}{N}\right)^2\left(\frac{1}{(\gamma-\alpha)^3}+\frac{1}{(\beta-\gamma)^3}+\frac{\sqrt{T}}{M^3}\right)\right). \quad (59)$$

Here the first term involving the values at the stationary point γ is the main term, and the boundary terms involving the values at the endpoints α and β are going to cancel out in Step 3 because of the periodicity.

Lemma 10. (Cf. [22, (25)], [2, §4]) *Assume (11); then on $[-\frac{\pi}{2\varepsilon}, \frac{\pi}{2\varepsilon}]$, the function $f_\pm(p)$ given by (57) has a unique critical point*

$$\gamma_\pm = \pm\frac{1}{\varepsilon}\arcsin\frac{m\varepsilon x}{\sqrt{t^2-x^2}}. \quad (60)$$

To estimate integral (56), we are going to apply Lemma 9 twice, for the functions $f(p) = \pm f_\pm(p)$ in appropriate neighborhoods of their critical points $\gamma_\pm$. In the case of $f(p) = -f_-(p)$, we perform complex conjugation of both sides of (59). Then the total contribution of the two resulting main terms is

$$\mathrm{MainTerm} := \frac{g(\gamma_+)e(f_+(\gamma_+)+1/8)}{\sqrt{f_+''(\gamma_+)}} - \frac{g(\gamma_-)e(f_-(\gamma_-)-1/8)}{\sqrt{-f_-''(\gamma_-)}}. \quad (61)$$

A direct but long computation (see [40, §2]) then gives the desired main term in the theorem:

Lemma 11. (See [40, §2]) *Assume* (11), (14), (57)–(58), (60)*; then expression* (61) *equals*

$$\mathrm{MainTerm} = \varepsilon\sqrt{\frac{2m}{\pi}}\left(t^2-(1+m^2\varepsilon^2)x^2\right)^{-1/4}\sin\theta(x,t,m,\varepsilon).$$

Step 2. To estimate the approximation error, we need to specify the particular values of parameters which Lemma 9 is applied for:

$$M = N = m, \qquad T = mt, \qquad U = \varepsilon. \tag{62}$$

Lemma 12. *If* $\varepsilon \le 1/m$ *then functions* (57)–(58) *and parameters* (62) *satisfy the inequalities*

$$\left|f_\pm^{(r)}(p)\right| \le 3T/M^r, \qquad \left|g^{(s)}(p)\right| \le 3U/N^s \qquad \text{for } p\in\mathbb{R},\ \ r=2,3,4,\ \ s=0,1,2,3.$$

We also need to specify the interval

$$[\alpha_\pm,\beta_\pm] := [\gamma_\pm - m\delta/2, \gamma_\pm + m\delta/2]. \tag{63}$$

To estimate the derivative $|f''_\pm(p)|$ from below, we make sure that we are apart its roots $\pm\pi/2\varepsilon$.

Lemma 13. *Assume* (11), (60)*; then interval* (63) *is contained in* $[-\pi/2\varepsilon+m\delta/2, \pi/2\varepsilon-m\delta/2]$.

The wise choice of the interval provides the following more technical estimate.

Lemma 14. *Assume* (11), (57), (60), *and* (63). *Then for each* $p\in[\alpha_\pm,\beta_\pm]$ *we have*

$$|f''_\pm(p)| \ge \frac{t\delta^{3/2}}{24\pi m}.$$

This gives $|f''_\pm(p)| \ge T/C_2M^2$ for $C_2 := 24\pi\delta^{-3/2}$ under notation (62). Now all the assumptions of Lemma 9 have been verified ($M\ge\beta_\pm-\alpha_\pm$ and $N\ge M/\sqrt{T}$ are automatic because $\delta\le 1$ and $t > C_\delta/m$ by (11)). Apply the lemma to $g(p)$ and $\pm f_\pm(p)$ on $[\alpha_\pm,\beta_\pm]$ (the minus sign before $f_-(p)$ guarantees the inequality $f''(p)>0$ and the factor of i inside $g(p)$ is irrelevant for application of the lemma). We arrive at the following estimate for the approximation error on those intervals.

Lemma 15. (See [40, §4]) *Parameters* (60) *and* (62)–(63) *satisfy*

$$\frac{M^4U}{T^2}\left(1+\frac{M}{N}\right)^2\left(\frac{1}{(\gamma_\pm-\alpha_\pm)^3}+\frac{1}{(\beta_\pm-\gamma_\pm)^3}+\frac{\sqrt{T}}{M^3}\right) = O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right).$$

Although that is only a part of the error, it is already of the same order as in the theorem.

Step 3. To estimate the approximation error outside $[\alpha_\pm,\beta_\pm]$, we use another known technical result.

Lemma 16 (Weighted first derivative test)**.** [20, Lemma 5.5.5] *Let* $f(p)$ *be a real function, three times continuously differentiable for* $\alpha\le p\le\beta$, *and let* $g(p)$ *be a real function, twice continuously differentiable for* $\alpha\le p\le\beta$. *Suppose that there are positive parameters* M,N,T,U, *with* $M\ge\beta-\alpha$, *and positive constants* C_r *such that, for* $\alpha\le p\le\beta$,

$$\left|f^{(r)}(p)\right| \le C_rT/M^r, \qquad \left|g^{(s)}(p)\right| \le C_sU/N^s,$$

for $r=2,3$ *and* $s=0,1,2$. *If* $f'(p)$ *and* $f''(p)$ *do not change sign on the interval* $[\alpha,\beta]$, *then*

$$\int_\alpha^\beta g(p)e(f(p))\,dp = \frac{g(\beta)e(f(\beta))}{2\pi i f'(\beta)} - \frac{g(\alpha)e(f(\alpha))}{2\pi i f'(\alpha)} + O_{C_0,\dots,C_3}\left(\frac{TU}{M^2}\left(1+\frac{M}{N}+\frac{M^3\min|f'(p)|}{N^2T}\right)\frac{1}{\min|f'(p)|^3}\right).$$

This lemma in particular requires the interval to be sufficiently small. By this reason we decompose the initial interval $[-\pi/2\varepsilon, \pi/2\varepsilon]$ into a large number of intervals by the points

$$-\frac{\pi}{2\varepsilon} = \alpha_{-K} < \beta_{-K} = \alpha_{-K+1} < \beta_{-K+1} = \alpha_{-K+2} < \dots = \alpha_i < \hat{\beta}_i = \alpha_\pm < \beta_\pm = \hat{\alpha}_j < \beta_j = \dots < \beta_{K-1} = \frac{\pi}{2\varepsilon}.$$

Here $\alpha_\pm$ and $\beta_\pm$ are given by (63) above. The other points are given by

$$\alpha_k = \frac{k\pi}{2\varepsilon K}, \quad \beta_k = \frac{(k+1)\pi}{2\varepsilon K}, \quad \text{where} \quad K = 2\left\lceil \frac{\pi}{m\varepsilon} \right\rceil \quad \text{and} \quad k = -K, \dots, i, j+1, \dots, K-1. \tag{64}$$

The indices i and j are the minimal ones such that $\frac{(i+1)\pi}{2\varepsilon K} > \alpha_\pm$ and $\frac{(j+1)\pi}{2\varepsilon K} > \beta_\pm$. Thus all the resulting intervals except $[\alpha_\pm, \beta_\pm]$ and its neighbors have the same length $\frac{\pi}{2\varepsilon K}$. (Although it is more conceptual to decompose using a geometric sequence rather than arithmetic one, this does not affect the final estimate here.)

We have already applied Lemma 9 to $[\alpha_\pm, \beta_\pm]$ and we are going to apply Lemma 16 to each of the remaining intervals in the decomposition for $f(p) = f_\pm(p)$ (this time it is not necessary to change the sign of $f_-(p)$). After summation of the resulting estimates, all the terms involving the values $f_\pm(\alpha_k)$ and $f_\pm(\beta_k)$ at the endpoints, except the leftmost and the rightmost ones, are going to cancel out. The remaining boundary terms give

$$\text{BoundaryTerm} := \frac{g(\frac{\pi}{2\varepsilon})e(f_+(\frac{\pi}{2\varepsilon}))}{2\pi i f'_+(\frac{\pi}{2\varepsilon})} - \frac{g(-\frac{\pi}{2\varepsilon})e(f_+(-\frac{\pi}{2\varepsilon}))}{2\pi i f'_+(-\frac{\pi}{2\varepsilon})} - \frac{g(\frac{\pi}{2\varepsilon})e(f_-(\frac{\pi}{2\varepsilon}))}{2\pi i f'_-(\frac{\pi}{2\varepsilon})} + \frac{g(-\frac{\pi}{2\varepsilon})e(f_-(-\frac{\pi}{2\varepsilon}))}{2\pi i f'_-(-\frac{\pi}{2\varepsilon})}. \tag{65}$$

Lemma 17. (See [40, §5]) *For $(x,t) \in \varepsilon\mathbb{Z}^2$ such that $(x+t)/\varepsilon$ is odd, expression* (65) *vanishes.*

It remains to estimate the error terms. We start estimates with the central intervals $[\alpha_0, \beta_0]$ and $[\alpha_{-1}, \beta_{-1}]$ (possibly without parts cut out by $[\alpha_\pm, \beta_\pm]$); they require a special treatment. Apply Lemma 16 to the intervals for the same functions (57)–(58) and the same values (62) of the parameters M, N, T, U as in Step 2. All the assumptions of the lemma have been already verified in Lemma 12; we have $\beta_0 - \alpha_0 \le \pi/2\varepsilon K = \pi/4\varepsilon\lceil\frac{\pi}{m\varepsilon}\rceil < m = M$ and $f''_\pm(p) \ne 0$ as well. We are thus left to estimate $|f'_\pm(p)|$ from below.

Lemma 18. *Assume* (11), (57), (60), (63)*; then for each $p \in [-\pi/2\varepsilon, \pi/2\varepsilon] \setminus [\alpha_\pm, \beta_\pm]$ we get*

$$|f'_\pm(p)| \ge t\delta^{5/2}/48\pi.$$

Then the approximation error on the central intervals is estimated as follows.

Lemma 19. (See [40, §6]) *Assume* (11), (60), *and* (63). *Then parameters* (62) *and functions* (57) *satisfy*

$$\frac{TU}{M^2}\left(1 + \frac{M}{N} + \frac{M^3 \min_{p\in[\alpha_\pm,\beta_\pm]} |f'_\pm(p)|}{N^2 T}\right) \frac{1}{\min_{p\in[\alpha_\pm,\beta_\pm]} |f'_\pm(p)|^3} = O\left(\frac{\varepsilon}{mt^2\delta^{15/2}}\right).$$

This value is $O_\delta\left(\varepsilon/m^{1/2}t^{3/2}\right)$ because $t > C_\delta/m$ by (11). Hence the approximation error on the central intervals is within the remainder of the theorem.

Step 4. To estimate the approximation error on the other intervals $[\alpha_k, \beta_k]$, where we assume that $k > 0$ without loss of generality, we apply Lemma 16 with slightly different parameters:

$$T = mt/k, \qquad M = mk, \qquad U = \varepsilon/k, \qquad N = mk. \tag{66}$$

Lemma 20. *For $0 < k < K$ and $\varepsilon \le 1/m$, parameters* (66) *and* (64)*, functions* (57)–(58) *on $[\alpha_k, \beta_k]$ satisfy all the assumptions of Lemma 16 possibly except the one on the sign of $f'(p)$.*

Since the neighborhood $[\alpha_\pm, \beta_\pm]$ of the root of $f'(p)$ is cut out, it follows that $f'(p)$ has constant sign on the remaining intervals, and by Lemma 16 their contribution to the error is estimated as follows.

Lemma 21. (See [40, §7]) *Assume* (11), (60), (63), $0 < k < K$. *Then functions* (57) *and parameters* (66) *satisfy*

$$\frac{TU}{M^2}\left(1+\frac{M}{N}+\frac{M^3\min_{p\notin[\alpha_\pm,\beta_\pm]}|f'_\pm(p)|}{N^2T}\right)\frac{1}{\min_{p\notin[\alpha_\pm,\beta_\pm]}|f'_\pm(p)|^3}=O\left(\frac{\varepsilon}{k^2mt^2\delta^{15/2}}\right).$$

Summation over all k gives the approximation error

$$\sum_{k=1}^{K}O\left(\frac{\varepsilon}{k^2mt^2\delta^{15/2}}\right)=O\left(\frac{\varepsilon}{mt^2\delta^{15/2}}\sum_{k=1}^{\infty}\frac{1}{k^2}\right)=O_\delta\left(\frac{\varepsilon}{m^{1/2}t^{3/2}}\right).$$

because the series inside big-O converges and $t > C_\delta/m$. Thus the total approximation error on all the intervals is within the remainder of the theorem, which completes the proof of Case (**A**) (namely, of (12)).

Case (B): $|x|/t > 1/\sqrt{1+m^2\varepsilon^2}+\delta$.

Step 5. The argument is analogous to Steps 3–4, but much simpler because there are no stationary points to worry about, and the main term vanishes. We decompose $[-\pi/2\varepsilon,-\pi/2\varepsilon]$ into $2\lceil\pi/m\varepsilon\rceil$ equal intervals $[\alpha_k,\beta_k]$. We apply Lemma 16 to each of them for the same functions (57)–(58) and the same values (62) and (66) of the parameters M,N,T,U as in Steps 3–4; parameters are chosen depending on if $[\alpha_k,\beta_k]$ contains the origin. All the assumptions of the lemma but one have been already verified in Lemmas 12 and 20. We are thus left to verify that $|f'_\pm(p)|$ has no zeroes and estimate it from below.

Lemma 22. *If* $|x|/t \ge 1/\sqrt{1+m^2\varepsilon^2}+\delta$ *then for each* $p\in\mathbb{R}$ *we have* $|f'_\pm(p)|\ge t\delta/2\pi>0$.

Lemma 23. (See [40, §8]) *If* $|x|/t \ge 1/\sqrt{1+m^2\varepsilon^2}+\delta$ *then parameters* (66) *and functions* (57) *satisfy*

$$\frac{TU}{M^2}\left(1+\frac{M}{N}+\frac{M^3\min|f'_\pm(p)|}{N^2T}\right)\frac{1}{\min|f'_\pm(p)|^3}=O\left(\frac{\varepsilon}{k^2mt^2\delta^3}\right).$$

Summation over all the intervals completes the proof of Case (**B**).

The derivation of the asymptotic formula for $a_2(x+\varepsilon,t+\varepsilon,m,\varepsilon)$ is analogous. By Proposition 12 for $(x+t)/\varepsilon$ even we get

$$\begin{aligned}a_2(x+\varepsilon,t+\varepsilon,m,\varepsilon)&=\frac{\varepsilon}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\left(1+\frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}}\right)e^{i\omega_pt-ipx}\,dp\\&=\int_{-\pi/2\varepsilon}^{\pi/2\varepsilon}\left[g_+(p)e(f_+(p))+g_-(p)e(f_-(p))\right]dp,\end{aligned}\tag{67}$$

where $f_\pm(p)$ are the same as above (see (57)) and

$$g_\pm(p)=\frac{\varepsilon}{2\pi}\left(1\pm\frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}}\right).\tag{68}$$

One repeats the argument of Steps 1–5 with $g(p)$ replaced by $g_\pm(p)$. The particular form of $g(p)$ was only used in Lemmas 11, 12, 17, 20. The analogues of Lemmas 11 and 17 for $g_\pm(p)$ are obtained by direct checking [40, §13]. Lemma 12 holds for $g_\pm(p)$: one needs not to repeat the proof because $g_\pm(p)=(\varepsilon/t)(f'_\pm(p)+(x+t)/2\pi)$ [40, §1]. But parameters (66) and Lemma 20 should be replaced by the following ones (then the analogues of Lemmas 21 and 23 hold):

$$T=mt/k,\qquad M=mk,\qquad U=\varepsilon,\qquad N=mk^{3/2}.\tag{69}$$

Lemma 24. *For* $0<k<K$ *and* $\varepsilon\le 1/m$, *parameters* (69) *and* (64), *functions* (57) *and* (68) *on* $[\alpha_k,\beta_k]$ *satisfy all the assumptions of Lemma 16 possibly except the one on the sign of* $f'(p)$.

This concludes the proof of Theorem 3 modulo the lemmas. □

Now we prove the lemmas. Lemmas 11, 15, 17, 19, 23, 21 are proved by direct checking [40]. The following expressions [40, §1,3] are used frequently in the proofs of the other lemmas:

$$f'_\pm(p) = \frac{1}{2\pi}\left(-x \pm \frac{t\sin p\varepsilon}{\sqrt{m^2\varepsilon^2+\sin^2 p\varepsilon}}\right); \tag{70}$$

$$f''_\pm(p) = \pm\frac{m^2\varepsilon^3 t\cos(p\varepsilon)}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{3/2}} \tag{71}$$

Proof of Lemma 10. Using (70) and solving the quadratic equation $f'_\pm(p) = 0$ in $\sin p\varepsilon$, we get (60). The assumption $|x|/t < 1/\sqrt{1+m^2\varepsilon^2}$ from (11) guarantees that the arcsine exists. □

Proof of Lemma 12. By the computation of the derivatives in [40, §3] and the assumption $m\varepsilon \le 1$ we get

$$|g(p)| = \frac{m\varepsilon^2}{2\pi\sqrt{m^2\varepsilon^2+\sin^2(p\varepsilon)}} \le \frac{m\varepsilon^2}{2\pi\sqrt{m^2\varepsilon^2+0}} \le \varepsilon = U,$$

$$|g^{(1)}(p)| = \frac{m\varepsilon^3|\sin(p\varepsilon)\cos(p\varepsilon)|}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{3/2}} \le \frac{m\varepsilon^3|\sin(p\varepsilon)\cos(p\varepsilon)|}{2\pi\left(m^2\varepsilon^2+0\right)\left(0+\sin^2(p\varepsilon)\right)^{1/2}} \le \frac{\varepsilon}{m} = \frac{U}{N},$$

$$|g^{(2)}(p)| = \frac{m\varepsilon^4\left|m^2\varepsilon^2+\sin^4(p\varepsilon)-2(1+m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{5/2}} \le \frac{m\varepsilon^4\left(3m^2\varepsilon^2+3\sin^2(p\varepsilon)\right)}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{5/2}} \le \frac{\varepsilon}{m^2} = \frac{U}{N^2},$$

$$|g^{(3)}(p)| = \frac{m\varepsilon^5|\sin(p\varepsilon)\cos(p\varepsilon)|\cdot\left|4m^4\varepsilon^4+9m^2\varepsilon^2+\sin^4(p\varepsilon)-(6+10m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{7/2}} \le \frac{3\varepsilon}{m^3} = \frac{3U}{N^3},$$

$$|f^{(2)}_\pm(p)| = \frac{m^2\varepsilon^3 t\cos(p\varepsilon)}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{3/2}} \le \frac{t}{m} = \frac{T}{M^2},$$

$$|f^{(3)}_\pm(p)| = \frac{m^2\varepsilon^4 t|\sin(p\varepsilon)|\left(m^2\varepsilon^2+\cos(2p\varepsilon)+2\right)}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{5/2}} \le \frac{4m^2\varepsilon^4 t|\sin(p\varepsilon)|}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{5/2}} \le \frac{t}{m^2} = \frac{T}{M^3},$$

$$|f^{(4)}_\pm(p)| = \frac{m^2\varepsilon^5 t\cos(p\varepsilon)\left|m^4\varepsilon^4+3m^2\varepsilon^2+4\sin^4(p\varepsilon)-2(6+5m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2+\sin^2(p\varepsilon)\right)^{7/2}} \le \frac{3t}{m^3} = \frac{3T}{M^4}.$$

□

Proof of Lemma 13. The lemma follows from the sequence of estimates

$$\begin{aligned}\frac{\pi}{2\varepsilon} - |\gamma_\pm| &= \frac{\sin(\pi/2)-\sin|\gamma_\pm\varepsilon|}{\varepsilon\cos(\theta\varepsilon)} \ge \frac{\sin(\pi/2)-\sin|\gamma_\pm\varepsilon|}{\varepsilon\cos(\gamma_\pm\varepsilon)} = \frac{1-m\varepsilon|x|/\sqrt{t^2-x^2}}{\varepsilon\sqrt{1-m^2\varepsilon^2x^2/(t^2-x^2)}}\\ &= \frac{\sqrt{1-x^2/t^2}-m\varepsilon|x|/t}{\varepsilon\sqrt{1-(1+m^2\varepsilon^2)x^2/t^2}} \ge \frac{1}{\varepsilon}\left(\sqrt{1-\frac{x^2}{t^2}}-\frac{m\varepsilon|x|}{t}\right)\\ &\ge \frac{1}{\varepsilon}\left(\sqrt{1-\frac{1}{1+m^2\varepsilon^2}}-\frac{m\varepsilon|x|}{t}\right) = m\left(\frac{1}{\sqrt{1+m^2\varepsilon^2}}-\frac{|x|}{t}\right) \ge m\delta.\end{aligned}$$

Here the first equality holds for some $\theta \in [|\gamma_\pm|, \pi/2\varepsilon]$ by the Lagrange theorem. The next inequality holds because the cosine is decreasing on the interval. The next one is obtained by substituting (60). The rest is straightforward because $|x|/t < 1/\sqrt{1+m^2\varepsilon^2} - \delta$ by (11). □

Proof of Lemma 14. Let us prove the lemma for $f_+(p)$ and $\gamma_+ \ge 0$; the other signs are considered analogously. Omit the index $+$ in the notation of $f_+, \alpha_+, \beta_+, \gamma_+$. The lemma follows from

$$\begin{aligned}|f^{(2)}(p)| &\overset{(*)}{\ge} |f^{(2)}(\beta)| = \frac{m^2\varepsilon^3 t\cos(\beta\varepsilon)}{2\pi\left(m^2\varepsilon^2+\sin^2(\beta\varepsilon)\right)^{3/2}} \overset{(**)}{\ge} \frac{m^2\varepsilon^3 t\cos(\gamma\varepsilon)}{4\pi\left(m^2\varepsilon^2+\sin^2(\gamma\varepsilon)+2m^2\varepsilon^2t^2/(t^2-x^2)\right)^{3/2}}\\ &\overset{(***)}{=} \frac{m^2\varepsilon^3 t\sqrt{t^2-(1+m^2\varepsilon^2)x^2(t^2-x^2)}}{4\pi\left(3m^2\varepsilon^2t^2\right)^{3/2}} \ge \frac{t\sqrt{1-(1+m^2\varepsilon^2)x^2/t^2(1-x^2/t^2)}}{24\pi m} \ge \frac{t\delta^{3/2}}{24\pi m}.\end{aligned}$$

Here inequality $(*)$ is proved as follows. By (71), $f^{(2)}(p)$ is increasing on $[-\pi/2\varepsilon, 0]$ and decreasing on $[0, \pi/2\varepsilon]$, because it is even, the numerator is decreasing on $[0, \pi/2\varepsilon]$ and the denominator is increasing on $[0, \pi/2\varepsilon]$. Thus $|f^{(2)}(p)| \geq \min\{|f^{(2)}(\beta)|, |f^{(2)}(\alpha)|\}$ for $p \in [\alpha, \beta]$ by Lemma 13. But since $f^{(2)}(p)$ is even and $\gamma \geq 0$, by (63) we get

$$|f^{(2)}(\alpha)| = |f^{(2)}(\gamma - m\delta/2)| = |f^{(2)}(|\gamma - m\delta/2|)| \geq |f^{(2)}(\gamma + m\delta/2)| = |f^{(2)}(\beta)|.$$

Inequality $(**)$ follows from the following two estimates. First, by Lemma 13 and the convexity of the cosine on the interval $[\gamma\varepsilon, \pi/2]$ we obtain

$$\cos(\beta\varepsilon) \geq \cos\left(\frac{\gamma\varepsilon}{2} + \frac{\pi}{4}\right) \geq \frac{1}{2}\left(\cos(\gamma\varepsilon) + \cos\frac{\pi}{2}\right) = \frac{\cos(\gamma\varepsilon)}{2}.$$

Second, using the inequality $\sin z - \sin w \leq z - w$ for $0 \leq w \leq z \leq \pi/2$, then $\delta \leq 1$ and (60)–(63), we get

$$\begin{aligned}\sin^2(\beta\varepsilon) - \sin^2(\gamma\varepsilon) &\leq \varepsilon(\beta-\gamma)\left(\sin(\beta\varepsilon) + \sin(\gamma\varepsilon)\right) \leq \varepsilon(\beta-\gamma)\left(\varepsilon(\beta-\gamma) + 2\sin(\gamma\varepsilon)\right) \\ &= \frac{m\varepsilon\delta}{2}\left(\frac{m\varepsilon\delta}{2} + \frac{2m\varepsilon x}{\sqrt{t^2-x^2}}\right) \leq \frac{m\varepsilon t}{2\sqrt{t^2-x^2}}\left(\frac{2m\varepsilon t}{\sqrt{t^2-x^2}} + \frac{2m\varepsilon t}{\sqrt{t^2-x^2}}\right) = \frac{2m^2\varepsilon^2 t^2}{t^2-x^2}.\end{aligned}$$

Equality $(***)$ is obtained from (60). The remaining estimates are straightforward. □

Proof of Lemma 18. By Lemmas 10 and 14, for $p \in [\beta_+, \pi/2\varepsilon]$ we have

$$f_+'(p) = f_+'(\gamma_+) + \int_{\gamma_+}^{p} f_+''(p)\,dp \geq 0 + (p - \gamma_+)\frac{t\delta^{3/2}}{24\pi m} \geq (\beta_+ - \gamma_+)\frac{t\delta^{3/2}}{24\pi m} = \frac{t\delta^{5/2}}{48\pi}$$

because $f_+''(p) \geq 0$ by (71). For $p \in [-\pi/2\varepsilon, \alpha_+]$ and for $f_-'(p)$ the proof is analogous. □

Proof of Lemma 20. Take $p \in [\alpha_k, \beta_k]$. By (64), the inequalities $\sin z \geq z/2$ for $z \in [0, \pi/2]$, and $m\varepsilon \leq 1$ we get

$$\begin{aligned}
|g(p)| &= \frac{m\varepsilon^2}{2\pi\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}} \leq \frac{m\varepsilon^2}{2\pi\sin(p\varepsilon)} \leq \frac{m\varepsilon^2}{\pi p\varepsilon} \leq \frac{2m\varepsilon^2 K}{\pi^2 k} = \frac{4m\varepsilon^2}{\pi^2 k}\left\lceil\frac{\pi}{m\varepsilon}\right\rceil = O\left(\frac{\varepsilon}{k}\right) = O\left(U\right),\\
|g^{(1)}(p)| &= \frac{m\varepsilon^3|\sin(p\varepsilon)\cos(p\varepsilon)|}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{3/2}} \leq \frac{m\varepsilon^3}{2\pi\sin^2(p\varepsilon)} = O\left(\frac{\varepsilon}{mk^2}\right) = O\left(\frac{U}{N}\right),\\
|g^{(2)}(p)| &= \frac{m\varepsilon^4\left|m^2\varepsilon^2 + \sin^4(p\varepsilon) - 2(1+m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{5/2}} \leq \frac{m\varepsilon^4\left(3m^2\varepsilon^2 + 3\sin^2(p\varepsilon)\right)}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{5/2}} = O\left(\frac{U}{N^2}\right),\\
|g^{(3)}(p)| &= \frac{m\varepsilon^5|\sin(p\varepsilon)\cos(p\varepsilon)|\cdot\left|4m^4\varepsilon^4 + 9m^2\varepsilon^2 + \sin^4(p\varepsilon) - (6+10m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{7/2}} = O\left(\frac{U}{N^3}\right),\\
|f_\pm^{(2)}(p)| &= \frac{m^2\varepsilon^3 t\cos(p\varepsilon)}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{3/2}} = O\left(\frac{t}{mk^3}\right) = O\left(\frac{T}{M^2}\right),\\
|f_\pm^{(3)}(p)| &= \frac{m^2\varepsilon^4 t|\sin(p\varepsilon)|\left(m^2\varepsilon^2 + \cos(2p\varepsilon) + 2\right)}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{5/2}} = O\left(\frac{t}{m^2k^4}\right) = O\left(\frac{T}{M^3}\right),\\
|f_\pm^{(4)}(p)| &= \frac{m^2\varepsilon^5 t\cos(p\varepsilon)\left|m^4\varepsilon^4 + 3m^2\varepsilon^2 + 4\sin^4(p\varepsilon) - 2(6+5m^2\varepsilon^2)\sin^2(p\varepsilon)\right|}{2\pi\left(m^2\varepsilon^2 + \sin^2(p\varepsilon)\right)^{7/2}} = O\left(\frac{t}{m^3k^5}\right) = O\left(\frac{T}{M^4}\right).
\end{aligned}$$

Further, $f_\pm''(p)$ does not change sign on the interval $[\alpha_k, \beta_k]$ because it vanishes only at $\pm\pi/2\varepsilon$. We also have $\beta_k - \alpha_k \leq \pi/2\varepsilon K = \pi/4\varepsilon\lceil\frac{\pi}{m\varepsilon}\rceil < m \leq mk = M$. □

Proof of Lemma 22. By (71) it follows that $f_\pm'(p)$ has extrema $p = \pi/2\varepsilon + \pi k/\varepsilon$, where $k \in \mathbb{Z}$. Thus by (70) and the assumption $|x|/t \geq 1/\sqrt{1+m^2\varepsilon^2} + \delta$, we get

$$|f_\pm'(p)| \geq \min_k\left|f_\pm'\left(\frac{\pi}{2\varepsilon} + \frac{\pi k}{\varepsilon}\right)\right| = \frac{1}{2\pi}\left(|x| - \frac{t}{\sqrt{1+m^2\varepsilon^2}}\right) \geq \frac{t\delta}{2\pi}.$$

□

Proof of Lemma 24. The assumptions on $f_\pm(p)$ have been checked in the proof of Lemma 20. The assumptions on $g_\pm^{(s)}(p)$ for $s = 1, 2, 3$ and $p \in [\alpha_k, \beta_k]$ follow from the ones on $f_\pm(p)$:

$$|g_\pm(p)| = \frac{\varepsilon}{2\pi}\left(1 \pm \frac{\sin(p\varepsilon)}{\sqrt{m^2\varepsilon^2 + \sin^2(p\varepsilon)}}\right) \le \varepsilon = U,$$
$$|g_\pm^{(s)}(p)| = \left|\frac{\varepsilon}{t} f_\pm^{(s+1)}(p)\right| = O\left(\frac{\varepsilon}{m^s k^{s+2}}\right) = O\left(\frac{\varepsilon}{m^s k^{3s/2}}\right) = O\left(\frac{U}{N^s}\right). \qquad \square$$

12.5 Large-time limit: the stationary phase method again (Corollaries 1–2)

In this section we prove Corollaries 1–2. First we outline the plan of the argument, then prove Corollary 1 modulo a technical lemma, then the lemma itself, and finally Corollary 2.

The plan of the proof of Corollary 1 (and results such as Problems 5–6) consists of 3 steps:

Step 1: computing the main contribution to the sum, using asymptotic formulae (12)–(13);

Step 2: estimating the contribution coming from a trigonometric sum;

Step 3: estimating the error coming from replacing sum by an integral;

Step 4: estimating the contribution coming from outside of the interval where (12)–(13) hold.

Proof of Corollary 1 modulo some lemmas. **Step 1.** Fix $m, \varepsilon, \delta > 0$, denote $n := 1 + m^2\varepsilon^2$, $V := 1/\sqrt{n} - \delta$, and fix $-V \le v \le V$. Let us prove that if t is sufficiently large in terms of δ, m, ε, then

$$\sum_{\substack{-Vt<x\le vt\\ x\in\varepsilon\mathbb{Z}}} P(x,t,m,\varepsilon) = F(v) - F(-V) + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right). \tag{72}$$

This follows from the sequence of asymptotic formulae:

$$\sum_{\substack{-V(t+\varepsilon)<x\le v(t+\varepsilon)\\ x\in\varepsilon\mathbb{Z}}} a_1^2(x,t+\varepsilon,m,\varepsilon) \overset{(*)}{=} \sum_{\substack{-V(t+\varepsilon)<x\le v(t+\varepsilon)\\ (x+t)/\varepsilon \text{ odd}}} \left(\frac{2m\varepsilon^2}{\pi t}\left(1-\frac{nx^2}{t^2}\right)^{-1/2} \sin^2\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon^2}{t^2}\right)\right)$$
$$\overset{(**)}{=} \sum_{\substack{-Vt<x<vt\\ (x+t)/\varepsilon \text{ odd}}} \frac{m\varepsilon^2}{\pi t}\left(1-\frac{nx^2}{t^2}\right)^{-1/2} - \sum_{\substack{-Vt<x<vt\\ (x+t)/\varepsilon \text{ odd}}} \frac{m\varepsilon^2}{\pi t}\left(1-\frac{nx^2}{t^2}\right)^{-1/2} \cos 2\theta(x,t,m,\varepsilon) + O_\delta\left(\frac{\varepsilon}{t}\right)$$
$$\overset{(***)}{=} \sum_{\substack{-Vt<x<vt\\ (x+t)/\varepsilon \text{ odd}}} \frac{m\varepsilon\cdot 2\varepsilon/t}{2\pi\sqrt{1-nx^2/t^2}} + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right) \overset{(****)}{=} \int_{-V}^{v} \frac{m\varepsilon\, dv}{2\pi\sqrt{1-nv^2}} + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right)$$
$$= m\varepsilon\frac{\arcsin(\sqrt{n}v) - \arcsin(-\sqrt{n}V)}{2\pi\sqrt{n}} + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right) \tag{73}$$

and an analogous asymptotic formula

$$\sum_{\substack{-V(t+\varepsilon)<x\le v(t+\varepsilon)\\ x\in\varepsilon\mathbb{Z}}} a_2^2(x,t+\varepsilon,m,\varepsilon) = \int_{-V}^{v} \frac{m\varepsilon(1+v)\,dv}{2\pi(1-v)\sqrt{1-nv^2}} + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right)$$
$$= F(v) - F(-V) - m\varepsilon\frac{\arcsin(\sqrt{n}v) - \arcsin(-\sqrt{n}V)}{2\pi\sqrt{n}} + O_{\delta,m,\varepsilon}\left(\sqrt{\frac{\varepsilon}{t}}\right).$$

Here $(*)$ follows from Theorem 3 because $|x|/t \le |v|(t+\varepsilon)/t < V + \delta/2 = 1/\sqrt{n} - \delta/2$ for large enough t; the product of the main term and the error term in (12) is estimated by ε^2/t^2.

Asymptotic formula $(**)$ holds because the number of summands is less than t/ε and the (possibly) dropped first and last summands are less than $m\varepsilon^2/t\sqrt{\delta}$.

Step 2. Let us prove formula $(***)$. We use the following simplified version of the stationary phase method.

Lemma 25. [25, Corollary from Theorem 4 in p. 17] *Under the assumptions of Lemma 9 (except the ones on $f'(p)$, $g^{(3)}(p)$, and $N \geq M/\sqrt{T}$), if $M = N$ and $M/C \leq T \leq CM^2$ for some $C > 0$, then*

$$\sum_{\alpha<p<\beta} g(p)e(f(p)) = O_{C,C_0,\ldots,C_4}\left(\frac{(\beta-\alpha)U\sqrt{T}}{M} + \frac{UM}{\sqrt{T}}\right).$$

For notational convenience, assume that t/ε is odd; otherwise the proof is analogous. Then the summation index $x = 2p\varepsilon$ for some integer p. We apply Lemma 25 for the functions

$$f_\pm(p) = \pm\frac{1}{\pi}\theta(2p\varepsilon, t, m, \varepsilon) \qquad \text{and} \qquad g(p) = \frac{m\varepsilon^2}{\pi t}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-1/2} \tag{74}$$

and the parameter values

$$M = N = T = t/\varepsilon, \qquad U = \varepsilon/t, \qquad \alpha = -Vt/2\varepsilon, \qquad \beta = vt/2\varepsilon. \tag{75}$$

Lemma 26. *For $\varepsilon \leq 1/m$ there exist $C, C_0, \ldots, C_4$ depending on δ, m, ε but not v, p such that parameters* (75) *and functions* (74) *satisfy all the assumptions of Lemma 25.*

Since parameters (75) satisfy $\frac{(\beta-\alpha)U\sqrt{T}}{M} + \frac{UM}{\sqrt{T}} = O\left(\sqrt{\frac{\varepsilon}{t}}\right)$, formula $(***)$ follows.

Step 3. Let us prove formula (****). We use yet another known result.

Lemma 27 (Euler summation formula)**.** [25, Remark to Theorem 1 in p. 3] *If $g(p)$ is continuously differentiable on $[\alpha, \beta]$ and $\rho(p) := 1/2 - \{p\}$, then*

$$\sum_{\alpha<p<\beta} g(p) = \int_\alpha^\beta g(p)\,dp + \rho(\beta)g(\beta) - \rho(\alpha)g(\alpha) + \int_\alpha^\beta \rho(p)g'(p)\,dp.$$

Again assume without loss of generality that t/ε is odd. Apply Lemma 27 to the same $\alpha, \beta, g(p)$ (given by (74)–(75)) as in Step 2. By Lemma 26 we have $g(p) = O_{\delta,m,\varepsilon}(\varepsilon/t)$ and $g'(p) = O_{\delta,m,\varepsilon}(\varepsilon^2/t^2)$. Hence by Lemma 27 the difference between the sum and the integral in (****) is $O_{\delta,m,\varepsilon}(\varepsilon/t)$, and (72) follows.

Step 4. Let us prove the corollary for arbitrary $v \in (-1/\sqrt{n}; 1/\sqrt{n})$. By (72), for each δ, m, ε there are $C_1(\delta, m, \varepsilon)$ and $C_2(\delta, m, \varepsilon)$ such that for each $v \in [-1/\sqrt{n} + \delta, 1/\sqrt{n} - \delta]$ and each $t \geq C_1(\delta, m, \varepsilon)$ we have

$$\left|\sum_{(-1/\sqrt{n}+\delta)t<x\leq vt} P(x, t, m, \varepsilon) - F(v)\right| \leq F\left(-\frac{1}{\sqrt{n}} + \delta\right) + C_2(\delta, m, \varepsilon)\sqrt{\frac{\varepsilon}{t}}.$$

Clearly, we may assume that $C_1(\delta, m, \varepsilon)$ and $C_2(\delta, m, \varepsilon)$ are decreasing functions in δ: the larger is the interval $[-\frac{1}{\sqrt{n}} + \delta, \frac{1}{\sqrt{n}} - \delta]$, the weaker is our error estimate in (*)–(****). Take $\delta(t)$ tending to 0 slowly enough so that $C_1(\delta(t), m, \varepsilon) \leq t$ for t sufficiently large in terms of m, ε and $C_2(\delta(t), m, \varepsilon)\sqrt{\frac{\varepsilon}{t}} \to 0$ as $t \to \infty$. Denote $V(t) := \frac{1}{\sqrt{n}} - \delta(t)$. Then since $F\left(-\frac{1}{\sqrt{n}} + \delta\right) \to F\left(-\frac{1}{\sqrt{n}}\right) = 0$ as $\delta \to 0$ by the definition of $F(v)$, it follows that

$$\sum_{-V(t)t<x\leq vt} P(x, t, m, \varepsilon) \rightrightarrows F(v) \qquad \text{as} \qquad t \to \infty \tag{76}$$

uniformly in $v \in (-\frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}})$. Similarly, since $F\left(\frac{1}{\sqrt{n}} - \delta\right) \to F\left(\frac{1}{\sqrt{n}}\right) = 1$ as $\delta \to 0$, we get

$$\sum_{-V(t)t<x\le V(t)t} P(x,t,m,\varepsilon) \to 1 \qquad \text{as} \qquad t \to \infty.$$

Then by Proposition 6 we get

$$\sum_{x\le -V(t)t} P(x,t,m,\varepsilon) = 1 - \sum_{x>-V(t)t} P(x,t,m,\varepsilon) \le 1 - \sum_{-V(t)t<x\le V(t)t} P(x,t,m,\varepsilon) \to 0.$$

With (76), this implies the corollary for $v \in (-1/\sqrt{n}; 1/\sqrt{n})$. For $v \le -1/\sqrt{n}$ and similarly for $v \ge 1/\sqrt{n}$, the corollary follows from $\sum_{x\le vt} P(x,t,m,\varepsilon) \le \sum_{x\le -V(t)t} P(x,t,m,\varepsilon) \to 0$. □

Now we prove the lemma and the remaining corollary.

Proof of Lemma 26. The inequalities $M/C \le T \le CM^2$ and $M \ge \beta - \alpha$ are automatic for $C = 1$ because t/ε is a positive integer and $|V|, |v| \le 1$. We estimate the derivatives (computed in [40, §9]) as follows, using the assumption $\varepsilon \le 1/m$, $\alpha \le p \le \beta$, and setting $C_2 := \max\{1/m\varepsilon, 2/\delta^{3/2}\}$:

$$\begin{aligned}
|g(p)| &= \frac{m\varepsilon^2}{\pi t}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-1/2} \le \frac{m\varepsilon^2}{\pi t\sqrt{1-nV^2}} \le \frac{m\varepsilon^2}{t\sqrt{\delta}} \le \frac{\varepsilon}{t\sqrt{\delta}} = O_\delta\left(U\right),\\
|g^{(1)}(p)| &= \frac{4m\varepsilon^4 n|p|}{\pi t^3}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-3/2} \le \frac{2m\varepsilon^3 nVt}{\pi t^3(1-nV^2)^{3/2}} \le \frac{m\varepsilon^3 n}{t^2\delta^{3/2}} \le \frac{2\varepsilon^2}{t^2\delta^{3/2}} = O_\delta\left(\frac{U}{N}\right),\\
|g^{(2)}(p)| &= \frac{4m\varepsilon^4 n(8\varepsilon^2np^2+t^2)}{\pi t^5}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-5/2} \le \frac{4m\varepsilon^4 n(2nV^2+1)t^2}{\pi t^5(1-nV^2)^{5/2}} = O_\delta\left(\frac{\varepsilon^3}{t^3}\right) = O_\delta\left(\frac{U}{N^2}\right),\\
|f^{(2)}(p)| &= \frac{4m\varepsilon^2}{\pi t}\left(1 - \frac{4\varepsilon^2p^2}{t^2}\right)^{-1}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-1/2} \ge \frac{m\varepsilon^2}{t} \ge \frac{T}{C_2M^2},\\
|f^{(2)}(p)| &= \frac{4m\varepsilon^2}{\pi t}\left(1 - \frac{4\varepsilon^2p^2}{t^2}\right)^{-1}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-1/2} \le \frac{4m\varepsilon^2}{\pi t\left(1-V^2\right)\sqrt{1-nV^2}} \le \frac{2m\varepsilon^2}{t\delta^{3/2}} \le \frac{C_2T}{M^2},\\
|f^{(3)}(p)| &= \frac{16m\varepsilon^4\left|(n+2)pt^2 - 12n\varepsilon^2p^3\right|}{\pi t^5}\left(1 - \frac{4\varepsilon^2p^2}{t^2}\right)^{-2}\left(1 - \frac{4n\varepsilon^2p^2}{t^2}\right)^{-3/2} = O_\delta\left(\frac{T}{M^3}\right),\\
|f^{(4)}(p)| &= \frac{16m\varepsilon^4\left|768n^2\varepsilon^6p^6 - 48n(2n+5)\varepsilon^4p^4t^2 + 8(n^2-n+3)\varepsilon^2p^2t^4 + (n+2)t^6\right|}{\pi t^9\left(1-4\varepsilon^2p^2/t^2\right)^3\left(1-4n\varepsilon^2p^2/t^2\right)^{5/2}} = O_\delta\left(\frac{T}{M^4}\right).
\end{aligned}$$

□

Proof of Corollary 2. We have $n_+(h\times w) - n_-(h\times w) = -2^{(w+h-1)/2}a_1(w-h, w+h)$ by the obvious bijection between the Young diagrams with exactly h rows and w columns, and checker paths from $(0,0)$ to $(w-h, w+h)$ passing through $(1,1)$ and $(w-h+1, w+h-1)$. Set $h := \lceil rw \rceil$. Apply Theorem 2 for

$$\delta = \tfrac{1}{2}\left|\tfrac{1}{\sqrt{2}} - \tfrac{r-1}{r+1}\right|, \quad m = \varepsilon = 1, \quad x = w-h, \quad t = w+h-1.$$

It remains to show that for $r < 3+2\sqrt{2}$ the value (14) is not bounded from $\frac{\pi}{2} + \pi\mathbb{Z}$ as $w \to \infty$.

Denote $v := \frac{h-w}{w+h-1}$ and $v_0 := \frac{r-1}{r+1}$. Write

$$\theta(vt,t,1,1) = t\left(\arcsin\frac{1}{\sqrt{2-2v^2}} - v\arcsin\frac{v}{\sqrt{1-v^2}}\right) + \frac{\pi}{4} =: t\theta(v) + \frac{\pi}{4}.$$

Since $\theta(v) \in C^2[0; 1/\sqrt{2}-\delta]$, by the Taylor expansion it follows that

$$\theta(vt,t,1,1) = \frac{\pi}{4} + t\theta(v_0) + t(v-v_0)\theta'(v_0) + O_\delta\left(t(v-v_0)^2\right).$$

Substituting

$$v - v_0 = \frac{h-w}{w+h-1} - \frac{r-1}{r+1} = \frac{2h-2rw+r-1}{(r+1)(w+h-1)} = \frac{2\{-rw\}+r-1}{(r+1)t},$$

where $h = \lceil rw \rceil = rw + \{-rw\}$, we get

$$\theta(vt,t,1,1) = \frac{\pi}{4} + (w+rw+\{-rw\}-1)\theta(v_0) + \frac{2\{-rw\}+r-1}{(r+1)}\theta'(v_0) + O_\delta\left(\frac{1}{w}\right)$$
$$= \frac{\pi}{4} - \theta(v_0) + v_0\theta'(v_0) + w(r+1)\theta(v_0) + \{-rw\}\left(\theta(v_0) + \tfrac{2}{(r+1)}\theta'(v_0)\right) + O_\delta\left(\tfrac{1}{w}\right)$$
$$=: \pi(\alpha(r)w + \beta(r)\{-rw\} + \gamma(r)) + O_\delta\left(\tfrac{1}{w}\right).$$

For almost every r, the numbers $1, r, \alpha(r)$ are linearly independent over the rational numbers because the graph of the function $\alpha(r) = (r+1)\theta\left(\frac{r-1}{r+1}\right)$ has a countable number of intersection points with rational lines. Hence by the Kronecker theorem for each $\Delta > 0$ there are infinitely many w such that

$$\{-rw\} < \Delta \qquad \text{and} \qquad \left|\{\alpha(r)w\} + \gamma(r) - \tfrac{1}{2}\right| < \Delta.$$

By (12), the corollary follows because those w satisfy

$$|\sin\theta(vt,t,1,1)| = 1 + O\left((1+\beta(r))\Delta\right) + O_\delta\left(\tfrac{1}{w}\right) \qquad \text{and} \qquad 2^{(r+1)w/2} \le 2^{(w+h)/2} \le 2^{(r+1)w/2+\Delta}.$$

□

12.6 The Feynman problem: Taylor expansions (Corollaries 3–4)

Here we deduce the solution of the Feynman problem from Theorem 3 by Taylor expansions.

Proof of Corollary 3. First derive an asymptotic formula for the function $\theta(x,t,m,\varepsilon)$ given by (14). Denote $n := 1 + m^2\varepsilon^2$. Since $1/\sqrt{1+z^2} = 1 + O(z^2)$, $\arcsin z = z + O(z^3)$ for $z \in [-1;1]$, and $t/\sqrt{t^2-x^2} < 1/\sqrt{1-\sqrt{n}x/t} < 1/\sqrt{\delta}$, we get

$$\arcsin\frac{m\varepsilon t}{\sqrt{n(t^2-x^2)}} = \frac{m\varepsilon t}{\sqrt{1+m^2\varepsilon^2}\sqrt{(t^2-x^2)}} + O\left(\frac{m^3\varepsilon^3}{n^{3/2}}\left(\frac{t}{\sqrt{t^2-x^2}}\right)^3\right) = \frac{m\varepsilon t}{\sqrt{t^2-x^2}} + O_\delta\left(m^3\varepsilon^3\right).$$

Combining with a similar asymptotic formula for $\arcsin\frac{m\varepsilon x}{\sqrt{t^2-x^2}}$, we get

$$\theta(x,t,m,\varepsilon) = \frac{mt^2}{\sqrt{t^2-x^2}} - \frac{mx^2}{\sqrt{t^2-x^2}} + \frac{\pi}{4} + \left(\frac{t+|x|}{\varepsilon}\right) O_\delta\left(m^3\varepsilon^3\right) = m\sqrt{t^2-x^2} + \frac{\pi}{4} + O_\delta\left(m^3\varepsilon^2 t\right).$$

Since

$$\left|\frac{\partial\sqrt{t^2-x^2}}{\partial t}\right| = \frac{t}{\sqrt{t^2-x^2}} < \frac{1}{\sqrt{\delta}} \qquad \text{and} \qquad \left|\frac{\partial\sqrt{t^2-x^2}}{\partial x}\right| = \frac{|x|}{\sqrt{t^2-x^2}} < \frac{1}{\sqrt{\delta}},$$

by the Lagrange theorem it follows that

$$\theta(x,t-\varepsilon,m,\varepsilon) = m\sqrt{t^2-x^2} + \frac{\pi}{4} + O_\delta\left(m\varepsilon + m^3\varepsilon^2 t\right),$$
$$\theta(x-\varepsilon,t-\varepsilon,m,\varepsilon) = m\sqrt{t^2-x^2} + \frac{\pi}{4} + O_\delta\left(m\varepsilon + m^3\varepsilon^2 t\right).$$

Consider the remaining factors in (12)–(13). By the Lagrange theorem, for some $\eta \in [0, nx^2/t^2]$ we get

$$\left(1-\frac{nx^2}{t^2}\right)^{-1/4} - 1 = \frac{nx^2}{t^2}\frac{(1-\eta)^{-5/4}}{4} \le \frac{nx^2}{t^2}\left(1-\frac{nx^2}{t^2}\right)^{-5/4}$$
$$\le \frac{x^2}{t^2}\left(\frac{1}{\sqrt{n}}-\frac{x}{t}\right)^{-5/4}\left(\frac{1}{\sqrt{n}}+\frac{x}{t}\right)^{-5/4} \le \frac{x^2}{t^2}\delta^{-5/2} = O_\delta\left(\frac{|x|}{t}\right).$$

Hence for $t \geq 2\varepsilon$ we get

$$\left(1 - \frac{nx^2}{(t-\varepsilon)^2}\right)^{-1/4} = 1 + O_\delta\left(\frac{|x|}{t}\right) \qquad \text{and} \qquad \left(1 - \frac{n(x-\varepsilon)^2}{(t-\varepsilon)^2}\right)^{-1/4} = 1 + O_\delta\left(\frac{|x|+\varepsilon}{t}\right).$$

We also have

$$\sqrt{\frac{t-\varepsilon+x-\varepsilon}{t-x}} = \sqrt{1+2\frac{x-\varepsilon}{t-x}} = 1 + O\left(\frac{x-\varepsilon}{t-x}\right) = 1 + O_\delta\left(\frac{|x|+\varepsilon}{t}\right).$$

Substituting all the resulting asymptotic formulae into (12)–(13), we get

$$\operatorname{Re} a\,(x,t,m,\varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi t}}\left(\sin\left(m\sqrt{t^2-x^2}+\frac{\pi}{4}\right) + O_\delta\left(\frac{1}{mt} + \frac{|x|+\varepsilon}{t} + m\varepsilon + m^3\varepsilon^2 t\right)\right),$$
$$\operatorname{Im} a\,(x,t,m,\varepsilon) = \varepsilon\sqrt{\frac{2m}{\pi t}}\left(\cos\left(m\sqrt{t^2-x^2}+\frac{\pi}{4}\right) + O_\delta\left(\frac{1}{mt} + \frac{|x|+\varepsilon}{t} + m\varepsilon + m^3\varepsilon^2 t\right)\right).$$

Since $m\varepsilon \leq \frac{1}{mt} + m^3\varepsilon^2 t$ and $\frac{\varepsilon}{t} \leq \frac{1}{mt}$ by the assumption $\varepsilon \leq 1/m$, the corollary follows. □

Proof of Corollary 4. This follows directly from Corollary 3 by plugging in the Taylor expansion

$$\sqrt{t^2-x^2} = t\left(1 - \frac{x^2}{2t^2} + O_\delta\left(\frac{x^4}{t^4}\right)\right) \qquad \text{for} \qquad \frac{|x|}{t} < 1-\delta.$$

□

12.7 Continuum limit: the tail-exchange method (Theorem 4 and Corollaries 5–6)

Proof of Theorem 4. The proof is based on the *tail-exchange method* and consists of 5 steps:

Step 1: dropping the normalization factor in (9)–(10), which is of order 1.

Step 2: dropping the summands in (9)–(10) starting from a number T (we take $T = \lceil \log \frac{\delta}{\varepsilon} \rceil$).

Step 3: replacing the binomial coefficients by powers in each of the remaining summands.

Step 4: replacing the resulting sum by infinite power series.

Step 5: combining the error bounds in the previous steps to get the total approximation error.

Let us derive the asymptotic formula for $a_1(x,t,m,\varepsilon)$; the argument for $a_2(x,t,m,\varepsilon)$ is analogous.

Step 1. Consider the 1st factor in (9). We have $0 \geq (1-t/\varepsilon)/2 \geq -t/\varepsilon$ because $t \geq \delta \geq \varepsilon$. Exponentiating, we get

$$1 \geq \left(1+m^2\varepsilon^2\right)^{(1-t/\varepsilon)/2} \geq \left(1+m^2\varepsilon^2\right)^{-t/\varepsilon} \geq e^{-m^2\varepsilon^2 t/\varepsilon} \geq 1 - m^2 t\varepsilon,$$

where in the latter two inequalities we used that $e^a \geq 1+a$ for each $a \in \mathbb{R}$. Thus

$$\left(1+m^2\varepsilon^2\right)^{(1-t/\varepsilon)/2} = 1 + O(m^2 t\varepsilon).$$

Step 2. Consider the T-th partial sum in (9) with $T = \lceil \log \frac{\delta}{\varepsilon} \rceil$ summands. The total number of summands is indeed at least T because $(t-|x|)/2\varepsilon \geq \delta/2\varepsilon \geq \log(\delta/\varepsilon)$ by the inequalities $t-|x| \geq \delta > \varepsilon$ and $e^a \geq 1+a+a^2/2 \geq 2a$ for each $a \geq 0$.

For $r \geq T$ the ratio of consecutive summands in (9) equals

$$(m\varepsilon)^2\,\frac{((t+x)/2\varepsilon-1-r)((t-x)/2\varepsilon-1-r)}{(r+1)^2} < (m\varepsilon)^2 \cdot \frac{(t+x)}{2\varepsilon T} \cdot \frac{(t-x)}{2\varepsilon T} = \frac{m^2 s^2}{4T^2} < \frac{1}{2},$$

where the latter inequality follows from $T = \lceil \log \frac{\delta}{\varepsilon} \rceil > \lceil \log e^{3ms} \rceil \geq 3ms$. Therefore, the error term (i.e., the sum over $r \geq T$) is less then the sum of geometric series with ratio $\frac{1}{2}$. Thus by Proposition 11 we get

$$a_1(x,t,m,\varepsilon) = m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot\left[\sum_{r=0}^{T-1}(-1)^r\binom{(t+x)/2\varepsilon-1}{r}\binom{(t-x)/2\varepsilon-1}{r}(m\varepsilon)^{2r}+\right.$$
$$\left.+O\left(\binom{(t+x)/2\varepsilon-1}{T}\binom{(t-x)/2\varepsilon-1}{T}(m\varepsilon)^{2T}\right)\right].$$

Step 3. To approximate the sum, take integers $L := (t\pm x)/2\varepsilon$, $r < T$, and transform binomial coefficients as follows:

$$\binom{L-1}{r} = \frac{(L-1)\cdots(L-r)}{r!} = \frac{L^r}{r!}\left(1-\frac{1}{L}\right)\cdots\left(1-\frac{r}{L}\right).$$

Here

$$\frac{r}{L} = \frac{2r\varepsilon}{t\pm x} < \frac{2T\varepsilon}{\delta} = \frac{2\varepsilon}{\delta}\left\lceil\log\frac{\delta}{\varepsilon}\right\rceil \leq \frac{2\varepsilon}{\delta}\left(\log\frac{\delta}{\varepsilon}+1\right) < \frac{1}{2},$$

because $\delta/\varepsilon \geq 16$, and $2(\log a+1)/a$ decreases for $a \geq 16$ and is less than $1/2$ for $a = 16$. Applying the inequality $1-a \geq e^{-2a}$ for $0 \leq a \leq 1/2$, then the inequalities $1-a \leq e^{-a}$ and $L \geq \delta/2\varepsilon$, we get

$$\left(1-\frac{1}{L}\right)\cdots\left(1-\frac{r}{L}\right) \geq e^{-2/L}e^{-4/L}\cdots e^{-2r/L} = e^{-r(r+1)/L} \geq e^{-T^2/L} \geq 1-\frac{T^2}{L} \geq 1-\frac{2T^2\varepsilon}{\delta}.$$

Therefore,

$$\frac{(t\pm x)^r}{r!(2\varepsilon)^r} \geq \binom{(t\pm x)/2\varepsilon-1}{r} \geq \frac{(t\pm x)^r}{r!(2\varepsilon)^r}\left(1-\frac{2T^2\varepsilon}{\delta}\right).$$

Inserting the result into the expression for $a_1(x,t,m,\varepsilon)$ from Step 2, we get

$$a_1(x,t,m,\varepsilon) = m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot$$
$$\cdot\left[\sum_{r=0}^{T-1}(-1)^r\left(\frac{m}{2}\right)^{2r}\frac{(t^2-x^2)^r}{(r!)^2}\left(1+O\left(\frac{T^2\varepsilon}{\delta}\right)\right)+O\left(\left(\frac{m}{2}\right)^{2T}\frac{(t^2-x^2)^T}{(T!)^2}\left(1+\frac{T^2\varepsilon}{\delta}\right)\right)\right].$$

The latter error term in the formula is estimated as follows. Since $T! \geq (T/3)^T$ and

$$T \geq \log\frac{\delta}{\varepsilon} \geq 3m\sqrt{t^2-x^2} \geq \frac{3m}{2}\sqrt{t^2-x^2}\sqrt{e},$$

it follows that

$$\frac{(t^2-x^2)^T}{(T!)^2}\cdot\left(\frac{m}{2}\right)^{2T} \leq \frac{(t^2-x^2)^T}{(T)^{2T}}\cdot\left(\frac{3m}{2}\right)^{2T} \leq e^{-T} \leq \frac{\varepsilon}{\delta}.$$

We have $(\varepsilon/\delta)\left(1+T^2\varepsilon/\delta\right) = O\left(T^2\varepsilon/\delta\right)$ because $T \geq 1$ and $\varepsilon < \delta$. Thus the error term in question can be absorbed into the 0-th summand $O\left(T^2\varepsilon/\delta\right)$. We get

$$a_1(x,t,m,\varepsilon) = m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot\sum_{r=0}^{T-1}(-1)^r\left(\frac{m}{2}\right)^{2r}\frac{(t^2-x^2)^r}{(r!)^2}\left(1+O\left(\frac{T^2\varepsilon}{\delta}\right)\right).$$

Notice that by our notational convention the constant understood in $O\left(T^2\varepsilon/\delta\right)$ does not depend on r.

Step 4. Now we can replace the sum with T summands by an infinite sum because the "tail" of alternating series with decreasing absolute value of the summands can be estimated by the first summand (which has just been estimated):

$$\left|\sum_{r=T}^{\infty}(-1)^r\left(\frac{m}{2}\right)^{2r}\frac{(t^2-x^2)^r}{(r!)^2}\right| \leq \frac{(t^2-x^2)^T}{(T!)^2}\cdot\left(\frac{m}{2}\right)^{2T} \leq \frac{\varepsilon}{\delta} = O\left(\frac{T^2\varepsilon}{\delta}\right).$$

Since the constant understood in each summand $O\left(\frac{T^2\varepsilon}{\delta}\right)$ is the same (see Step 3), we get

$$\begin{aligned} a_1(x,t,m,\varepsilon) &= m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot\sum_{r=0}^{\infty}(-1)^r\left(\frac{m}{2}\right)^{2r}\frac{(t^2-x^2)^r}{(r!)^2}\left[1+O\left(\frac{T^2\varepsilon}{\delta}\right)\right] \\ &= m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot\left(J_0(ms)+O\left(\frac{T^2\varepsilon}{\delta}I_0(ms)\right)\right), \end{aligned}$$

where we use the *modified Bessel functions of the first kind*:

$$I_0(z):=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2}, \qquad I_1(z):=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+1}}{k!(k+1)!}.$$

Step 5. We have $m^2t\delta\le m^2(t+|x|)(t-|x|)=m^2s^2\le 9m^2s^2\le T^2$. Thus $m^2t\varepsilon\,J_0(ms)\le T^2\varepsilon\,I_0(ms)/\delta$ and $m^2t\varepsilon\le T^2\varepsilon/\delta<(\log(\delta/\varepsilon)+1)^2\varepsilon/\delta<2$ because $(a+1)^2/2<e^a$ for $a\ge 0$. We arrive at the formula

$$a_1(x,t,m,\varepsilon)=m\varepsilon\left(J_0(ms)+O\left(\frac{\varepsilon}{\delta}\log^2\frac{\delta}{\varepsilon}\,I_0(ms)\right)\right).$$

Analogously,

$$\begin{aligned} a_2(x,t,m,\varepsilon) &= m\varepsilon\left(1+O\left(m^2t\varepsilon\right)\right)\cdot\frac{t+x}{\sqrt{t^2-x^2}}\cdot\sum_{r=1}^{T-1}(-1)^r\left(\frac{m}{2}\right)^{2r-1}\frac{(t^2-x^2)^{\frac{2r-1}{2}}}{(r-1)!r!}\left[1+O\left(\frac{T^2\varepsilon}{\delta}\right)\right] \\ &= -m\varepsilon\cdot\frac{t+x}{s}\left(J_1(ms)+O\left(\frac{\varepsilon}{\delta}\log^2\frac{\delta}{\varepsilon}\,I_1(ms)\right)\right). \end{aligned}$$

This gives the required asymptotic formula for $a(x,t,m,\varepsilon)$ because

$$I_0(ms)\le\sum_{k=0}^{\infty}\frac{(ms/2)^{2k}}{k!}=e^{m^2s^2/4}\le e^{m^2t^2},$$

$$\frac{t+x}{s}I_1(ms)\le\frac{t+x}{s}\cdot\frac{ms}{2}\sum_{k=0}^{\infty}\frac{(ms/2)^{2k}}{k!}=m\frac{t+x}{2}\,e^{m^2s^2/4}\le mt\,e^{m^2t^2/4}\le e^{m^2t^2/2}e^{m^2t^2/4}\le e^{m^2t^2}.$$ ☐

Proof of Corollary 5. This follows from Theorem 4 because the right-hand side of (18) is uniformly continuous on each compact subset of the angle $|x|<t$. ☐

Proof of Corollary 6. Since the right-hand side of (18) is continuous on $[-t+\delta;t-\delta]$, it is bounded there. Since a sequence uniformly converging to a bounded function is uniformly bounded, by Corollary 5 the absolute value of the left-hand side of (18) is less than some constant $C_{t,m,\delta}$ depending on t,m,δ but not on x,ε. Then by Proposition 6 for $t/2\varepsilon\in\mathbb{Z}$ we get

$$\begin{aligned} 1-\sum_{x\in\varepsilon\mathbb{Z}:|x|\ge t-\delta}P(x,t,m,\varepsilon) &= \sum_{x\in\varepsilon\mathbb{Z}:|x|<t-\delta}P(x,t,m,\varepsilon)=\sum_{x\in\varepsilon\mathbb{Z}:|x|<t-\delta}4\varepsilon^2\left|\frac{1}{2\varepsilon}a(x,t,m,\varepsilon)\right|^2 \\ &< 4\varepsilon^2C^2_{t,m,\delta}\frac{t-\delta}{\varepsilon}\to 0\quad\text{as }\varepsilon\to 0. \end{aligned}$$ ☐

12.8 Probability of chirality flip: combinatorial identities (Theorem 5)

Although Theorem 5 can be deduced from (73), we give a direct proof relying on §12.1 only.

Proof of Theorem 5. Denote $S_1(t)=\sum_x a_1^2(x,t)$; $S_2(t)=\sum_x a_2^2(x,t)$; $S_{12}(t)=\sum_x a_1(x,t)a_2(x,t)$.
By Propositions 1, 8, and 9 we have

$$a_1(0,2t)=\frac{1}{\sqrt{2}}\sum_x a_1(x,t)(a_2(x,t)-a_1(x,t))+a_2(x,t)(a_2(x,t)+a_1(x,t))=\frac{1}{\sqrt{2}}(S_2(t)+2S_{12}(t)-S_1(t)).$$

By definition and Proposition 1 we have

$$S_1(t+1) - S_2(t+1) = 2S_{12}(t).$$

Hence,

$$S_1(t+1) - S_2(t+1) = S_1(t) - S_2(t) + a_1(0,2t)\sqrt{2}.$$

Since $S_1(t) + S_2(t) = 1$ by Proposition 2, we have the recurrence relation $S_1(t+1) = S_1(t) + \frac{1}{\sqrt{2}}a_1(0,2t)$; cf. [22, (33)]. Then Proposition 4 implies by induction that

$$S_1(t) = \frac{1}{2}\sum_{k=0}^{\lfloor t/2\rfloor -1} \frac{1}{(-4)^k}\binom{2k}{k}.$$

By the Newton binomial theorem we get $\sum_{k=0}^{\infty}\binom{2k}{k}x^k = \frac{1}{\sqrt{1-4x}}$ for each $x \in \left[-\frac{1}{4}, \frac{1}{4}\right)$. Setting $x = -\frac{1}{4}$ we obtain $\lim_{t\to\infty} \frac{1}{2}\sum_{k=0}^{\infty}\binom{2k}{k}\left(-\frac{1}{4}\right)^k = \frac{1}{2\sqrt{2}}$. Using the Stirling formula we estimate the convergence rate:

$$\left|\sum_{x\in\mathbb{Z}} a_1(x,t)^2 - \frac{1}{2\sqrt{2}}\right| < \frac{1}{2\cdot 4^{\lfloor t/2\rfloor}}\binom{2\lfloor t/2\rfloor}{\lfloor t/2\rfloor} < \frac{e}{2\pi\sqrt{2\lfloor t/2\rfloor}} < \frac{1}{2\sqrt{t}}. \qquad \square$$

Underwater rocks

Finally, let us warn a mathematically-oriented reader. The outstanding papers [2, 28, 29] are well-written, insomuch that the physical theorems and proofs there could be carelessly taken for mathematical ones, although some of them are wrong as written. The main source of those issues is actually application of a wrong theorem from a mathematical paper [9, Theorem 3.3].

A simple counterexample to [9, Theorem 3.3] is $a = b = \alpha = \beta = x = 0$ and n odd. Those values automatically satisfy the assumptions of the theorem, that is, condition (ii) of [9, Lemma 3.1]. Then by Remark 3 and Proposition 4, the left-hand side of [9, (2.16)] vanishes. Thus it cannot be equivalent to the nonvanishing sequence in the right-hand side. Here we interpret the "$\approx$" sign in [9, (2.16)] as the equivalence of sequences, following [28]. An attempt to interpret the sign so that the difference between the left- and the right-hand sides of [9, (2.16)] tends to zero would void the result because each of the sides clearly tends to zero separately. A similar counterexample ($n = -1$, t even) shows that [2, Theorem 2] is wrong as stated.

Although [2, 28, 29] report minor errors in [9], the issue is more serious. The known asymptotic formulae for Jacobi polynomials are never stated as an equivalence but rather contain an additive error term. Estimating the error term is hard even in particular cases studied in [30], and the case from Remark 3 is reported as more difficult [30, bottom p. 198]. Thus [9, Theorem 3.3] should be viewed as an interesting physical but not mathematical result.

Acknowledgements

This work has been presented as courses at Summer Conference of Tournament of Towns in Arandelovac, Summer School in Contemporary Mathematics in Dubna, Faculty of Mathematics in Higher School of Economics in Moscow, and Independent Moscow University in 2019. The authors are grateful to all participants of those events for their contribution, especially to I. Novikov, F. Ozhegov, A. Voropaev for numerous remarks and the Russian translation, to E. Akhmedova, R. Valieva for typesetting some parts of the text, to I. Bogdanov, A. Daniyarkhodzhaev, M. Fedorov, I. Gaidai-Turlov, T. Kovalev, F. Kuyanov, G. Minaev, I. Russkikh, V. Skopenkova for some of the figures, to A. Kudryavtsev, A. Lvov for writing appendices (these two authors were less than 16 years old that time). The authors are grateful to V. Akulin, M. Blank, A. Borodin, V. Buchstaber, G. Chelnokov, I. Dynnikov, I. Ibragimov, I. Ivanov, T. Jacobson, D.U. Kim, E. Kolpakov, A.B.J. Kuijlaars, S. Lando, S. Nechaev, S. Novikov, G. Olshanski, I. Polekhin, P. Pylyavskyy, A. Rybko, I. Sabitov, A. Semenov, L. Schulman, S. Shlosman, D. Treschev, A. Vershik for useful discussions.

A A. Kudryavtsev. Alternative "explicit" formulae

Set $\binom{n}{k} := 0$ for integers $k < 0 < n$ or $k > n > 0$. Denote $\theta(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

Proposition 18 ("Explicit" formula)**.** *For each integers $|x| < t$ such that $x+t$ is even we have:*

$$\textbf{(A)} \qquad a_1(x,t) = 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-2)^r \binom{(x+t-2)/2}{r} \binom{t-r-2}{(x+t-2)/2},$$

$$a_2(x,t) = 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-2)^r \binom{(x+t-2)/2}{r} \binom{t-r-2}{(x+t-4)/2};$$

$$\textbf{(B)} \qquad a_1(x,t) = 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-1)^r \binom{(t-|x|-2)/2}{r} \binom{|x|}{(t+|x|-4r-2)/2},$$

$$a_2(x,t) = 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-1)^r \binom{(t-|x|-2)/2}{r-\theta(x)} \binom{|x|}{(t+|x|-4r)/2}.$$

Proof of Proposition 18(A). Introduce the generating functions

$$\hat{a}_1(p,q) := 2^{n/2} \sum_{n>k\geq 0} a_1(2k-n+1, n+1) p^k q^n \quad \text{and} \quad \hat{a}_2(p,q) := 2^{n/2} \sum_{n>k\geq 0} a_2(2k-n+1, n+1) p^k q^n.$$

By Proposition 1 we get

$$\begin{cases} \hat{a}_1(p,q) - \hat{a}_1(p,0) = q \cdot (\hat{a}_2(p,q) + \hat{a}_1(p,q)); \\ \hat{a}_2(p,q) - \hat{a}_2(p,0) = pq \cdot (\hat{a}_2(p,q) - \hat{a}_1(p,q)). \end{cases}$$

Since $\hat{a}_1(p,0) = 0$ and $\hat{a}_2(p,0) = 1$, the solution of this system is

$$\hat{a}_2(p,q) = \frac{1-q}{1-q-pq+2pq^2}, \qquad \hat{a}_1(p,q) = \frac{q}{1-q-pq+2pq^2} = q + q^2(1+p-2pq) + q^3(1+p-2pq)^2 + \dots$$

The coefficient before $p^k q^n$ in $\hat{a}_1(p,q)$ equals

$$\sum_{j=\max(k,n-k)}^{n} (-2)^{n-j-1} \cdot \binom{j}{n-j-1 \quad k-n+j+1 \quad j-k},$$

because we must take exactly one combination of factors from every summand of the form $q^{j+1}(1+p-2pq)^j$:

- for the power of q to be equal to n, the number of factors $-2pq$ must be $n-j-1$;
- for the power of p to be equal to k, the number of factors p must be $k-(n-j-1)$;
- the number of remaining factors 1 must be $j-(k-(n-j-1))-(n-j-1) = j-k$.

Changing the summation variable to $r = n-j-1$, we arrive at the required formula for $a_1(x,t)$.

The formula for $a_2(x,t)$ follows from the one for $a_1(x,t)$, Proposition 1, and the Pascal rule:

$$\begin{aligned} a_2(x,t) &= \sqrt{2}\, a_1(x-1,t+1) - a_1(x,t) \\ &= 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-2)^r \binom{(x+t-2)/2}{r} \left(\binom{t-r-1}{(x+t-2)/2} - \binom{t-r-2}{(x+t-2)/2} \right) \\ &= 2^{(1-t)/2} \sum_{r=0}^{(t-|x|)/2} (-2)^r \binom{(x+t-2)/2}{r} \binom{t-r-2}{(x+t-4)/2}. \end{aligned}$$

□

Proof of Proposition 18(B). (by A. Voropaev) By Proposition 4, for each $|x|<t$ the numbers $a_1(x,t)$ and $a_2(x,t)$ are the coefficients before $z^{(t-x-2)/2}$ and $z^{(t-x)/2}$ respectively in the expansion of the polynomial

$$2^{(1-t)/2}(1+z)^{(t-x-2)/2}(1-z)^{(t+x-2)/2}=\begin{cases}2^{(1-t)/2}(1-z^2)^{\frac{t-x-2}{2}}(1-z)^x, & \text{for } x\ge 0;\\ 2^{(1-t)/2}(1-z^2)^{\frac{t+x-2}{2}}(1+z)^{-x}, & \text{for } x<0.\end{cases}$$

For $x<0$, this implies the required proposition immediately. For $x\ge 0$, we first change the summation variable to $r'=(t-x-2)/2-r$ or $r'=(t-x)/2-r$ for $a_1(x,t)$ and $a_2(x,t)$ respectively. □

B A. Lvov. Pointwise continuum limit

Theorem 7 (Pointwise continuum limit)**.** *For each real $m\ge 0$ and $|x|<t$ we have*

$$\lim_{n\to\infty} n\,a_1\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor,\frac{2}{n}\left\lfloor\frac{nt}{2}\right\rfloor,m,\frac{1}{n}\right)=m\,J_0(m\sqrt{t^2-x^2});$$
$$\lim_{n\to\infty} n\,a_2\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor,\frac{2}{n}\left\lfloor\frac{nt}{2}\right\rfloor,m,\frac{1}{n}\right)=-m\frac{x+t}{\sqrt{t^2-x^2}}J_1(m\sqrt{t^2-x^2}).$$

Proof of Theorem 7. Denote $A:=\lfloor\frac{nx}{2}\rfloor+\lfloor\frac{nt}{2}\rfloor$ and $B:=\lfloor\frac{nt}{2}\rfloor-\lfloor\frac{nx}{2}\rfloor$. The first limit is computed as follows:

$$n\,a_1\left(\frac{2}{n}\left\lfloor\frac{nx}{2}\right\rfloor,\frac{2}{n}\left\lfloor\frac{nt}{2}\right\rfloor,m,\frac{1}{n}\right)=n\left(1+\frac{m^2}{n^2}\right)^{\lfloor\frac{nt}{2}\rfloor-\frac12}\cdot\sum_{r=0}^{\infty}(-1)^r\binom{A-1}{r}\binom{B-1}{r}\left(\frac{m}{n}\right)^{2r+1}$$
$$\sim\sum_{r=0}^{\infty}(-1)^r\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}=\sum_{r=0;2|r}^{\infty}\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}-\sum_{r=0;2\nmid r}^{\infty}\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}$$
$$\to\sum_{r=0;2|r}^{\infty}\frac{(x+t)^r(t-x)^r m^{2r+1}}{2^{2r}(r!)^2}-\sum_{r=0;2\nmid r}^{\infty}\frac{(x+t)^r(t-x)^r m^{2r+1}}{2^{2r}(r!)^2}=m\,J_0(m\sqrt{t^2-x^2})\qquad\text{as } n\to\infty.$$

Here the equality in the 1st line is Proposition 11. The equivalence in the 2nd line follows from

$$1\le\left(1+\frac{m^2}{n^2}\right)^{\lfloor\frac{nt}{2}\rfloor-\frac12}\le\left(1+\frac{m^2}{n^2}\right)^{nt}=\sqrt[n]{\left(1+\frac{m^2}{n^2}\right)^{n^2t}}\sim\sqrt[n]{e^{m^2t}}\to 1\qquad\text{as } n\to\infty,$$

by the squeeze theorem. The equality in the 2nd line holds because $\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}=0$ for $r>\max\{A,B\}$, hence as all the three sums involved are finite. The convergence in the 3rd line is established in Lemmas 28–30 below. The second limit in the theorem is computed analogously. □

Lemma 28. *For each positive integer r we have $\lim\limits_{n\to\infty}\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}=\frac{(x+t)^r(t-x)^r m^{2r+1}}{2^{2r}(r!)^2}$.*

Proof. We have

$$\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}}=\frac{(A-1)\dots(A-r)\cdot(B-1)\dots(B-r)}{(r!)^2}\cdot\frac{m^{2r+1}}{n^{2r}}\to\left(\frac{x+t}{2}\right)^r\left(\frac{t-x}{2}\right)^r\frac{m^{2r+1}}{(r!)^2}$$

as $n\to\infty$ because for each $1\le i\le r$

$$\lim_{n\to\infty}\frac{A-i}{n}=\lim_{n\to\infty}\frac{A}{n}=\lim_{n\to\infty}\frac{\lfloor\frac{nx}{2}\rfloor+\lfloor\frac{nt}{2}\rfloor}{n}=\lim_{n\to\infty}\frac{\frac{nx}{2}+\frac{nt}{2}+o(n)}{n}=\frac{x+t}{2}$$

and analogously, $\lim\limits_{n\to\infty}\frac{B-i}{n}=\frac{t-x}{2}$. □

Lemma 29. *For each positive integer r we have* $\binom{A-1}{r}\binom{B-1}{r}\frac{m^{2r+1}}{n^{2r}} \leq \frac{(x+t)^r(t-x)^r m^{2r+1}}{2^{2r}(r!)^2}$.

Proof. This follows analogously because for each $1 \leq i \leq r < \min\{A, B\}$ we have

$$(A-i)(B-i) \leq \left(\left\lfloor \frac{nx}{2} \right\rfloor + \left\lfloor \frac{nt}{2} \right\rfloor - 1\right)\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nx}{2} \right\rfloor - 1\right) \leq \left(\frac{nx}{2} + \frac{nt}{2}\right)\left(\frac{nt}{2} - \frac{nx}{2}\right).$$

□

Lemma 30. *Suppose $(\{a_0(n)\}, \{a_1(n)\} \dots)$ is a sequence of nonnegative sequences such that $\lim\limits_{n\to\infty} a_k(n) = b_k$ for each k; $a_k(n) \leq b_k$ for each k, n; and $\sum\limits_{k=0}^{\infty} b_k$ is finite. Then $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_k(n) = \sum\limits_{k=0}^{\infty} b_k$.*

Proof. Denote $b := \sum\limits_{k=0}^{\infty} b_k$. Then for each n we have $\sum\limits_{k=0}^{\infty} a_k(n) \leq b$. Take any $\varepsilon > 0$. Take such N that $\sum\limits_{k=0}^{N} b_k > b - \varepsilon$. For each $0 \leq k \leq N$ take M_k such that for each $n \geq M_k$ we have $a_k(n) > b_k - \frac{\varepsilon}{2^{k+1}}$. Then for each $n > \max\{M_0, M_1, \dots, M_N\}$ we have $\sum\limits_{k=0}^{\infty} a_k(n) > b - 2\varepsilon$. Therefore, $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_k(n) = b$. □

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